# A lower bound for the number of Vassiliev knot invariants

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**Abstract**

We prove that the number of primitive Vassiliev knot invariants of degree $d$ grows at least as $d^{\log(d)}$ when $d$ tends to infinity. In particular it grows faster than any polynomial in $d$. The proof is based on the explicit construction of an ample family of linearly independent primitive elements in the corresponding graded Hopf algebra.

*Keywords*: knots, Vassiliev invariants, Hopf algebras, Feynman diagrams.

*AMS classification*: primary 57M25, secondary 17B35, 16W30.

## 1 Introduction

Vassiliev knot invariants [V] form a vector space $\mathcal{V}$ filtered by finite dimensional vector spaces $\mathcal{V}_d$ of invariants of order $\leq d$:

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \ldots \subseteq \mathcal{V}_d \subseteq \ldots \subset \mathcal{V}.$$

M. Kontsevich [K] gave a combinatorial description of the corresponding graded space

$$\mathcal{A} = \operatorname{gr} \mathcal{V} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \ldots \oplus \mathcal{A}_d \oplus \ldots \quad ,$$

where $\mathcal{A}_i = \mathcal{V}_i / \mathcal{V}_{i-1}$. This space turns out to be a graded, associative, coassociative, commutative and cocommutative Hopf algebra. Therefore it coincides with the algebra of polynomials over the primitive subspace $\mathcal{P} \subset \mathcal{A}$. The primitive space is also graded:

$$\mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \mathcal{P}_3 \oplus \ldots$$

D. Bar-Natan [BN1] has calculated $\dim \mathcal{P}_d$ up to $d = 9$:

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $\dim \mathcal{P}_d$ | 1 | 1 | 1 | 2 | 3 | 5 | 8 | 12 | 18 |

The knowledge of $\dim \mathcal{P}_i$ for $i \leq d$ is equivalent to the knowledge of $\dim \mathcal{A}_i$, $i \leq d$, or $\dim \mathcal{V}_i$, $i \leq d$. At present, the exact asymptotic behaviour of these numbers as $d$ tends to infinity is not known. Below, we summarize the known results on the asymptotic lower and upper bounds.

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In our paper [CD] we proved that $\dim \mathcal{A}_d < (d-1)!$ — an upper bound which is certainly far from being sharp; it was later improved by K. Ng [Ng] who replaced $(d-1)!$ by $(d-2)!/2$. In a recent preprint [St1] A. Stoimenow proves that $\dim \mathcal{A}_d$ grows slower than $d!/a^d$ for a constant $a > 1$.

The story of lower bounds for the number of Vassiliev knot invariants is more enigmatic. The strongest estimate that appeared in the published literature is contained in the following passage from [K], section 3:

"Using this construction [1], one can obtain the estimate

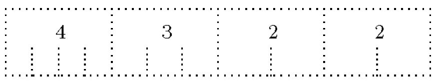
$$\dim(\mathcal{V}_n) > e^{c\sqrt{n}}, \quad n \to +\infty$$

for any positive constant $c < \pi\sqrt{2/3}$ (see [BN3], Exercise 6.14)."

We assume that here $\mathcal{V}_n$ is a slip of the pen, instead of $\mathcal{P}_n$, because of the reference to Exercise 6.14 where primitive elements are considered. Exercise 6.14 was present, however, only in the first edition of Bar-Natan's preprint and eliminated in the following editions as well as in the final published version of his text [BN1]. In [BN3] it reads as follows (page 43):

"*Exercise* 6.14. (Kontsevich, [24]) Let $P_{\geq 2}(m)$ denote the number of partitions of an integer $m$ into a sum of integers bigger than or equal to 2. Show that $\dim \mathcal{P}_m \geq P_{\geq 2}(m+1)$.

*Hint* 6.15. Use a correspondence like




$$\longleftrightarrow \qquad 10 + 1 = 4 + 3 + 2 + 2,$$

and ..."

The reference [24] is to "M. Kontsevich. Private communication."! Thus, although this estimate might be true, both authors refer to each other, and none of them gives any proof.

Among the facts that are related to the lower bound for the number of Vassiliev invariants and proved in the existing literature, let us mention the following estimates for the dimension of $d$-th grading of the primitive space:

1. $\dim \mathcal{P}_d \geq 1$ ("forest elements" found in [CDL3]).
2. $\dim \mathcal{P}_d \geq [d/2]$ (given by coloured Jones function — see [MM] and [CV]).
3. $\dim \mathcal{P}_d \gtrsim d^2/96$ (see [Du]).

This paper has grown from our attempts to carry out Bar-Natan's exercise 6.14. Although we could not attain the hypothetical Kontsevich–Bar-Natan lower bound mentioned above, we prove the following lower bound: $\dim \mathcal{P}_d \gtrsim d^{\log d}$ as $d \to +\infty$. In particular, $\dim \mathcal{P}_d$ grows faster than any polynomial in $d$. We use Bar-Natan's hint 6.15 and interpret his picture as a Primitive Chinese Character (see Sec. 2). We prove that a certain family of such Primitive Chinese Characters are linearly independent using the so called universal $gl_N$ weight system.

Using our bound for $\dim \mathcal{P}_d$ one can prove that the dimension of the $d$-th component of the graded algebra of Vassiliev invariants $\dim \mathcal{A}_d$ grows at least as $\exp(de^{-\sqrt{\log d}})$.

It is our pleasure to thank A. Stoimenow and O. Dasbach who read the draft version of the manuscript and made numerous valuable remarks. We are also grateful to A. Kaishev for his help in the struggle with asymptotic estimates.

[1] Bar-Natan's construction of Lie algebra weight systems.

# 2 Underlying combinatorics and main result

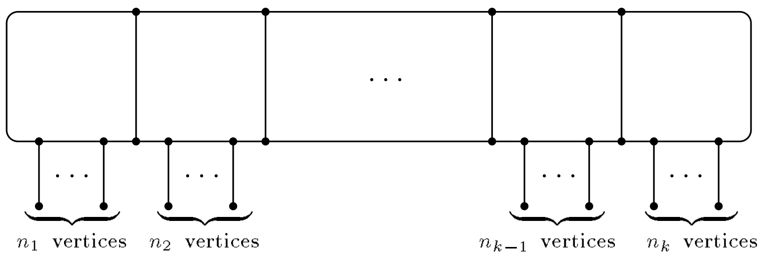
We begin by defining the main combinatorial object used in this paper (cf. Chinese Character diagrams [BN1], Japanese character diagrams [CV] and Feynman diagrams [Ch1, Ch2]).

**2.1. Definition.** A *Primitive Chinese Character* (PCC) is a connected graph with only trivalent and univalent vertices and with a chosen cyclic ordering of the edges meeting at a trivalent vertex for each such vertex.

We will draw PCCs using their plane diagrams similar to those adopted in knot theory. A vertex is pictured by a fat point and at each trivalent vertex the counterclockwise ordering of the edges is chosen. Here is an important example which will be used in the main theorem.

**2.2. Example.** The *baguette diagram*

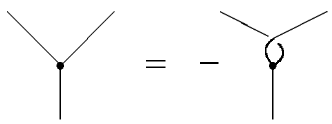
$B_{n_1,\dots,n_k}$ =




$B_{n_1,\dots,n_k}$ has $2(n_1+\dots+n_k+k-1)$ vertices, out of which $n_1+\dots+n_k$ are univalent.

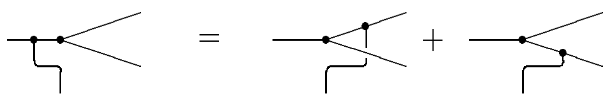
**2.3. The space of Primitive Chinese Characters.** We consider the space $\mathcal{C}$ generated by all Primitive Chinese Characters modulo two types of relations:
*Antisymmetry relation:*

(AS):




*Kirchhof law*[2]*:*

(IHX):




Note that every PCC has an even number of vertices. We will call the half of this number the *degree* of a PCC and denote it by $d$. Both relations that we use are homogeneous with respect to $d$. Moreover, they are homogeneous with respect to the number of univalent vertices $n$. The space $\mathcal{C}$ is thus bigraded: $\mathcal{C} = \oplus\, \mathcal{C}_{d,n}$, where $C_{d,n}$ is a subspace of $\mathcal{C}$ generated by PCCs with a total of $2d$ vertices, precisely $n$ of which are univalent. The baguette diagram of Example 2.2 has $d = n_1+\dots+n_k+k-1$ and $n = n_1+\dots+n_k$.

Since all PCCs are connected by definition, a standard graph theoretic argument shows that $n$ and $d$ always satisfy $n \le d+1$. It is easy to check that $\mathcal{C}_{d,d+1} = 0$, if $d > 1$.

