Varchenko's View of Quantum Groups

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Main theorem. *The diagram*

$$\begin{array}{ccc} V(M_1\otimes\ldots\otimes M_n)_{m-2k} & \stackrel{S}{\longrightarrow} & V(M_1^*\otimes\ldots\otimes M_n^*)_{m-2k} \\ \downarrow\varphi_C & & \downarrow\varphi_C \\ H_k(P_{n,k}(z)\setminus U_{n,k}(z);\mathcal{F}(z))_{ant} & \stackrel{I}{\longrightarrow} & H_k(P_{n,k}(z),U_{n,k}(z);\mathcal{F}(z))_{ant} \end{array}$$

is commutative. The isomorphisms φ_C are horizontal with respect to the Gauss-Manin connection and send the R-matrix representation of the pure braid group in the upper line to its monodromy representation in the lower line.

Remark. The upper line is the 'contravariant operator' between 'vacuum spaces'— classical objects in the theory of quantum groups. The lower line features their geometric interpretation discovered by Varchenko.

Data that define the diagram.

The *upper line* is defined by the choice of a natural number n, a non-negative integer k, a non-zero complex number $q=\exp(2\pi i/\kappa)$, $\kappa\in\mathbf{C}^*$ (parameter of the quantum group U_qsl_2), and a set $m_1,\ldots,m_n$ of arbitrary complex numbers (parameters of Verma modules).

To define the *lower line*, one has to fix, moreover, a set $z_1,\ldots,z_n$ of different complex numbers.

The *isomorphisms* φ_C are defined by a comb C with boundary $z_1,\ldots,z_n$, i.e. a collection on n non-intersecting curves in $\mathbf{C}$ such that the j-th curve coincides with $\mathrm{Re}(z)=j$ in a neighbourhood of infinity and ends at z_j.

Explanation of the upper line.

$M=M_1\otimes\ldots\otimes M_n$ is the tensor product (over $\mathbf{C}$) of Verma modules $M_i=M(m_i)$. $M^*=M_1^*\otimes\ldots\otimes M_n^*$ is the tensor product of corresponding dual modules. M_i^* is defined as the space of linear functionals on M_i that take non-zero values only on a finite number of the standard basis elements. Module structure is given by $h\psi(\cdot)=\psi(h\cdot)$, $e\psi(\cdot)=\psi(f\cdot)$, $f\psi(\cdot)=\psi(e\cdot)$.

$V(M)_{m-2k}=(M)_{m-2k}\cap\bigcap_{l=1}^{k}\ker[e^l]_q$ is the *vacuum subspace* of level k. Here $(M)_{m-2k}$ is the eigenspace of h with eigenvalue $m-2k$ (note: the entire M is the direct sum of M_{m-2k} over all non-negative integers k) and the operator $[e^l]_q$ is defined as $(1/c_{l,q})e^l$, if $c_{l,q}=(q^{1/2}+q^{-1/2})(q+1+q^{-1})\ldots(q^{(l-1)/2}+\ldots q^{-(l-1)/2})\neq 0$, otherwise one has to consider the limit of an appropriate family of operators $[e^l]_{\tilde q}$ as $\tilde q\to q$. It may happen that, for some $v\in M$, $ev=0$ but $[e^l]_qv\neq 0$, although $[e^l]_q$ looks like a multiple of e^l. If q is not a root of 1, then $V(M)_{m-2k}=(M)_{m-2k}\cap\ker e$. If q is a primitive root of $q^p=1$, then $V(M)_{m-2k}=(M)_{m-2k}\cap\ker e\cap\ker[e^p]_q$.

For the dual module, the vacuum subspace is simply $V(M^*)_{m-2k}=(M^*)_{m-2k}\cap\ker e$.

The operator $S:M_1\otimes\ldots\otimes M_n\to M_1^*\otimes\ldots\otimes M_n^*$ takes $x_1\otimes\ldots\otimes x_n$ into $B(x_1,\cdot)\otimes\ldots\otimes B(x_n,\cdot)$ where the bilinear form B is symmetric and satisfies $B(v,v)=1$, $B(fx,y)=B(x,ey)$.

Explanation of the lower line.

