On the Kleinian weight systems

Sergei Duzhin

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Abstract

Generalizing the construction of [Du], we introduce a family of weight systems in a sense dual to the family of Lie algebra weight systems. The basic component in our construction is a skew-symmetric function of three variables $f(x,y,z)$ that satisfies the following equation (we call it the *Klein equation*): $f(x,y,z)f(u,v,z) - f(x,u,z)f(y,v,z) + f(x,v,z)f(y,u,z) = 0$, which is the counterpart of the Jacobi identity for the structure tensor of a Lie algebra. We prove that analytic Kleinian functions lead to weight systems expressible through the classical Lie algebraic $\mathfrak{sl}_2$ weight system. Non-analytic Kleinian functions do exist, but the nature of the corresponding weight systems is yet unclear. Chemin faisant, we study the multivariate analog of the Klein equation and prove the criterion of decomposability of skew-symmetric functions in an arbitrary number of variables.

1 Weight systems

Let V_n be the space of Vassiliev invariants for framed oriented knots of degree no greater than n. According to the theorem of Vassiliev and Kontsevich, we have

$$V_n/V_{n-1} \cong \mathcal{A}_n^*,$$

where $\mathcal{A}_n^*$, the space of *weight systems of degree* n, is defined as dual to the space of diagrams

$$\mathcal{A}_n = \frac{\langle \text{diagrams of degree } n \rangle}{\langle \text{AS and IHX relations} \rangle}.$$

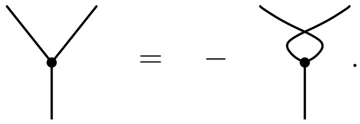
Here are the exact definitions.

Definition 1 *A (Jacobi) diagram (also called Chinese character, see [BN1]) is a 1-3-valent graph*[1] *with a fixed rotation. The rotation is the choice of a cyclic order of edges at every 3-valent vertex, i. e. one of the two cyclic permutations in the set of three edges adjacent to this vertex.*

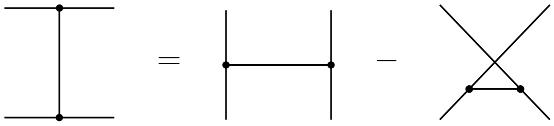
[1]Note that the number of vertices of such a graph is always even. Half of this number is referred to as the *degree* of a diagram.

Definition 2 *The space $\mathcal{A}_n$ is the quotient space of the linear space over $\mathbb{Q}$ generated by connected diagrams of degree n (i. e. having $2n$ vertices), modulo the following relations.*

AS *(antisymmetry) relation:*



IHX *relation:*



Definition 3 *A weight system of degree n is a function on the space $\mathcal{A}_n$ with values in some Abelian group. In other words, a weight system is a function of a diagram which satisfies the AS and IHX relations.*

A well known construction is Kontsevich's construction of the weight system with values in the universal enveloping algebra of a Lie algebra equipped with an ad-invariant non-degenerate symmetric bilinear form (see [Kon] for the original construction, [BN1] for the specialization to linear representations and [CD2, CDK] for more details). In this construction, one associates a copy of the structure tensor of the Lie algebra $\mathfrak{g}$ with every 3-valent vertex and then makes the contraction over all edges. The structure tensor, moved into the space $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ by means of the metric, is a totally antisymmetric element of this space, which ensures the compatibility with the AS relation. The IHX relation follows from the Jacobi identity.

2 Matiyasevich's polynomial

This construction, introduced in [Du], was inspired by [YuM].

