Wave propagation in a heteromodular elastic medium

Serge N. Gavrilov

Institute for Problems in Mechanical Engineering RAS, St.Petersburg, RUSSIA

Abstract

Rocks, soils, and oil and tar sands are complex materials containing pores, cracks, and other defects. If this is the case, their constitutive behaviour can be nonlinear and stress-dependent, which implies that loading changes the properties of the material. If materials react differently to compression and tension, this can have a strong influence on the propagation of seismic waves. A possible approach to describe such materials is heteromodular elastic theory: a piece-wise linear theory with different elastic moduli depending on the stress state. The formulation of such a theory for the 2D and 3D cases is a difficult task. Even for the 1D case, there are only a few dynamical problems solved. One needs a simple model problem to imagine how the signal behaves when passing through such a medium. We consider a 1D problem with a small heteromodularity, obtain its analytical (asymptotical) solution for the case of a suddenly applied harmonic load, and compare it with numerical results. This gives us a general idea of the character of the wave propagation we may expect in such media, and the technique to apply in more complex cases.

Introduction

In this paper we deal with heteromodular elastic materials, i.e., materials that demonstrate different stiffness in tension and compression tests. It is known that many natural and artificial materials possess this property. Examples are: concrete, or cracked and defective rocks. Most of the studies concerning heteromodular media only consider the statical properties, not wave propagation. This is due to the complexity of the corresponding dynamical problems. The equations of heteromodular elasticity are always strongly nonlinear and there is as yet no 3D heteromodular theory in canonical form. Assume that

\[ W = \frac{\lambda}{2} I_2^2 + \mu I_2 - \nu I_1 \sqrt{I_2}, \]

\[ u = u \delta(x_1, t). \]  

(1)

Here \( W(\varepsilon) \) is the elastic potential for material, \( I_1 = \varepsilon_{ij} \delta_{ij}, \)
\( I_2 = \varepsilon_{ij} \varepsilon_{ij} \) are invariants of strain tensor \( \varepsilon \), and \( u \) are the displacements. The corresponding governing equation is

\[ \rho \ddot{u} - (\lambda + 2\mu) u'' = -2(\nu |u'|)'. \]  

(2)

The dimensionless form of this equation is

\[ \tilde{u} - u'' = -a|\tilde{u}'|', \quad 0 < a \leq 1. \]  

(3)

Equation (3) also describes the longitudinal oscillation of a heteromodular bar. The parameter \( a \) is the heteromodularity parameter. The dimensionless Young’s modulus of the bar in compression tests equals \( 1 + a \), whereas in tension tests it equals \( 1 - a \). If \( a = 0 \), then Eq. (3) simplifies to the classical wave equation. If \( a = 1 \), Eq. (3) can be used to describe a one-dimensional non-cohesive granular medium.

In this paper, we consider wave propagation in a semi-infinite heteromodular bar subjected to a harmonic force at its free end. We obtain the analytical solution with a simple structure and investigate the spectral properties of the solution. This could enable a comparison with experimental results obtained from dynamical experiments with heteromodular materials.

Problem formulation and asymptotic solution

Consider the semi-infinite \((x \geq 0)\) heteromodular bar subjected to a harmonic force \( F \) at its free end \( x = 0 \). The governing equation is Eq. (3). The boundary condition at \( x = 0 \) is

\[ u'|_{x=0} = -\frac{F}{1 + a \text{sign} F}, \quad F = H(t) \sin \Omega t. \]  

(4)

Here, \( H(t) \) is the Heaviside function. For simplicity, we assume that \( \Omega = 1 \). The speeds of sound for the compressed and tensile sections of the bar are denoted by \( (c_+) \) and \( (c_-) \), respectively, given by \( c_{\pm} = \sqrt{1 \mp a} \). The initial conditions and the boundary condition at \( x = +\infty \) are zero.