[2]D. Bar-Natan referred to this relation as IHX. We prefer the name "Kirchhof law" because of the following analogy. Consider the left vertex in the left hand side picture as an electron moving through another vertex. Then the relation expresses the law: the sum of currents coming in a node is equal to the sum of currents coming out of the node. This electro-technical analogy is quite useful. Another acceptable name is Jacoby identity — by the reason that it is closely related to the Jacoby identity in Lie algebras (see below).

**Theorem** (Theorem 8 of [BN1]). *The primitive space $\mathcal{P}_d$ of degree $d > 1$ is isomorphic to the space $\mathcal{C}_d = \bigoplus_{n=1}^{d} \mathcal{C}_{d,n}$ of Primitive Chinese Characters of degree $d$ with a nonzero number of univalent vertices.*

This theorem, together with some ideas contained in its proof, leads to the following interesting consequence.

**Proposition.** *If there exists an odd $n$ such that $\mathcal{C}_{d,n} \neq 0$ for some $d$ then there exists a Vassiliev knot invariant $f$ of order $\leq d$ and a knot $K$ such that $f$ distinguishes $K$ from its inverse (that is the same knot $K$ only with the opposite orientation) and vice versa.*

For small $d$ dimensions of $\mathcal{C}_{d,n}$ were calculated by D. Bar-Natan [BN2] (empty entries in the table are zeros):

| $n \backslash d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 | 2 | 2 | 3 | 4 | 5 | ? | ? |
| 4 | | | | 1 | 1 | 2 | 3 | 4 | 6 | ? | ? |
| 6 | | | | | | 1 | 2 | 3 | 5 | ? | ? |
| 8 | | | | | | | | 1 | 2 | 4 | 8 |
| 10 | | | | | | | | | | 1 | 2 |

It is also known [BN2] that

- $\mathcal{C}_{d,1} = 0$ for all $d$.
- $\dim \mathcal{C}_{d,d} = 0$ for odd $d$ and 1 for even $d$. A proof of this fact can be found [CV]; its relation to the Melvin–Morton conjecture was elucidated in [Ch1, Ch2].
- $\dim \mathcal{C}_{d,d-1} = [d/6] + 1$ for odd $d$ and 0 for even $d$. See [Da1] for a proof.

Using the polynomial invariant $P_{gl_N}$ constructed in the present paper, O. Dasbach [Da2] managed to prove that $\dim \mathcal{C}_{d,d-2} = [\frac{(d-2)^2+12(d-2)}{48}] + 1$ for even $d$.

**2.4. Main Theorem.** *Let $n = n_1 + \ldots + n_k$ and $d = n + k - 1$. Elements $B_{n_1,\ldots,n_k}$ from Example 2.2 are linearly independent in $\mathcal{C}_{d,n}$ if $n_1, \ldots, n_k$ are all even and satisfy the following conditions:*

$$\begin{aligned}
&n_1 < n_2 \\
&n_1 + n_2 < n_3 \\
&n_1 + n_2 + n_3 < n_4 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&n_1 + n_2 + \ldots + n_{k-2} < n_{k-1} \\
&n_1 + n_2 + \ldots + n_{k-2} + n_{k-1} < n/3.
\end{aligned}$$

**2.5. Outline of the proof of Main Theorem.** The proof of Main Theorem 2.4 spans three sections, 3–5. First of all, we construct a linear mapping $P_{\mathcal{G}}$, associated with a Lie algebra $\mathcal{G}$, from the space $C_{d,n}$ to the algebra of polynomials in $n+1$ variables $x_0, x_1, \ldots, x_n$. This is done in subsection 3.1 and then, in 3.2–3.5, specialized for the case $\mathcal{G} = gl_N$. Section 4 is devoted to finding an explicit formula for $P_{gl_N}(B_{n_1,n_2,\ldots,n_k})$ where $B_{n_1,n_2,\ldots,n_k}$ is the baguette element presented in Example 2.2. In Section 5, we prove the linear independence of $P_{gl_N}(B_{n_1,n_2,\ldots,n_k})$ when the numbers $n_1, n_2, \ldots, n_k$ satisfy the assumptions of the theorem.

**2.6. Theorem.** *For any fixed value of $k = d-n+1$ we have the following asymptotical inequality as $d$ tends to infinity:*

$$\dim \mathcal{C}_{d,n} \gtrsim \frac{1}{2^{\frac{k(k-1)}{2}} 3^{k-1}(k-1)!} \, (d-k+1)^{k-1}.$$

**Proof.** This theorem follows from Main Theorem 2.4. Indeed, we have to count the number of integer points with even coordinates belonging to the body in $\mathbf{R}^{k-1}$ described by the set of inequalities above. Asymptotically the number of such points is equal to the volume of the body divided by $2^{k-1}$.

To find the volume, note that the condition $n_1+n_2+\ldots+n_{k-1} < n/3$ specifies the inner part of a $(k-1)$-dimensional simplex in $\mathbf{R}^{k-1}$ having $(k-1)$ sides of length $n/3$ at right angles to each other. Its volume is equal to $\frac{(n/3)^{k-1}}{(k-1)!}$. The inequality $n_1 < n_2$ cuts exactly one half of this body, the next inequality $n_1+n_2 < n_3$ — exactly one fourth of the already chosen part, etc., the inequality $n_1+n_2+\ldots+n_{k-2} < n_{k-1}$ cuts $1/2^{k-2}$-th part of the result obtained at the previous step.

Summing up, we get the expression given in the statement of the Theorem.

**2.7. Corollary.** $\dim \mathcal{P}_d$ *grows faster than any polynomial in $d$.*

This is an immediate consequence of Theorem 2.6, because the value of $k$ can be arbitrarily large.

**2.8. Theorem.** *Asymptotically, as $d$ tends to infinity,* $\dim \mathcal{P}_d > d^{\log_b d}$*, where $b$ is an arbitrary number greater than 4.*

**Proof.** The desired estimate can be achieved using Theorem 2.6 with an appropriate choice of $k$.

We will prove that for $k = [\log_2 d]+1$ the right hand side of the inequality stated in Theorem 2.6 is asymptotically greater than $d^{\log_b d}$ for any $b > 4$. Taking the logarithm to base 2, we see that the inequality to prove becomes

$$[\log_2 d]\log_2(d-[\log_2 d]) - \frac{1}{2}[\log_2 d]([\log_2 d]+1) - [\log_2 d]\log_b d - \log_2 3[\log_2 d] - \log_2([\log_2 d]!) \gtrsim 0.$$

We will show that under the assumptions of the theorem, the expression on the left is not only asymptotically positive, it even tends to $+\infty$. In fact, as $d$ tends to infinity, each of the three first terms of the sum outweighs the remaining two. Their sum is

$$[\log_2 d]\Big(\log(d-[\log_2 d]) - \frac{1}{2}([\log_2 d]+1) - \log_b d\Big).$$

Since $\log_2(d - [\log_2 d])$ is asymptotically equivalent to $\log_2 d$, the desired fact follows from the inequality $\frac{1}{2}\log_2 d > \log_b d$ which holds for any $b > 4$. The theorem is proved.

A simple argument (taking the maximum over $k$) shows that the assertion of Theorem 2.8 is the best lower bound that can be deduced from Theorem 2.6 by the techniques employed. More details and relevant discussions can be found in [St2].

# 3 Lie algebra invariants

In this section for a Lie algebra $\mathcal{G}$ we construct a polynomial valued function $P_{\mathcal{G}}$ on the set of PCCs. This function satisfies the Antisymmetry relation and the Kirchhof law and thus descends to the space $\mathcal{C}$.

**3.1. Universal Lie algebra weight systems.** Our definition of functions $P_{\mathcal{G}}$ is related to the universal Lie algebra weight systems — a construction invented by M. Kontsevich [K] for chord diagrams and, in a modified form, used in [CV] for Japanese Characters. In distinction with the Japanese Characters of [CV], the Primitive Chinese Characters studied here have no numbering of their univalent vertices, therefore we can correctly define only the highest graded part of the mapping with values in the universal enveloping algebra.

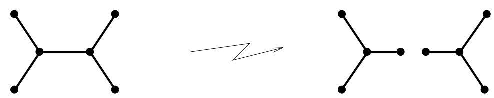
Suppose that $\mathcal{G}$ is a Lie algebra supplied with an $ad$-invariant symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ and $C$ is a PCC with $n$ univalent vertices.

We proceed in four steps:

1. The construction of a tensor $T_{\mathcal{G}}(C) \in \bigotimes_{v\in V} \mathcal{G}$.

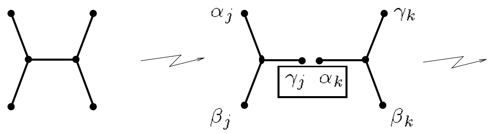
   First of all for a PCC $C$ with the set $V$ of $n$ univalent vertices we construct an element $T_{\mathcal{G}}(C)$ of the space $\bigotimes_{v\in V}\mathcal{G}$ whose tensor factors correspond to the univalent vertices.

