Dedicated to V. I. Arnold on his 65th birthday

Decomposable skew-symmetric functions

S. Duzhin*

Abstract

A skew-symmetric function F in several variables is said to be decomposable if it can be represented as a determinant $\det(f_i(x_j))$ where f_i are univariate functions. We give a criterion of the decomposability in terms of an identity imposed on the function F.

1 Introduction

Vladimir Igorevich Arnold to whom I dedicate this paper, is famous for his ability to discover new properties of everyday mathematical objects, like plane curves, continuous fractions or binary forms, the properties that Euler or Gauss might have well discovered in days bygone, but which for some reason were concealed from mathematicians until now. In the present note, I am trying to mimic the style of the great teacher, presenting to the reader a simple problem, complete with solution, which sounds fairly natural and old-fashioned, but, to the best of my knowledge, was not treated in due time by the classics like Lagrange [Lag], Cayley [Cay] or Sylvester [Syl].

I have arrived to the algebraic problem discussed in this article from my studies of finite type knot invariants — more precisely, weight systems, see [BN]. (It is my pleasure to mention — in view of the dedication — that the theory of finite type knot invariants [Vas] was born in Arnold's mathematical school.)

The relation of the problem with weight systems is explained in my papers [Du1] and [Du2]. It relies on the observation that Equation 1 (see below), imposed on a skew-symmetric function of 3 variables, assures that this function can be used to construct a weight system.

2 Problem and theorem

We formulate the problem in its simplest setting, for real-valued functions of real variables. The reader will see right away that the main result generalizes without any change

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to multivariate functions $F : X^n \to \mathbb{R}$ (or $\mathbb{C}$) on irreducible real (or complex) analytic manifolds X. Further generalizations are possible, as long as two facts are valid: (1) there are no zero divisors in the ring of functions under study (2) the ground field admits extraction of roots from any element or its opposite.

Definition. A function of n variables $F : \mathbb{R}^n \to \mathbb{R}$ is said to be *skew-symmetric*, if it changes sign under any odd permutation of the arguments:

$$F(x_1, \dots, x_i, \dots, x_j, \dots, x_n) = -F(x_1, \dots, x_j, \dots, x_i, \dots, x_n).$$

It is easy to construct a skew-symmetric function in n variables from n uni-variate functions $f_i : \mathbb{R} \to \mathbb{R}$ by taking the determinant of $f_i(x_j)$:

$$F(x_1, \dots, x_n) = \begin{vmatrix} f_1(x_1) & \dots & f_1(x_n) \\ \vdots & & \vdots \\ f_n(x_1) & \dots & f_n(x_n) \end{vmatrix}.$$

We will call such skew-symmetric functions *decomposable.*

The problem that naturally arises is to describe the class of decomposable skew-symmetric functions in an inner way, through an identity imposed on the function under study.

Theorem. *A skew-symmetric analytic function $F(x_1, ..., x_n)$, $n \geq 2$ is decomposable if and only if it satisfies the identity*

$$\begin{aligned} F(x_1, x_2, ...)F(x_3, x_4, ...) - F(x_1, x_3, ...)F(x_2, x_4, ...) \\ + F(x_1, x_4, ...)F(x_2, x_3, ...) = 0, \end{aligned} \tag{1}$$

where the dots stand for a set of variables, the same for each instance of the function F.

Remark. The assumption of analyticity is essential. There exist non-analytic functions that satisfy Equation 1, but are not decomposable. A simple example is provided by the function

$$F(x, y, z) = E_{0,1,2} + E_{3,4,5},$$

where $E_{a,b,c}$ denotes the function equal to 1 at the point (a, b, c), equal to ± 1 at the points obtained from (a, b, c) by permutations of the coordinates, and 0 elsewhere.

3 Proof of the theorem

The theorem consists of two parts: easy part (sufficiency) and difficult part (necessity).