The main assumption, which allows us to obtain the analytical solution, is that \( a = o(1) \), implying small heteromodularity. We are interested in evaluation of the solution to an accuracy of \( O(a) \).

The structure of the asymptotic solution

We are particularly interested in the investigation of the near-field of the wavefield in the bar. Therefore, we assume that \( x \to \infty \) such that \( ax = O(1) \). All asymptotics obtained are valid under the following assumption: \( 2\pi - ax = O(1) > 0 \). Skipping the details, we only describe here the results of asymptotic investigation of the problem.

In Figure 1, we illustrate the structure of the solution. This is the characteristic plane for Eq. (3). The shaded sine-like domain shows the value of \( u'(t) \) at the free end of the bar (in fact this domain should be represented out-of-plane). It can be shown that the solution of the problem possesses the following properties:

- At every instant of time, when \( u'|_{x=0} \) changes its value from negative to positive, two wave fronts appear. We call these fronts simple shocks. Simple shocks move at constant speeds \( c_- \) and \( c_+ \). Between the wave fronts, the strain is zero: \( u' \equiv 0 \).
At every instant of time, when \( u'_{|x=0} \) changes its value from positive to negative, one shock wave appears. One has \( u_+/u_- < 0 \), where \( u'_+ \) and \( u'_- \) are the limiting values of the strain just before and just behind the shock wave. The speed \( l \) of the shock wave is not constant and satisfies the following inequality: \( c_- < l < c_+ \). Here, \( l(t) \) is the position of the shock wave. It can be shown that \( l(t) = L(t - t_0) \), where

\[
L(t) = t - \frac{a}{4} \tan \frac{at}{4} - \frac{3}{8} a^2 t + o(a),
\]

and where \( t_0 \) is the instant of the shock wave formation.

The solution in the domains between a simple shock and a shock wave can be described in the form of the D’Alembert wave moving at a speed, which is equal to \( c_- \) in the domain before the shock wave and \( c_+ \) in the domain behind the shock wave. Therefore, one has

\[
u' = -\frac{1}{1 \pm a} \sin \left( t - t_0 - \frac{x}{c_{\pm}} \right) + O(a^2).
\]

Figure 2 shows the comparison between the analytical and numerical solutions of the problem for \( a = 0.1 \). One can see that, in the near-field approximation, the asymptotic solution approximates the numerical solution quite well.

Spectral properties of the solution

The non-stationary solution obtained for the problem under consideration does not converge to a steady-state solution. Nevertheless, it can be shown that, for a fixed \( x = x_0 \) and \( t \) greater than a certain value \( t_* \), the strain \( u'(x_0, t) \) has the form

\[
u'(x_0, t) = \gamma(ax_0, t) + O(a), \quad t > t_* = 2\pi + x_0/c_+,
\]

where \( \gamma(ax_0, t) \) is a known periodic function of \( t \) with period \( 2\pi \). Therefore the strain can be expanded into the Fourier series. The Fourier coefficients are

\[
C_n = \pi^{-1} \int_{2\pi + t_*}^{-1} \gamma(t) \cos nt \, dt, \quad S_n = \pi^{-1} \int_{2\pi + t_*}^{-1} \gamma(t) \sin nt \, dt.
\]

Note that \( C_0 = 0 \). In Figure 3, we show the dependence of the amplitudes \( A_n = \sqrt{S_n^2 + C_n^2} \), and the phases \( \phi_n = \arctan C_n/S_n \) on the quantity \( x_0a \).

These results can be compared with experimental data obtained from dynamical experiments with materials displaying pronounced heteromodular properties. This would help us to verify the applicability of the theory to describe wave propagation in heteromodular materials.

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\[ \text{Figure 1: Characteristic plane for Eq.}(3) \]
\[ \text{Figure 2: Analytical & numerical solutions} \]
\[ \text{Figure 3: Amplitude } A_n \text{ and phase } \phi_n \text{ for } \gamma(x_0a, t) \]