Coherent Configurations: answers to Exercises

Contents

1.4. Exercises

1.4.1. For a set S of relations on Ω, denote by S^∞ the union of all finite compositions $r \cdot s \cdots$ with $r, s, \ldots$ belonging to S. Then given $s \subseteq \Omega^2$,

$$\langle s \rangle = \{1_{\Omega(s)}, s, s^*\}^\infty. \tag{1.4.1}$$

Proof. Set $T := \{1_{\Omega(s)}, s, s^*\}$ and $e := T^\infty$. Then

$$\Omega(e) = \Omega(s).$$

Therefore, e is reflexive as $1_{\Omega(s)} \subseteq e$, symmetric as $T^* = T$, and transitive by the definition of T^∞. As a consequence, e is a partial equivalence relation on Ω which contains s. Also, if e' is a partial equivalence relation containing s, then e' contains $1_{\Omega(s)}$ and s^*, which yields that

$$e = T^\infty \subseteq \{e'\}^\infty = e'.$$

Thus, $\langle s \rangle = e$. □

1.4.2. Let $s \subseteq \Omega^2$. Then the points α and α' belong to the same class of $\langle s \rangle$ if and only if $\alpha \overset{s \cup s^*}{\to} \alpha'$.

Proof. By Exercise 1.4.1, $\alpha' \in \alpha\langle s \rangle$ if and only if (α, α') belongs to the composition of some $s_1, \ldots, s_k \in \{1_{\Omega(s)}, s, s^*\}$. Without loss of generality, we can assume that each s_i is not reflexive. Then the existence of such a composition exactly means that $\alpha \overset{s \cup s^*}{\to} \alpha'$. □

1.4.3. Let $s \subseteq \Omega^2$. Then $\mathrm{rad}(s)$ is equal to the largest partial equivalence relation e on $\Omega(s)$ for which

$$s = \bigcup_{\substack{\Delta, \Gamma \in \Omega/e: \\ \Delta \times \Gamma \subseteq s}} \Delta \times \Gamma. \tag{1.4.2}$$

Proof. Without loss of generality, we assume that s has full support.

First, we prove that $\mathrm{rad}(s)$ is the union of all equivalence relations e such that

$$e \cdot s = s = s \cdot e. \tag{1.4.3}$$

To this end, choose equivalence relations e_1 and e_2 satisfying this condition. We claim that

$$(e_1 \vee e_2) \cdot s = s = s \cdot (e_1 \vee e_2). \tag{1.4.4}$$

Obviously,

$$s = s \cdot e_1 \subseteq s \cdot (e_1 \vee e_2).$$

To prove the inverse inclusion, let $(\alpha, \beta) \in s$ and $(\beta, \gamma) \in e_1 \vee e_2$. Then there exists a sequence

$$\gamma_0 = \beta, \gamma_1, \cdots, \gamma_m = \gamma$$

such that $(\gamma_{i-1}, \gamma_i) \in e_1$ or e_2. Then $(\alpha, \gamma_1) \in s \cdot e_1$ or $s \cdot e_2$ and hence (α, γ_1) belongs to s. Proceeding in this way, we conclude that $(\alpha, \gamma) \in s$, as required. Thus, the right-hand side equality of formula (1.4.4) follows and the left-hand side equality can be proved similarly.

To complete the proof, it suffices to show that an equivalence relation e on Ω satisfies (1.4.3) if and only if e satisfies (1.4.2).

If e satisfies formula (1.4.2), then given $(\alpha,\beta)\in e$ and $(\beta,\gamma)\in s$,

$$(\beta,\gamma)\in s_{\Delta\times\Gamma}=\Delta\times\Gamma$$

where Δ and Γ are the classes of e containing α and γ respectively. Since $\Delta\times\Gamma\subseteq s$, $(\alpha,\gamma)\in s$ and hence $e\cdot s\subseteq s$. Similarly, $s\subseteq e\cdot s$. Thus, $e\cdot s=s=s\cdot e$.

Conversely, suppose that the partial equivalence relation e statisfies formula (1.4.3). For any $\Delta,\Gamma\in\Omega/e$ with $s_{\Delta\times\Gamma}$ nonempty, choose $(\alpha,\beta)\in s_{\Delta\times\Gamma}$ arbitrarily. Then for any $\beta'\in\Gamma$,

$$(\alpha,\beta')\in s\cdot e=s.$$

This implies that $\{\alpha\}\times\Gamma\subseteq s$ and analogously $\Delta\times\{\beta\}\subseteq s$. When (α,β) runs over $s_{\Delta\times\Gamma}$, the union of all relations $\{\alpha\}\times\Gamma$ and $\Delta\times\{\beta\}$ equals $\Delta\times\Gamma$, which is then contained in s. Therefore e satisfies formula (1.4.2). □

1.4.4. Let e be an equivalence relation on Ω. Then the mapping π_e induces a surjection from the set of (partial) equivalence relations on Ω to the set of (partial) equivalence relations on Ω/e.

Proof. Since $\Omega(\pi_e(s))=\pi_e(\Omega(s))$ for any relation s on Ω, it suffices to consider equivalence relations only. One can see that π_e preserves the composition of relations. Therefore $\pi_e(e_1)$ is an equivalence relation on Ω/e for any equivalence relation e_1 on Ω.

To prove the surjectivity, let $\overline{e}_1$ be an equivalence relation on Ω/e. Denote by e_1 the equivalence relation whose classes are of the form $\bigcup_{\Lambda\in\overline{\Delta}}\Lambda$ with $\overline{\Delta}$ running over the classes of $\overline{e}_1$. Then, $\pi_e(e_1)=\overline{e}_1$ as required. □

1.4.5. Let $e\subseteq\Omega^2$ be an equivalence relation and s a relation on Ω/e. Then

$$e\cdot\pi_e^{-1}(s)\cdot e=\pi_e^{-1}(s).$$

In particular, $e\subseteq\operatorname{rad}(\pi_e^{-1}(s))$.

Proof. Obviously,

$$e\cdot\pi_e^{-1}(s)\cdot e\supseteq 1_\Omega\cdot\pi_e^{-1}(s)\cdot 1_\Omega=\pi_e^{-1}(s).$$

To prove the reverse inclusion, let $(\alpha,\alpha')\in e$, $(\alpha',\beta')\in\pi_e^{-1}(s)$, and $(\beta',\beta)\in e$. Then $\alpha e=\alpha' e$ and $\beta e=\beta' e$. Therefore,

$$(\alpha',\beta')\in\pi_e^{-1}(s)\quad\Rightarrow\quad(\alpha' e,\beta' e)\in s\quad\Rightarrow\quad(\alpha e,\beta e)\in s\quad\Rightarrow\quad(\alpha,\beta)\in\pi_e^{-1}(s).$$

Thus $e\cdot\pi_e^{-1}(s)\cdot e\subseteq\pi_e^{-1}(s)$. □

1.4.6. Let r and s be thin relations on Ω. Then so are the relations s^* and $r\cdot s$. Furthermore, if t is a thin relation on Δ, then $s\otimes t$ is a thin relation on $\Omega\times\Delta$.

Proof. By the definition of thin relation, $|\alpha s|\le 1$ and $|\alpha s^*|\le 1$ for all $\alpha\in\Omega$. By the symmetry, these conditions hold for s^*. Hence s^* is thin.

By the assumption, for any $\alpha\in\Omega_-(r\cdot s)$ there exist uniquely determined β and γ such that $(\alpha,\beta)\in r$ and $(\beta,\gamma)\in s$. In particular,

$$(\alpha,\gamma)\in r\cdot s\quad\text{and}\quad|\alpha(r\cdot s)|=1.$$

It follows that $|\alpha(r\cdot s)|\le 1$ for any $\alpha\in\Omega$. Similarly, $|\alpha(r\cdot s)^*|\le 1$ and hence $r\cdot s$ is thin.

To prove the last statement, choose $(\alpha,\beta)\in\Omega\times\Delta$. Obviously,

$$(\alpha,\beta)(s\otimes t)=\alpha s\otimes\beta t.$$

Since $|\alpha s|\le 1$ and $|\beta t|\le 1$, it follows that

$$|(\alpha,\beta)(s\otimes t)|\le 1.$$

Similarly, $|(\alpha,\beta)(s\otimes t)^*|\le 1$. Thus, $s\otimes t$ is thin. □

1.4.7. The mapping $s\mapsto A_s$ defines a 1-1 correspondence between the relations on Ω and {0,1}-matrices of Mat_Ω.

Proof. The mapping $s\mapsto A_s$ is a well-defined injection because the adjacency matrix of any relation is a uniquely determined {0,1}-matrix. The mapping is surjective because any {0,1}-matrix $A\in\mathrm{Mat}_\Omega$ is the adjacent matrix of a relation s defined as

$$(\alpha,\beta)\in s\quad\Leftrightarrow\quad A_{\alpha,\beta}=1.$$

□

1.4.8. Given relations $r,s\subseteq\Omega^2$,
(1) $A_{r^*}=(A_r)^T$,
(2) $A_{r\cap s}=A_r\circ A_s$,
(3) $A_{r\cup s}=A_{r\setminus s}+A_{s\setminus r}+A_{r\cap s}$; in particular, $A_{r\cup s}=A_r+A_s$ if $r\cap s=\varnothing$,
(4) $|\alpha r\cap\beta s^*|=(A_rA_s)_{\alpha,\beta}$ for all $\alpha,\beta\in\Omega$.

Proof. For statement (1),

$$(A_{r^*})_{\alpha,\beta}=1\quad\Leftrightarrow\quad(\alpha,\beta)\in r^*\quad\Leftrightarrow\quad(\beta,\alpha)\in r\quad\Leftrightarrow\quad(A_r)^T_{\alpha,\beta}=1.$$

For statement (2),

$$(A_r\circ A_s)_{\alpha,\beta}=1\quad\Leftrightarrow\quad(A_r)_{\alpha,\beta}=1\quad\text{and}\quad(A_s)_{\alpha,\beta}=1\quad\Leftrightarrow\quad(\alpha,\beta)\in r\cap s.$$

For statement (3), note that $r\cup s$ is the disjoint union of $r\setminus s$, $s\setminus r$, and $r\cap s$. It follows that

$$A_{r\cup s}=A_{r\setminus s}+A_{s\setminus r}+A_{r\cap s}.$$

For statement (4),

$$(A_rA_s)_{\alpha,\beta}=\sum_{\gamma\in\Omega}(A_r)_{\alpha,\gamma}(A_s)_{\gamma,\beta}=|\alpha r\cap\beta s^*|.$$

□

1.4.9. For any relations r and s, we have $A_{r\otimes s}=A_r\otimes A_s$.

Proof. If r and s are relations on Ω and Δ respectively, then $A_{r\otimes s}$ and $A_r\otimes A_s$ are of the same order. Observe that for any $(\alpha,\rho),(\beta,\tau)\in\Omega\times\Delta$,

$$(A_r\otimes A_s)_{(\alpha,\rho),(\beta,\tau)}=1\Leftrightarrow(A_r)_{\alpha,\beta}=1\quad\text{and}\quad(A_s)_{\rho,\tau}=1\Leftrightarrow(A_{r\otimes s})_{(\alpha,\rho),(\beta,\tau)}=1.$$

Thus, $A_{r\otimes s}=A_r\otimes A_s$. □

1.4.10. For any permutations $k,k'\in\mathrm{Sym}(\Omega)$,

$$P_{kk'}=P_kP_{k'}. \tag{1.4.5}$$

In particular, $P_{k^{-1}}=(P_k)^{-1}$, and the mapping $k\mapsto P_k$ is a linear representation of the group $\mathrm{Sym}(\Omega)$.

Proof. Set $P_k = (x_{\alpha,\beta})$ and $P_{k'} = (y_{\alpha,\beta})$. Then for any $\alpha, \beta \in \Omega$,

$$(P_k P_{k'})_{\alpha,\beta} = \sum_{\gamma\in\Omega} x_{\alpha,\gamma} y_{\gamma,\beta}.$$

Hence, $(P_k P_{k'})_{\alpha,\beta} \neq 0 \iff x_{\alpha,\gamma} = y_{\gamma,\beta} = 1$ for some $\gamma \iff \gamma = \alpha^k$ and $\beta = \gamma^{k'}$ $\iff \beta = \alpha^{kk'} \iff (P_{kk'})_{\alpha,\beta} = 1$. It follows that $(P_k P_{k'})_{\alpha,\beta} \neq 0 \Leftrightarrow (P_k P_{k'})_{\alpha,\beta} = 1$ and

$$(P_k P_{k'})_{\alpha,\beta} = (P_{kk'})_{\alpha,\beta}\,,$$

as required. □

1.4.11. For a relation $s \subseteq \Omega^2$ and a permutation $k \in \mathrm{Sym}(\Omega)$,

$$A_{s^k} = P_k^{-1} A_s P_k. \tag{1.4.6}$$

Proof. Given $\alpha, \beta \in \Omega$, let us verify that the (α,β)-entry of A_{s^k} is equal to that of $P_k^{-1} A_s P_k$. Note that

$$(A_{s^k})_{\alpha,\beta} = 1 \quad\Leftrightarrow\quad (\alpha,\beta) \in s^k \quad\Leftrightarrow\quad (\alpha^{k^{-1}}, \beta^{k^{-1}}) \in s$$

and

$$(P_k^{-1} A_s P_k)_{\alpha,\beta} = \sum_{\gamma,\tau\in\Omega} (P_k^{-1})_{\alpha,\gamma} (A_s)_{\gamma,\tau} (P_k)_{\tau,\beta}.$$

The last sum is nonzero if and only if $\gamma = \alpha^{k^{-1}}, \tau = \beta^{k^{-1}}$, and $(\gamma,\tau) \in s$ if and only if $(\alpha,\beta) \in s^k$. It follows that

$$(A_{s^k})_{\alpha,\beta} = \left(P_k^{-1} A_s P_k\right)_{\alpha,\beta}\,,$$

as desired. □

1.4.12. For any relation $s \subseteq \Omega^2$,

$$A_s \alpha = \underline{\alpha s^*}, \qquad \alpha \in \Omega. \tag{1.4.7}$$

Proof. Note that α is a $\{0,1\}$-vector of size $|\Omega|$ with all but the α-entry zero. Thus equality (1.4.7) follows, because

$$(A_s\alpha)_\beta = 1 \quad\Leftrightarrow\quad (\beta,\alpha) \in s \quad\Leftrightarrow\quad (\alpha,\beta) \in s^*.$$

□

1.4.13. For any group G,

$$\langle G_{left}, G_{right}\rangle = G_{left}\,\mathrm{Inn}(G) = G_{right}\,\mathrm{Inn}(G).$$

Proof. We prove the first equality only and the second can be proved similarly.