$z=(z_1,\ldots,z_n)$ is a point in the complex space $\mathbf{C}^n$. $P_{n,k}$ is the fibre over z of the projection $\mathbf{C}^{n+k}\to\mathbf{C}^n:(t_1,\ldots,t_{n+k})\mapsto(t_1,\ldots,t_n)$. $U_{n,k}=P_{n,k}\cap U_{n+k}$ where U_{n+k} is the union of all diagonal hyperplanes in $\mathbf{C}^{n+k}$.

$\mathcal{F}$ is the local system of coefficients defined by the function

$$F(t_1,\ldots,t_k)=\Pi(z_i-z_j)^{m_im_j/2\kappa}\Pi(z_i-t_j)^{-m_i/\kappa}\Pi(t_i-t_j)^{2/\kappa}.$$

Subscript *ant* denotes the totally antisymmetric part of the homology spaces with respect to the natural action of the permutation group of coordinates in $P_{n,k}=\mathbf{C}^k$. Operator I of the lower line is the natural operator in homology arising from the identity chain mapping.

Help for beginners

Hopf algebras.

A Hopf algebra (=bialgebra) over a basic ring K is a K-module A equipped with two coherent sructures: that of an associative algebra with unity and that of an associative coalgebra with counity. The coalgebra structure consists of comultiplication $\delta : A \to A \otimes A$ and counity $\varepsilon : A \to K$. To obtain the axioms that govern δ and ε, one has to write the standard algebra axioms in the form of commutative diagrams involving multiplication $\mu : A \otimes A \to A$ and unity $\iota : K \to A$ and take the dual diagrams. The algebra and coalgebra structures of a Hopf algebra must agree in the sense that the following 4 conditions hold:

(1) $\varepsilon(1) = 1$,

(2) $\delta(1) = 1 \otimes 1$,

(3) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$,

(4) $\delta(ab) = \delta(a)\delta(b)$.

Observing that

(1) & (3) $\Leftrightarrow$ ε is an algebra homomorphism,

(2) & (4) $\Leftrightarrow$ δ is an algebra homomorphism,

(1) & (2) $\Leftrightarrow$ ι is a coalgebra homomorphism,

(3) & (4) $\Leftrightarrow$ μ is a coalgebra homomorphism,

we can state the definition of a Hopf algebra in either of the two forms:

(A) ε and δ are algebra homomorphisms, or

(B) ι and μ are coalgebra homomorphisms.

Examples.

In the space of functions on a finite group, there are two natural Hopf algebra structures:

(1) The commutative functional algebra with comultiplication $\delta(f)(x, y) = f(xy)$.

(2) The (convolution) group algebra with $\delta(g) = g \otimes g$ on generators.

The universal enveloping algebra $U(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ is a Hopf algebra with $\varepsilon(1) = 1$, $\varepsilon(x) = 0$, $\delta(1) = 0$, $\delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathcal{G}$.

Quantum groups and Verma modules. The quantum group $A = U_q sl_2$ is an infinite dimensional associative $\mathbf{C}$–algebra whose elements are formal power series in 3 non-commutative variables e, f and h subject to relations $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = (h)_q$, where $(a)_q = q^{a/2} - q^{-a/2}$ and $q^b = \exp(2\pi i b/\kappa)$ (both definitions equally apply to formal series and complex numbers). A can be viewed as a deformation of the universal enveloping algebra $U(sl_2)$, the latter appearing when the series $(h)_q$ is replaced by its first term.

The coalgebra structure in A is defined by counity $\varepsilon(1) = 1$, $\varepsilon(x) = 0$, $x = e, f, h$ and comultiplication $\delta(h) = h \otimes 1 + 1 \otimes h$, $\delta(e) = e \otimes q^{h/4} + q^{-h/4} \otimes e$, $\delta(f) = f \otimes q^{h/4} + q^{-h/4} \otimes f$.

A Verma module over A with highest weight $m \in \mathbf{C}$ is the A–module $M(m)$ with one generator v satisfying $ev = 0$, $hv = mv$. (Choosing $m \in \mathbf{N}$ and taking $U(sl_2)$ instead of A gives the usual irreducible finite dimensional representations of sl_2.) Vectors $v_k = f^k v$, $k = 0, 1, 2, \ldots$ form a $\mathbf{C}$–basis of $M(m)$ satisfying $fv_k = v_{k+1}$, $hv_k = (m - 2k)v_k$, $ev_k = (m - k + 1)_q (k)_q / (1)_q v_{k-1}$.