   We cut all the edges connecting trivalent vertices of $C$. This splits $C$ into a union of elementary pieces (tripods), each consisting of one trivalent vertex and three univalent vertices. Here is an example:



   Now with each elementary piece we associate a tensor $-J \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}$ whose factors correspond to the univalent vertices in agreement with the cyclic ordering of the edges. This tensor is defined as follows. Consider the Lie bracket $[\cdot,\cdot]$ as an element of $\mathcal{G}^*\otimes\mathcal{G}^*\otimes\mathcal{G}$. Identification of $\mathcal{G}^*$ and $\mathcal{G}$ via $\langle\cdot,\cdot\rangle$ provides the tensor $J \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}$. Due to the properties of $[\cdot,\cdot]$ and $\langle\cdot,\cdot\rangle$ this tensor $J$ is skew-symmetric under the permutations of the three factors which ensures that the previous construction makes sense.

   The tensor corresponding to $C$ is combined from these elementary ones in the following way. Let us restore $C$ from the tripod pieces. Each time two univalent vertices are put together, we contract the two tensor factors corresponding to these vertices by taking the form $\langle\cdot,\cdot\rangle$ on the factors. For example, if $J = \sum \alpha_j\otimes\beta_j\otimes\gamma_j$, then the tensor we relate to the union of two pieces will be:




$$\sum_{i,j,k} \langle \gamma_j, \alpha_k \rangle \alpha_j \otimes \beta_j \otimes \beta_k \otimes \gamma_k.$$

Gluing and contracting in such a way over all the edges previously cut we obtain a tensor whose factors correspond to the univalent vertices of $C$. This is $T_{\mathcal{G}}(C)$.

2. The construction of an element $W^{\nu}_{\mathcal{G}}(C) \in ZU(\mathcal{G})$.

   A numbering of the set of univalent vertices $V$ yields an isomorphism between $\bigotimes_{v\in V} \mathcal{G}$ and the conventional tensor power $\bigotimes_{i=1}^{n} \mathcal{G} = \mathcal{G}^{\otimes n}$. Choosing such a numbering $\nu$, we set $W^{\nu}_{\mathcal{G}}(C)$ to be the image of the tensor thus obtained under the natural projection $\bigotimes_{v\in V} \mathcal{G} \to U(\mathcal{G})$. The element $W^{\nu}_{\mathcal{G}}(C)$ always belongs to the center $ZU(\mathcal{G})$ of the universal enveloping algebra.

3. Passing to the quotient $\overline{W}_{\mathcal{G}}(C) \in U_n(\mathcal{G})/U_{n-1}(\mathcal{G})$.

   Elements of $U(\mathcal{G})$ are represented by linear combinations of words in the elements of $\mathcal{G}$. The projection $\mathcal{G}^{\otimes n} \to U(\mathcal{G})$ is given by forgetting the symbols of tensor product. Let $U_m(\mathcal{G})$ be the subspace of $U(\mathcal{G})$ generated by words of length no greater than $m$. This gives a natural filtration in $U(\mathcal{G})$. By construction $W^{\nu}_{\mathcal{G}}(C) \in U_n(\mathcal{G})$. This element depends on the choice of the numbering of univalent vertices used to identify $\bigotimes_{v\in V} \mathcal{G}$ with $\mathcal{G}^{\otimes n}$. However, the projection $\overline{W}_{\mathcal{G}}(C)$ of $W_{\mathcal{G}}(C)$ in $U_n(\mathcal{G})/U_{n-1}(\mathcal{G})$ is well defined due to the identity $xy - yx = [x,y]$.

4. The construction of a polynomial $P_{\mathcal{G}}(C) \in \mathbf{Z}[x_0, x_1, \ldots, x_n]$.

   Actually $W^{\nu}_{\mathcal{G}}(C)$ belongs to the center $ZU(\mathcal{G})$ of $U(\mathcal{G})$. By Harish-Chandra theorem $ZU(\mathcal{G})$ is isomorphic to an algebra of polynomials. These polynomials are weighted by the natural filtration in $U(\mathcal{G})$. So $\overline{W}_{\mathcal{G}}(C)$ is equal to the (maximal possible) weight $n$ part of the polynomial $W_{\mathcal{G}}(C)$. We put $P_{\mathcal{G}}(C)$ to be equal to the highest degree homogeneous part of the polynomial $\overline{W}_{\mathcal{G}}(C)$.

**3.2. Lie algebra $gl_N$.** Consider the Lie algebra $\mathcal{G} = gl_N$ of $N \times N$ matrices. Fix the trace of the product of matrices as the $ad$-invariant form: $\langle x, y\rangle = \mathrm{Tr}(xy)$.

The algebra $gl_N$ is generated by matrices $e_{ij}$ with 1 on the intersection of $i$-th row with $j$-th column and zero elsewhere. We have $\langle e_{ij}, e_{kl}\rangle = \delta^l_i \delta^k_j$, where $\delta$ is the Kronecker delta. Therefore, duality between $gl_N$ and $(gl_N)^*$ defined by $\langle\cdot,\cdot\rangle$ is given by the formula $e^*_{ij} = e_{ji}$.

One can check that $[e_{ij}, e_{kl}] \neq 0$ only in the following cases:

- $[e_{ij}, e_{jk}] = e_{ik}$, if $i \neq k$,
- $[e_{ij}, e_{ki}] = -e_{kj}$, if $j \neq k$,
- $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, if $i \neq j$,

This gives the following formula for the Lie bracket:

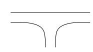
$$[\cdot,\cdot] = \sum_{i,j,k=1}^{N} (e^*_{ij} \otimes e^*_{jk} \otimes e_{ik} - e^*_{ij} \otimes e^*_{ki} \otimes e_{kj}).$$

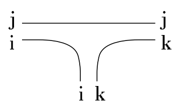
Using the identification of $(gl_N)^*$ with $gl_N$ we obtain

$$J = \sum_{i,j,k=1}^{N} (e_{ji} \otimes e_{kj} \otimes e_{ik} - e_{ji} \otimes e_{ik} \otimes e_{kj}).$$

**3.3. Graphical notation for tensors.** Here we develop a graphical notation for a special class of the elements of $(gl_N)^{\otimes n}$ and $U(gl_N)$ similar to that invented by R. Penrose [Pe] (see also [BN1]). Consider a plane diagram (*$T$-diagram*) that consists of $n$ pairs of mutually close points arranged along a circle and connected pairwise by $n$ lines. Strictly speaking, a $T$-diagram is a set $S$ of cardinality $2n$ endowed with three structures: a cyclic ordering and two involutions without fixed points (or, in other words, two partitions of $S$ into $n$ two-point subsets). The first partition should be consistent with the cyclic order in the sense that the two points of each pair must be neighbours with respect to that order. In the pictures below, $S$ is the set of all protruding endpoints of the lines which indicate the second partition, while the pairs of the first partition are the endpoints that are drawn close to each other. The cyclic order is the one defined by going counterclockwise along an imagined circle that encompasses the whole picture.

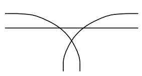
Given a $T$-diagram, we can write down an element of $n$-th tensor power of $gl_N$ in the following way. We put the same index at either end of each line, and then, travelling around the encompassing circle, write a factor $e_{ij}$ each time we encounter a pair of neighbouring points, the first of which is assigned the index $i$ and the second, the index $j$. Then, we consider the sum of all such tensors when all the indices range from 1 to $N$. In a sense, this procedure gives a graphical substitute for the formal Einstein summation rule in multi-index expressions. We must admit that in general the element of $\otimes^n gl_N$ thus obtained may depend on the choice of the starting point of the circle, — but the corresponding element of the universal enveloping algebra will not depend on this choice.

For example, to find the tensor corresponding to the diagram  we take three indices $i$, $j$ and $k$ and write:

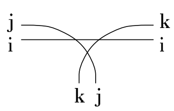



$$\rightsquigarrow \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{ik} \otimes e_{kj}.$$

Here is another example:

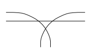
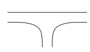


$\rightsquigarrow$

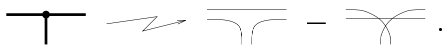



$$\rightsquigarrow \sum_{i,j,k=1}^{N} e_{ji} \otimes e_{kj} \otimes e_{ik}.$$

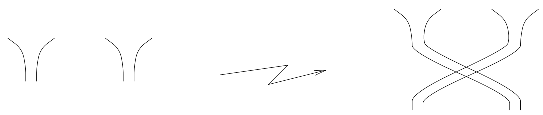
These two examples are sufficient to graphically express the tensor $J$:

$J =$  $-$  .

