1 A Reduced Cosserat Model for the Flow of Granular Materials

Dilatant, Rotational Shear

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Cosserat models in granular materials

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REMARKS ON MODELLING

- Simple model - may explain trends and general features
- Complex model - may obscure understanding
- Continuum model - good for simple theories and explanations
- No standard continuum model has met with universal acceptance
- Retain as much of the classical models as possible
- Choose the simplest model possible
- Evidence for existence of couple stresses?
- Rotation in a granular material is self evident! - not present in classical models

STRESS

$\sigma-$ Cauchy stress tensor (σ_{ij})

Planar flows

Planar invariants

$$\begin{aligned} p &= -\frac{1}{2}\left(\sigma_{11}+\sigma_{22}\right) \\ q &= \frac{1}{2}\left[\left(\sigma_{11}-\sigma_{22}\right)^2+\left(\sigma_{12}+\sigma_{21}\right)^2\right]^{1/2} \\ r &= \frac{1}{2}\left(\sigma_{12}-\sigma_{21}\right) \end{aligned}$$

Angle greater principal direction (symmetric part) of the stress makes with x_1-axis

$$\tan 2\psi_\sigma = \frac{\sigma_{12}+\sigma_{21}}{\sigma_{11}-\sigma_{22}}$$

$$\begin{aligned} \sigma_{11} &= -p+q\cos 2\psi_\sigma, \qquad \sigma_{22}=-p-q\cos 2\psi_\sigma \\ \sigma_{12} &= r+q\sin 2\psi_\sigma, \qquad \sigma_{21}=-r+q\sin 2\psi_\sigma \end{aligned}$$

YIELD CONDITION

σ^s- symmetric part of the stress

$$f\left(\sigma^s\right) \leq 0$$

Coulomb-Mohr yield condition

$$q \leq p \sin\phi + c \cos\phi$$

ϕ- angle of internal friction
c- coefficient of cohesion
σ^a- anti-symmetric part of the stress

Rotational yield condition

$$|r| \leq M$$

STRESS REPRESENTATION

Prescribe $\sigma_{22} = -\sigma$

$$\begin{aligned} \sigma &= p + q\cos 2\psi_\sigma \\ p &= \frac{\sigma - c\cos\phi\cos 2\psi_\sigma}{1+\sin\phi\cos 2\psi_\sigma} \\ q &= \frac{\sigma\sin\phi + c\cos\phi}{1+\sin\phi\cos 2\psi_\sigma} \end{aligned}$$

$$\sigma_{11} = \frac{-\sigma\left(1-\sin\phi\cos 2\psi_\sigma\right) + 2c\cos\phi\cos 2\psi_\sigma}{1+\sin\phi\cos 2\psi_\sigma}$$

$$\frac{\sigma_{12}+\sigma_{21}}{2} = \frac{\left(\sigma\sin\phi + c\cos\phi\right)\sin 2\psi_\sigma}{1+\sin\phi\cos 2\psi_\sigma}$$

$$\frac{|\sigma_{12}-\sigma_{21}|}{2} \le M$$

This is a one parameter representation of the stress

KINEMATICS

Non-Cosserat Models

Rectangular Cartesian axes Ox_i

$\mathbf{v}-$ Eulerian velocity

$\mathbf{\Gamma}-$ velocity gradient tensor$\left(\frac{\partial v_i}{\partial x_j}\right)$

$\mathbf{d}-$ deformation-rate tensor: symmetric part of $\mathbf{\Gamma}$

$\mathbf{s}-$ spin tensor: anti-symmetric part of $\mathbf{\Gamma}$

Reduced-Cosserat Models

$\omega-$ intrinsic spin

$\mathbf{I}-$ moment of inertia density

NON-COSSERAT MODELS

plastic potential model

$$d_{ij} = \dot{\lambda}\frac{\partial g}{\partial \sigma_{ij}}$$

d_{ij} − deformation-rate tensor components
g − plastic potential
σ_{ij} − stress components
$\dot{\lambda}$ − scalar multiplier

Planar flows

(a) dilatancy

$$\begin{aligned} & d_{11} + d_{22} \\ = \; & \sin\chi\,[(d_{11} - d_{22})\cos 2\psi_\sigma + 2d_{12}\sin 2\psi_\sigma] \end{aligned}$$