For any $g \in G$, let g_l, g_r, and c_g denote respectively g_{left}, g_{right}, and the inner automorphism of G induced by g. It is straightforward to verify that

$$g_l c_h = c_h (g^h)_l, \quad g_r = g_l^{-1} c_g, \quad c_g = g_l g_r$$

for all $g, h \in G$. The first equality implies that $G_{left}\,\mathrm{Inn}(G)$ is a subgroup of $\mathrm{Sym}(G)$, and together the second one that $\langle G_{left}, G_{right}\rangle$ is contained in $G_{left}\,\mathrm{Inn}(G)$. The reverse inclusion follows from the third equality. □

1.4.14. For any group G, the mapping

$$\tau : \mathbb{C}G \to \mathrm{Mat}_G(\mathbb{C}), \qquad g \mapsto P_{g_{left}}, \tag{1.4.8}$$

is an algebra monomorphism. Moreover,

(1) $\tau(1) = I_G$ and $\tau(\underline{G}) = J_G$,
(2) $\tau(\xi^{-1}) = \tau(\xi)^T$ for all $\xi \in \mathbb{C}G$,
(3) $\tau(\xi \circ \eta) = \tau(\xi) \circ \tau(\eta)$ for all $\xi, \eta \in \mathbb{C}G$.

Proof. For any $g, h \in G$, observe that $(gh)_{left} = g_{left}h_{left}$. Then,

$$\tau(gh) = P_{(gh)_{left}} = P_{g_{left}h_{left}} = P_{g_{left}}P_{h_{left}} = \tau(g)\tau(h),$$

where the third equality holds by formula (1.4.5). It follows that τ is compatible with multiplication in $\mathbb{C}G$. Since it respects the addition, τ is an algebra homomorphism and it is easy to check that it is injective.

To prove statement (1), observe that $\left(P_{1_{left}}\right)_{\alpha,\beta} = 1$ if and only if $\beta = \alpha$. Thus,

$$\tau(1) = P_{1_{left}} = I_G.$$

To verify the second equality, let $x, y \in G$. Then

$$\tau(\underline{G})_{x,y} = \sum_{g \in G} \left(P_{g_{left}}\right)_{x,y} = (P_{(xy^{-1})_{left}})_{x,y} = 1.$$

Thus, $\tau(\underline{G}) = J_G$.

To prove the remaining statements, without loss of generality we may assume that $\xi = x$ and $\eta = y$ are elements of G.

To prove statement (2), as $\tau(x^{-1}) = P_{x^{-1}_{left}}$,

$$\tau(x^{-1})_{g,h} = 1 \quad \Leftrightarrow \quad h = xg \quad \Leftrightarrow \quad g = x^{-1}h \quad \Leftrightarrow \quad \tau(x)_{h,g} = 1,$$

which yields that $\tau(x^{-1}) = \tau(x)^T$.

To prove statement (3), assume first that $x = y$. Then $x \circ y = x$, and

$$\tau(x) \circ \tau(y) = \tau(x)$$

as $\tau(x)$ is a {0,1}-matrix. Now let $x \neq y$. Then $x \circ y = 0$ and for any $g, h \in G$, one of the numbers $\tau(x)_{g,h}$ and $\tau(y)_{g,h}$ equals zero. Thus,

$$\tau(x) \circ \tau(y) = 0 = \tau(x \circ y).$$

□

1.4.15. For any group G and any set $X \subseteq G$,

$$\tau(\underline{X}^{-1}) = A_s$$

where

$$s = \{(g, xg) : x \in X, g \in G\}. \tag{1.4.9}$$

This relation is G_{right}-invariant. The mapping

$$\rho : X \mapsto s \tag{1.4.10}$$

is a bijection between the subsets of G and the G_{right}-invariant relations on G. The inverse of ρ is defined by formula

$$\rho^{-1}(s) = \alpha s \tag{1.4.11}$$

where α is the identity of G.

Proof. For any $X \subseteq G$, the relation (1.4.9) is obviously G_{right}-invariant. Also, for any $g, h \in G$

$$(A_s)_{g,h} = 1 \quad \Leftrightarrow \quad hg^{-1} \in X \quad \Leftrightarrow \quad \tau(\underline{X})_{g,h} = 1.$$

Thus, $\tau(\underline{X}) = A_s$. Furthermore, any G_{right}-invariant relation s has the form (1.4.9) with $X = \alpha s$. Thus mapping (1.4.10) is a bijection between the subsets of G and the G_{right}-invariant relations on G, which satisfies (1.4.11). □

1.4.16. Let G be a group, and let ρ be the mapping from Exercise 1.4.15. Then for any sets $X, Y \subseteq G$,

(1) $\rho(X) = 1_G$ if and only if X consists of the identity of G,
(2) $\rho(X) = G \times G$ if and only if $X = G$,
(3) $\rho(X^{-1}) = \rho(X)^*$,
(4) $\rho(X) \subseteq \rho(Y)$ if and only if $X \subseteq Y$,
(5) $\langle \rho(X) \rangle = \rho(\langle X \rangle)$,
(6) $X \leq G$ if and only if $e = \rho(X)$ is an equivalence relation and $G/e = G/X$,
(7) $\mathrm{rad}(\rho(X)) = \rho(\mathrm{rad}((X))$, where $\mathrm{rad}(X) = \{g \in G : gX = Xg = X\}$.

Proof. Statements (1), (2) and (4) immediately follow from formulas (1.4.9) and (1.4.11).

To prove statement (3), observe that

$$\rho(X^{-1}) = \{(g, x^{-1}g) : g \in G, x \in X\} = \{(xh, h) : h \in G, x \in X\} = \rho(X)^*,$$

where $h = x^{-1}g$.

To prove statement (5), we claim that $\rho(\langle X \rangle)$ is an equivalence relation. Indeed, it is reflexive by statement (4) as $\langle X \rangle$ contains the identity element, symmtric by statement (3) as $\langle X \rangle = \langle X \rangle^{-1}$, and transitive by formula (1.4.9) because $\langle X \rangle$ is closed with respect to multiplication.

Thus $\rho(\langle X \rangle)$ is an equivalence relation, which contains $\rho(X)$ by statement (4). It follows that

$$\langle \rho(X) \rangle \subseteq \rho(\langle X \rangle).$$

To prove the reverse inclusion, let $(g, h) \in \rho(\langle X \rangle)$. Then $hg^{-1} = x_r \cdots x_1$, where $x_i \in X$ or $x_i^{-1} \in X$ for each i. It follows that each of the following pairs

$$(g, x_1 g),\ (x_1 g, x_2 x_1 g),\ \cdots,\ (x_1 \cdots x_{r-1} g, hg^{-1} g)$$

lies in $\rho(X)$ or $\rho(X)^*$. Thus, $(g, h) \in \langle \rho(X) \rangle$ as required.

For statement (6), the necessity follows from statement (5) and the fact that any class of e is of the form

$$\{xg : x \in X\} = Xg \in G/X.$$

Conversely, if $e = \rho(X)$ is an equivalence relation, then by statement (5),

$$\rho(X) = \langle \rho(X) \rangle = \rho(\langle X \rangle).$$

By statement (4), $X = \langle X \rangle$ is a subgroup of G.

To prove statement (7), let $g, h \in G$. Then for all $x \in X$,

$$(h, gh) \in \rho(g) \quad \text{and} \quad (gh, xgh) \in \rho(X),$$

where $\rho(g) := \rho(\{g\})$. If $g \in \mathrm{rad}(X)$, then $xg \in X$. Hence,

$$(h, xgh) \in \rho(X).$$

It follows that
$$\rho(g)\rho(X) \subseteq \rho(X).$$
Similarly
$$\rho(X)\rho(g) \subseteq \rho(X).$$
Thus, $\rho(\mathrm{rad}(X)) \subseteq \mathrm{rad}(\rho(X))$.

To prove the reverse inclusion, let $(g, h) \in \mathrm{rad}(\rho(X))$. Since $(h, yh) \in \rho(X)$ for any $y \in X$, we have
$$(g, yh) \in \mathrm{rad}(\rho(X))\rho(X) = \rho(X).$$
This yields that
$$yhg^{-1} \in X.$$
When y runs over X, the above formula implies that $Xhg^{-1} = X$. Similarly, we obtain $hg^{-1}X = X$. Thus, $hg^{-1} \in \mathrm{rad}(X)$ and hence $(g, h) \in \rho(\mathrm{rad}(X))$. □

1.4.17. For an abelian group G of order n, the center of $\mathrm{Aut}(G)$ consists of all mappings
$$\sigma_m : G \to G,\ g \mapsto g^m, \tag{1.4.12}$$
where m is coprime to n.

Proof. Without loss of generality, we may assume that G is not cyclic. Denote the center of $\mathrm{Aut}(G)$ by $Z(\mathrm{Aut}(G))$. We need the following lemma.

LEMMA 1.4.18. *Let p be a prime and G a finite noncyclic abelian p-group. Then for any $\sigma \in Z(\mathrm{Aut}(G))$ and any $g \in G$ such that $G = \langle g\rangle \times H$ for some subgroup H of G, there exists an integer m_g coprime to p such that $\sigma(g) = g^{m_g}$.*

Proof. Otherwise, there exist $\sigma \in Z(\mathrm{Aut}(G)$ and $g \in G$ such that $G = \langle g\rangle \times H$ and $\sigma(g) \notin \langle g\rangle$. Then there exists $h \in H$ such that $H = \langle h\rangle \times H'$ and
$$\sigma(g) = g^{n_g} h^{n_h} h',$$
where $h^{n_h} \neq 1$ and $h' \in H'$. Observe that
$$G = \langle g\rangle \times \langle h\rangle \times H'. \tag{1.4.13}$$
Hence, there exists $k' \in H'$ such that
$$\sigma(h) := g^{l_g} h^{l_h} k'.$$
By formula (1.4.13), every automorphism of $\langle g\rangle \times \langle h\rangle$ can be extended to an automorphism of G. Denote by $\tau_{x,y}$ the automorphism of G, where $x, y \in \langle g\rangle \times \langle h\rangle$ such that
$$\tau_{x,y}(g) = x, \quad \tau_{x,y}(h) = y, \quad \text{and} \quad \tau_{x,y}(h') = h',$$
for all $h' \in H'$. Since $\sigma \in Z(\mathrm{Aut}(G))$, one can see that
$$\sigma\tau_{g,h^{-1}} = \tau_{g,h^{-1}}\sigma \quad \Rightarrow \quad \sigma\tau_{g,h^{-1}}(g) = \tau_{g,h^{-1}}\sigma(g) \quad \Rightarrow \quad (h^{n_h})^2 = 1.$$
Thus, if p is odd then the lemma follows. From now on, we assume that $p = 2$ and $o(h^{n_h}) = 2$. Set
$$o(g) := 2^d \quad \text{and} \quad o(h) := 2^e. \tag{1.4.14}$$
Then $n_h = 2^{e-1}$.

If $d > e$, then
$$\sigma\tau_{g,g^{2^{d-e}}h} = \tau_{g,g^{2^{d-e}}h}\sigma \quad \Rightarrow \quad \sigma\tau_{g,g^{2^{d-e}}h}(g) = \tau_{g,g^{2^{d-e}}h}\sigma(g) \quad \Rightarrow \quad g^{2^{d-1}} = 1,$$

a contradiction to formula (1.4.14).

If $d \leq e$, then

$$\sigma\tau_{g,gh} = \tau_{g,gh}\sigma \quad \Rightarrow \quad \sigma\tau_{g,gh}(h) = \tau_{g,gh}(h) \quad \Rightarrow \quad h^{n_h} = h^{2^{e-1}} = 1$$

a contradiction to formula (1.4.14). We are done. □

To prove the statement, we assume first that G is an abelian p-group. Let σ and $g \in G$ be as in Lemma (1.4.18). Then, there exists an integer m_g such that $\sigma(g) = g^{m_g}$. Assume that $h \in G$ such that G has the decomposition as in (1.4.13) for some subgroup H'. By the lemma, there exists an integer m_h such that $\sigma(h) = h^{m_h}$. Without loss of generality, we may assume that $o(g) \geq o(h)$. Then

$$\tau_{gh,h}\sigma = \sigma\tau_{gh,h} \quad \Rightarrow \quad \tau_{gh,h}\sigma(g) = \tau_{gh,h}\sigma(g) \quad \Rightarrow \quad h^{m_g} = h^{m_h}.$$

Since this is true for any such h, $\sigma = \sigma_{m_g}$, as required.

In general, let $G = \prod_{p\in\mathcal{P}} G_p$ where $\mathcal{P}$ is the set of prime divisors of n and G_p is the Sylow p-subgroup of G. Let $\sigma \in Z(\mathrm{Aut}(G))$. Since $\mathrm{Aut}(G) = \prod_{p\in\mathcal{P}} \mathrm{Aut}(G_p)$, by the previous paragraph we have

$$\sigma = \prod_{p\in\mathcal{P}} \sigma_{m_p},$$

where m_p is an integer comprime to p for each $p \in \mathcal{P}$. By Chinese Remainder Theorem, $\sigma = \sigma_m$ for some positive integer coprime to n. □

1.4.19. The identity element of the wreath product $G \wr K$ is the pair $(f_1, 1)$, where the function f_1 takes any element to the identity of G. The element inverse to (f, k) is given by $(f, k)^{-1} = ((f^k)^{-1}, k^{-1})$.

Proof. For any $(f, k) \in G \wr K$,

$$(f, k)(f_1, 1) = (ff_1^{k^{-1}}, k)$$

where

$$\left(ff_1^{k^{-1}}\right)(\alpha) = f(\alpha)f_1(\alpha^k) = f(\alpha)$$

for any $\alpha \in \Omega$. It follows that

$$(f, k)(f_1, 1) = (f, k).$$

Thus, $(f_1, 1)$ is the identity of the wreath product.

In addition,

$$(f, k)((f^k)^{-1}, k^{-1}) = (f((f^k)^{-1})^{k^{-1}}, 1),$$

where

$$\left(f((f^k)^{-1})^{k^{-1}}\right)(\alpha) = f(\alpha)(f^k)^{-1}(\alpha^k) = f(\alpha)f(\alpha^{kk^{-1}})^{-1} = 1.$$

Thus $f((f^k)^{-1})^{k^{-1}} = f_1$ and

$$(f, k)^{-1} = ((f^k)^{-1}, k^{-1}).$$

□

1.4.20. Let e be a partial equivalence relation on Ω. Assume that e is invariant with respect to a group $K \leq \mathrm{Sym}(\Omega)$. Then the natural action of K on Ω/e induces the homomorphism $k \mapsto k^{\Omega/e}$ from K to $\mathrm{Sym}(\Omega/e)$ with the image and kernel equal to

$$K^{\Omega/e} = \{k^{\Omega/e} : \ k \in K\} \quad \text{and} \quad K_e = \bigcap_{\Delta \in \Omega/e} K_{\{\Delta\}}, \tag{1.4.15}$$

respectively.

Proof. Note that

$$e = \bigcup_{\Delta \in \Omega/e} \Delta \times \Delta.$$

Hence, e is invariant with respect to K if and only if K permutes the classes of e. In this case we have a homomorphsim $k \mapsto k^{\Omega/e}$ from K to $\mathrm{Sym}(\Omega/e)$, whose image is exactly $K^{\Omega/e}$.