If M_1, M_2 are modules over the quantum group A, then $M_1 \otimes_{\mathbf{C}} M_2$ also has the structure of an A–module: the action of an element $a \in A$ is the natural action of $\delta(a) \in A \otimes A$. Modules $M_1 \otimes M_2$ and $M_2 \otimes M_1$ are isomorphic. The isomorphism is established by PR where P is the transposition of factors, while R is the operator of multiplication by the so called R–matrix

$$R = q^{h \otimes h/4} \sum_{k \geq 0} q^{-k(k+1)/2} (1)_q^k / (k)_q! q^{kh/4} e^k \otimes q^{-kh/4} f^k,$$

where $(k)_q! = (1)_q \cdot \ldots \cdot (k)_q$.

The R-matrix representation of the pure braid group on n strings in the tensor product of n A–modules is defined by the action of the j-th braid generator σ_j as PR on the j-th and $(j + 1)$-th factors.

Homology with local coefficients. A *local system* of abelian groups on a topological space X consists in assigning a group G_x to every point $x \in X$ and an isomorphism $\tau_s : G_x \to G_y$ to every path in X beginning at x and ending at y in such a way that (i) τ_s depends only on the homotopy class of s, (ii) $\tau_s = \mathrm{id}$, if s is a constant path, and (iii) $\tau_{s's''} = \tau_{s'}\tau_{s''}$ whenever both sides make sense. A section of the local system $\mathcal{G}$ over a subset $Y \subset X$ is a family $\{ g_y \in G_y \mid y \in Y \}$ of elements going into one another under isomorphisms over the paths contained entirely in Y. This shows that the notion of the local system is equivalent to that of a locally constant sheaf.

Let F be a multivalued complex analytic function in a domain $U \subset \mathbf{C}^n$. Suppose that (i) any two branches of F differ by a non-zero multiplicative constant, (ii) F has no zeros in U. Then F defines a local system of additive groups $\mathbf{C}$ in U.

To define *homology* with coefficients in a local system, one has to modify the standard definition of singular chains by considering formal combinations $\sum g_i\varphi_i$, where $\varphi : Q^k \to X$ is a singular k–cube and g_i is a section of the local system $\varphi^*(\mathcal{G})$ over Q^k. In other words, the homology with local coefficients is nothing but the homology with coefficients in the corresponding locally constant sheaf.

Examples.

(i) Take $X = S^1$ and fix a point $\star \in S^1$. Let $G_x = \mathbf{Z}$ for every $x \in S^1$ and $\tau_s = \mathrm{id}$ or $-\mathrm{id}$ according to the parity of the number of times the path s goes through $\star$. The homology is $H_0(S^1;\mathcal{G}) = \mathbf{Z}_2$, $H_1(S^1;\mathcal{G}) = 0$.

(ii) If $\mathcal{F}$ is the system of local coefficients in $\mathbf{C}\backslash 0$ defined by the function $F(z) = z^{1/2}$, then $H_0(\mathbf{C}\backslash 0;\mathcal{F}) = 0$, $H_1(\mathbf{C} \setminus 0;\mathcal{F}) = 0$.

Relative homology with local coefficients. If $Y \subset X$, where X is equipped with a local system $\mathcal{G}$, then one can restrict $\mathcal{G}$ to Y and obtain a pair of chain complexes with local coefficients $C_k(Y;\mathcal{F}) \subset C_k(X;\mathcal{F})$ whose quotient in a standard way defines the relative homology with local coefficients.

This construction is valid only if the local system is defined on the entire space X. However, if we consider the local system $\mathcal{F}$ associated with an analytic function F as above, we can define the relative homology of the pair $\bar{U}, Z$, where $\bar{U} = U \cup Z$, $Z = F^{-1}(0)$, with coefficients in $\mathcal{F}$, as the homology of the corresponding sheaf extended to $\bar{U}$ in a natural way. Note that (i) the extended sheaf is no longer locally constant and (ii) the relative homology modulo Z coincides with absolute homology of $\bar{U}$, since all the chains of Z appear with zero coefficients.

Example. If $\mathcal{F}$ is defined by the function $F(z) = z^{1/2}$, then $H_0(\mathbf{C}, 0;\mathcal{F}) = 0$, $H_1(\mathbf{C}, 0;\mathcal{F}) = 0$.