χ− dilatancy parameter (obtained from g)
(b) coaxiality of $\mathbf{d}, \sigma$

$$(d_{11} - d_{22})\sin 2\psi_\sigma - 2d_{12}\cos 2\psi_\sigma = 0$$

double-shearing model

(a) dilatancy

$$\begin{aligned} & (d_{11}+d_{22})\cos(\phi-\nu) \\ = \;& \sin\nu\,[(d_{11}-d_{22})\cos 2\psi_\sigma + 2d_{12}\sin 2\psi_\sigma] \end{aligned}$$

(b) non-coaxiality

$$\begin{aligned} & 2\left(\dot{\psi}_\sigma - s_{21}\right)\sin(\phi-\nu) \\ = \;& \cos\nu\,[(d_{11}-d_{22})\sin 2\psi_\sigma - 2d_{12}\cos 2\psi_\sigma] \end{aligned}$$

$\phi-$ angle of internal friction
$\nu-$ angle of dilatancy

double-sliding free rotating model

$$\begin{aligned} & 2\left(\Omega - s_{21}\right)\sin(\phi-\nu) \\ = \;& \cos\nu\,[(d_{11}-d_{22})\sin 2\psi_\sigma - 2d_{12}\cos 2\psi_\sigma] \end{aligned}$$

$\Omega-$ free rotation

REDUCED COSSERAT MODEL

(a) Linear momentum

$$\rho\left(\partial_t v_1 + v_1\partial_1 v_1 + v_2\partial_2 v_1\right) = \partial_1\sigma_{11} + \partial_2\sigma_{21} + \rho F_1$$
$$\rho\left(\partial_t v_2 + v_1\partial_1 v_2 + v_2\partial_2 v_2\right) = \partial_1\sigma_{12} + \partial_2\sigma_{22} + \rho F_2$$

F_1, F_2- body force components

(b) Rotational momentum

$$\rho I\left(\partial_t\omega + v_1\partial_1\omega + v_2\partial_2\omega\right) - 2r - \rho G = 0$$

$I-$moment of inertia density

$G-$body couple

(c) Coulomb yield condition

$$q \leq p\sin\phi + c\cos\phi$$

(d) Rotational yield condition

$$\left|\sigma_{12} - \sigma_{21}\right| \leq M\left(\rho\right)$$

(e) Continuity

$$\partial_t\rho + v_1\partial_1\rho + v_2\partial_2\rho + \rho\partial_1 v_1 + \rho\partial_2 v_2 = 0$$

(f) dilatancy

$$\begin{aligned} & \left(d_{11} + d_{22}\right)\cos\left(\phi - \nu\right) \\ = \; & \sin\nu\left[\left(d_{11} - d_{22}\right)\cos 2\psi_\sigma + 2d_{12}\sin 2\psi_\sigma\right] \end{aligned}$$

(g) rotational/coaxiality equation

$$\begin{aligned} & 2\left(\omega - s_{21}\right)\sin\left(\phi - \nu\right) \\ = \; & \cos\nu\left[\left(d_{11} - d_{22}\right)\sin 2\psi_\sigma - 2d_{12}\cos 2\psi_\sigma\right] \end{aligned}$$

$\omega-$intrinsic spin, $s_{21}-$spin component

DILATANT, ROTATIONAL SHEAR

Material parameters ϕ, I constant.
angle of dilatancy: ν variable.
Not perfect plasticity.
Rate independent material.
Quasi-static loading.

Stresses homogeneous in space.
No body forces/ couples.
$\sigma_{ij} = \sigma_{ij}(t)$
Flow: dilatant shear

$$v_1 = \alpha x_2, \qquad v_2 = \beta(t) x_2, \qquad \omega = \omega(t).$$

Initial conditions

$$\alpha > 0, \qquad \beta(0) = \beta_0, \qquad \omega(0) = \omega_0$$

$|\beta(t)|$ decreasing function of t, $\beta(t) \to 0$.
$\beta = 0$ simple shear: $\nu = 0$
$\alpha \gg |\beta_0|$
$\beta > 0$ dilatation
$\beta = 0$ isochoric flow
$\beta < 0$ consolidation