The kernel of this homomorphism is the set of permutations which fix each class of e, as described in the second equality in formula (1.4.15). □

1.4.21. Any abelian permutation group is quasiregular, and is regular if and only if it is transitive.

Proof. Let K be an abelian permutation group on Ω. To prove the first assertion, let $\Delta \in \mathrm{Orb}(K, \Omega)$. For any $\alpha, \beta \in \Delta$, there exists $k \in K$ such that $\beta = \alpha^k$. It follows that

$$K_\beta = k^{-1} K_\alpha k = K_\alpha,$$

because K is abelian. Thus, if $k \in K_\alpha$ then k fixes every point in Δ. Hence, K acts regularily on Δ. We conclude that K is quasiregular. The second statement follows easily. □

1.4.22. A normal subgroup of a transitive group is 1/2-transitive.

Proof. Let N be a normal subgroup of a transitive group K on Ω. For any $\alpha, \beta \in \Omega$, there exists $k \in K$ such that $\beta = \alpha^k$. It follows that $K_\beta = K_\alpha^k$. Thus,

$$|N : N_\beta| = |N : N \cap K_\beta| = |N : N \cap K_\alpha^k| = |N : N \cap K_\alpha| = |N : N_\alpha|,$$

where the third equality holds because N is normal in K. It follows that the length of orbits of N containing α and β are the same, i.e., N is 1/2-transitive, as required. □

2.7. Exercises

In what follows, unless otherwise stated, $\mathcal{X}$ is a coherent configuration on Ω and $S = S(\mathcal{X})$, $F = F(\mathcal{X})$, and $E = E(\mathcal{X})$. The notations $\mathcal{X}'$ and Ω', S', F', and E' have the same meaning.

2.7.1. [**?**] The conditions (CC1), (CC2), and (CC3) are independent.

Proof. (CC1), (CC2) $\nRightarrow$ (CC3). Let $\Omega = \{1, 2, 3\}$ and

$$s_1 = \Omega^2 \backslash 1_\Omega, \quad s_2 = \{(1,1)\}, \quad \text{and} \quad s_3 = \{(2,2),(3,3)\}.$$

If $S = \{s_1, s_2, s_3\}$, then (Ω, S) satisfies the conditions (CC1) and (CC2) but not the condition (CC3): indeed, $(1,2) \in s_1$, $(2,1) \in s_1$, but

$$|1s_1 \cap 2s_2^*| = 0, \quad |2s_1 \cap 1s_2^*| = 1.$$

(CC2), (CC3) $\nRightarrow$ (CC1). Let Ω be a nonempty set and $S = \{\Omega^2\}$. Then (Ω, S) satisfies the conditions (CC2) and (CC3) but not the condition (CC1).

(CC1), (CC3) $\nRightarrow$ (CC2). Let $M = \{1, 2, 3\}$ and $\Omega = M^2 \backslash 1_M$. Set

$$\begin{aligned} B_1 &:= \{((j,i),(i,k)) : i,j,k \in M, j \neq k\}, \\ B_2 &:= \{((i,k),(j,i)) : i,j,k \in M, j \neq k\}, \\ B_3 &:= \{((j,i),(k,i)) : i,j,k \in M, j \neq k\}, \\ B_4 &:= \{((i,j),(i,k)) : i,j,k \in M, j \neq k\}, \\ B_5 &:= \{((i,j),(j,i)) : i,j \in M, i \neq j\}. \end{aligned}$$

Let

$$s_1 = 1_\Omega, \quad s_2 = B_1 \cup B_3, \quad s_3 = B_2 \cup B_4, \quad \text{and} \quad s_4 = B_5.$$

Denote $\{s_i : 1 \leq i \leq 4\}$ by S. Note that S is a partition of Ω^2 and (Ω, S) satisfies the condition (CC1), but not the condition (CC2) since

$$s_2^* = B_1^* \cup B_3^* = B_2 \cup B_3 \quad \Rightarrow \quad s_2^* \notin S.$$

However, (Ω, S) statisfies the condition (CC3) as the “intersection numbers” exist. Indeed, if we denote the adjacency matrix of s_i by A_i for $i = 1, \ldots, 4$, then it is straightforward to check that

$$\begin{aligned} A_2^2 &= A_1 + A_3 + A_4, & A_2A_3 &= A_1 + A_3 + A_4, & A_2A_4 &= A_2, \\ A_3A_2 &= A_1 + A_2 + A_4, & A_3^2 &= A_1 + A_2 + A_4, & A_3A_4 &= A_3, \\ A_4A_2 &= A_3, & A_4A_3 &= A_2, & A_4^2 &= A_1. \end{aligned}$$

□

2.7.2. Find all coherent configurations of degree at most 4.

Proof. Coherent configurations of degree at most 2 are the discrete or the trivial coherent configurations. Up to isomorphism, the amounts of other nontrivial and nondiscrete coherent configurations of degree at most 4 are as follows.

Degree	Homogeneouss	Non-homogeneous
3	1	1
4	3	5

The irreflexive basis graphs of the four homogeneous coherent configurations are given in Figures (2.1) and (2.2). Here $\mathcal{X}_1$ is the regular scheme corresponding to the cyclic group of order 3; $\mathcal{X}_2$ and $\mathcal{X}_3$ are respectively the regular schemes corresponding to the cyclic group of order 4 and the Klein four-group; the scheme $\mathcal{X}_4$ is the scheme of an undirected 4-cycle.

FIGURE 2.1. Irreflexive Basis Graphs of $\mathcal{X}_1$ and $\mathcal{X}_4$

FIGURE 2.2. Irreflexive Basis Graphs of $\mathcal{X}_2$ and $\mathcal{X}_3$

Let $\mathcal{X}$ be a non-homogeneous coherent configuration and $F(\mathcal{X}) = \{\Delta_1, \ldots \Delta_m\}$ with $m > 1$. The isomorphism type of $\mathcal{X}$ is an $m \times m$ matrix whose (i, j)-entry equals $|S_{\Delta_i,\Delta_j}|$. In this terminology, non-homogeneous coherent configurations of degree 3 and 4 are uniquely determined (up to isomorphism) by their isomorphism types. The isomorphism types of non-disctrete coherent configurations are given in Table (1).

Degree	Number of Fibers	Cardinalities of Fibers	Isomorphism Type
3	2	1,2	$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
4	2	1,3	$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ or $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$
	2	2,2	$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
	3	1,1,2	$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

TABLE 1. Nonhomogeneous Cases: Degree 3 and 4

2.7.3. Denote by s_i the relation on the vertex set Ω of a three-dimensional cube that is defined by the property "to be at distance i", $i = 0, 1, 2, 3$. Then the pair (Ω, S) with $S = \{s_0, s_1, s_2, s_3\}$, is a coherent configuration.

Proof. Obviously, S is a partition of Ω^2. Observe that

$$s_0 = 1_\Omega.$$

This implies that (Ω, S) satisfies condition (CC1). It also statisfies condition (CC2) because each s_i is symmetric.

Finally, we know that the rotation group R of the cube has order 24 since after a rotation, a face of the cube can coincide with any of the six faces of the original cube, and in each location. Now choose arbitrarily

$$s_i, s_j \quad \text{and} \quad s_k \in S$$

and any

$$(\alpha, \beta) \quad \text{and} \quad (\alpha', \beta') \in s_i.$$

There exists a rotation $r \in R$ such that

$$(\alpha, \beta)^r = (\alpha', \beta') \quad \text{and} \quad \alpha' s_j \cap \beta' s_k = (\alpha s_j \cap \beta s_k)^r.$$

This yields that (Ω, S) satisfies condition (CC3). We are done. □

2.7.4. Let $\Delta, \Gamma \in F$ and $s \in S_{\Delta,\Gamma}$. Then $\Omega_-(s) = \Delta$ and $\Omega_+(s) = \Gamma$. In particular, $\Omega_-(r)$, $\Omega_+(r)$, and $\Omega(r)$ are homogeneity sets of $\mathcal{X}$ for all $r \in S^{\cup}$.

Proof. Since $s \in S_{\Delta,\Gamma}$, we have

$$\Omega_-(s) \subseteq \Delta \quad \text{and} \quad c_{ss^*}^{1_\Delta} > 0.$$

The latter implies that $\Delta \subseteq \Omega_-(s)$, which proves the first equality. Similarly $\Omega_+(s) = \Gamma$. Thus, $\Omega(s)$ is a homogeneity set. □

2.7.5. Let $M \subset \mathbb{N}$ and $T \subseteq S^{\cup}$. Then $\{\alpha \in \Omega : \ |\alpha s| \in M \text{ for all } s \in T\}$ is a homogeneity set of $\mathcal{X}$.

Proof. Observe that for any $\alpha \in \Omega$ and any $s \in S$,

$$|\alpha s| = |\beta s|, \quad \beta \in \Delta$$

where Δ is the fiber containing α. Now if $|\alpha s| \in M$ for all $s \in T$, then the set in question contains Δ. Hence it is a homogeneity set. □

2.7.6. Let $r, s, t \in S$ and $\Delta \in F$. Then

(1) $c_{rs}^{1_\Delta} \neq 0$ if and only if $s = r^*$ and $\Omega_-(r) = \Delta$,
(2) $c_{rs}^t \leq \min\{n_r, n_{s^*}\}$,
(3) $\sum_{s \in S_{\Gamma,\Delta}} n_s = |\Delta|$ for all $\Gamma \in F$,
(4) $\sum_{w \in S} c_{rs}^w c_{tu}^v = \sum_{w \in S} c_{rw}^v c_{su}^w$ for all $u, v \in S$.

Proof. For statement (1), the sufficency is straightforward. To prove the necessity, assume $c_{rs}^{1_\Delta} \neq 0$. Then $\Delta \subseteq \Omega_-(r)$. In fact, here we have equality since $\Omega_-(r)$ is a fiber by Exercise (2.7.5). Obviously, $s = r^*$.

Statement (2) follows, because for any $(\alpha, \beta) \in t$ we have

$$c_{rs}^t = |\alpha r \cap \beta s^*| \leq \min\{|\alpha r|, |\beta s^*|\}.$$

For statement (3), fix $\alpha \in \Gamma$. Then,

$$\bigcup_{s \in S_{\Gamma,\Delta}} \alpha s \subseteq \Delta.$$

The equality holds as $r(\alpha, \beta) \in S_{\Gamma,\Delta}$ for any $\beta \in \Delta$. Since $|\alpha s| = n_s$, we are done.

For statement (4), fix a pair $(\alpha, \tau) \in v$. Let

$$W := \{(\beta, \gamma) : (\alpha, \beta) \in r, \quad (\beta, \gamma) \in s \quad \text{and} \quad (\gamma, \tau) \in u\}.$$

Assume that $(\alpha,\gamma)\in w$, then

$$|W| = \sum_{w\in rs\cap vu^*} c_{rs}^w c_{wu}^v = \sum_{w\in S} c_{rs}^w c_{wu}^v.$$

Also,

$$|W| = \sum_{w\in su\cap r^*v} c_{rw}^v c_{su}^w = \sum_{w\in S} c_{rw}^v c_{su}^w$$

by assuming $(\beta,\tau)\in w$. The proof is complete. □

2.7.7. [**?**, p.28] Let $\mathcal{X}$ be a scheme and $r,s,t\in S$. Then

(1) c_{rs}^t is a multiple of $\frac{n_s n_t\,\mathrm{GCD}(n_r,n_s,n_t)}{\mathrm{GCD}(n_r,n_s)\,\mathrm{GCD}(n_s,n_t)\,\mathrm{GCD}(n_t,n_r)}$.
(2) $n_t c_{rs}^t = 0 \pmod{m}$, where $m=\mathrm{LCM}(n_r,n_s)$.

Proof. By formula (2.1.14),

$$n_t c_{rs}^t = n_s c_{t^*r}^{s^*} = n_r c_{st^*}^{r^*}.$$

This yields that

$$n_s | n_t c_{rs}^t \quad \text{and} \quad n_r | n_t c_{rs}^t.$$

Thus, m divides $n_t c_{rs}^t$, which proves statement (2) and shows that

$$\frac{n_s}{\mathrm{GCD}\,(n_s,n_t)} \,|\, c_{rs}^t \quad \text{and} \quad \frac{n_r}{\mathrm{GCD}\,(n_r,n_t)} \,|\, c_{rs}^t.$$

Hence, the lowest common multiple of these two numbers divides c_{rs}^t. Now statement (1) follows since this multiple, as easily seen, equals

$$\frac{n_s n_t\,\mathrm{GCD}(n_r,n_s,n_t)}{\mathrm{GCD}(n_r,n_s)\,\mathrm{GCD}(n_s,n_t)\,\mathrm{GCD}(n_t,n_r)}.$$

□

2.7.8. Let $s\in S^\cup$. Then

(1) $e(s)=\{(\alpha,\beta)\in\Omega^2:\ \alpha s=\beta s\}$ belongs to E,
(2) $s\cdot s^*\in E$ if $s\in S$ and $n_s=1$.

Proof. Without loss of generality, we assume that s has full support.

To prove statement (1), observe that $e(s)$ is an equivalence relation. To prove that $e(s)\in E$, it suffices to show that for any $t\in S$,

$$t\cap e(s)\neq\varnothing \quad\Rightarrow\quad t\subseteq e(s).$$

To this end, take $t\in S$ and assume that

$$(\alpha,\beta)\in t\cap e(s).$$

Set $s:=s_1\cup\cdots\cup s_m$ with each $s_i\in S$. Then for every $1\le j\le m$,

$$\alpha s_j = \alpha s_j\cap\beta s = \bigcup_{i=1}^m (\alpha s_j\cap\beta s_i).$$

This implies that

$$n_{s_j} = \sum_{i=1}^m c_{s_j s_i^*}^t. \tag{2.7.1}$$

Take an arbitrary pair $(\alpha', \beta') \in t$. Since

$$|\alpha' s_j| = n_{s_j} \quad \text{and} \quad c^t_{s_j s_i^*} = |\alpha' s_j \cap \beta' s_i|,$$

formula (2.7.1) yields that

$$|\alpha' s_j| = \sum_{i=1}^{m} |\alpha' s_j \cap \beta' s_i| = |\alpha' s_j \cap \beta' s|.$$

Thus,

$$\alpha' s_j \subseteq \beta' s, \quad j = 1, \cdots, m.$$

Therefore, $\alpha' s \subseteq \beta' s$. Similarly, $\beta' s \subseteq \alpha' s$. It follows that $(\alpha', \beta') \in e(s)$ and hence $t \subseteq e(s)$, as required.

To prove statement (2) observe that the relation $e = s \cdot s^*$ is reflexive and symmetric. To prove transitivity, let $(\alpha, \beta), (\beta, \gamma) \in e$. There exist β_1, β_2 such that

$$(\alpha, \beta_1) \in s \quad \text{and} \quad (\beta_1, \beta) \in s^*; \quad (\beta, \beta_2) \in s \quad \text{and} \quad (\beta_2, \gamma) \in s^*.$$

Because $n_s = 1$, $|\beta s| = 1$ and thus $\beta_1 = \beta_2$. It follows that $(\alpha, \beta_1) \in s$ and $(\beta_1, \gamma) \in s^*$, which yields that $(\alpha, \gamma) \in e$, as wanted. □

2.7.9. Let $e \in E$. For $\alpha \in \Omega$ and $\Delta \in \Omega/e$, set $S(\alpha, \Delta) = \{s \in S : \ \alpha s \cap \Delta \neq \varnothing\}$. Then

1. for any $\alpha' \in \Omega$, the sets $S(\alpha, \Delta)$ and $S(\alpha', \Delta)$ are equal or disjoint,
2. for any $\Delta' \in \Omega/e$, the sets $S(\alpha, \Delta)$ and $S(\alpha, \Delta')$ are equal or disjoint.