3.1 Proof of sufficiency

To prove the easy part of the theorem, we only must check the identity

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \end{vmatrix} \cdot \begin{vmatrix} a_{31} & \dots & a_{3n} \\ a_{41} & \dots & a_{4n} \\ \vdots & & \vdots \end{vmatrix} - \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{31} & \dots & a_{3n} \\ \vdots & & \vdots \end{vmatrix} \cdot \begin{vmatrix} a_{21} & \dots & a_{2n} \\ a_{41} & \dots & a_{4n} \\ \vdots & & \vdots \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{41} & \dots & a_{4n} \\ \vdots & & \vdots \end{vmatrix} \cdot \begin{vmatrix} a_{21} & \dots & a_{2n} \\ a_{31} & \dots & a_{3n} \\ \vdots & & \vdots \end{vmatrix} = 0.$$

Geometrically, this means that

$$V(v_1, v_2, w)V(v_3, v_4, w) - V(v_1, v_3, w)V(v_2, v_4, w) + V(v_1, v_4, w)V(v_2, v_3, w) = 0,$$

where v_i is the vector $(a_{i1}, \dots, a_{in})$, w stands for the set of $n-2$ vectors appearing in rows 3 through n of each matrix, and V denotes the Euclidean volume. Now choose a 2-plane P in $\mathbb{R}^n$ complementary to the subspace W spanned by w. The volumes will not change if we replace all v_i's by their orthogonal projections v_i' onto P parallel to W. Then we have: $V(v_i, v_j, w) = V(v_i', v_j')V(w)$, and the assertion follows from the 2-dimensional identity

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 0.$$

3.2 k-term relations

In what follows, we refer to Equation 1 as the *3-term relation*. The idea of the proof is that 3-term relations for a skew-symmetric function in n arguments imply k-term relations for any $k \le n+1$ (Lemma 1), and the $(n+1)$-term relations imply complete decomposability (Lemma 2).

A 4-term relation is the equation of the form

$$\begin{aligned} &F(x_1, x_2, x_3, ...)F(x_4, x_5, x_6, ...) - F(x_1, x_2, x_4, ...)F(x_3, x_5, x_6, ...) \\ +&F(x_1, x_2, x_5, ...)F(x_3, x_4, x_6, ...) - F(x_1, x_2, x_6, ...)F(x_3, x_4, x_5, ...) = 0 \end{aligned} \tag{2}$$

where the dots stand for one and the same set of variables, if $n > 3$. Below we will use shorthand notation for equations of this sort, writing $[i_1, \dots, i_k]$ instead of $F(x_{i_1}, \dots, x_{i_k}, \dots)$ (in particular, suppressing the trailing dots). In this notation, Equation 2 reads as follows:

$$[1,2,3] \cdot [4,5,6] - [1,2,4] \cdot [3,5,6] + [1,2,5] \cdot [3,4,6] - [1,2,6] \cdot [3,4,5] = 0.$$

Using skew-commutativity, this can be rewritten as

$$[1,2,3] \cdot [4,5,6] = [1,2,4] \cdot [3,5,6] + [1,2,5] \cdot [4,3,6] + [1,2,6] \cdot [4,5,3].$$

Note that the three summands on the right are obtained from the expression on the left by all possible transpositions of the number 3 and the numbers in the second pair of brackets.

In general, by a $(k+1)$-term relation for a function in $n \geq k$ variables we understand the identity (in shorthand notation)

$$\sum_{i=k}^{2k} (-1)^i [1, \ldots, k-1, i] \cdot [k, k+1, \ldots, \widehat{i}, \ldots, 2k] = 0$$

or, equivalently,

$$[1, \ldots, k] \cdot [k+1, \ldots, 2k] = \sum_{i=k+1}^{2k} S_k^i [1, \ldots, k] \cdot S_i^k [k+1, \ldots, 2k], \tag{3}$$

where S_i^j denotes the substitution of j instead of i in a sequence of numbers. Of course, the numbers $1, 2, \ldots, 2k$ in these relations can be replaced by another set of $2k$ different numbers. The choice of a specific expansion of type (3) depends on the choice of the *jumper*, i.e. the number that changes places with its counterparts in another pair of square brackets. In Equation 3, the role of the jumper belongs to the number k, and we can rewrite it in short as follows:

$$[1, \ldots, k] \cdot [k+1, \ldots, 2k] = R_k(1, \ldots, k; k+1, \ldots, 2k),$$

In general, suppose that two sets $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_k\}$ have $s < k$ common elements and the number m belongs to one and only one of these sets. Set

$$R_m(I; J) = \begin{cases} \sum\limits_{j \in J \setminus I} S_m^j [I] \cdot S_j^m [J], & \text{if } m \in I, \\ \sum\limits_{i \in I \setminus J} S_i^m [I] \cdot S_m^i [J], & \text{if } m \in J. \end{cases}$$

Then the equality

$$[i_1, \ldots, i_k] \cdot [j_1, \ldots, j_k] = R_m(i_1, \ldots, i_k; j_1, \ldots, j_k), \tag{4}$$

is the general form of writing a $(k-s+1)$-term relation.

3.3 Proof of necessity

The proof that any skew-symmetric function in n variables that satisfies 3-term relations also satisfies k-term relations for any $k \leq n+1$, is inductive on k.

Lemma 1. *If F is a skew-symmetric analytic function in $n \geq k$ variables that satisfies s-term relations for all $s \leq k$, then F satisfies $(k+1)$-term relations.*

Proof. We start with two examples.

Example 1: How to deduce 4-term relations from 3-term relations.
Multiplying the left-hand side of Equation 2 by $[1,3,5]$, we get:

$$\begin{gathered}[1,3,5]\cdot[1,2,3]\cdot[4,5,6]-[1,3,5]\cdot[1,2,4]\cdot[3,5,6]\\+[1,3,5]\cdot[1,2,5]\cdot[3,4,6]-[1,3,5]\cdot[1,2,6]\cdot[3,4,5].\end{gathered}$$

By 3-term relations,

$$\begin{aligned}[1,3,5]\cdot[1,2,3]\cdot[4,5,6]&=[1,2,3]([1,4,5]\cdot[3,5,6]+[1,6,5]\cdot[4,5,3])\\ [1,3,5]\cdot[1,2,4]\cdot[3,5,6]&=[3,5,6]([1,4,5]\cdot[1,2,3]+[1,3,4]\cdot[1,2,5])\\ [1,3,5]\cdot[1,2,5]\cdot[3,4,6]&=[1,2,5]([1,3,4]\cdot[3,5,6]+[1,3,6]\cdot[3,4,5])\\ [1,3,5]\cdot[1,2,6]\cdot[3,4,5]&=[3,4,5]([1,6,5]\cdot[1,2,3]+[1,3,6]\cdot[1,2,5]),\end{aligned}$$

and the alternating sum of the right-hand parts is 0.

Example 2: How to deduce 5-term relations from 3-term and 4-term relations.
In shorthand notation, a 5-term relation says that the combination

$$\begin{gathered}[1,2,3,4]\cdot[5,6,7,8]-[1,2,3,5]\cdot[4,6,7,8]+[1,2,3,6]\cdot[4,5,7,8]\\-[1,2,3,7]\cdot[4,5,6,8]+[1,2,3,8]\cdot[4,5,6,7]\end{gathered}$$

is equal to 0. Multiply this combination by $[1,3,5,7]$ and then express each summand through alternatively a 3-term or a 4-term relation, using the numbers 4, 5, 6, 7, 8 as the respective jumper:

$$\begin{aligned}[1,3,5,7]\cdot[1,2,3,4]\cdot[5,6,7,8]&=[5,6,7,8]([1,3,4,7]\cdot[1,2,3,5]+[1,3,5,4]\cdot[1,2,3,7])\\ [1,3,5,7]\cdot[1,2,3,5]\cdot[4,6,7,8]&=[1,2,3,5]([1,3,4,7]\cdot[5,6,7,8]+[1,3,6,7]\cdot[4,5,7,8]\\&\qquad+[1,3,8,7]\cdot[4,6,7,5])\\ [1,3,5,7]\cdot[1,2,3,6]\cdot[4,5,7,8]&=[4,5,7,8]([1,3,5,2]\cdot[1,7,3,6]+[1,3,5,6]\cdot[1,2,3,7])\\ [1,3,5,7]\cdot[1,2,3,7]\cdot[4,5,6,8]&=[1,2,3,7]([1,3,5,4]\cdot[7,5,6,8]+[1,3,5,6]\cdot[4,5,7,8]\\&\qquad+[1,3,5,8]\cdot[4,5,6,7])\\ [1,3,5,7]\cdot[1,2,3,8]\cdot[4,5,6,7]&=[4,5,6,7]([1,3,8,7]\cdot[1,2,3,5]+[1,3,5,8]\cdot[1,2,3,7])\end{aligned}$$

It is easy to see that the right-hand parts of these equations, taken with alternating signs, sum up to 0.

In general, we must prove the relation

$$\sum_{i=k}^{2k}(-1)^i[1,\ldots,k-1,i]\cdot[k,k+1,\ldots,\widehat{i},\ldots,2k]=0$$

provided that the function under study satisfies s-term relations for all $s\le k$. We multiply the left-hand part of this equation by $[1,3,\ldots,2k-1]$ and prove that the result is 0. In the sum of all terms with an odd i we put the factor $[1,\ldots,k-1,i]$ on the left, and express the product of the two remaining brackets by an appropriate s-term relation (in the form of Equation 4), using i as the jumper:

$$\sum_{\text{odd } i\in[k,2k]} [1,\ldots,k-1,i]\cdot R_i(1,3,\ldots,2k-1;k,k+1,\ldots,\widehat{i},\ldots,2k).$$

In the terms with an even i we factor out $[k,k+1,\ldots,\widehat{i},\ldots,2k]$, and rewrite the product of the two remaining brackets by an s-term relation, also using i as the jumper:

$$\sum_{\text{even } i\in[k,2k]} [k,k+1,\ldots,\widehat{i},\ldots,2k]\cdot R_i(1,3,\ldots,2k-1;1,\ldots,k-1,i).$$

(The value of s is defined by the number of coinciding elements in the two groups of indices of length k in the argument of R_i; it depends on the parity of k and is equal to $k/2+1$ for even k and to $(k+1)/2$ or $(k+3)/2$ for odd k.)

It does not take long to see that the two sums actually consist of the same terms, only with some sequences differing one from another by a cyclic permutation of odd length. Therefore, the alternating sum is 0. Dividing by $[1,3,\ldots,2k-1]$ (which is possible due to analiticity), we get the result. $\square$

Lemma 2. *If $F:\mathbb{R}^n\to\mathbb{R}$ is a skew-symmetric analytic function of n variables that satisfies $(n+1)$-term relations, then F is decomposable, i.e. can be expressed through n functions of one variable as the determinant* $\det(f_i(x_j))$.

Proof. Let p_i^j, $1\le i\le n$, $1\le j\le n-1$ be a collection of fixed numbers. Setting

$$f_i(x)=F(p_i^1,\ldots,p_i^{n-1},x).$$

for $i=1,\ldots,n$, we obtain n univariate functions. We will prove that the constants p_i^j can be chosen in such a way that

$$F(x_1,\ldots,x_n)=c\begin{vmatrix} f_1(x_1) & \ldots & f_1(x_n)\\ \vdots & & \vdots\\ f_n(x_1) & \ldots & f_n(x_n)\end{vmatrix}$$

for an appropriate constant $c\in\mathbb{R}$. To this end, consider the $(n+1)\times(n+1)$-matrix

$$M=\begin{pmatrix} F(x_1,\ldots,x_n) & -F(x_0,x_2\ldots,x_n) & \ldots & (-1)^nF(x_0,\ldots,x_{n-1})\\ f_1(x_0) & f_1(x_1) & \ldots & f_1(x_n)\\ \vdots & \vdots & & \vdots\\ f_n(x_0) & f_n(x_1) & \ldots & f_n(x_n)\end{pmatrix}$$