NON-COAXIAL FLOW

Dilatant rotational shear

$$v_1 = \alpha x_2, \qquad v_2 = \beta(t)\,x_2, \qquad \omega = \omega(t)\,.$$

where $\alpha = \alpha_0$ or $\alpha(t)$
Deformation-rate/spin tensor

$$\begin{aligned} d_{11} &= d_{21} = 0, \qquad d_{22} = \beta, \\ 2d_{12} &= \alpha, \qquad\qquad 2s_{21} = -\alpha \end{aligned}$$

Constitutive equations

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \beta \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ -2\omega \sin(\phi - \nu) \end{bmatrix}$$

$$\begin{aligned} a_{11} &= \cos(\phi - \nu) + \sin\nu \cos 2\psi_\sigma \\ a_{12} &= -\sin\nu \sin 2\psi_\sigma \\ a_{21} &= \cos\nu \sin 2\psi_\sigma \\ a_{22} &= \sin(\phi - \nu) + \cos\nu \cos 2\psi_\sigma \end{aligned}$$

Unique solutions for α, β unless

$$\cos(\phi - 2\nu)(\sin\phi + \cos 2\psi_\sigma) = 0$$

i.e.

$$\cos 2\psi_\sigma = -\sin\phi, \qquad \sin 2\psi_\sigma = \cos\phi.$$

Case (1) $\cos 2\psi_\sigma = -\sin\phi$: solution is

$$\omega = 0, \qquad \beta = \alpha \tan\nu, \qquad .$$

where α is arbitrary. Only true for quasi-static flows.
Case (2) $\cos 2\psi_\sigma \neq -\sin\phi$ solution is

$$\begin{aligned} \frac{\beta}{\sin\nu \sin 2\psi_\sigma} &= \frac{\alpha}{\cos(\phi-\nu) + \sin\nu\cos 2\psi_\sigma} \\ &= \frac{-2\omega \sin(\phi-\nu)}{\cos(\phi-2\nu)(\sin\phi + \cos 2\psi_\sigma)} \end{aligned}$$

Special cases
(1) $\nu = 0 \Rightarrow \beta = 0$

$$\alpha = \frac{-2\sin\phi}{\sin\phi + \cos 2\psi_\sigma}\omega$$

shear strength depends on ω

(2) $\cos 2\psi_\sigma \to -\sin\phi,\ \sin 2\psi_\sigma \to \cos\phi$

$$\beta \to \alpha \tan\nu$$

Also

$$\begin{aligned}
\beta &= \frac{-2\sin\nu\sin(\phi-\nu)\sin 2\psi_\sigma}{\cos(\phi-2\nu)}\frac{\omega}{\sin\phi+\cos 2\psi_\sigma} \\
\alpha &= \frac{-2\sin(\phi-\nu)\left[\cos(\phi-\nu)+\sin\nu\cos 2\psi_\sigma\right]}{\cos(\phi-2\nu)} \\
&\quad \times \frac{\omega}{\sin\phi+\cos 2\psi_\sigma}
\end{aligned}$$

$\omega \to 0,\ \sin\phi + \cos 2\psi_\sigma \to 0$ simultaneously
$\Rightarrow \alpha$ arbitrary.
Finally

$$\begin{aligned}
\beta &= \frac{\alpha\sin\nu\sin 2\psi_\sigma}{\cos(\phi-\nu)+\sin\nu\cos 2\psi_\sigma} \\
&= \frac{-2\sin\nu\sin(\phi-\nu)\sin 2\psi_\sigma}{\cos(\phi-2\nu)}\frac{\omega}{\sin\phi+\cos 2\psi_\sigma}
\end{aligned}$$

$$\begin{aligned} & 2(\omega - s_{21})\sin(\phi - \nu) \\ = \; & \cos\nu\left(-\beta\sin 2\psi_\sigma - \alpha\cos 2\psi_\sigma\right). \end{aligned}$$

Case (1)