Proof. To prove statement (1), let $s \in S$ be such that

$$s \in S(\alpha, \Delta) \cap S(\alpha', \Delta).$$

Then there exist $\beta, \beta' \in \Delta$ such that

$$r(\alpha, \beta) = s = r(\alpha', \beta'). \tag{2.7.2}$$

Furthermore, for any $s_1 \in S(\alpha, \Delta)$, one can find $\beta_1 \in \Delta$ such that $r(\alpha, \beta_1) = s_1$. Since $\beta, \beta_1 \in \Delta$, the relation $t := r(\beta, \beta_1)$ is contained in e. Thus,

$$(\alpha, \beta_1) \in s_1 \quad \text{and} \quad (\beta_1, \beta) \in t^* \quad \Rightarrow \quad |\alpha s_1 \cap \beta t| \neq \varnothing.$$

This yields that $c^s_{s_1 t^*} \neq 0$. By formula (2.7.2),

$$|\alpha' s_1 \cap \beta' t| \neq \varnothing.$$

Since $\beta' \in \Delta$ and $t \subseteq e$, this implies that $s_1 \in S(\alpha', \Delta)$. Thus,

$$S(\alpha, \Delta) \subseteq S(\alpha', \Delta).$$

Similarly the reverse inclusion can be proved.

To prove statement (2), assume that

$$s \in S(\alpha, \Delta) \cap S(\alpha, \Delta').$$

Then there exist $\beta \in \Delta$ and $\beta' \in \Delta'$ such that

$$r(\alpha, \beta) = s = r(\alpha, \beta'). \tag{2.7.3}$$

Furthermore, for any $s_1 \in S(\alpha, \Delta)$, one can find $\beta_1 \in \Delta$ such that $r(\alpha, \beta_1) = s_1$. Thus,

$$t := r(\beta, \beta_1) \subseteq e \quad \text{and} \quad |\alpha s_1 \cap \beta t| \neq \varnothing.$$

By formula (2.7.3),

$$|\alpha s_1 \cap \beta' t| \neq \varnothing.$$

It follows that $s_1 \in S(\alpha, \Delta')$. We deduce that

$$S(\alpha, \Delta) \subseteq S(\alpha, \Delta').$$

The reverse inclusion can be proved similarly. □

2.7.10. Let $e \in E$ and $\Delta \in F$ be such that $e_\Delta \neq \varnothing$. Then $e \cdot 1_\Delta \cdot e$ is an indecomposable component of e.

Proof. By the assumption, there exist $\alpha, \beta \in \Delta$ such that $(\alpha, \beta) \in e$. It follows that

$$(\alpha, \alpha) \subseteq r(\alpha, \beta) \cdot r(\alpha, \beta)^* \subseteq e.$$

Denote $e_1 := e \cdot 1_\Delta \cdot e$. Then obviously,

$$1_\Delta \subseteq e_1 \subseteq e.$$

Suppose on the contrary that there exist disjoint nonempty partial parabolics e'_1 and e'_2 such that

$$e_1 = e'_1 \cup e'_2.$$

Since $1_\Delta \subseteq e_1$, we may assume without loss of generality that $1_\Delta \subseteq e'_1$. In view of $e'_1 \cdot e'_2 = \varnothing$, we have

$$e_1 \cdot e'_1 \cdot e_1 = (e'_1 \cup e'_2) \cdot e'_1 \cdot (e'_1 \cup e'_2) = e'_1.$$

Taking into account that $e_1 = e \cdot e_1 \cdot e$, we obtain

$$e \cdot e'_1 \cdot e = e \cdot (e_1 \cdot e'_1 \cdot e_1) \cdot e = e_1 \cdot e'_1 \cdot e_1 = e'_1.$$

Thus,

$$e_1 = e \cdot 1_\Delta \cdot e \subseteq e \cdot e'_1 \cdot e = e'_1,$$

a contradiction. □

2.7.11. Let $s \in S$ and $e \in E$. Then

(1) $|\alpha s \cap \Delta|$ does not depend on $\alpha \in \Omega$ and $\Delta \in \Omega/e$ for which $\alpha s \cap \Delta \neq \varnothing$,
(2) if $\Omega(s) \subseteq \Omega(e)$ and $e \cdot s = s \cdot e$, then $n_{s_{\Omega/e}}$ divides n_s.

Proof. To prove statement (1), let $\alpha \in \Omega$ and $\Delta \in \Omega/e$ be such that $\alpha s \cap \Delta \neq \varnothing$. Then

$$(\alpha, \beta) \in s \quad \text{and} \quad \Delta = \beta e$$

for some $\beta \in \Omega$. Denote by T the set of basis relations contained in e. Then

$$\Delta = \bigcup_{r \in T} \beta r \quad \text{and} \quad \alpha s \cap \Delta = \bigcup_{r \in T} (\alpha s \cap \beta r).$$

Thus,

$$|\alpha s \cap \Delta| = \sum_{r \in T} c^s_{sr^*}.$$

Since the number on the right-hand side does not depend on the choice of α and Δ, we are done.

To prove statement (2), fix a point $\alpha \in \Omega(e)$. We claim that for any $\Delta \in \Omega/e$,

$$(\alpha e,\, \Delta) \in s_{\Omega/e} \quad \Leftrightarrow \quad \alpha s \cap \Delta \neq \varnothing. \tag{2.7.4}$$

To prove the implication "⇐", assume that $\alpha s \cap \Delta \neq \varnothing$. Then there exists $\beta \in \Delta$ such that $(\alpha, \beta) \in s$. It follows that

$$(\alpha, \beta) \in s \cap (\alpha e \times \Delta),$$

i.e., $(\alpha e, \Delta) \in s_{\Omega/e}$.

To prove the implication "⇒", assume that $(\alpha e, \Delta) \in s_{\Omega/e}$. Then there exist $\beta \in \alpha e$ and $\gamma \in \Delta$ such that $(\beta, \gamma) \in s$. It follows that

$$(\alpha, \gamma) \in e \cdot s = s \cdot e.$$

Consequently, one can find $\beta' \in \Omega$ such that

$$(\alpha, \beta') \in s \quad \text{and} \quad (\beta', \gamma) \in e.$$

Thus, $\Delta = \gamma e = \beta' e$ and hence $\beta' \in \alpha s \cap \Delta$, as required. The claim is proved.

Taking into account $\Omega(s) = \Omega(e)$ and claim (2.7.4), we obtain

$$\alpha s = \bigcup_{\Delta \in \Omega/e} (\alpha s \cap \Delta) = \bigcup_{(\alpha e, \Delta) \in s_{\Omega/e}} (\alpha s \cap \Delta).$$

The number of summands on the right-hand side equals $n_{s_{\Omega/e}}$, and any two of them have the same cardinalities (statement (1)). Since $|\alpha s| = n_s$, we are done. □

2.7.12. Let $\mathcal{X}$ be a regular scheme. Then

(1) the closed subsets of S and the subgroups of S_1 are in a 1-1 correspondence,
(2) any fission of $\mathcal{X}$ is semiregular.

Proof. By definition, the mapping

$$T \quad \mapsto \quad \bigcup_{t \in T} t$$

establishes a 1-1 correspondence between the closed subsets of S and the parabolics of $\mathcal{X}$. Since $\mathcal{X}$ is regular, $S = S_1$. Thus statement (1) follows from statement (4) of Theorem 2.1.26.

To prove statement (2), let $\mathcal{X}'$ be a fission of $\mathcal{X}$. Then any $s' \in S(\mathcal{X}')$ is contained in some $s \in S$. It follows that given $\alpha \in \Omega$, we have $\alpha s' \subseteq \alpha s$. Since $\mathcal{X}$ is regular,

$$|\alpha s'| \le |\alpha s| = 1.$$

This implies that the coherent configuration $\mathcal{X}'$ is semiregular. □

2.7.13. Let $\mathcal{X}$ be a semiregular coherent configuration. Then

(1) $|\Omega| = |\Delta| \cdot |F|$ and $|S| = |F|^2 \cdot |\Delta|$ for all $\Delta \in F$,
(2) if $\Delta, \Gamma \in F$ and $s \in S_{\Delta,\Gamma}$, then $f_s \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_\Gamma)$,
(3) there exists a system of distinct representatives of the family $\{S_{\Delta,\Gamma}\}_{\Delta,\Gamma \in F}$ that is closed with respect to the composition of relations.

Proof. To prove statement (1), choose $\Delta \in F$ and fix a point $\alpha \in \Delta$. For any $\Gamma \in F$, the map

$$\Gamma \to S_{\Delta,\Gamma}, \quad \beta \mapsto r(\alpha, \beta)$$

is a surjection since $\Omega_+(r) = \Gamma$ for any $r \in S_{\Delta,\Gamma}$. Because $\mathcal{X}$ is semiregular, $r(\alpha, \beta) \ne r(\alpha, \beta')$ for all distinct $\beta, \beta' \in \Gamma$. Hence the above map is also an injection. It follows that

$$|S_{\Delta,\Gamma}| = |\Gamma|.$$

Since this is true for any $\Gamma \in F$,

$$|\Gamma| = |(S_{\Delta,\Gamma})^*| = |S_{\Gamma,\Delta}| = |\Delta|.$$

Thus,

$$|\Omega| = \sum_{\Gamma\in F} |\Gamma| = |\Delta|\cdot|F|$$

and

$$|S| = \sum_{\Lambda,\Gamma\in F} |S_{\Lambda,\Gamma}| = |\Delta|\cdot|F|^2.$$

To prove statement (2), let $\Delta,\Gamma\in F$ and $s\in S_{\Delta,\Gamma}$. Since $\mathcal{X}$ is semiregular, the relation s is a matching and the mapping

$$f_s:\,\Delta\to\Gamma$$

is a bijection. From the definition of f_s, it easily follows that

$$s^*\cdot r\cdot s = \{(\alpha^{f_s},\beta^{f_s}):\quad (\alpha,\beta)\in r\} = r^{f_s},$$

see also Fig. 2.3.

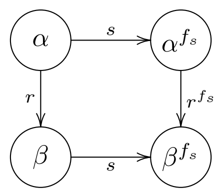

FIGURE 2.3. Configuration for Exercise 2.7.13.

Since r,s are thin, $r^{f_s}\in S_\Gamma$ by Lemma 2.1.25. Thus, f_s is the required isomorphism.

To prove statement (3), let $m=|F|$. Denote the fibers of $\mathcal{X}$ by $\Delta_1,\ldots,\Delta_m$ and set $S_{ij}=S_{\Delta_i,\Delta_j}$ for all $i,j=1,\ldots,m$. For each i, fix a basis relation

$$s_{1i}\in S_{1i}\quad\text{and}\quad s_{11}=1_{\Delta_1}.$$

Now the relations,

$$s_{ij}=s_{1i}^*s_{1j},\quad 1\le i,j\le m,$$

form a required system of representatives. Indeed, for all i,j,k,

$$s_{ij}\cdot s_{jk} = (s_{1i}^*\cdot s_{1j})(s_{1j}^*s_{1k}) = s_{1i}^*\cdot(s_{1j}\cdot s_{1j}^*)s_{1k} = s_{1i}^*\cdot s_{11}\cdot s_{1k} = s_{1i}^*s_{1k} = s_{ik}.$$

□

2.7.14. Let $s\in S$ be such that ss^* consists of thin relations. Then $ss^*s=\{s\}$.

Proof. Let $t\in ss^*$. By formula (2.1.9), $c_{t^*s}^s\ne 0$ if and only if $c_{ss^*}^t\ne 0$. This implies that

$$s\subseteq t^*\cdot s.$$

As t^* is thin, $t^*\cdot s$ is a basis relation. Hence,

$$s=t^*\cdot s.$$

Together with the obvious fact $(ss^*)^*=ss^*$, we obtain

$$ss^*s=\bigcup_{t\in ss^*} ts = \bigcup_{t\in ss^*} t^*s=\{s\}.$$

□

2.7.15. Let e be the equivalence relation on Ω such that $\Omega/e = F$. Then $e \in E$ and $e \cdot s = s \cdot e$ for all $s \in S$.

Proof. By the assumption,

$$e = \bigcup_{\Delta \in F} \Delta^2 = \bigcup_{\Delta \in F} \bigcup_{t \in S_{\Delta,\Delta}} t.$$

This implies that $e \in S^{\cup}$ and hence $e \in E$.

To prove the second assertion, let $s \in S_{\Delta,\Gamma}$ with $\Delta, \Gamma \in F$. Set u to be the union of all basis relations in $S_{\Delta,\Gamma}$. It suffices to verify that

$$e \cdot s = u = s \cdot e.$$

We prove the first equality. The second one can be proved similarly.

On one hand, for any $(\alpha, \beta) \in e \cdot s$, there exists γ such that $(\alpha, \gamma) \in e$ and $(\gamma, \beta) \in s$. It follows that

$$\gamma \in \Omega_-(s) = \Delta \quad \text{and} \quad \beta \in \Omega_+(s) = \Gamma.$$

By the definition of e the first equality implies that $\alpha \in \gamma e = \Delta$. This together with the second one yield that $r(\alpha, \beta) \subseteq u$. Thus,

$$e \cdot s \subseteq u.$$

On the other hand, for any $(\alpha, \beta) \in u$, there exists $t \in S_{\Delta,\Gamma}$ such that $(\alpha, \beta) \in t$. By the choice of s, one can find $\alpha' \in \Delta$ satisfying $(\alpha', \beta) \in s$. Thus,

$$\alpha,\, \alpha' \in \Delta \quad \Rightarrow \quad (\alpha, \alpha') \in e \quad \Rightarrow \quad (\alpha, \beta) \in e \cdot s.$$

It follows that $u \subseteq e \cdot s$. □

2.7.16. Let $\mathcal{X}$ be a cyclotomic scheme over a field $\mathbb{F}$. Then $\mathrm{A\Gamma L}(1, \mathbb{F}) \leq \mathrm{Iso}(\mathcal{X})$.