Let $\mu : E(D) \to \{1, 2, \ldots, m\}$ be a numbering of the set of edges of a diagram D. We assign an independent variable x_i, $i = 1, \ldots, m$, to the edge number i and, with every vertex $v \in V(D)$, we associate the polynomial

$$(v_1 - v_2)(v_2 - v_3)(v_3 - v_1), \tag{1}$$

if v_1, v_2, v_3 are the variables assigned to the three edges meeting at v, taken in the order consistent with the rotation at v. Set

$$\mathrm{M}^\mu(D) = \prod_{v\in V(D)} (v_1 - v_2)(v_2 - v_3)(v_3 - v_1).$$

This is the *numbered* Matiyasevich's polynomial. To obtain an invariant object, symmetrize $\mathrm{M}^\mu(D)$ over all numberings μ, or over all permutations of x_1, ..., x_m:

$$\mathrm{M}(D) = \frac{1}{m!} \sum_{\sigma\in\mathcal{S}(m)} \prod_{v\in V(D)} (\sigma(v_1) - \sigma(v_2))(\sigma(v_2) - \sigma(v_3))(\sigma(v_3) - \sigma(v_1)). \tag{2}$$

Theorem 1 *The Matiyasevich polynomial* $\mathrm{M} : \mathcal{A}_n \to S\mathbb{Q}[x_1, \ldots, x_m]$ *is a weight system on the space* $\mathcal{A}_n$ *with values in the space of symmetric polynomials in* m *variables.*

Proof. The AS relation, as well as the correctness of the definition of M, follow from the fact that expression (1) is totally antisymmetric with respect to the permutations of v_1, v_2 and v_3. The IHX relation is a consequence of the following remarkable polynomial identity:

$$(a-b)(c-d) + (b-c)(a-d) + (c-a)(b-d) = 0. \tag{3}$$

Remark. The set of all edges of a diagram splits into two subsets: $E(D) = E_i(D) \cup E_o(D)$, where $E_i(D)$ is the set of all *inner* edges, connecting two 3-valent vertices, and $E_o(D)$ is the set of all outer edges, having one univalent vertex. In the construction of the Matiyasevich polynomial (eq. (2)), instead of symmetrizing over *all* permutations of the edges, we can symmetrize only over the subgroup that permutes inner edges separately and outer edges separately, thus arriving at the *modified Matiyasevich's polynomial* $\widetilde{\mathrm{M}}(D)$. The modified polynomial also satisfies the AS and IHX relations — the last fact follows from the observation that IHX relations never mix inner and outer vertices. Note that $\widetilde{\mathrm{M}}(D)$ is a stronger invariant of D than $\mathrm{M}(D)$, because $\mathrm{M}(D)$ is the image of $\widetilde{\mathrm{M}}(D)$ under a ring homomorphism.

Relation with the $so(3)$ *weight system.* In general, a weight system for 3-graphs can be constructed from any object which is skew-commutative in 3 variables. Both in the case of $so(3)$ and in the case of Matiyasevich, we assign a certain element of $\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ to every vertex of the graph, and this element is totally antisymmetric with respect to some action of the permutation group on 3 symbols. But the group actions in question are different:

- In the $so(3)$ case, the space $\mathbb{R}^3$ is identified with $so(3)$ and the group $\mathcal{S}_3$ acts in
$$\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$$
by permutations of the three factors of the tensor product.
- In Matiyasevich's case, $\mathbb{R}^3$ is the linear span of the formal variables v_1, v_2, v_3, while the group $\mathcal{S}_3$ acts in
$$\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$$
by the same permutation of the bases in each of the three factors.

3 Kleinian weight systems

We want to generalize the construction of Matiyasevich's weight system given in section 2, using an arbitrary function $F(v_1, v_2, v_3)$ instead of the polynomial (1).

What are the restrictions on the function F in order that the result of the construction be a weight system?

For the AS relation to be satisfied, F must be skew-symmetric:

$$F(x, y, z) = F(y, z, x) = -F(y, x, z). \tag{4}$$

The IHX relation is equivalent to the following identity:

$$F(x,y,z)F(u,v,z) - F(x,u,z)F(y,v,z) + F(x,v,z)F(y,u,z) = 0, \tag{5}$$

which we call the *Klein* equation.