According to Sec. 3.1, with every trivalent vertex of a PCC we have to associate the tensor $-J$:



We thus consider two resolutions of each trivalent vertex — the *positive* resolution with the plus sign and the *negative* resolution with the minus sign. After all such resolutions we get a linear combination of $2^t$ diagrams, where $t$ is the number of trivalent vertices, representing an element $(gl_N)^{\otimes n}$. This linear combination is the graphical notation for $W_{gl_N}(C)$. Each diagram consists of $n$ pairs of points and $2n$ lines connecting them. Permuting the tensor factors in $(gl_N)^{\otimes n}$ corresponds to interchanging the pairs of points in the diagram:



Therefore, if we consider a $T$-diagram up to an arbitrary permutation of the pairs, it correctly defines an element of $U_n(gl_N)/U_{n-1}(gl_N)$. On the combinatorial level, a $T$-diagram considered up to an arbitrary permutation of pairs of legs is a set of cardinality $2n$ with a splitting into $n$ ordered pairs and a splitting into $n$ unordered pairs. Thus, the linear combination of $T$-diagrams which is considered up to such permutations represents $\overline{W}_{gl_N}(C)$.

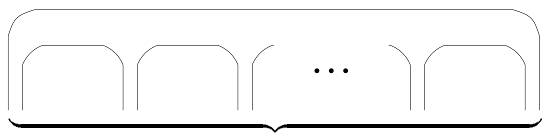
Multiplication in $U(gl_N)$ corresponds to the disjoint union of diagrams.

**3.4. Center of $U(gl_N)$.** It is known [Zh] that $ZU(gl_N)$ is generated by $N$ variables $x_m$, $1 \le m \le N$ (*generalized Casimir elements*):

$$x_m = \sum_{i_1,\ldots,i_m=1}^{N} e_{i_1 i_2} e_{i_2 i_3} e_{i_3 i_4} \ldots e_{i_{m-1} i_m} e_{i_m i_1}$$

as a free commutative polynomial algebra.

In the graphical notation of Sec. 3.3

$x_m =$ 

$m$ pairs

In particular, $x_1 = \bigcap = \sum_{i=1}^{N} e_{ii}$ is the unit matrix.

It is convenient to extend the list $x_1, \ldots, x_N$ of our variables by setting $x_0 = N$. The graphical notation for $x_0$ will be a circle.

According to Sec. 3.1 for a PCC $C$ with $n$ univalent vertices $W_{gl_N}(C)$ is a polynomial in $x_m$'s. The element $x_m$ has weight $m$ with respect to the natural filtration: $x_m \in U_m(gl_N)$. So $\overline{W}_{gl_N}(C)$ is a part of the polynomial $W_{gl_N}(C)$ consisting of all monomials in $x_m$'s such that the sum of indices of their variables is equal to $n$. $P_{gl_N}(C)$ is a part of $\overline{W}_{gl_N}(C)$ consisting of monomials of the maximal possible degree.

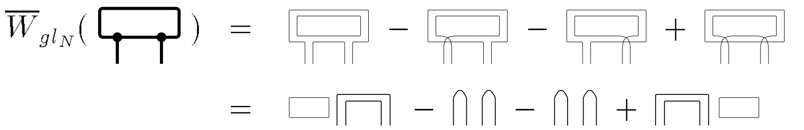
**3.5. Remark.** O. Dasbach [Da2] has rightly observed that one can *define* the polynomial $\overline{W}_{gl_N}$ by the rule we have arrived at:

$$\overline{W}_{gl_N}(C) = \sum_r (-1)^{|r|} x_{i_1} x_{i_2} \ldots x_{i_{k(r)}},$$

where the sum is taken over all the $2^n$ resolutions of the $n$ triple points of $C$, $|r|$ is the number of points resolved in the negative sense, and variables $x_i$ correspond to the connected components of the diagram obtained from $C$ after the resolution (see examples in the next section). The correctness of this definition can be verified directly, without even mentioning the Lie algebra $gl_N$.

# 4 Calculation of $P_{gl_N}(B_{n_1,\ldots,n_k})$

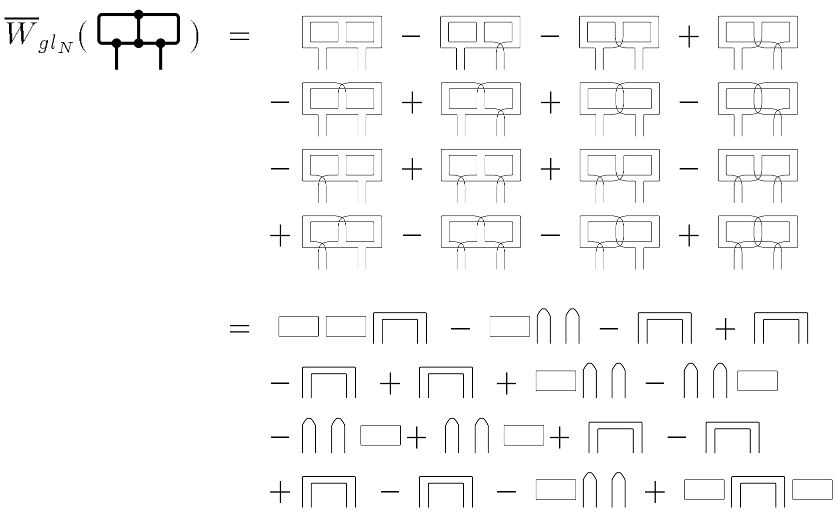
**4.1. Example.** To calculate $\overline{W}_{gl_N}(B_2)$ we resolve the two trivalent vertices of $B_2$:



$$= 2(x_0x_2 - x_1^2).$$

Both monomials have the same degree 2, so $P_{gl_N}(B_2) = 2(x_0x_2 - x_1^2)$.

**4.2. Example.** To calculate $\overline{W}_{gl_N}(B_{1,1})$ we have to do sixteen resolutions of the four trivalent vertices of $B_{1,1}$:

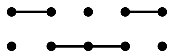


$$\begin{aligned} = \ & x_0^2x_2 - x_0x_1^2 - x_2 + x_2 - x_2 + x_2 + x_0x_1^2 - x_0x_1^2 \\ & -x_0x_1^2 + x_0x_1^2 + x_2 - x_2 + x_2 - x_2 - x_0x_1^2 + x_0^2x_2. \end{aligned}$$

So $P_{gl_N}(B_{1,1}) = 2(x_0^2x_2 - x_0x_1^2)$.

**4.3. Definition of a scheme.** A *scheme* is a combinatorial object defined as follows. Consider $k$ pairs of vertices arranged in two rows: $:\ :\ \cdots\ :$. Choose one of $2^{k-1}$ subsets of the set $\{1, \dots, k-1\}$. If $s$ belongs to the chosen subset then we connect the lower vertices of $s$-th and $(s+1)$-th pairs, otherwise we connect the upper vertices.

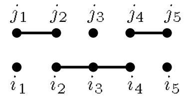
Here is an example of a scheme for $k = 5$ and the subset $\{2, 3\}$:

 .

For a given $k$ there exist $2^{k-1}$ different schemes. Each of them has $k+1$ connected components.

**4.4. Definition of the monomial** associated with a scheme.

Let $\sigma$ be a scheme; $i_1, \dots, i_k$ be non negative integers: $0 \le i_1 \le n_1, \dots, 0 \le i_k \le n_k$. We assign $i_s$ to the lower vertex of the $s$-th pair of $\sigma$ and $j_s = n_s - i_s$ — to the upper vertex. For example



.

With the scheme $\sigma$ and integers $i_1, \dots, i_k$ we associate the *monomial* $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ where $\sigma_t$ is the sum of integers assigned to the vertices of $t$-th connected component of $\sigma$.

In our example the monomial is $x_{i_1} x_{j_1+j_2} x_{i_2+i_3+i_4} x_{j_3} x_{j_4+j_5} x_{i_5}$.

**4.5. Proposition.** *If $N > n_1 + \dots + n_k$ then*

$$P_{gl_N}(B_{n_1,\dots,n_k}) = \sum_{i_1,\dots,i_k} (-1)^{j_1+\dots+j_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} \sum_{\sigma} x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k},$$

*where the external sum ranges over all integers $i_1, \dots, i_k$ such that $0 \le i_1 \le n_1, \dots, 0 \le i_k \le n_k$; the internal sum ranges over all $2^{k-1}$ schemes and $x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$ is the monomial associated with the scheme $\sigma$ and integers $i_1, \dots, i_k$.*

**4.6. Example.** For $B_2$: $k = 1$, $n_1 = 2$. There is only one scheme:  . The corresponding monomial is $x_{i_1} x_{j_1}$. According to Proposition 4.5,

$$\begin{aligned} P_{gl_N}(B_2) &= \sum_{i_1=0}^{2} (-1)^{j_1} \binom{2}{i_1} x_{i_1} x_{j_1} \\ &= x_0 x_2 - 2 x_1 x_1 + x_2 x_0 \quad = \quad 2(x_0 x_2 - x_1^2) \end{aligned}$$

which agrees with Example 4.1.