Proof. Let $\tau \in \mathrm{A\Gamma L}(1, \mathbb{F})$, i.e.,

$$\tau : \quad \alpha \mapsto a + \alpha^{\sigma} b,\ \alpha \in \mathbb{F},$$

for some $a \in \mathbb{F}$, $b \in \mathbb{F}^{\times}$, and $\sigma \in \mathrm{Aut}(\mathbb{F})$. Keeping the notation cercerning cyclotomic schemes (page 43), let s_u be a basis reltaion of $\mathcal{X}$ with $u \in \mathbb{F}$. For any $(\alpha, \beta) \in s_u$, we have $\alpha - \beta \in uM$, where M is a subgroup of $\mathbb{F}^{\times}$ associated with $\mathcal{X}$. Obviously, $M^{\sigma} = M$. Then,

$$\alpha^{\tau} - \beta^{\tau} = (\alpha^{\sigma} - \beta^{\sigma})b = (\alpha - \beta)^{\sigma} b \in (uM)^{\sigma} b = u^{\sigma} M b = (u^{\sigma} b)M.$$

Thus, $s_u^{\tau} \subseteq s_{u^{\sigma} b}$. Because s_u and $s_{u^{\sigma} b}$ have the same cardinalities, we have

$$s_u^{\tau} = s_{u^{\sigma} b}.$$

This equality holds for all s_u. Hence, $\tau \in \mathrm{Iso}(\mathcal{X})$, as required. □

2.7.17. Let $K \leq \mathrm{Sym}(\Omega)$ and $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. Then

(1) $S^{\cup}$ equals the set of all K-invariant relations on Ω,
(2) if $e \in E$ and $\Delta \in \Omega/e$, then $\mathcal{X}_{\Delta} = \mathrm{Inv}(K^{\Delta}, \Delta)$,
(3) K is of odd oder if and only if $\mathcal{X}$ is antisymmetric,
(4) K is a p-group for a prime p if and only if $|s|$ is a p-power for each $s \in S$.

Proof. Every basis relation of $\mathcal{X}$ is a 2-orbit of K. Thus, a relation on Ω belongs to $S^{\cup}$ if and only if it is a union of some 2-orbits of K if and only if it is K-invariant. Statement (1) follows.

To prove statement (2), without loss of generality assume that e has full support. Let $s_{\Delta} \in S_{\Delta}$. Obviously K^{Δ} acts on s_{Δ}. Let (α,β), $(\alpha',\beta') \in s_{\Delta}$. Since s is a K-orbit, there exists $k \in K$ such that

$$(\alpha',\beta') = (\alpha,\beta)^k = (\alpha^k,\beta^k).$$

The parabolic e is K-invariant by statement (1). By Exercise (1.4.20), this implies that $\Delta^k \in \Omega/e$. However, the set $\Delta \cap \Delta^k$ is not empty (it contains α' and β'). Thus,

$$\Delta^k = \Delta$$

and hence $k \in K_{\{\Delta\}}$. It follows that s_{Δ} is a K^{Δ}-orbit. Therefore,

$$S_{\Delta} = \mathrm{Orb}(K^{\Delta},\Delta^2).$$

To prove statement (3), first let K be of odd order. Assume on the contrary that $\mathcal{X}$ is not antisymmetric. Then there exists an irreflexive symmetric $s \in S$. It follows that

$$(\alpha,\beta) \in s \quad \Leftrightarrow \quad (\beta,\alpha) \in s.$$

This yields that $|s|$ is even. However, if $(\alpha,\beta) \in s$, then the number

$$|s| = |(\alpha,\beta)^K| = |K : K_{(\alpha,\beta)}|$$

is odd, a contradiction.

Second, let $\mathcal{X}$ be antisymmetric. Assume on the contrary that K is of even order. Then there exists an involution $k \in K$. Thus there exist $\alpha \neq \beta \in \Omega$ such that

$$\alpha^k = \beta \quad \text{and} \quad \beta^k = \alpha.$$

It follows that

$$(\beta,\alpha) = (\alpha,\beta)^k \in r(\alpha,\beta)^k = r(\alpha,\beta).$$

Consequently, the irreflexive basis relation $r(\alpha,\beta)$ is symmetric, a contradiction.

To prove statement (4), suppose that K is a p -group. Then the cardinality of each 2-orbit of K is a p-power. Since the 2-orbits are exactly the basis relations of $\mathcal{X}$, the necessity follows. To prove the sufficiency, we will use the technique developed in section 3.1 of Chapter 3.

If K is nontransitive on Ω, then the assertion follows by induction since

$$K \cong \prod_{\Delta \in \mathrm{Orb}(K,\Omega)} K^{\Delta}.$$

So one can assume that K is transitive. Then $\mathcal{X}$ is a p-scheme in the sense of Exercise (3.7.17). Suppose that $\mathcal{X}$ is imprimitive. Let $\Omega^2 \neq e \neq 1_{\Omega}$ be a parabolic of $\mathcal{X}$. Then the quotient $\mathcal{X}_{\Omega/e}$ is still a p-scheme by (4) of Exercise (3.7.17). By formula (3.1.8), we have

$$\mathcal{X}_{\Omega/e} = \mathrm{Inv}(K^{\Omega/e},\Omega/e).$$

Hence, $K^{\Omega/e}$ is a p-group by induction.

Let $\Delta \in \Omega/e$. For any $t \in S_{\Delta}$, there exists $s \in S$ such that $t = s_{\Delta}$. Then $|t|$ divides $|s|$ (Proposition 2.1.18). Thus, $|t|$ is a p-power. It follows that $\mathcal{X}_{\Delta}$ is also a

p-scheme. Since $\mathcal{X}_\Delta = \mathrm{Inv}(K^\Delta, \Delta)$ (statement (2)), K^Δ is a p-group by induction. Note that K is isomorphic to a subgroup of

$$K^{\Omega/e} \wr K^\Delta.$$

Hence, K is a p-group as both $K^{\Omega/e}$ and K^Δ are p-groups.

If the scheme is primitive, by statement (2) of Exercise (3.7.17), $S = S_1$. It follows that the scheme is regular. Thus, $|K| = |1_\Omega|$ is a p-power. □

2.7.18. Let $\mathcal{X}$ be a schurian coherent configuration. Then the group $\mathrm{Iso}(\mathcal{X})$ equals the normalizer of $\mathrm{Aut}(\mathcal{X})$ in $\mathrm{Sym}(\Omega)$.

Proof. Set

$$N = N_{\mathrm{Sym}(\Omega)}(K),$$

where $K = \mathrm{Aut}(\mathcal{X})$. Since $\mathcal{X}$ is schurian, every basis relation of $\mathcal{X}$ is a 2-orbit of K. Therefore for any $g \in N$, $s \in S$, and $(\alpha, \beta) \in s$, we have

$$s^g = (\alpha, \beta)^{Kg} = (\alpha, \beta)^{gK} = r(\alpha^g, \beta^g).$$

It follows that $g \in \mathrm{Iso}(\mathcal{X})$ and hence $N \subseteq \mathrm{Iso}(\mathcal{X})$.

Conversely, for any $h \in \mathrm{Iso}(\mathcal{X})$, $k \in K$, and $s \in S$,

$$s^{h^{-1}kh} = (s^{h^{-1}})^{kh} = (s^{h^{-1}})^h = s,$$

which yields that $h^{-1}kh \in K$. Thus, $h \in N$. Therefore, $\mathrm{Iso}(\mathcal{X}) \subseteq N$. □

2.7.19. Let $\mathcal{X}$ be a *quasiregular* coherent configuration, i.e., every its homogeneous component is regular. Then the group $\mathrm{Aut}(\mathcal{X})$ is abelian if each homogeneous component of $\mathcal{X}$ is commutative. The converse is true if $\mathcal{X}$ is schurian.

Proof. For a coherent configuration $\mathcal{X}$, denote $\mathrm{Aut}(\mathcal{X})$ by K. Since each $s \in S$ is K-invariant, each $\Delta \in F$ is also K-invariant. It follows that

$$K^\Delta \le \mathrm{Aut}(\mathcal{X}_\Delta).$$

Then there is a group monomorphism:

$$\psi : K \to \prod_{\Delta \in F} \mathrm{Aut}(\mathcal{X}_\Delta).$$

Now assume further that $\mathcal{X}$ is quasiregular. Let $\Delta \in F$. Then $\mathcal{X}_\Delta$ is regular. It follows that $S(\mathcal{X}_\Delta)$ is a group isomorphic to $\mathrm{Aut}(\mathcal{X}_\Delta)$. If $\mathcal{X}_\Delta$ is commutative, then $\mathrm{Aut}(\mathcal{X}_\Delta)$ is abelian. If this is true for all $\Delta \in F$, then $K \cong \mathrm{Im}(\psi)$ is abelian.

Conversely, suppose that $\mathcal{X}$ is schurian and K is abelian. Choose $\Delta \in F$ arbitrarily. Obviously,

$$K^\Delta \le \mathrm{Aut}(\mathcal{X}_\Delta). \tag{2.7.5}$$

Since $\mathcal{X}_\Delta$ is regular, $\mathrm{Aut}(\mathcal{X}_\Delta)$ is regular on Δ. As $\mathcal{X}$ is schurian, 1_Δ is a K-orbit. This yields that K^Δ is transitive on Δ. These facts together with formula (2.7.5) show that $\mathrm{Aut}(\mathcal{X}_\Delta)$ is abelain. Hence, $\mathcal{X}_\Delta$ is commutative. □

2.7.20. [**?**] In the notation of Theorem 2.2.7, assume that the group K is transitive and H is a point stabilizer of K. Then for any $r, s, t \in S$, the number $|H|c_{rs}^t$ is equal to the number of occurrences of the double coset D_{t^*} in the the product $D_{r^*}D_{s^*}$.

Proof. Without loss of generality, we may assume that

$$\Omega = \{Hk : \quad k \in K\}$$

and K acts on Ω by right multiplications. Let

$$(H, Hx) \in t, \quad (H, Hy) \in r, \quad \text{and} \quad (H, Hz) \in s.$$

Then

$$D_t = HxH, \quad D_r = HyH, \quad \text{and} \quad D_s = HzH,$$

and

$$D_{t^*} = Hx^{-1}H, \quad D_{r^*} = Hy^{-1}H, \quad \text{and} \quad D_{s^*} = Hz^{-1}H.$$

Observe that

$$c_{rs}^t = |\{Hu \in \Omega : (H, Hu) \in r, \quad (Hu, Hx) \in s\}|.$$

Furthermore,

$$(H, Hu) \in r \quad \Leftrightarrow \quad Hu \subseteq D_r \quad \Leftrightarrow \quad HuH = D_r,$$

and

$$(Hu, Hx) \in s \Leftrightarrow (H, Hux^{-1}) \in s^* \Leftrightarrow Hux^{-1}H = D_{s^*} \Leftrightarrow HuH \subseteq D_{s^*}D_t.$$

Thus,

$$|H|c_{rs}^t = |\{(g, h) \in D_{s^*} \times D_t : \quad y^{-1} = gh\}|.$$

Since the right-hand side equals the number of occurences of the double coset D_{t^*} in the product $D_{r^*}D_{s^*}$, we are done. □

2.7.21. Let $e \in E$ and $\Delta \in \Omega/e$. Then

(1) the mapping $S_\Delta \to S$, $s_\Delta \mapsto s$ is an injection; it induces injections from $F(\mathcal{X}_\Delta)$ and $E(\mathcal{X}_\Delta)$ into F and E, respectively,
(2) the coherent configuration $\mathcal{X}_\Delta$ is schurian whenever so is $\mathcal{X}$,
(3) the restriction of a schurian coherent configuration to any homogeneity set is schurian.

Proof. To prove statement (1), denote the mapping $s_\Delta \mapsto s$ by φ. Clearly for $s_\Delta, t_\Delta \in S_\Delta$,

$$s_\Delta = t_\Delta \quad \Leftrightarrow \quad \varphi(s_\Delta) = \varphi(t_\Delta).$$

This implies that φ is an injection. The induced injection from $S_\Delta^\cup$ to $S^\cup$ is also denoted by φ. It is easily seen that φ maps reflexive relations to reflexive relations. Hence, one can extend φ to an injection from $F(\mathcal{X}_\Delta)^\cup$ to $F^\cup$ such that

$$\varphi(1_\Gamma) = 1_{\Gamma^\varphi}, \quad \Gamma \in F(\mathcal{X}_\Delta)^\cup. \tag{2.7.6}$$

In addition, for any $s \in S(\mathcal{X}_\Delta)^\cup$, it is easy to see that

$$(\Omega_\pm(s))^\varphi = \Omega_\pm(\varphi(s)).$$

Now let $e \in E(\mathcal{X}_\Delta)$. It suffices to verify that $\varphi(e)$ belongs to E. By formula (2.7.6), we obtain

$$(\Omega(e))^\varphi = \Omega_-(\varphi(e)) = \Omega_+(\varphi(e)).$$

This together with the obvious fact that $\varphi(s^*) = \varphi(s)^*$ yield that $\varphi(e)$ is a reflexive and symmetric relation on $\Omega(\varphi(e))$. Since φ preserves the composition of relations, for any $r, s \in S(\mathcal{X}_\Delta)$

$$r \cdot s \subseteq e \quad \Rightarrow \quad \varphi(r) \cdot \varphi(s) \subseteq \varphi(e).$$

Thus, $\varphi(e)$ is transitive on $\Omega(\varphi(e))$. This shows that $\varphi(e) \in E$.

Statement (2) follows immediately from statement (2) of Exercise (2.7.17). Statement (3) is a special case of statement (2): for a homogeneity set Δ, take $e = \Delta^2$. □

2.7.22. For any 2-orbit s of the group $\mathrm{Sym}(\Omega)$ acting on Ω^m $(m \geq 1)$, there exists an equivalence relation e on $\{1, \ldots, 2m\}$ such that

$$s = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \ (\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j \ \Leftrightarrow \ (i, j) \in e\}.$$

Conversely, any such s is a 2-orbit of $\mathrm{Sym}(\Omega)$ acting on Ω^m.

Proof. Let s be a 2-orbit of $\mathrm{Sym}(\Omega)$ on Ω^m. Fix $(\alpha, \beta) \in s$. Then

$$s = \{(\alpha^k, \beta^k) : \quad k \in \mathrm{Sym}(\Omega)\}.$$

Observe that for any $1 \leq i, j \leq 2m$ and any $k \in \mathrm{Sym}(\Omega)$

$$(\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j \quad \Leftrightarrow \quad (\alpha^k \cdot \beta^k)_i = (\alpha^k \cdot \beta^k)_j. \tag{2.7.7}$$

Let e be the relation on $\{1, \ldots, 2m\}$ defined as follows:

$$(i, j) \in e \quad \Leftrightarrow \quad (\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j.$$

It is easily seen that e is an equivalence relation on $\{1, \ldots, 2m\}$. Furthermore, by statement (2.7.7) we have

$$s \subseteq \{(\gamma, \tau) \in \Omega^m \times \Omega^m : (\gamma \cdot \tau)_i = (\gamma \cdot \tau)_j \Leftrightarrow (i, j) \in e\}. \tag{2.7.8}$$

To prove the reverse inclusion, let (γ, τ) be an arbitrary element in the set on the right-hand side in (2.7.8). Set

$$n := |\{(\gamma \cdot \tau)_i : \ 1 \leq i \leq 2m\}|.$$

Note that for any $1 \leq i, j \leq 2m$,

$$(\gamma \cdot \tau)_i = (\gamma \cdot \tau)_j \quad \Leftrightarrow \quad (\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j. \tag{2.7.9}$$

It follows that there exist indices $1 \leq i_1 < \ldots < i_n \leq 2m$ such that

$$u \neq u' \quad \Rightarrow \quad (\gamma \cdot \tau)_{i_u} \neq (\gamma \cdot \tau)_{i_{u'}}.$$

Note that $n \leq |\Omega|$. Therefore $\mathrm{Sym}(\Omega)$ is n-transitive on Ω. Thus, there exists $k \in \mathrm{Sym}(\Omega)$ such that

$$(\gamma \cdot \tau)_{i_u} = ((\alpha \cdot \beta)_{i_u})^k, \quad u = 1, \ldots, n. \tag{2.7.10}$$

For any $1 \leq l \leq 2m$, there exists $1 \leq u \leq n$ such that $(\gamma \cdot \tau)_l = (\gamma \cdot \tau)_{i_u}$. By formulas (2.7.9) and (2.7.10), we have

$$(\gamma \cdot \tau)_l = (\gamma \cdot \tau)_{i_u} = ((\alpha \cdot \beta)_{i_u})^k = ((\alpha \cdot \beta)_l)^k.$$

It follows that $(\gamma, \tau) = (\alpha, \beta)^k \in s$. We are done. □

2.7.23. Let $K \leq \mathrm{Sym}(\Omega)$. Then

(1) $K^{(1)}$ equals the direct product of $\mathrm{Sym}(\Delta)$, $\Delta \in \mathrm{Orb}(K, \Omega)$,
(2) if K is 2-transitive, then $K^{(2)} = \mathrm{Sym}(\Omega)$,
(3) $(K^{(a)})^{(b)} = K^{(m)}$, where $m = \min\{a, b\}$,
(4) if $L \leq K$, then $L^{(m)} \leq K^{(m)}$.