Theorem 2 *Any function F of three variables satisfying the two relations 4 and 5, gives rise to weight systems K_F (symmetrization over the complete permutation group on the edges of a diagram) and $\widetilde{K}_F$ (symmetrization over the subgroup that preserves inner and outer edges).*

Example. For arbitrary univariate functions f, g, h the determinant

$$F(x,y,z) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(y) & g(y) & h(y) \\ f(z) & g(z) & h(z) \end{vmatrix}$$

is a Kleinian function. In particular, the choice $f(x) = 1$, $g(x) = x$, $h(x) = x^2$ leads to the Matiyasevich's polynomial studied above.

Definition 4 *A function of three variables representable as a determinant above will be referred to as a decomposable skew-symmetric function.*

Remark. (D. Bar-Natan [BN3]). The Kleinian weight system defined by a decomposable skew-symmetric function is reduced to the Lie algebraic weight system for the Lie algebra $\mathfrak{sl}_2$.

4 Decomposable skew-symmetric function

Theorem 3 *An analytic skew-symmetric function $F(x_1, x_2, x_3)$ is decomposable if and only if it satisfies the identity*

$$\begin{aligned} F(x_1,x_2,x_5)\,F(x_3,x_4,x_5) - F(x_1,x_3,x_5)\,F(x_2,x_4,x_5)& \\ + F(x_1,x_4,x_5)\,F(x_2,x_3,x_5) = 0,& \end{aligned} \tag{6}$$

where the dots stand for a set of variables, the same for every instance of the function F.

Proof. In what follows, we refer to Equation 3 as *3-term relation.* The idea of the proof is that 3-term relations for a skew-symmetric function imply 4-term relations, and the 4-term relations imply decomposability.

A 4-term relation is an equation of the form

$$\begin{aligned} F(x_1,x_2,x_3)\,F(x_4,x_5,x_6) - F(x_1,x_2,x_4)\,F(x_3,x_5,x_6)& \\ +F(x_1,x_2,x_5)\,F(x_3,x_4,x_6) - F(x_1,x_2,x_6)\,F(x_3,x_4,x_5) = 0\,.& \end{aligned} \tag{7}$$

Step 1: 4-term relations follow from 3-term relations. Indeed, multiplying the left-hand side of Equation 4 by $F(x_1, x_3, x_5)$, we get:

$$\begin{gathered} F(x_1,x_3,x_5)\,F(x_1,x_2,x_3)\,F(x_4,x_5,x_6) - F(x_1,x_3,x_5)\,F(x_1,x_2,x_4)\,F(x_3,x_5,x_6) \\ +F(x_1,x_3,x_5)\,F(x_1,x_2,x_5)\,F(x_3,x_4,x_6) - F(x_1,x_3,x_5)\,F(x_1,x_2,x_6)\,F(x_3,x_4,x_5)\,. \end{gathered}$$

Applying a 3-term relation to each summand of this expression, we get:

$$\begin{gathered} F(x_1,x_2,x_3)[F(x_1,x_4,x_5)F(x_3,x_5,x_6) - F(x_1,x_5,x_6)F(x_3,x_4,x_5)] \\ +F(x_3,x_5,x_6)[-F(x_1,x_2,x_3)F(x_1,x_4,x_5) - F(x_1,x_3,x_4)F(x_1,x_2,x_5)] \\ +F(x_1,x_2,x_5)[F(x_1,x_3,x_4)F(x_3,x_5,x_6) + F(x_1,x_3,x_6)F(x_3,x_4,x_5)] \\ +F(x_3,x_4,x_5)[F(x_1,x_2,x_3)F(x_1,x_5,x_6) - F(x_1,x_3,x_6)F(x_1,x_2,x_5)], \end{gathered}$$

which is 0, because all terms cancel in pairs.

Since the function is analytical, we can divide by $F(x_1, x_3, x_5)$ and obtain the 4-term relation.

