The Matiyasevich polynomial, four colour theorem and weight systems

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Started on 00.01.06. This version 00.01.06

Abstract

We show that a modification of the polynomial introduced by Yu. Matiyasevich for sphere triangulations, can be defined to be an invariant of 3-graphs and thus provide a weight system for finite type knot invariants. The relation of this construction with the four-colour theorem is also discussed.

1 Weight systems

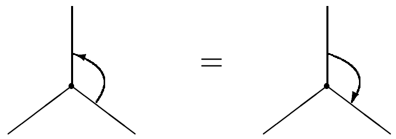
Weight systems are crucial in the combinatorial study of finite type (Vassiliev) knot invariants. They are defined as functions on the spaces generated by graphs with all vertices of valency 1 or 3, satisfying certain relations. For the sake of this exposition, I will give a precise definition of a weight system on the space of 3-graphs (see [CDK]). Relation of the space of 3-graphs to other spaces of diagrams is clarified in [CDK].

Definition 1 *A 3-graph is a regular 3-valent graph*[1] *with a fixed rotation. The rotation is the choice of a cyclic order of edges at every vertex, i. e. one of the two cyclic permutations in the set of three edges adjacent to this vertex.*

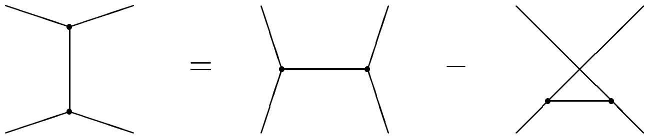
[1]Note that the number of vertices of such a graph is always even, and the number of edges a multiple of 3. Half the number of the vertices is referred to as the *degree* of a 3-graph.

Definition 2 *The space Γ_n is the quotient space of the linear space over $\mathbb{Q}$ generated by connected 3-graphs of degree n (i. e. having $2n$ vertices), modulo the following relations.*

AS (antisymmetry) relation:



IHX relation:



Definition 3 *A weight system of degree n is a function on the space Γ_n with values in some Abelian group. In other words, a weight system is a function of a 3-graph which satisfies the AS and IHX relations.*

A well known construction of weight systems is Kontsevich's construction of the invariant with values in the universal enveloping algebra of a Lie algebra equipped with a fixed ad-invariant non-degenerate symmetric bilinear form (see [Kon] for the original construction, [BN1] for the specialization to linear representations and [CDK] for the case of the algebra Γ). In this construction, one associates a copy of the structure tensor of the Lie algebra $\mathfrak{g}$ to every vertex and then makes the contraction over all edges. The structure tensor, moved into the space $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ by means of the metric, is a totally antisymmetric element of this space, which ensures the compatibility with the AS relation. The IHX relation follows from the Jacobi identity.

2 Matiyasevich's polynomial

Let $\mu : V(G) \to \{1, 2, \ldots, 3n\}$ be a numbering of the set of edges of the graph G. We assign an independent variable x_i, $i = 1, \ldots, 3n$, to the edge number i and, with every vertex $v \in V(G)$, we associate the polynomial

$$(v_1 - v_2)(v_2 - v_3)(v_3 - v_1), \tag{1}$$

if v_1, v_2, v_3 are the variables assigned to the three edges meeting at v, taken in the order consistent with the rotation at v. Set

$$\mathrm{M}_\mu(G) = \prod_{v \in V(G)} (v_1 - v_2)(v_2 - v_3)(v_3 - v_1).$$

This is the *numbered* Matiyasevich's polynomial. To obtain an invariant object, symmetrize $\mathrm{M}_\mu(G)$ over all numberings μ, or over all permutations of x_1, ..., x_{3n}:

$$\mathrm{M}(G) = \frac{1}{(3n)!} \sum_{\sigma \in \mathrm{Perm}(3n)} \prod_{v \in V(G)} (\sigma(v_1) - \sigma(v_2))(\sigma(v_2) - \sigma(v_3))(\sigma(v_3) - \sigma(v_1)).$$

Theorem 1 *The Matiyasevich's polynomial* $\mathrm{M} : \Gamma_n \to S\mathbb{Q}[x_1, \ldots, x_{3n}]$ *is a weight system on the space* Γ_n *with values in the space of symmetric polynomials in* $3n$ *variables.*

Proof. The AS relation, as well as the correctness of the definition of M, follow from the fact that expression (1) is totally antisymmetric with respect to the permutations of v_1, v_2 and v_3. The IHX relation is a consequence of the following remarkable polynomial identity:

$$(a - b)(c - d) + (b - c)(a - d) + (c - a)(b - d) = 0. \qquad (2)$$

Relation with the $so(3)$ *weight system.* In general, a weight system for 3-graphs can be constructed from any object which is skew-commutative in 3 variables. Both in the case of $so(3)$ and in the case of Matiyasevich, we assign a certain element of $\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$ to every vertex of the graph, and this element is totally antisymmetric with respect to some action of the permutation group on 3 symbols. But the group actions in question are different:

- In the $so(3)$ case, the space $\mathbb{R}^3$ is identified with $so(3)$ and the group $\mathrm{Perm}(3)$ acts in
$$\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$$
by permutations of the three factors of the tensor product.