Proof. For statement (1), denote the direct product of $\mathrm{Sym}(\Delta)$, $\Delta \in \mathrm{Orb}(K,\Omega)$ by L. Obviously $\mathrm{Orb}(L,\Omega) = \mathrm{Orb}(K,\Omega)$. This implies that L and K are 1-equivalent. Hence,

$$L \leq K^{(1)}.$$

The reverse inclusion holds because

$$K^{(1)} \leq \prod_{\Delta\in\mathrm{Orb}(K,\Omega)} (K^{(1)})^{\Delta} \leq L.$$

For statement (2), since K is 2-transitive, $\mathrm{Inv}(K) = \mathcal{T}_\Omega$. This implies that

$$K^{(2)} = \mathrm{Aut}(\mathrm{Inv}(K)) = \mathrm{Aut}(\mathcal{T}_\Omega) = \mathrm{Sym}(\Omega).$$

For statement (3), first assume $b \leq a$. Fix a point $\beta \in \Omega$. Then, for any $(\alpha_1,\ldots,\alpha_b) \in \Omega^b$, we have $(\alpha_1,\ldots,\alpha_b,\beta,\ldots,\beta) \in \Omega^a$. Since K and $K^{(a)}$ are a-equivalent,

$$(\alpha_1,\ldots,\alpha_b,\beta,\ldots,\beta)^K = (\alpha_1,\ldots,\alpha_b,\beta,\ldots,\beta)^{K^{(a)}}.$$

This yields that

$$(\alpha_1,\ldots,\alpha_b)^K = (\alpha_1,\ldots,\alpha_b)^{K^{(a)}}.$$

It follows that K and $K^{(a)}$ are b-equivalent. Since K and $K^{(b)}$ are b-equivalent, so are $K^{(a)}$ and $K^{(b)}$. In other words,

$$\mathrm{Orb}(K^{(a)},\Omega^b) = \mathrm{Orb}(K^{(b)},\Omega^b).$$

Thus,

$$(K^{(a)})^{(b)} = \mathrm{Aut}(\mathrm{Orb}(K^{(a)},\Omega^b)) = \mathrm{Aut}(\mathrm{Orb}(K^{(b)},\Omega^b)) = K^{(b)},$$

which completes the proof of the case in question. In particular,

$$K^{(b)} = (K^{(a)})^{(b)} \geq K^{(a)}$$

and

$$\mathrm{Aut}(\mathrm{Orb}(K,\Omega^b)) \geq \mathrm{Aut}(\mathrm{Orb}(K,\Omega^a)). \tag{2.7.11}$$

Next assume $b > a$. Applying formula (2.7.11) to $K = K^{(a)}$ and with a and b interchanged, we obtain

$$(K^{(a)})^{(b)} = \mathrm{Aut}(\mathrm{Orb}(K^{(a)},\Omega^b)) \leq \mathrm{Aut}(\mathrm{Orb}(K^{(a)},\Omega^a)) = K^{(a)}.$$

Since $K^{(a)}$ is contained in its b-closure, $(K^{(a)})^{(b)} = K^{(a)}$, which completes the proof.

For statement (4), by the Galois correspondence one can see that

$$\mathrm{Orb}(L,\Omega^m) \supseteq \mathrm{Orb}(K,\Omega^m) \quad \Rightarrow \quad \mathrm{Aut}(\mathrm{Orb}(L,\Omega^m)) \leq \mathrm{Aut}(\mathrm{Orb}(K,\Omega^m)),$$

i.e., $L^{(m)} \leq K^{(m)}$. □

2.7.24. Given a matrix $A \in \mathrm{Mat}_\Omega$, set

$$e(A) = \{(\alpha,\beta) \in \Omega^2 : \ A\alpha = A\beta \neq 0\}. \tag{2.7.12}$$

Then $e(A) \in E$, whenever $A \in \mathrm{Adj}(\mathcal{X})$.

Proof. Let

$$\Delta =: \{\alpha \in \Omega : A\alpha \neq 0\}.$$

It is straightforward to see that $e(A)$ is an equivalence relation on Δ. It suffices to show that $e(A)$ belongs to $S^{\cup}$.

There exists $T \subseteq S^{\cup}$ such that

$$A = \sum_{t\in T} a_t A_t,$$

where each $a_t \neq 0$ and $a_t \neq a_{t'}$ for any $t \neq t' \in T$. For any point $\alpha \in \Omega$, Exercise (1.4.7) implies that

$$A\alpha = \sum_{t\in T} a_t(A_t\alpha) = \sum_{t\in T} a_t\,\underline{\alpha t^*}\,. \tag{2.7.13}$$

It follows that $\alpha \in \Delta$ if and only if $\alpha t^* \neq \varnothing$ for at least one $t \in T$. Thus,

$$\Delta = \bigcup_{t\in T} \Omega_-(t^*) \in F^{\cup}.$$

Let $e(s)$ be defined as in Exercise (2.7.8) for $s \in S^{\cup}$. Since the set $\{\underline{\alpha t^*} : t \in T\}$ in (2.7.13) consists of pariwise orthogonal {0,1}-vectors, we have

$$e(A) = (\bigcap_{t\in T} e(t)) \cap \Delta^2.$$

However, $e(t) \in E$ for each $t \in T$ by Exercise (2.7.8). Thus, $e(A) \in S^{\cup}$, as required. □

2.7.25. Let $m \geq 2$ be an integer, $r \in S$, and $s_1, \ldots, s_{m-1} \in S^{\cup}$. Then the number $p_r(\alpha, \beta; s_1, \ldots, s_{m-1})$ of all tuples $(\alpha_1, \ldots, \alpha_m) \in \Omega^m$ such that

$$(\alpha_1, \alpha_m) = (\alpha, \beta) \quad \text{and} \quad r(\alpha_i, \alpha_{i+1}) = s_i,\ i = 1, \ldots, m-1,$$

does not depend on the choice of $(\alpha, \beta) \in r$.

Proof. Since $\mathrm{Adj}(\mathcal{X})$ is a coherent algebra, there exists a nonnegative integer a such that

$$(A_{s_1} \cdot \ldots \cdot A_{s_{m-1}}) \circ A_r = aA_r.$$

According to the rule of matrix multiplication (see also statement (4) of Exercise (1.4.8)), one can easily check that for each $(\alpha, \beta) \in r$,

$$a = p_r(\alpha, \beta; s_1, \ldots, s_{m-1}).$$

Since a does not depend on the choice of $(\alpha, \beta) \in r$, we are done. □

2.7.26. The scalar product on the adjacency algebra $\mathrm{Adj}(\mathcal{X})$ defined by the formula

$$\langle \sum_{s\in S} c_s A_s, \sum_{s\in S} b_s A_s \rangle = \frac{1}{|\Omega|} \sum_{s\in S} c_c b_s |s|$$

is associative, i.e., $\langle AB, C\rangle = \langle B, A^*C\rangle$ for all $A, B, C \in \mathrm{Adj}(\mathcal{X})$.

Proof. Since the scalar product is linear in each arguement, without loss of generality, one can assume that

$$A = A_r, \quad B = A_s, \quad \text{and} \quad C = A_t,$$

where $r, s, t \in S$.

On one hand,

$$\langle AB, C\rangle = \langle \sum_{u\in S} c_{rs}^u A_u, A_t\rangle = \frac{1}{|\Omega|}|t|c_{rs}^t.$$

On the other hand,

$$\langle B, A^*C\rangle = \langle A_s, \sum_{v\in S} c_{r^*t}^v A_v\rangle = \frac{1}{|\Omega|}|s|c_{r^*t}^s.$$

Note that $|t| = |t^*|$ and $c_{s^*r^*}^{t^*} = c_{rs}^t$ by formula (2.1.3). These equalities together with formula (2.1.9) yield that

$$|s|c_{r^*t}^s = |t^*|c_{s^*r^*}^{t^*} = |t|c_{rs}^t.$$

We are done. □

2.7.27. [**?**, Lemma 2.3] Let $\mathcal{X}$ be a shceme and $r, s \in S^{\#}$. Then $rr^*\cap ss^* = \{1_\Omega\}$ if and only if $c_{r^*s}^t \le 1$ for all $t \in S$.

Proof. The scalar product defined in Exercise (2.7.26) is applied here. Observe that

$$\langle A_rA_{r^*}, A_sA_{s^*}\rangle = \frac{1}{|\Omega|}\sum_{u\in rr^*\cap ss^*} c_{rr^*}^u c_{ss^*}^u |u|.$$

Thus,

$$rr^*\cap ss^* = \{1_\Omega\} \quad\Leftrightarrow\quad \langle A_rA_{r^*}, A_sA_{s^*}\rangle = \frac{1}{|\Omega|}c_{rr^*}^{1_\Omega}c_{ss^*}^{1_\Omega}|1_\Omega| = n_rn_s. \tag{2.7.14}$$

Furthermore,

$$\begin{aligned}\langle A_rA_{r^*}, A_sA_{s^*}\rangle &= \langle A_{r^*}A_s, A_{r^*}A_s\rangle\\ &= \frac{1}{|\Omega|}\sum_{t\in S}(c_{r^*s}^t)^2|t|\\ &\ge \frac{1}{|\Omega|}\sum_{t\in S}c_{r^*s}^t|t|\\ &= \frac{1}{|\Omega|}\sum_{t\in S}c_{r^*s}^t|\Omega|n_t\\ &= \sum_{t\in S}c_{r^*s}^t n_t = n_{r^*}n_s = n_rn_s.\end{aligned}$$

Here the equality is attained if and only if

$$c_{r^*s}^t \le 1, \quad \text{for} \quad \text{all} \quad t \in S.$$

Together with formula (2.7.14), these complete the proof. □

2.7.28. Let s be a relation of $\mathcal{X}$. Then so is $\{(\alpha,\beta)\in\Omega^2 :\ \alpha \stackrel{s}{\to} \beta\}$.

Proof. By using the notation of Exercise (2.7.25), one can see that, for any pair $(\alpha,\beta)\in\Omega^2$,

$$p_t(\alpha,\beta;\underbrace{s,\ldots,s}_{m-1}) > 0 \text{ for some } m \ge 2 \quad\Leftrightarrow\quad \alpha \stackrel{s}{\to} \beta.$$

To prove that

$$s' := \{(\alpha, \beta) \in \Omega^2 : \ \alpha \xrightarrow{s} \beta\}$$

is a relation of $\mathcal{X}$, let t be a basis relation intersecting s'. It suffices to verify that $t \subseteq s'$. To this end, take an arbitrary pair $(\alpha, \beta) \in t \cap s'$. Then $p_t(\alpha, \beta; s, \ldots, s) > 0$. Since this is true for any $(\alpha', \beta') \in t$ (Exercise (2.7.25)), we obtain $(\alpha', \beta') \in s'$, i.e., $t \subseteq s'$. □

2.7.29. Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ and $r, s \in S^{\cup}$. Then

(1) $\varphi(r \cup s) = \varphi(r) \cup \varphi(s)$ and $\varphi(r \cap s) = \varphi(r) \cap \varphi(s)$,

(2) $\varphi(\langle s \rangle) = \langle \varphi(s) \rangle$ and $\varphi(\mathrm{rad}(s)) = \mathrm{rad}(\varphi(s))$.

Proof. To prove statement (1), decompose r and s respectively as unions of basis relations as follows:

$$r = r_1 \cup \ldots \cup r_k \quad \text{and} \quad s = s_1 \cup \ldots \cup s_l,$$

where k and l are nonnegative integers. By formula (2.3.16) on page 67, we obtain

$$\begin{aligned} \varphi(r \cup s) =& \varphi(r_1 \cup \ldots \cup r_k \cup s_1 \cup \ldots \cup s_l) \\ =& (\varphi(r_1) \cup \ldots \cup \varphi(r_k)) \cup (\varphi(s_1) \cup \ldots \cup \varphi(s_l)) \\ =& \varphi(r) \cup \varphi(s). \end{aligned}$$

Similarly, one can prove that $\varphi(r \cap s) = \varphi(r) \cap \varphi(s)$.

To prove statement (2), note that by Exercise (1.4.1),

$$\langle s \rangle = \{1_{\Omega(s)}, s, s^*\}^{\infty}.$$

In addition, statement (2) of Corollary 2.3.23 and statement (4) of Proposition 2.3.18 respectively imply that

$$\varphi(1_{\Omega(s)}) = 1_{\Omega(\varphi(s))} \quad \text{and} \quad \varphi(s^*) = \varphi(s)^*.$$

Together with statement (2) of Proposition 2.3.18, this yields that

$$\varphi(\langle s \rangle) = \{1_{\Omega(\varphi(s))}, \varphi(s), \varphi(s)^*\}^{\infty} = \langle \varphi(s) \rangle. \tag{2.7.15}$$

To prove the second equality of statement (2), note that by the first part of Propostion 2.3.25, $\varphi(\mathrm{rad}(s)) \in E'$. Since

$$\mathrm{rad}(s) \cdot s = s = s \cdot \mathrm{rad}(s),$$

by statement (2) of Proposition 2.3.18, we obtain

$$\varphi(\mathrm{rad}(s)) \cdot \varphi(s) = \varphi(s) = \varphi(s) \cdot \varphi(\mathrm{rad}(s)).$$

This implies that

$$\varphi(\mathrm{rad}(s)) \subseteq \mathrm{rad}(\varphi(s)).$$

This formula for $\varphi = \varphi^{-1}$ and $s = \varphi(s)$ proves the reverse inclusion. □

2.7.30. Every algebraic isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$ induces a lattice isomorphism from E to E'.

Proof. By the first part of Propostion 2.3.25, φ induces a bijection from E to E'. To prove that φ induces a lattice isomorphism, we use the partial orders of E and E' defined by inclusion of relations (in both cases, the smallest elements are the empty sets and the largest elements are respectively Ω^2 and Ω'^2). By statement (1) of Proposition 2.3.18, φ respects these partial orders.