**4.7. Example.** For $B_{1,1}$: $k = 2$, $n_1 = n_2 = 1$. There are two schemes:  and  . The corresponding monomial are $x_{i_1} x_{i_2} x_{j_1+j_2}$ and $x_{i_1+i_2} x_{j_1} x_{j_2}$. According to Proposition 4.5,

$$\begin{aligned} P_{gl_N}(B_{1,1}) &= \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{j_1+j_2} x_{i_1} x_{i_2} x_{j_1+j_2} + \sum_{i_1=0}^{1} \sum_{i_2=0}^{1} (-1)^{j_1+j_2} x_{i_1+i_2} x_{j_1} x_{j_2} \\ &= x_0 x_0 x_2 - x_0 x_1 x_1 - x_1 x_0 x_1 + x_1 x_1 x_0 + x_0 x_1 x_1 - x_1 x_0 x_1 - x_1 x_1 x_0 + x_2 x_0 x_0 \\ &= 2(x_0^2 x_2 - x_0 x_1^2) \end{aligned}$$

which agrees with Example 4.2.

**4.8. Remark.** The right hand side of the above formula admits the following symmetry. Let $\sigma'$ be a scheme obtained from $\sigma$ by interchanging the lower and upper rows of vertices. Set $i'_s = j_s$ and $j'_s = i_s$. Then the monomial associated with $\sigma$ and numbers $i_1, \ldots, i_k$ is equal to the monomial associated with $\sigma'$ and $i'_1, \ldots, i'_k$. The binomial coefficients also coincide: $\binom{n_s}{i_s} = \binom{n_s}{i'_s}$. But the sign $(-1)^{j_1+\ldots+j_k}$ is replaced by $(-1)^{j'_1+\ldots+j'_k} = (-1)^{n-(j_1+\ldots+j_k)}$, where $n = n_1 + \ldots + n_k$. So, for even $n$ the signs coincide too and for odd $n$ the signs are opposite. Therefore, the right hand side vanishes for odd $n$.

**4.9. Plan of the proof of Proposition 4.5.** The PCC $B_{n_1,\ldots,n_k}$ has $k$ parts separated by $k-1$ walls. Each wall is an edge connecting trivalent vertices to which we will refer as *wall vertices*. The $s$-th part has $n_s$ outgoing legs. We will refer to the corresponding trivalent vertices as *leg vertices*.

The proof of Proposition 4.5 consists of three steps.

At the first step we study the effect of resolutions of wall vertices. We prove that the monomial obtained by certain resolutions of these vertices has the maximal possible degree if and only if for each wall both resolutions of its vertices have the same sign. These signs are related to the above defined schemes in the following way. If we take the positive resolutions at both endpoints of $s$-th wall, then we connect the lower vertices of $s$-th and $(s+1)$-th pairs in the scheme. If we take the negative resolutions, then we connect the upper vertices.

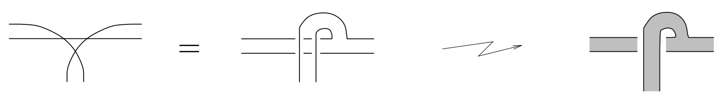
At the second step we study the effect of resolutions of leg vertices. We show that the result depends only on the numbers of positive resolutions of leg vertices in each part and does not depend on which vertices in a part were resolved positively and which negatively. We denote by $i_s$ the number of positive resolutions in part $s$. This yields the binomial coefficients $\binom{n_s}{i_s}$ in the formula of Proposition 4.5. The total number $j_1 + \ldots + j_k$ of negative resolutions of leg vertices gives the sign $(-1)^{j_1+\ldots+j_k}$ in Proposition 4.5.

The first two steps allow us to consider only those cases where the resolutions of the left $i_s$ leg vertices in the part $s$ are positive, the rest $j_s$ resolutions are negative and both resolutions at the ends of each wall have the same sign. At the third step we prove that such resolutions of wall vertices lead to monomials associated with corresponding schemes according to Definition 4.4.

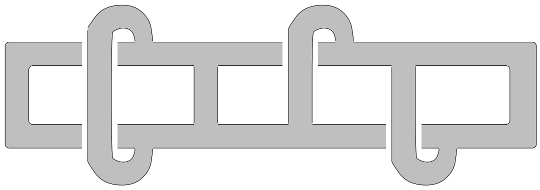
**4.10. The first step of the proof.** Let us fix certain resolutions of all trivalent vertices of $B_{n_1,\ldots,n_k}$. We denote the obtained $T$-diagram by $T$. According to Sec. 3.3 (see also Examples 4.1 and 4.2) it consists of $n = n_1 + \ldots + n_k$ pairs of points and a number of lines connecting them. The Lie algebra construction of Sec. 3 implies that after a suitable permutation of the pairs $T$ will look like a disjoint union of certain $x_m$'s. Hence it defines a monomial in $x_m$'s which we denote by $m(T)$.

Let us close all lines in the diagram by connecting the two points in every pair with an additional short line. We obtain a number of closed curves, and we can draw them in such a way that they have 3 intersection points in the vicinity of each negative resolution and do not have other intersections. Each variable $x_m$ gives precisely one closed curve. Thus the degree of $m(T)$ is equal to the number of these closed curves.

Now represent the lines of $T$ by curves in 3-dimensional space without intersections which in the neighbourhood of every negative resolution behaves as shown in the picture:

 .

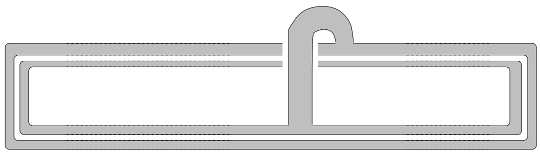
Consider an oriented surface $S$ which has our family of curves as its boundary (the Seifert surface). The degree of $m(T)$ is equal to the number of boundary components $b$ of $S$. The whole surface $S$ consists of an annulus corresponding to the big circle in $B_{n_1,\ldots,n_k}$ and $k-1$ bands corresponding to the walls. Here is an example:



where each of the two walls on the left has the same resolutions at its endpoints, while the two walls on the right have different resolutions at their endpoints. The resolutions of the leg vertices do not influence the surface $S$.

The Euler characteristic $\chi$ of $S$ can be easily computed. The surface $S$ is contractible to a circle with $k-1$ chords, thus $\chi = 2(k-1) - (k-1+2k-2) = -k+1$. On the other hand $\chi = 2-2g-b$, where $g$ and $b$ are the genus and the number of boundary components of $S$. Hence $b = k+1-2g$. Therefore, the degree of $m(T)$, equal to $b$, attains its maximal value $k+1$ only for such resolutions which give a zero genus surface $S$.

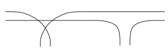
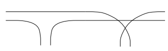
We claim that if there exists a wall whose ends are resolved with the opposite signs then the genus of $S$ is not zero. Indeed, in this case we can draw a closed curve in $S$ which does not separate the surface (independently on the remaining resolutions):



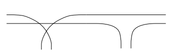
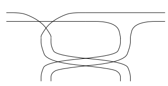
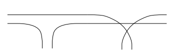
So, the contribution to $P_{gl_N}(B_{n_1,\ldots,n_k})$ is given by only those monomials which come from equal resolutions at the ends of each wall.

### 4.11. The second step of the proof.

**Lemma.** *Suppose that two neighbouring leg vertices have a negative and a positive resolution, respectively. Then this diagram gives the same contribution to $\overline{W}_{gl_N}$ as the diagram with switched resolutions of these two vertices:*

 =  .

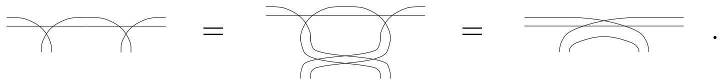
**Proof of Lemma.** Since we can freely interchange the pairs of legs we have:

 =  =  .

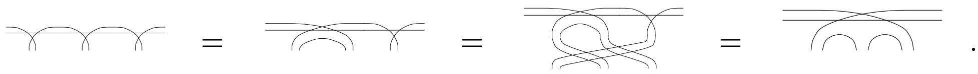
The lemma is proved.

Using the Lemma we can collect all positive resolutions on the left side of a part and all negative resolutions on the right side. Let $i_s$ (resp. $j_s$) be the number of positive (resp. negative) resolutions in a part $s$. The lemma implies that such a diagram appears in $P_{gl_N}(B_{n_1,\ldots,n_k})$ with the coefficient $(-1)^{j_1+\ldots+j_k}\binom{n_1}{i_1}\ldots\binom{n_k}{i_k}$ as in Proposition 4.5.