Due to $(n+1)$-term relations, the first row of this matrix is orthogonal to all other rows. Below we will see that for a clever choice of the constants p_i^j and generic values of x_i's rows number 2 through $(n+1)$ form a matrix of rank n; therefore, their vector product must be proportional to the first row of the matrix:

$$F(x_0, \ldots, \widehat{x_i}, \ldots, x_n) = c \begin{vmatrix} f_1(x_1) & \ldots & f_1(x_n) \\ \vdots & & \vdots \\ f_n(x_1) & \ldots & f_n(x_n) \end{vmatrix}$$

A priori, c may depend on $x_0, \ldots, x_n$. However, the above relation for a given i assures that c does not depend on x_i. Since i takes all values from 0 to n, we conclude that $c \in \mathbb{R}$ is a constant. Note, finally, that the multiplier c can be eliminated using two operations: change of sign (equivalent to the transposition of, say, f_1 and f_2) and multiplication by λ^n, $\lambda \in \mathbb{R}$ (equivalent to the change $f_1 \mapsto \lambda f_1$).

It remains to check that the rows number 2 through $(n+1)$ of the matrix M can be assumed to be linearly independant in $\mathbb{R}^n$. This assertion is quite evident, and we only illustrate it in the particular case $n = 4$. We claim that the leftmost 4×4 minor in rows 2–5:

$$\begin{vmatrix} F(p_1^1, p_1^2, p_1^3, x_1) & F(p_1^1, p_1^2, p_1^3, x_2) & F(p_1^1, p_1^2, p_1^3, x_3) & F(p_1^1, p_1^2, p_1^3, x_4) \\ F(p_2^1, p_2^2, p_2^3, x_1) & F(p_2^1, p_2^2, p_2^3, x_2) & F(p_2^1, p_2^2, p_2^3, x_3) & F(p_2^1, p_2^2, p_2^3, x_4) \\ F(p_3^1, p_3^2, p_3^3, x_1) & F(p_3^1, p_3^2, p_3^3, x_2) & F(p_3^1, p_3^2, p_3^3, x_3) & F(p_3^1, p_3^2, p_3^3, x_4) \\ F(p_4^1, p_4^2, p_4^3, x_1) & F(p_4^1, p_4^2, p_4^3, x_2) & F(p_4^1, p_4^2, p_4^3, x_3) & F(p_4^1, p_4^2, p_4^3, x_4) \end{vmatrix}$$

can be assumed to be non-zero. Indeed, setting

$$\begin{aligned} x_1 &= p_4^1 = p_3^2 = p_2^3, \\ x_2 &= p_4^2 = p_3^3, \\ x_3 &= p_4^3, \end{aligned}$$

we make the matrix triangular, and its determinant is

$$F(p_1^1, p_1^2, p_1^3, p_2^3) F(p_2^1, p_2^2, p_2^3, p_3^3) F(p_3^1, p_2^3, p_3^3, p_4^3) F(p_2^3, p_3^3, p_4^3, x_4),$$

which cannot be identically zero.

This completes the proof of the lemma and thus of the theorem.

4 Open Problems

Besides the notion of (complete) decomposability, introduced above, one can define *partially decomposable* skew-symmetric functions. If $\nu = (n_1, ..., n_k)$ is a partition of n, then by a ν-decomposable skew-symmetric function of n variables we understand the complete antisymmetrization of the product of k arbitrary functions of n_1, ..., n_k variables. The partition $(1, 1, ..., 1)$ leads to completely decomposable functions, while the partition (n) yields the class of all skew-symmetric functions in n variables.

Open Problem 1. For a given ν, find a criterion of ν-decomposability.

The second problem refers to Equation 1 which is meaningful by itself, because it comes from a construction of weight systems in the theory of finite type knot invariants. For non-analytic functions, the theorem that we have proved is valid only one way, and the following problem arises.

Open Problem 2. Find a complete description of the solutions to equation 1 for non-analytic functions F. For example, it is interesting to study functions with finite support, or whose support is a hyperplane arrangement in $\mathbb{R}^n$ and the restriction of the function to each hyperplane is polynomial.

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