If the join and meet are defined respectively by equivalence closure and intersection, i.e.,

$$e_1 \vee e_2 = \langle e_1 \cup e_2 \rangle \quad \text{and} \quad e_1 \wedge e_2 = e_1 \cap e_2,$$

then by Exercise (2.7.29), for any $e_1, e_2 \in E$, we have

$$\varphi(e_1 \vee e_2) = \varphi(\langle e_1 \cup e_2 \rangle) = \langle \varphi(e_1 \cup e_2) \rangle = \langle \varphi(e_1) \cup \varphi(e_2) \rangle = \varphi(e_1) \vee \varphi(e_2)$$

and

$$\varphi(e_1 \wedge e_2) = \varphi(e_1 \cap e_2) = \varphi(e_1) \wedge \varphi(e_2).$$

Consequently, φ induces a lattice isomorphism. □

2.7.31. Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, e an indecomposable partial parabolic of $\mathcal{X}$, $\Delta \in \Omega/e$, and $e' = \varphi(e)$. Then for any $\Delta' \in \Omega'/e'$, the bijection

$$\varphi_{\Delta,\Delta'} : S_\Delta \to S'_{\Delta'}, \; s_\Delta \mapsto \varphi(s)_{\Delta'}$$

is an algebraic isomorphism from $\mathcal{X}_\Delta$ onto $\mathcal{X}'_{\Delta'}$.

Proof. For any $s \in S^\cup$, set $s' := \varphi(s)$. By the assumption and statement (2) of Proposition 2.3.25, e' is indecomposable. Take an arbitrary class $\Delta' \in \Omega'/e'$. Then by statement (1) of Theorem 2.1.22, for any $s \in S$,

$$s_\Delta \neq \varnothing \quad \Leftrightarrow \quad s \subseteq e \quad \Leftrightarrow \quad s' \subseteq e' \quad \Leftrightarrow \quad s'_{\Delta'} \neq \varnothing. \tag{2.7.16}$$

Thus, for any basis relations $r, s, t \subseteq e$, formula (2.1.16) shows that

$$c_{r_\Delta s_\Delta}^{t_\Delta} = c_{rs}^t = c_{r's'}^{t'} = c_{r'_{\Delta'} s'_{\Delta'}}^{t'_{\Delta'}}. \tag{2.7.17}$$

Formulas (2.7.16) and (2.7.17) prove that the mapping $s_\Delta \mapsto s'_{\Delta'}$ is an algebraic isomorphism from $\mathcal{X}_\Delta$ to $\mathcal{X}'_{\Delta'}$. □

2.7.32. If one of two algebraically isomorphic coherent configurations is half-homogeneous (respectively, homogeneous, equivalenced, regular, semiregular, quasiregular), then so is the other.

Proof. Let $\mathcal{X}$ be a coherent configuration. For any positive integer k, set

$$F_k(\mathcal{X}) := \{\Delta \in F : |\Delta| = k\}.$$

Suppose $\mathcal{X}'$ is a coherent configuration algebraically isomorphic to $\mathcal{X}$. Statement (2) of Proposition 2.3.22 implies that there is a bijection between $F_k(\mathcal{X})$ and $F_k(\mathcal{X}')$ for each k. Since $\mathcal{X}$ is half-homogeneous if and only if $F_k(\mathcal{X}) \neq \varnothing$ for exactly one k and $\mathcal{X}$ is homogeneous if further this k equals the degree of $\mathcal{X}$, we are done.

For any positive integer k, set

$$S_k(\mathcal{X}) := \{s \in S : n_s = k\}.$$

If $\mathcal{X}'$ is a coherent configuration algebraically isomorphic to $\mathcal{X}$, then Corollary 2.3.20 shows that there is a bijection from $S_k(\mathcal{X})$ to $S(\mathcal{X}')$ for each k. One can see that $\mathcal{X}$ is equivalenced if and only if it is a scheme and there exists at most one $k > 1$ such that $S_k(\mathcal{X}) \neq \varnothing$. This proves the statement in the equivalenced case. The regular case follows from it since $\mathcal{X}$ is regular if and only if $\mathcal{X}$ is equivalenced and $S_k(\mathcal{X}) \neq \varnothing$ only if $k = 1$.

Statement (1) of Corollary 2.3.22 implies that each homogeneous component of $\mathcal{X}$ is algebraically isomorphic to some homogeneous component of $\mathcal{X}'$. Since $\mathcal{X}$ is quasiregular if and only if each homogeneous component of $\mathcal{X}$ is regular, the statement in the quasiregular case follows from that of the regular case. Since $\mathcal{X}$ is semiregular if and only if $\mathcal{X}$ is quasiregular and $S_k \neq \varnothing$ only if $k = 1$, we are done. □

2.7.33. The coherent configuration of a dihedral group $D_{2n} \leq \mathrm{Sym}(n)$ is separable for all $n \geq 1$.

Proof. Let $\Omega = \{1, \ldots, n\}$, $\mathcal{X} = \mathrm{Inv}(D_{2n})$, and $d = \lfloor \frac{n}{2} \rfloor$. Without loss of generality, we may assume that the n-cycle $k := (1, \ldots, n) \in D_{2n}$.

One can see that $\mathrm{rk}(\mathcal{X}) = d + 1$ and the basis relations $s_0, s_1, \ldots, s_d$ can be chosen so that

$$A_0 = A_{s_0} = I_n, \quad A_i = A_{s_i} = P_{k^i} + P_{k^{-i}},\ i = 1, \ldots, d-1,$$

and $A_{s_d} = P_{k^d}$ if n is even and $A_{s_d} = P_{k^d} + P_{k^{-d}}$ if n is odd. Then by inducition on i we can prove that

$$A_{s_1} A_{s_i} = \begin{cases} 2A_0 + A_2 & \text{if} \quad i = 1, \\ A_{i-1} + A_{i+1} & \text{if} \quad 1 < i \leq d-1. \end{cases}$$

These equalities imply that

$$s_1 \cdot s_i = s_{i-1} \cup s_{i+1}, \quad i = 1, 2, \ldots, d-1. \tag{2.7.18}$$

Let φ be an algebraic isomorphism from $\mathcal{X}$ to a coherent configuration $\mathcal{X}'$ on Ω'. Set $s_i' := \varphi(s_i)$. Then, by formula (2.7.18) we have

$$s_1' \cdot s_i' = s_{i-1}' \cup s_{i+1}', \quad i = 1, 2, \ldots, d-1. \tag{2.7.19}$$

One can see that s_1 is an undirected cycle of length n. In particular, $n_{s_1} = 2$ and $\langle s_1 \rangle = \Omega^2$. Since $n_{s_1'} = n_{s_1}$ (Corollary 2.3.20) and $\langle s_1' \rangle = \Omega'^2$ (statement (2) of Exercise (2.7.29)), the relation s_1' is a undirected cycle of length n. It follows that there exists a bijection $f : \Omega \to \Omega'$ such that

$$s_1' = (s_1)^f = \varphi(s_1).$$

By induction, formulas (2.7.18) and (2.7.19) show that

$$(s_i)^f = \varphi(s_i), \quad i = 1, \ldots, d,$$

i.e., f induces φ. Thus, $\mathcal{X}$ is separable. □

2.7.34. [**?**] Every quasiregular coherent configuration with at most three fibers is schurian and separable.

Proof. For any coherent configuration $\mathcal{X}$, denote by $\mathcal{F} := \mathcal{F}(\mathcal{X})$ the set of all systems of distinct representatives of F in Ω.

Let $\mathcal{X}$ be a quasiregular coherent configuration with $|F| \leq 3$. Choose $\Delta \in \mathcal{F}$. For each $\alpha \in \Omega$, there exist a unique point $\overline{\alpha} \in \Delta$ such that α and $\overline{\alpha}$ belong to the same fiber and a unique basis relation $s_\alpha \in S$ and such that $(\overline{\alpha}, \alpha) \in s_\alpha$. Since $\mathcal{X}$ is quasiregular, the basis relation s_α is thin and

$$\overline{\alpha} s_\alpha = \{\alpha\}. \tag{2.7.20}$$

In pariticular, for any $\alpha, \beta \in \Omega$, $(\alpha, \overline{\alpha}) \in s_\alpha^*$ and $(\overline{\beta}, \beta) \in s_\beta$. Since s_α and s_β are thin, one can see that

$$r(\alpha, \beta) = s_\alpha^* \cdot r(\overline{\alpha}, \overline{\beta}) \cdot s_\beta. \tag{2.7.21}$$

Let $\varphi : \mathcal{X} \to \mathcal{X}'$ be an algebraic isomorphism. Then the coherent configuration $\mathcal{X}'$ is also quasiregular (Exercise (2.7.32)) and $|F(\mathcal{X}')| = |F|$ (Corollary 2.3.24).

Therefore, there exists an injection f from Δ into Ω' such that $\Delta' := \mathrm{Im}(f)$ with $\Delta' \in \mathcal{F}(\mathcal{X}')$ and

$$r(\delta,\gamma)^{\varphi} = r(\delta^f,\gamma^f), \quad \delta,\gamma \in \Delta, \tag{2.7.22}$$

here we use the fact that $|F| \leq 3$. By the same reason, $\mathcal{X}$ satisfies the assumption of the lemma below. The rest of the proof immediately follows from this lemma.

LEMMA 2.7.35. *Let $\mathcal{X}$ be a quasiregular coherent configuration. Suppose that for any $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{X}')$, any two distinct $\alpha,\beta \in \Delta \in \mathcal{F}$, and any $\alpha',\beta' \in \Omega'$ with $\varphi(r(\alpha,\beta)) = r(\alpha',\beta')$, there exists an injection $f : \Delta \to \Omega'$ such that $(\alpha,\beta)^f = (\alpha',\beta')$ and condition* (2.7.22) *is satisfied. Then $\mathcal{X}$ is separable and schurian.*

REMARK 2.7.36. *To prove the separability of $\mathcal{X}$, it suffices to assume the weaker condition, namely for any $\Delta \in \mathcal{F}$, there exists an injection $f : \Delta \to \Omega'$ satisfying condition* (2.7.22).

Proof. If $|F| = 1$, then $\mathcal{X}$ is regular. Hence $\mathcal{X}$ is schurian and separable (Theorem 2.2.11, Theorem 2.3.33). We may assume without loss of generality that $|F| > 1$.

Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{X}')$. Take $\Delta \in \mathcal{F}$ (for arbitrary α and β). Then by the assumption of the lemma, there exists an injection $f : \Delta \to \Omega'$. One can extend f to a bijection $\Omega \to \Omega'$, also denoted by f. Namely, for any $\alpha \in \Omega$, set α^f to be the unique point of Ω' such that

$$\overline{\alpha}^f \varphi(s_\alpha) = \{\alpha^f\}, \tag{2.7.23}$$

(here we use the facts that $\varphi(s_\alpha)$ is thin and $\overline{\alpha}^f \in \Omega_-(\varphi(s_\alpha))$). In particular,

$$(\alpha^f,\overline{\alpha}^f) \in \varphi(s_\alpha)^* \quad \text{and} \quad (\overline{\beta}^f,\beta^f) \in \varphi(s_\beta).$$

Since $\varphi(s_\alpha)$ and $\varphi(s_\beta)$ are thin, we obtain

$$r(\alpha^f,\beta^f) = \varphi(s_\alpha)^* \cdot r(\overline{\alpha}^f,\overline{\beta}^f) \cdot \varphi(s_\beta).$$

Together with formula (2.7.22), this imples that for any $\alpha,\beta \in \Omega$,

$$\begin{aligned} r(\alpha,\beta)^{\varphi} =& (s_\alpha^* \cdot r(\overline{\alpha},\overline{\beta}) \cdot s_\beta)^{\varphi} \\ =& \varphi(s_\alpha)^* \cdot r(\overline{\alpha}^f,\overline{\beta}^f) \cdot \varphi(s_\beta) \\ =& r(\alpha^f,\beta^f) \\ =& r(\alpha,\beta)^f. \end{aligned}$$

It follows that $f \in \mathrm{Iso}(\mathcal{X},\mathcal{X}',\varphi)$. Hence, $\mathcal{X}$ is separable.

To prove the schurity of $\mathcal{X}$, take α and β from different fibers of $\mathcal{X}$ and arbitrary $\alpha',\beta' \in \Omega$ such that

$$r(\alpha,\beta) = r(\alpha',\beta').$$

Let $\mathcal{X}' = \mathcal{X}$, $\varphi = \mathrm{id}_S$, and $\Delta \in \mathcal{F}$ be such that $\alpha,\beta \in \Delta$. By the assumption of the lemma, there exists an injection $f : \Delta \to \Omega$ such that $(\alpha,\beta)^f = (\alpha',\beta')$ and condition (2.7.22) is satisfied. One can extend f according to formula (2.7.23) to a bijection from Ω to itself. By the arguement of the previous paragraph, f induces $\varphi = \mathrm{id}_S$. This yields that $f \in \mathrm{Aut}(\mathcal{X})$.

It follows that the group K generated by all such f for all possible (α,β) and (α',β') is transitive on $r(\alpha,\beta)$. Therefore,

$$\mathrm{Orb}(K,\Gamma\times\Lambda) = S_{\Gamma,\Lambda}, \tag{2.7.24}$$

for all distinct $\Gamma, \Lambda \in F$. In particular, K acts transitively on each $\Gamma \in F$. However, K^Γ is a subgroup of the regular group $\mathrm{Aut}(\mathcal{X}_\Gamma)$ (the regularity holds because $\mathcal{X}$ is quasiregular). Thus, $K^\Gamma = \mathrm{Aut}(\mathcal{X}_\Gamma)$. This proves formula (2.7.24) for $\Gamma = \Lambda$. We are done. □

2.7.37. Every semiregular coherent configuration is schurian and separable.

Proof. Let $\mathcal{X}$ be a semiregular coherent configuration. Since $\mathcal{X}$ is obviously quasiregular, it suffices to verify the assumptions of Lemma 2.7.35 for $\mathcal{X}$.

Let $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. By Exercise 2.7.32, $\mathcal{X}'$ is semiregular. In particular, all basis relations of $\mathcal{X}$ and $\mathcal{X}'$ are thin.