- In Matiyasevich's case, $\mathbb{R}^3$ is the linear span of the formal variables v_1, v_2, v_3, while the group $\mathrm{Perm}(3)$ acts in
$$\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \mathbb{R}^3$$
by the same permutaion of the bases in each of the three factors.

Remark. Polynomial identity 2 is the analogue of Jacobi identity in some sense which still needs to be understood.

3 Four colour theorem

Definition 4 *An edge 3-colouring of a three-valent graph is a colouring of the set of its edges with three colours, such that the three edges meeting at any vertex have different colours.*

Definition 5 *The number of colourings of a 3-graph G is defined as*

$$\chi(G) = \sum_{\substack{\text{edge} \\ \text{3-colourings of } G}} \prod_{v \in V} \epsilon_v \,,$$

where ϵ_v is 1 or -1, depending on whether the order of the three colours meeting at v agrees with the rotation at v.

The famous four colour theorem is equivalent to the assertion that $\chi(G) \neq 0$ for any planar 3-graph G. As explained in [BN2] and [CDK], the number $\chi(G)$ is equal to the value of the $so(3)$ weight system on G.

To explain the relation of the Matiyasevich's polynomial to the four colour theorem, we introduce the *reduced Matiyasevich's polynomial* $\mathrm{RM}(G)$ as follows:

$$\mathrm{RM}(G) = \mathrm{M}(G)|_{x_i^3 \mapsto 1},$$

In other words, $\mathrm{RM}(G)$ is obtained from $\mathrm{M}(G)$ by reducing all the exponents modulo 3.

Theorem 2 *The constant term of the reduced Matiyasevich polynomial is equal to the number of edge colourings of the 3-graph:*

$$\mathrm{RM}(G)|_{x_i=0} = \chi(G).$$

PROOF. We will prove that the constant term of every numbered reduced Matiyasevich's polynomial (before symmetrization) is equal to $\chi(G)$. Indeed, for any edge numbering μ, the polynomial $\mathrm{M}_\mu(G)$ is the product of polynomials

$$v_1^2 v_2 + v_2^2 v_3 + v_3^2 v_1 - v_2^2 v_1 - v_3^2 v_2 - v_1^2 v_3.$$

The choice of a term in every such polynomial is equivalent to a colouring of the set of half-edges of the given graph by three colours: 0, 1 and 2. For example, the term $v_1^2 v_2$ means putting a 2 on the half-edge corresponding to v_1, a 1 on the half-edge corresponding to v_2 and a 0 on the half-edge corresponding to v_3. Contributions to the constant term of $M_\mu(G)$ arise from such colourings where the two ends of every edge either have colours $\{0,0\}$ or colours $\{1,2\}$. We claim that the algebraic number of such colourings is equal to the algebraic number of conventional edge 3-colourings, where both endpoints of every edge have the same colour. Indeed, take an half-edge 3-colouring and delete all edges of colour 0. What remains is the disjoint union of several cycles. If all the cycles are of even length, then, for each of them, there are exactly two edge colourings and exactly two half-edge colourings, and all the signs agree. If there is at least one cycle of odd length, then there is no edge colouring, while the half-edge colourings appear in mutually cancelling pairs.

4 Open problems

1. Is the polynomial M independent of classical Lie algebra weight systems?

2. Is it possible to detect knot inversion by polynomial M? In other words, is there a Chinese character (in the sense of [BN1]) with an odd number of univalent vertices whose M-polynomial is different from zero?

3. The polynomial M is related to Lie algebra $so(3)$ in the sense described above. Are there any generalizations of M for other Lie algebras?

5 Acknowledgments

I thank S. Chmutov for permanent fruitful discussions and S. Mechveliani for his help in computations using computer algebra system DoCon2 ([DoC]). I am also indebted to T. Kohno for encouraging me to prepare this paper for the “Art of Low Dimensional Topology VI”, Kyoto, January 2000.

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