Step 2: 4-term relations imply decomposability. Pick some generic numbers a_1, a_2, b_1, b_2, c_1, c_2 and put

$$\begin{aligned} f(x) &= F(a_1,a_2,x), \\ g(x) &= F(b_1,b_2,x), \\ h(x) &= F(c_1,c_2,x). \end{aligned}$$

The 4-term relation says that the vector

$$(F(x_4,x_5,x_6), -F(x_3,x_5,x_6), F(x_3,x_4,x_6), -F(x_3,x_4,x_5))$$

is orthogonal to each of the three vectors

$$\begin{gathered} (f(x_3), f(x_4), f(x_5), f(x_6)), \\ (g(x_3), g(x_4), g(x_5), g(x_6)), \\ (h(x_3), h(x_4), h(x_5), h(x_6)) \end{gathered} \tag{8}$$

and therefore proportional to their vector product. This means that the function F is decomposable. The theorem is proved.

Remark. The assumption of analyticity is essential. There exist non-analytic Kleinian functions which are not decomposable. A simple example is provided by

$$F(x,y,z) = E_{0,1,2} + E_{3,4,5},$$

where $E_{a,b,c}$ denotes the function equal to 1 at the point (a, b, c), equal to ± 1 at the points obtained from (a, b, c) by permutations of the coordinates, and 0 elsewhere.

Theorem 3 can be generalized to the case of skew-symmetric functions of an arbitrary number of variables. We are going to devote a detailed publication to these algebraic questions.

5 Klein functions and Lie algebras

There is a simple way (suggested by S. Chmutov) to construct a Lie algebra associated with a Kleinian function f: one sets $c_{ij}^k = f(a_i, a_j, a_k)$, where a_i are fixed real numbers, then the two axioms (4) and (5) translate into the skew-commutativity and the Jacobi identity for the algebra with structure constants c_{ij}^k. This looks, however, as a rather artificial construction.

In the case when the function f is a polynomial:

$$F(x, y, z) = \sum_{i,j,k} p_{ij}^k x^i y^j z^k,$$

there seems to exist a more natural association between Klein functions and Lie algebras. Indeed, the two equations 4 and 5 can be rewritten as identities for the coefficients:

$$p_{ij}^k = p_{jk}^i = -p_{ji}^k,$$
$$\sum_{k+n=\mathrm{const}} (p_{ij}^k p_{lm}^n - p_{il}^k p_{jm}^n + p_{im}^k p_{jl}^n) = 0.$$

These equations very much resemble the two defining properties of the structure constants of a Lie algebra:

$$c_{ij}^k = c_{jk}^i = -c_{ji}^k,$$
$$\sum_k (c_{ij}^k c_{lm}^k - c_{il}^k c_{jm}^k + c_{im}^k c_{jl}^k) = 0.$$

It seems that, in general, the first set of equations does not imply the second, so a simple assignment $c_{ij}^k = p_{ji}^k$ does not give a construction of a Lie algebra starting from a Kleinian polynomial. However, I have a feeling that every Klein polynomial does correspond to a Lie algebra via a certain natural procedure, and in this way one can obtain a certain class of Lie algebras (*Kleinian Lie algebras*), whose properties are to be studied.

6 Open problems

1. Are there Kleinian weight systems which are independent of classical Lie algebra weight systems?

2. Is it possible to detect knot inversion by Kleinian weight systems? In other words, is there a diagram D with an odd number of univalent vertices and a Kleinian function $F(x, y, z)$ such that $K_F(D) \neq 0$?

3. Understand the relation between the Kleinian systems and Lie algebras: what is the exact procedure to obtain Lie algebras from Klein polynomials and what Lie algebras appear in this way? Also, investigate the relation between the Kleinian weight system and the Lie algebraic weight system computed with respect to the associated Lie algebra.

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e-mail `duzhin@pdmi.ras.ru`
`http://www.pdmi.ras.ru/~duzhin`