(large negative) intrinsic spin = half the vorticity

$$\omega = s_{21} = -\frac{1}{2}\alpha$$

Coaxiality

$$(d_{11} - d_{22})\sin 2\psi_\sigma - 2d_{12}\cos 2\psi_\sigma = 0$$

Parameter

$$\frac{\cos\nu}{\sin(\phi - \nu)}$$

is arbitrary

($\phi \neq \nu$ non-associated flow rule).

$$\begin{aligned} \left[\cos(\phi - \nu) + \sin\nu\cos 2\psi_\sigma\right]\beta - \left(\sin\nu\sin 2\psi_\sigma\right)\alpha &= 0 \\ \sin 2\psi_\sigma\beta + \cos 2\psi_\sigma\alpha &= 0 \end{aligned}$$

Non-trivial solutions if

$$\cos 2\psi_\sigma = -\frac{\sin\nu}{\cos(\phi - \nu)}$$

then

$$\sin 2\psi_\sigma = \frac{\sqrt{\cos(\phi - 2\nu)\cos\phi}}{\cos(\phi - \nu)}$$

Hence

$$\begin{aligned}\beta &= -\alpha \cot 2\psi_\sigma \\ &= \alpha \frac{\sin \nu}{\sqrt{\cos \phi \cos(\phi - 2\nu)}}\end{aligned}$$

$\nu = 0$

$$\psi_\sigma = \frac{1}{4}\pi$$

Case (2)

$\phi = \nu$ large dilatancy (associated flow rule)

$$\omega - s_{21}$$

is arbitrary!

$$d_{11} + d_{22} = \sin \phi \left[(d_{11} - d_{22}) \cos 2\psi_\sigma + 2d_{12} \sin 2\psi_\sigma \right],$$

$$0 = (d_{11} - d_{22}) \sin 2\psi_\sigma - 2d_{12} \cos 2\psi_\sigma.$$

Metal plasticity ($\phi = \nu = 0$) loses friction, dilatation and intrinsic spin!

$$\cot 2\psi_\sigma = -\frac{\beta}{\alpha} = -\tan \phi$$

STRESS IN TERMS OF FLOW PARAMETERS

Case (1) $\nu = 0$

$$\begin{aligned} \beta &= 0 \\ \cos 2\psi_\sigma &= -\left(1 + \frac{2\omega}{\alpha}\right)\sin\phi \end{aligned}$$

Thus the principal stress direction depends on the ratio intrinsic spin/shear strength

Classical stress solutions:

- Non-coaxial: $\omega = 0$.

  $$\psi_\sigma = \frac{\pi}{4} + \frac{\phi}{2}.$$

  Unstable solution.

- Coaxial: $\omega = -\frac{1}{2}\alpha$ gives

  $$\psi_\sigma = \frac{\pi}{4}.$$

  Unstable solution.

- restriction on spin

  $$-\left(1 + \csc\phi\right) \leq \frac{2\omega}{\alpha} \leq \csc\phi - 1$$

Case (2) $\nu \neq 0$

$$\begin{aligned} \frac{\beta}{\sin\nu\sin 2\psi_\sigma} &= \frac{\alpha}{\cos(\phi-\nu)+\sin\nu\cos 2\psi_\sigma} \\ &= \frac{-2\omega_0\sin(\phi-\nu)}{\cos(\phi-2\nu)(\sin\phi+\cos 2\psi_\sigma)} \end{aligned}$$

$$\cos 2\psi_\sigma = -\frac{\alpha\cos(\phi-2\nu)\sin\phi+\omega\sin 2(\phi-\nu)}{\alpha\cos(\phi-2\nu)-2\omega\sin(\phi-\nu)\sin\nu}$$

Regard α as given (or arbitrary) (ϕ known constant) then equations determine ψ_σ in terms of ω, ν (or ρ).