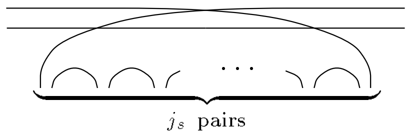
Several consecutive positive resolutions look like a fragment of $x_m$. Indeed, after resolutions of wall vertices they participate in a certain $x_m$ (see Sec. 4.12). Let us look at several consecutive negative resolutions. Two such resolutions can be transformed as:



If we have three such resolutions then:

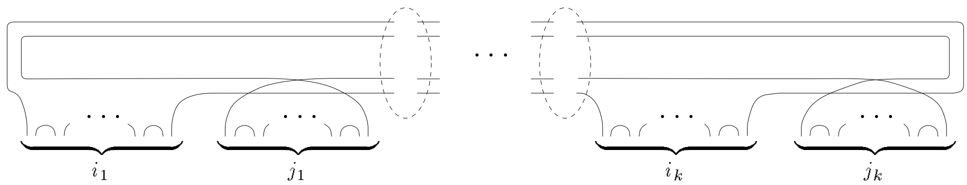


Analogously $j_s$ consecutive negative resolutions can be transformed to




— and this picture is also a fragment of some $x_m$.

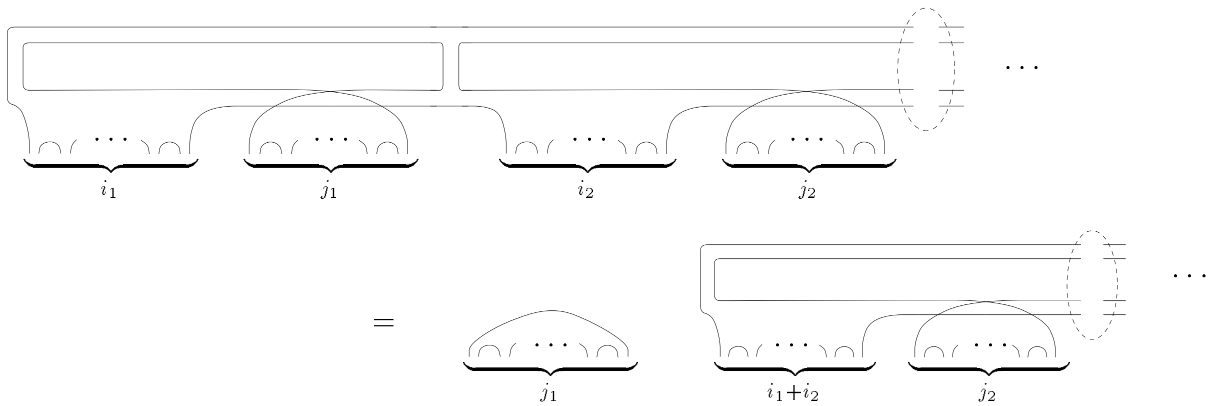
**4.12. The third step of the proof.** In the previous subsection we showed that $P_{gl_N}(B_{n_1,\ldots,n_k})$ consists of diagrams




with coefficients $(-1)^{j_1+\cdots+j_k}\binom{n_1}{i_1}\ldots\binom{n_k}{i_k}$. Here we assume the wall vertices in the dashed ovals have the same resolutions at their respective endpoints.

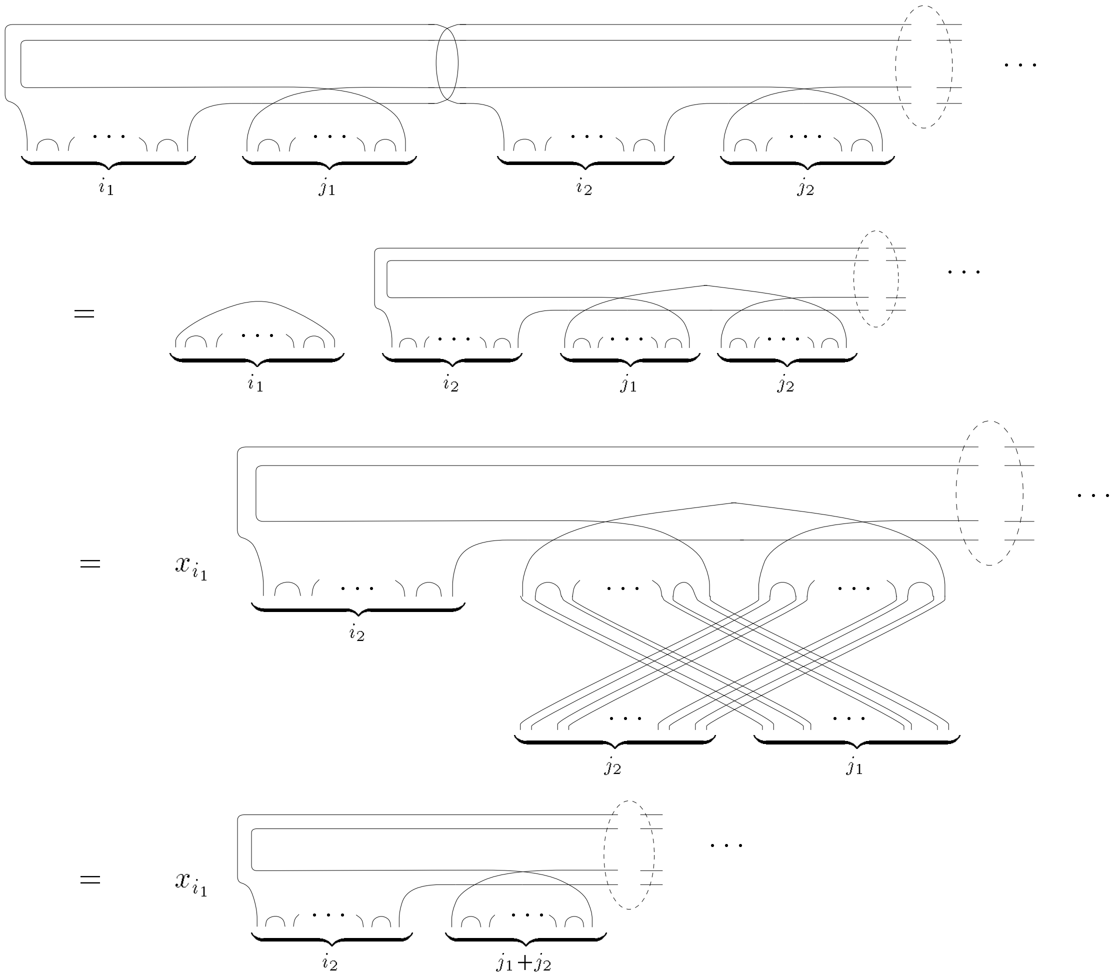
Now we will prove that such a diagram yields the monomial associated with a scheme according to Definition 4.4. To do so let us look at positive and negative resolutions of the ends of the first wall on the left.

For positive resolutions we have:




In this case we obtain the product of $x_{j_1}$ with the diagram of a similar type having one wall less. This corresponds to the connection of lower vertices of the first and second pairs of our scheme (see Sec. 4.9).

For negative resolutions we have:




We again obtain the product of $x_{i_1}$ with the diagram of a similar type having one wall less. This case corresponds to the connection of upper vertices of the first and second pairs of our scheme.

Continuing this process, we obviously obtain the monomial associated with the scheme. Proposition 4.5 is proved.

# 5 Linear independence of $P_{gl_N}(B_{n_1,\dots,n_k})$

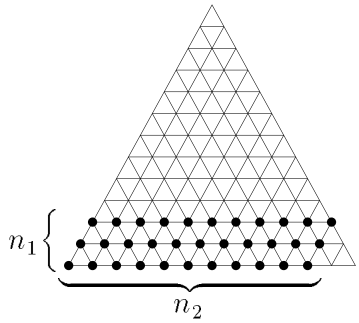
**5.1. Proposition.** *Polynomials $P_{gl_N}(B_{n_1,\dots,n_k})$ from Proposition 4.5 are linearly independent if $n_1,\dots,n_k$ satisfy the conditions of Main Theorem 2.4.*

Before giving the formal proof of the proposition for the arbitrary $k$, we will explain the main geometric idea in the case of small values of $k$. As $k$ grows, geometry becomes more and more complicated; this is why in the eventual proof we have to resort to algebra.