DENSITY

Constant $tr\mathbf{d}$: homogeneous $\rho = \rho(t) = \rho(\beta) = \rho(\nu)$.

$$\partial_t \rho + v_1 \partial_1 \rho + v_2 \partial_2 \rho + \rho \partial_1 v_1 + \rho \partial_2 v_2 = 0.$$

$$\partial_t \rho + \rho \beta = 0.$$

Density cannot indefinitely decrease/increase. Let ρ_c denote a critical density (dependent upon pressure p) such that $\rho \to \rho_c$ as $\beta \to 0$ Initial density $\rho_0 > \rho_c$: material dilates $(\beta > 0)$,
Initial density $\rho_0 < \rho_c$: material consolidates $(\beta < 0)$
Volumetric strain

$$e = \int tr\mathbf{d}dt = \int \beta dt$$

Write

$$\int_{\rho_0}^{\rho} \frac{d\rho'}{\rho' \beta} = -\int_0^t dt'$$

Let $\beta = \beta(\rho)$: $\beta = 0$ in the asymptotic simple shear

$$\beta = k(\rho - \rho_c)$$

$k-$ material parameter

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho(\rho - \rho_c)} = -k \int_0^t dt$$

Evolution of the density:

$$\frac{\rho(t)}{\rho_0} = \frac{\rho_c}{\rho_0 - (\rho_0 - \rho_c)\exp(-k\rho_c t)}$$

$t = 0$: $\rho(0) = \rho_0$
$t \to \infty, \rho \to \rho_c$

Simple shear reached asymptotically as $t \to \infty$: truly translationally quasi-static!

ROTATION

$$\rho I\left(\partial_t\omega+v_1\partial_1\omega+v_2\partial_2\omega\right)=\sigma_{12}-\sigma_{21}$$

Rotationally quasi-static conditions: $\sigma_{12}=\sigma_{21}$:

$$\omega=\omega_0.$$

Choose $M(\rho)$ to be such that $M(t)\rightarrow 0$, as $t\rightarrow\infty$, e.g.

$$M(\rho)=M_0\left(\rho-\rho_c\right)$$

ρ_c is the "critical" density in the asymptotic steady simple shear.

$$\rho I\frac{d\omega}{dt}=\sigma_{12}-\sigma_{21}$$

subject to $t=0, \omega=\omega_0$

$$\int_{\omega_0}^{\omega}d\omega'=\pm\frac{M_0}{I}\int_0^t\frac{\rho-\rho_c}{\rho}d\tau.$$

But

$$\frac{\rho(t)}{\rho_0}=\frac{\rho_c}{\rho_0-\left(\rho_0-\rho_c\right)\exp\left(-k\rho_c t\right)}$$

$$\begin{aligned}&\int_0^t\frac{\rho-\rho_c}{\rho}d\tau\\ =\;&\frac{\rho_0-\rho_c}{\rho_0}\int_0^t\exp\left(-k\rho_c\tau\right)d\tau\end{aligned}$$

$$\omega(t)-\omega_0=\mp\frac{M_0}{I}\frac{\rho_0-\rho_c}{k\rho_c\rho_0}\left[1-\exp\left(-k\rho_c t\right)\right]$$

As $t\rightarrow\infty$ if ρ_c known

$$\omega\rightarrow\omega_0\mp\frac{M_0}{I}\frac{\rho_0-\rho_c}{k\rho_c\rho_0}$$

As $t \to \infty$ if ω_∞ known

$$\rho_c \to \rho_0 \left[\frac{M_0}{M_0 \mp Ik\rho_0 \left(\omega_\infty - \omega_0\right)} \right]$$

Classical "double-shearing" type solution, $\omega_\infty = 0$.
Classical "coaxial" type solution, $\omega_\infty = -\alpha$.

- Either ρ evolves to a known ρ_c and ω evolves in accordance with above formula.
- Or ω evolves to a known value and ρ_c evolves in accordance with above formula.
- But a prescribed evolution of both ρ and ω is not possible.

CONCLUSIONS

New Model:

- for flow of granular materials based on reduced Cosserat continuum
- extends classical models of plastic potential and double shearing
- distinguishes between solutions for stress under simple shear by ascribing different rotation regimes to each solution (double shearing $\omega = 0$; plastic potential $\omega = -\alpha$)

Assuming

- translational quasi-static conditions
- homogeneous stress states
- dilatant shear flow

we have shown

- flow determines the stress; is determined by density ρ, intrinsic spin ω
- evolution of ρ, ω using continuity, rotational equation of motion, yields asymptotic steady shear flow
- Single equation relates asymptotic values of $\rho_\infty, \omega_\infty$: the material may find it impossible to evolve to a state corresponding to the classical stress solutions in simple shear.