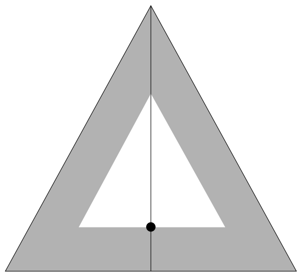
**5.2. Example:** $k=2$. We will represent the monomials of this polynomial by points of the triangular lattice. All these points will lie inside a triangle with side $d-1$. A monomial $x_a x_b x_c$ corresponds to the point which has distances $(a,b,c)$ from the sides of this triangle, or, more precisely, to the orbit of such a point with respect to the group of permutations of the three coordinates. For fixed values of $n_1$ and $n_2$ the set of all points $(i_1, i_2, j_1 + j_2)$ that correspond to nonvanishing terms in

$$P_{gl_N}(B_{n_1,n_2}) = 2\sum_{i_1,i_2}(-1)^{j_1+j_k}\binom{n_1}{i_1}\binom{n_2}{i_2}x_{i_1}x_{i_2}x_{j_1+j_2}$$

forms a parallelogram:




If $n_1 < n_2/2$, then the union of six such parallelograms obtained by the action of the symmetry group is a triangular annulus whose width is proportional to $n_1$:

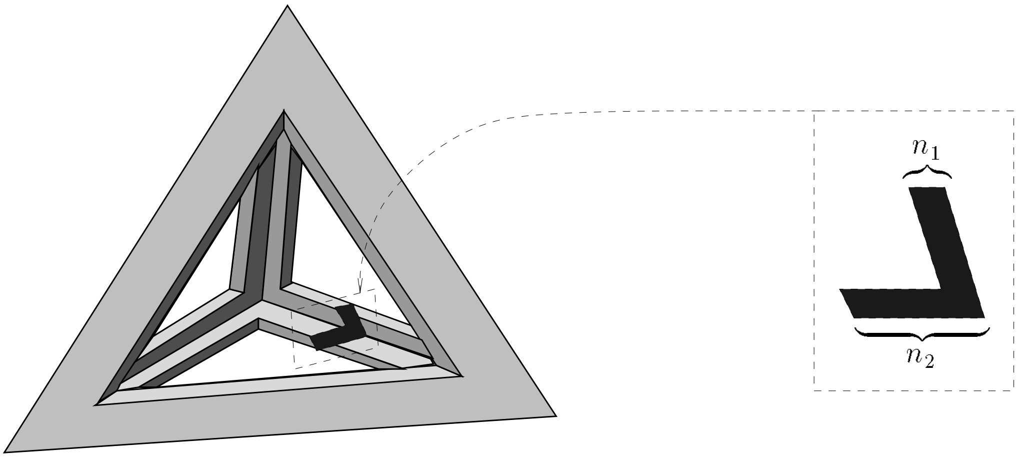


It can be verified (this will be done in the formal proof) that for every even value of $n_1$ the polynomial $P_{gl_N}(B_{n_1,n_2})$ has a monomial which does not appear in such polynomials for smaller values of $n_1$. This monomial lies on the height of the triangle; it is shown by a fat point in the picture above. Hence, all such polynomials form an independent family with one degree of freedom.

**5.3. Example:** $k = 3$. In this case the set of all non-vanishing monomials $x_\alpha x_\beta x_\gamma x_\delta$ in $P_{gl_N}(B_{n_1,n_2,n_3})$ can be viewed as a subset of the regular tetrahedron

$$\{(\alpha, \beta, \gamma, \delta) \,|\, \alpha, \beta, \gamma, \delta \geq 0,\ \alpha + \beta + \gamma + \delta = \mathrm{const}\} \subset \mathbf{Z}^3 \subset \mathbf{Z}^4,$$

equal to the union of 48 parallelepipeds with sides $n_1$, $n_2$, $n_3$. If $n_1$ and $n_2$ are small enough with respect to $n_3$, then the body obtained lives along the edges of the tetrahedron and has the following shape:




The intersection of this body with the mirror plane of the tetrahedron in the neighbourhood of its edge is an $L$-shaped figure whose height and width are proportional to $n_1$ and $n_2$, respectively. When both $n_1$ and $n_2$ vary, we obtain a 2-parametric family of independent polynomials.

**5.4. Proof of Proposition.** Fix $n = n_1 + \ldots + n_k$ and $d = n + k - 1$.

Suppose that $P$ is a polynomial in $n+1$ variables $x_0, \ldots, x_n$ which is double homogeneous in the following sense: (1) the degree of each monomial is the same number $k+1$, and (2) the sum of subscripts of variables in each monomial is the same number $n$.

**Definition.** The *midsection support* of $P$ is the set of integer points $(a_1, \ldots, a_{k-1}) \in \mathbf{Z}^{k-1}$ such that the coefficient of $x_{a_1} \ldots x_{a_{k-1}} x^2_{(n-a_1-\ldots-a_{k-1})/2}$ in $P$ does not vanish. We denote by $M_{n_1,\ldots,n_k}$ the midsection support of the polynomial $P_{gl_N}(B_{n_1,\ldots,n_k})$

**Definition.** A countable family $\{M_\alpha\}$ of subsets of a common set is said to be *independent* if these sets can be linearly ordered as $M_1, M_2, M_3, \ldots$ in such a way that $M_i \setminus (M_1 \cup \ldots \cup M_{i-1}) \neq \emptyset$ for all $i$.

It is obvious that if the midsection supports $M_{n_1,\ldots,n_k} \subset \mathbf{Z}^{k-1}$ are independent then the polynomials $P_{gl_N}(B_{n_1,\ldots,n_k})$ are linearly independent.

To prove the independence $\{M_{n_1,\ldots,n_k}\}$ we order the collections $(n) = (n_1, \ldots, n_k)$ lexicographically using the sequence of linear functions:

$$\begin{aligned}
&l_1(n_1, \ldots, n_k) = n_1 + n_2 + \ldots + n_{k-2} + n_{k-1};\\
&l_2(n_1, \ldots, n_k) = n_1 + n_2 + \ldots + n_{k-2};\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&l_{k-2}(n_1, \ldots, n_k) = n_1 + n_2;\\
&l_{k-1}(n_1, \ldots, n_k) = n_1\ .
\end{aligned}$$

This means that $(n') < (n)$ if and only if there exists a $t$ $(1 \leq t \leq k-1)$ such that $l_i(n'_1, \ldots, n'_k) = l_i(n_1, \ldots, n_k)$ for $i < t$ and $l_t(n'_1, \ldots, n'_k) < l_t(n_1, \ldots, n_k)$.

Now the independence of $\{M_{n_1,\ldots,n_k}\}$ follows from the fact that the set

$$M_{n_1,\ldots,n_k} \setminus \bigcup_{(n')<(n)} M_{n'_1,\ldots,n'_k}$$

contains the point $(n_1, \dots, n_{k-1})$ and thus is not empty. We divide the proof of this fact into two lemmas.

**5.5. Lemma.** $(n_1, \dots, n_{k-1}) \in M_{n_1,\dots,n_k}$.

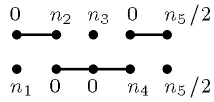
**Proof.** We have to prove that the coefficients of all monomials $x_{n_1} x_{n_2} \dots x_{n_{k-1}} x^2_{n_k/2}$ that appear in the right-hand side of

$$P_{gl_N}(B_{n_1,\dots,n_k}) = \sum_{i_1,\dots,i_k} (-1)^{j_1+\dots+j_k} \binom{n_1}{i_1} \dots \binom{n_k}{i_k} \sum_\sigma x_{\sigma_0} x_{\sigma_1} \dots x_{\sigma_k}$$

sum up to a number different from zero.

We will show that for an arbitrary scheme $\sigma$ there is one and only one choice of $i_1, \dots, i_k$ which yields the given monomial $x_{n_1} x_{n_2} \dots x_{n_{k-1}} x^2_{n_k/2}$, and all the coefficients have the same sign.

In fact, for each scheme $\sigma$ there exists at least one collection $i_1, \dots, i_k$ such that the corresponding monomial is equal to $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$. To see this, let us assign zero to those vertices of $\sigma$ which are not right hand endpoints of a connected component of $\sigma$. If the right endpoint of a connected component of $\sigma$ belongs to the $s$-th pair ($s < k$) then we assign $n_s$ to it. Finally, assign $n_k/2$ to either vertex of the $k$-th pair. Here is an example of such an assignment:




.

So, for this scheme $\sigma$ we put $i_1 = n_1$, $i_2 = i_3 = 0$, $i_4 = n_4$, $i_5 = n_5/2$. In general, the sum of indices assigned to the $t$-th component of $\sigma$ is equal to $n_t$ with two exceptions of components containing the two vertices of the $k$-th pair. For these two exceptional components the sum will be $n_k/2$. Therefore, such an assignment always gives the monomial $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$. For these assignments $j_s$ ($s < k$) is either 0 or $n_s$. Since all $n_s$ are even, we have one and the same sign $(-1)^{n_k/2}$ for all monomials corresponding to our assignments. So the sum of the coefficients is not zero.

Now we would like to show that the collections $i_1, \dots, i_k$ constructed above for a scheme $\sigma$ are the only collections that can lead to the monomial $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$. Indeed, fix a scheme $\sigma$ and consider a collection $i_1, \dots, i_k$ such that the associated monomial is $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$. In this way the indices $j_1, \dots, j_k$ get assigned to the upper vertices of $\sigma$ and the indices $i_1, \dots, i_k$ — to the lower vertices. Since the variable $x_{n_k/2}$ enters in the monomial twice, there must be precisely two connected components of $\sigma$ such that the sum of indices assigned to their vertices is equal to $n_k/2$. The last inequality given in the premises of Main Theorem 2.4 means that $n_1 + \dots + n_{k-1} < n_k/2$. This implies that each of the two components giving $n_k/2$ must contain one of the two vertices of the last column in $\sigma$. By the definition of a scheme (4.3) one of these components consists of only one vertex, so the number assigned to this vertex must be $n_k/2$. Since the sum of the two is $n_k$, the number assigned to another vertex must also be equal to $n_k/2$ and thus all the remaining vertices of its connected component have zeros. Therefore, the numbers assigned to all vertices of the two rightmost components of $\sigma$ are exactly the same as in the previous paragraph. The second biggest index of variables in the monomial under study is $n_{k-1}$. To match it, there must be a component in $\sigma$ with the sum of indices equal to $n_{k-1}$. This component is obliged to contain a vertex from $k-1$-th column because of the assumptions of Main Theorem 2.4. One of the two numbers in this column is zero, according to previous argument. Hence the numbers assigned to the vertices of this component follow the same pattern: all are zero except for the last one which is equal to $n_{k-1}$. Continuing this process, we deduce that the distribution of numbers assigned to all vertices must be the same that we had in the previous paragraph.

The lemma is proved.

**5.6. Lemma.** *If* $(n') < (n)$ *then* $(n_1, \dots, n_k) \notin M_{n'_1,\dots,n'_k}$.

**Proof.** We have to prove that the polynomial $P_{gl_N}(B_{n'_1,\dots,n'_k})$ contains no monomial $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$.

By our definition of order (Sec. 5.2) there exists $t$ $(1 \le t \le k-1)$ such that

$$\begin{array}{rcl} n'_1 + \ldots + n'_{k-1} & = & n_1 + \ldots + n_{k-1}; \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ n'_1 + \ldots + n'_{k-t+1} & = & n_1 + \ldots + n_{k-t+1}; \\ n'_1 + \ldots + n'_{k-t} & < & n_1 + \ldots + n_{k-t}\ . \end{array}$$

It is sufficient to show that each monomial of $P_{gl_N}(B_{n'_1,\dots,n'_k})$ contains $k-t$ variables such that the sum of their indices is no greater than $n'_1 + \ldots + n'_{k-t}$. Indeed, the smallest possible sum of indices of $k-t$ variables in $x_{n_1} \dots x_{n_{k-1}} x^2_{n_k/2}$ is $n_1 + \ldots + n_{k-t}$ which is strictly greater than $n'_1 + \ldots + n'_{k-t}$.

Consider a scheme $\sigma$. By Definition 4.3, in $s$-th pair $(1 \le s \le k-1)$ there is a unique vertex which is the right hand endpoint of a connected component of $\sigma$. So, for $s = k-t$ there exists exactly $k-t$ connected components of $\sigma$ whose right endpoints belong to at most $(k-t)$-th pair. According to Definition 4.4 of a monomial, the sum of indices of variables corresponding to these $k-t$ components is no greater than $n'_1 + \ldots + n'_{k-t}$.

The lemma, Proposition 5.1 and Main Theorem 2.4 are proved.

# 6 Appendix

Here we explain how to find an asymptotical lower bound for $\dim \mathcal{A}_d$ from an asymptotical lower bound for $\dim \mathcal{P}_d$, where $\mathcal{A} = \oplus_d \mathcal{A}_d$ is a graded Hopf algebra and $\mathcal{P} = \oplus_d \mathcal{P}_d$ is its space of primitive elements.

Suppose that $\dim \mathcal{P}_n = p_n$ and choose a linear basis in every $\mathcal{P}_n$. The basis of $\mathcal{A}_d$ is formed by all monomials $x_1 x_2 \dots x_s$ where $x_i$'s are basic primitive elements the sum of whose degrees equals $d$. The idea is to take $d = mn$ and count the monomials which are products of $m$ basic elements of only one graded component $\mathcal{P}_n$. Their number is $\binom{p_n+m-1}{m} > (\frac{p_n}{m})^m = (\frac{np_n}{d})^{d/n}$. For a fixed $d$ we then find the maximum of this expression as a function of $n$.

The first entry of the following table is a well-known result about the asymptotical behaviour of the number of partitions. The rest is obtained by the above described trick. Columns 2 and 3 represent the bounds for the number of Vassiliev invariants actually proved in this paper; the numbers in column 4 correspond to the conjecture of Kontsevich–Bar-Natan. For a detailed discussion of all these estimates we refer the reader to [St2].

| If $\dim \mathcal{P}_d \gtrsim$ | $1$ | $d^k$ | $d^{\log d}$ | $e^{C\sqrt{d}}$ |
|---|---|---|---|---|
| Then $\dim \mathcal{A}_d \gtrsim$ | $e^{C\sqrt{d}}$ | $e^{Cd^{1-1/k}}$ | $e^{de^{-\sqrt{\log d}}}$ | $e^{d/\log d}$ |

**Note added in proof.** After this paper was submitted to the journal O. Dasbach and M. Kontsevich independently found improvements of our estimate for $\dim(\mathcal{P}_n)$. M. Kontsevich (private communication) proved the estimate $\dim(\mathcal{P}_n) > e^{\pi\sqrt{n/3}}$ for $n \to +\infty$. In fact Kontsevich reinvented a method quite similar to the one given in this paper but he considered a different family of Primitive Chinese Characters and looked at monomials different from ours. O. Dasbach got a better estimate $\dim(\mathcal{P}_n) > \frac{C}{n\sqrt{n}} e^{\pi\sqrt{2n/3}}$ as $n \to +\infty$, for some positive constant $C$ and proved that asymptotically it is the best possible (up to a multiplication by a rational function in $n$) one can get using the Lie algebra $gl_N$ (O. Dasbach, *'On the Combinatorial Structure of Primitive Vassiliev Invariants, III — A Lower Bound'*, preprint, Universität Düsseldorf, 1997). Dasbach's proof basically relies on the method worked out in this paper, but the family of Primitive Chinese Characters he uses is different both from ours and from the Kontsevich's one.

# References


[BN1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423–472.

[BN2] D. Bar-Natan, *Some computations related to Vassiliev knot invariants*, Preprint (edition: May 5, 1996).

[BN3] D. Bar-Natan, *On the Vassiliev knot invariants*, Preprint (edition: August 21, 1992).

[CD] S. V. Chmutov, S. V. Duzhin, *An upper bound for the number of Vassiliev knot invariants*, Journal of Knot Theory and its Ramifications. **3** (1994) 141–151.

[CDL3] S. V. Chmutov, S. V. Duzhin, S. K. Lando, *Vassiliev knot invariants III. Forest algebra and weighted graphs*, Advances in Soviet Math., **21** (1994) 135-145.

[CV] S. Chmutov, A. Varchenko, *Remarks on the Vassiliev knot invariants coming from* $sl_2$, Topology **36** (1997) 153–178.

[Ch1] S. V. Chmutov, *A proof of Melvin–Morton conjecture and Feynman diagrams*, preprint 6-96 of the University of Liverpool (May, 1996), available via anonymous file transfer from `ftp.botik.ru`, subdirectory `pub/local/zmr`, file name `memo_ps.gz`.

[Ch2] S. V. Chmutov, *Combinatorial analog of the Melvin–Morton conjecture*, Preprint, (September, 1996), available via anonymous file transfer from `ftp.botik.ru`, subdirectory `pub/local/zmr`, file name `mm-waseda.ps.gz`. to appear in *Proceedings of KNOT'96.*

[Da1] O. Dasbach. *On subspaces of the space of Vassiliev invariants*, Ph.D. thesis. Düsseldorf, August 1996.

[Da2] O. Dasbach. *On the combinatorial structure of primitive Vassiliev invariants II*, Preprint, December 1996.

[Du] S. Duzhin. *A quadratic lower bound for the number of primitive Vassiliev invariants*, Extended abstract, KNOT'96 Conference/Workshop report, Waseda University, Tokyo, July 1996, p. 52–54.

[K] M. Kontsevich, *Vassiliev's knot invariants*, Adv. in Soviet Math., **16** Part 2 (1993) 137–150.

[MM] P. M. Melvin, H. R. Morton, *The coloured Jones function*, Comm. Math. Physics. **169** (1995) 501–520.

[Ng] K. Y. Ng, *Groups of Ribbon Knots*, Preprint (1995), to appear in Topology.

[Pe] R. Penrose, *Applications of negative dimensional tensors*, In: Combinatorial mathematics and its applications (ed. D. J. A. Welsh), New York, London: Academic Press, 1971.

[St1] A. Stoimenow, *Vassiliev invariants are bounded by factorial over exponential in the degree*, Preprint, September 1996.

[St2] A. Stoimenow, *The dimension of a graded commutative algebra and asymptotics of Vassiliev invariants*, Preprint, December 1996.

[V] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and Its Applications (ed. V. I. Arnold), Advances in Soviet Math., **1** (1990) 23–69.

[Zh] D. P. Zhelobenko, *Compact Lie groups and their representations*, Moscow, Nauka, 1970 (AMS translation, Providence, 1973).

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