Take two distinct points α and β belonging to some $\Delta \in \mathcal{F}$ and two points $\alpha', \beta' \in \Omega'$ such that

$$\varphi(r(\alpha, \beta)) = r(\alpha', \beta').$$

For any $\gamma \in \Delta$, the fiber containing α' is equal to $\Omega_-(\varphi(r(\alpha, \gamma)))$ (statement (2) of Corollary 2.3.23). Since every basis relation of $\mathcal{X}'$ is thin, there exists a unique point $\gamma' \in \Omega'$ such that

$$\{\gamma'\} = \alpha' \varphi(r(\alpha, \gamma)). \tag{2.7.25}$$

Note that

$$\gamma = \alpha \Rightarrow \gamma' = \alpha' \quad \text{and} \quad \gamma = \beta \Rightarrow \gamma' = \beta'.$$

Moreover, if $\gamma \neq \lambda \in \Delta$, then $r(\alpha, \gamma) \neq r(\alpha, \lambda)$. Hence, $\gamma' \neq \lambda'$. It follows that formula (2.7.25) defines an injection

$$f : \Delta \to \Omega', \quad \gamma \mapsto \gamma'$$

such that$(\alpha, \beta)^f = (\alpha', \beta')$. Since every basis relation in $\mathcal{X}$ is thin, we deduce that for any $\gamma, \lambda \in \Delta$,

$$r(\gamma, \lambda) = r(\alpha, \gamma)^* \cdot r(\alpha, \lambda).$$

It follows that for any $\gamma, \lambda \in \Delta$,

$$\begin{aligned} \varphi(r(\gamma, \lambda)) &= \varphi(r(\alpha, \gamma)^* \cdot r(\alpha, \lambda)) \\ &= \varphi(r(\alpha, \gamma)^*) \cdot \varphi(r(\alpha, \lambda)) \\ &= \varphi(r(\alpha, \gamma))^* \cdot \varphi(r(\alpha, \lambda)) \\ &= r(\alpha^f, \gamma^f)^* \cdot r(\alpha^f, \lambda^f) \\ &= r(\gamma^f, \lambda^f). \end{aligned}$$

This implies that the injection f satisfies condition (2.7.22). We are done. □

2.7.38. Let $K \leq \mathrm{Iso}(\mathcal{X})$. Then $K \leq \mathrm{Aut}(\mathcal{X}^K)$.

Proof. Choose an arbitrary basis relation $t \in S(\mathcal{X}^K)$. There exists $s \in S$ such that

$$t = \bigcup_{k \in K} s^k.$$

This implies that t is K-invariant. Since this is true for any basis relation t of $S(\mathcal{X}^K)$, we deduce that $K \leq \mathrm{Aut}(\mathcal{X}^K)$, as required. □

2.7.39. Let $\Psi \leq \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$, $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, and $\Psi' = \varphi \Psi \varphi^{-1}$. Then

(1) $\Psi' \leq \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}')$,

(2) $\varphi_\Psi : S(\mathcal{X}^\Psi) \to S(\mathcal{X}'^{\Psi'})$, $s^\Psi \mapsto \varphi(s)^{\Psi'}$ is a well-defined bijection,
(3) $\varphi_\Psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}^\Psi, \mathcal{X}'^{\Psi'})$.

Proof. By the definition of algebraic isomorphisms, both the inverse of an algebraic isomorphism and the composition of two algebraic isomorphisms are algebraic isomorphisms. This proves statement (1).

To prove the other statements, observe that $\mathcal{X}^\Psi \leq \mathcal{X}$. By Corollary 2.3.21, the algebraic isomorphism φ induces by restriction an algebraic isomorphism from $\mathcal{X}^\Psi$ to the coherent configuration

$$(\mathcal{X}^\Psi)^\varphi = (\mathcal{X}^\varphi)^{\varphi^{-1}\Psi\varphi} = (\mathcal{X}')^{\Psi'},$$

here we use statement (1). However, this induced algebraic isomorphism maps a basis relation $s^\Psi \in S(\mathcal{X}^\Psi), s \in S$ to $\varphi(s)^{\Psi'}$ because

$$\varphi(s^\Psi) = \varphi(\bigcup_{\psi\in\Psi} \psi(s)) = \bigcup_{\psi\in\Psi} \varphi\psi(s) = \bigcup_{\psi\in\Psi} (\varphi\psi\varphi^{-1})(\varphi(s)) = \varphi(s)^{\Psi'}.$$

This proves statements (2) and (3). □

2.7.40. Find a schurian algebraic fusion of a non-schurian scheme.

Proof. Let $\mathcal{X}$ be the unique non-schurian scheme of degree 15. Then $\mathcal{X}$ is an antisymmetric commutative scheme of rank 3. Thus,

$$\varphi : S \to S, \quad s \mapsto s^*$$

is an algebraic isomorphism of $\mathcal{X}$. Set

$$\Phi := \langle\varphi\rangle.$$

Then the algebraic fusion $\mathcal{X}^\Phi = \mathcal{T}_\Omega$ is obviously schurian. □

2.7.41. Let G be a group, $K = \langle G_{right}, G_{left}\rangle$, and $\mathcal{X} = \mathrm{Inv}(K, G)$. Then
(1) the stabilizer K_1 of the identity of G in K equals $\mathrm{Inn}(G)$,
(2) $\mathrm{Orb}(K_1, G) = \{x^G : x \in G\}$,
(3) $\mathrm{Adj}(\mathcal{X})$ is isomorphic to the center of $\mathbb{C}G$,
(4) the scheme $\mathcal{X}$ is commutative.

Proof. For any $x \in G$, set $x_r := x_{right}$ and c_x to be the conjugation mapping of G induced by x,

$$c_x : \quad G \quad \to \quad G, \quad g \mapsto x^{-1}gx.$$

To prove statement (1), we make use of Exercise 1.4.13 showing that

$$\mathrm{Inn}(G) \subseteq K_1.$$

Conversely, for any $k \in K_1$, this exercise implies that there exist $y, x \in G$ such that $k = y_r c_x$. Thus,

$$1 = 1^k = 1^{y_r c_x} = y^x.$$

It follows that $y = 1$. Hence, $k = c_x \in \mathrm{Inn}(G)$. We obtain

$$K_1 \subseteq \mathrm{Inn}(G).$$

We are done.

Statement (2) follows directly from statement (1).

Note that $\mathcal{X}$ is a Cayley scheme over G. For each $s \in S$, by formula (1.4.11)

$$\rho^{-1}(s) = \alpha s,$$

where ρ is defined as in Exercise 1.4.15 and $\alpha = 1$ is the identity element of G. Furthermore, if $(\alpha, x) \in s$ for some $x \in G$, then

$$\alpha s = x^G.$$

Hence, the S-ring corresponding to $\mathcal{X}$ (see formual (2.4.9)) is equal to

$$\mathfrak{A} = \operatorname{Span}\{\underline{x^G} : x \in G\}.$$

This yields that $\mathfrak{A}$ is the center of $\mathbb{C}G$.

Note that the mapping ρ induces a linear isomorphism from $\mathfrak{A}$ to $\operatorname{Adj}(\mathcal{X})$, which takes $\underline{\mathcal{S}}(\mathfrak{A})$ to the standard basis of $\operatorname{Adj}(\mathcal{X})$. Moreover, this linear isomorphism preserves the structure constants with respect to these bases (page 83). It follows that

$$\operatorname{Adj}(\mathcal{X}) \cong \mathfrak{A}.$$

This proves statement (3). Since $\mathfrak{A}$ is commutative, statement (4) follows. □

2.7.42. Let $\mathcal{X}$ be a Cayley scheme and $\mathcal{X} \geq \mathcal{X}'$. Then $\mathcal{X}$ is normal, whenever so is $\mathcal{X}'$.

Proof. Denote by G the underline group of $\mathcal{X}$. Then,

$$G_{right} \leq \operatorname{Aut}(\mathcal{X}).$$

Since $\mathcal{X}' \leq \mathcal{X}$, by formula (2.2.5) we have

$$\operatorname{Aut}(\mathcal{X}) \leq \operatorname{Aut}(\mathcal{X}'). \tag{2.7.26}$$

It follows that

$$G_{right} \leq \operatorname{Aut}(\mathcal{X}').$$

Thus, $\mathcal{X}'$ is a Cayley scheme. Now assume that $\mathcal{X}'$ is normal, then

$$G_{right} \trianglelefteq \operatorname{Aut}(\mathcal{X}').$$

By formula (2.7.26),

$$G_{right} \trianglelefteq \operatorname{Aut}(\mathcal{X}),$$

which yields that $\mathcal{X}$ is normal, as required. □

2.7.43. Let $\mathcal{X}$ be a cyclotomic scheme over a group G, H a characteristic subgroup of G, and ρ the mapping defined in Exercise 1.4.15. Then $H^\rho \in E$.

Proof. Let $M \leq \operatorname{Aut}(G)$ be such that $\mathcal{X} = \operatorname{Inv}(G_{right}M, G)$. Observe that for any $x, y \in G$,

$$(x, y) \in H^\rho \quad \Leftrightarrow \quad yx^{-1} \in H.$$

As H is a characteristic subgroup of G, for any $m \in M$

$$yx^{-1} \in H \quad \Leftrightarrow \quad (yx^{-1})^m \in H \quad \Leftrightarrow \quad (x^m, y^m) \in H^\rho.$$

It follows that H^ρ is M-invariant. Obviously, H^ρ is G_{right}-invariant. Therefore, H^ρ is $G_{right}M$-invariant. Thus, $H^\rho \in S^\cup$. Since H^ρ is an equivalence relation on G (statement (6) of Exercise 1.4.16), we are done. □

2.7.44. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be S-rings over groups G and G', respectively. Then

(1) a ring isomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is an algebraic isomorphism if and only if $\underline{X}^\varphi \in \underline{\mathcal{S}}(\mathfrak{A}')$ for all $X \in \mathcal{S}(\mathfrak{A})$,

(2) a bijection $f : G \to G'$ is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$ if and only if there exists an algebraic isomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}'$ such that

$$f(Xy) = X^{\varphi} y^f \quad \text{for all } \ X \in \mathcal{S}(\mathfrak{A}), \ y \in G.$$

Proof. Let $\mathcal{X} = \mathcal{X}(G, \mathfrak{A})$ and $\mathcal{X}' = \mathcal{X}(G', \mathfrak{A}')$. Then

$$\mathcal{S}(\mathfrak{A}) = \{s^{\rho^{-1}} : s \in S(\mathcal{X})\} \quad \text{and} \quad \mathcal{S}(\mathfrak{A}') = \{s'^{\rho'^{-1}} : s' \in S(\mathcal{X}')\}, \tag{2.7.27}$$

where ρ and ρ' are defined according to formula (1.4.10). Suppose the bijection $\varphi : \mathcal{S}(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A}')$ is an algebraic isomorphism of S-rings. By definition, this means that

$$\psi : \ S(\mathcal{X}) \to S(\mathcal{X})', \quad s \mapsto (s^{\rho^{-1}})^{\varphi\rho'^{-1}}$$

is an algebraic isomorphism of schemes. In other words, we have

$$X^{\varphi} = (\psi(X^{\rho}))^{\rho'^{-1}}, \quad X \in \mathcal{S}(\mathfrak{A}).$$

To prove the necessity of statement (1), suppose that the ring isomorphism φ is an algebraic isomorphim (of S-rings). This means that there exists an algebraic isomorphism $\tilde{\varphi}$ from $\mathcal{X}$ to $\mathcal{X}'$ such that

$$\varphi(\underline{s^{\rho^{-1}}}) = \underline{\tilde{\varphi}(s)}^{\rho'^{-1}}, \quad s \in S.$$

If the left-hand side $X := s^{\rho^{-1}}$ runs over $\mathcal{S}(\mathfrak{A})$, then the right-hand side runs over $\mathcal{S}(\mathfrak{A}')$ (see formula (2.7.27)), as required.

To prove the sufficency, assume that $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a ring isomorphism such that $\underline{X}^{\varphi} \in \underline{\mathcal{S}}(\mathfrak{A}')$ for any $X \in \mathcal{S}(\mathfrak{A})$. It follows that there exists $X' \in \mathcal{S}(\mathfrak{A}')$ satifying

$$\underline{X'} = \underline{X}^{\varphi}.$$

Since φ is obviously a linear isomorphism, $\mathfrak{A}$ and $\mathfrak{A}'$ have identical dimensions. Taking into account that $\mathcal{S}(\mathfrak{A})$ and $\mathcal{S}(\mathfrak{A}')$ are linear bases of $\mathfrak{A}$ and $\mathfrak{A}'$ respectively, we conclude that the mapping $X \mapsto X'$ is a bijection. Thus,

$$\tilde{\varphi} : \rho(X) \mapsto \rho'(X')$$

is a bijection from $S(\mathcal{X})$ to $S(\mathcal{X}')$ (see formula (2.7.27)).

Moreover, for any $\rho(X), \rho(Y), \rho(Z) \in S(\mathcal{X})$,

$$c_{\rho(X),\rho(Y)}^{\rho(Z)} = c_{X,Y}^{Z} = c_{X',Y'}^{Z'} = c_{\rho(X'),\rho(Y')}^{\rho(Z')},$$

where the first and the third equalities follow from formulas on structure constants on page 83 and the second equality follows from the fact that φ is a ring isomorphism. We deduce that $\tilde{\varphi}$ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$.

To prove the necessity of statement (2), assume first that $f : G \to G'$ is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}'$. This implies that

$$S(\mathcal{X}') = \{\rho(X)^f : \ X \in \mathcal{S}(\mathfrak{A})\}.$$

Hence, for any $X \in \mathcal{S}(\mathfrak{A})$ there exists a unique $X^{\varphi} \in \mathcal{S}(\mathfrak{A}')$ such that

$$\rho(X)^f = \rho'(X^{\varphi}). \tag{2.7.28}$$

Moreover,

$$\varphi : \ \mathcal{S}(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A}'), \quad X \mapsto X^{\varphi}$$

is an algebraic isomorphism of S-rings. Thus, for any $y \in G$ and any $x \in X$,

$$(y, xy) \in \rho(X) \quad \Rightarrow \quad (y, xy)^f \in \rho'(X^{\varphi}) \quad \Rightarrow \quad (xy)^f \in X^{\varphi} y^f.$$

This yields that

$$f(Xy) \subseteq X^{\varphi} y^{f}.$$

Since f is a bijection and $|X| = |X^{\varphi}|$, we conclude that for all $X \in \mathcal{S}$ and $y \in G$

$$f(Xy) = X^{\varphi} y^{f}. \qquad (2.7.29)$$

Conversely, if there exists an algbraic isomorphism φ from $\mathfrak{A}$ to $\mathfrak{A}'$ satisfying (2.7.29) for all $X \in \mathcal{S}$ and all $y \in G$, then it is easy to see that equality (2.7.28) holds for all $X \in \mathcal{S}$. Thus, the bijection $f : G \to G'$ is an isomorphism induced the algebraic isomorphism φ. □

2.7.45. Let Ω be the set of flags of a projective plane of order q, where the flag is a pair of a point and a line incident to it. Every two flags (p, l) and (p', l') belongs to one of the relations in the set $S = \{s_0, \ldots, s_5\}$ that are defined as in Fig. 2.4, where the double line and arrow denote the equality and incidence, respectively,

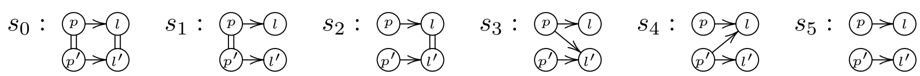

FIGURE 2.4. The scheme on flags of a projective plane: basis relations.

and the absence of any line means general position. For example, $s_0 = 1_{\Omega}$ and s_1 consists the pairs of flags having common point. Then

(1) $s_i = s_i^*$ if and only if $i \neq 3, 4$, and $s_3^* = s_4$,
(2) $(s_3, s_4) = (s_1 \cdot s_2,\, s_2 \cdot s_1)$ and $s_5 = s_1 \cdot s_2 \cdot s_1 = s_2 \cdot s_1 \cdot s_2$,
(3) the rainbow (Ω, S) is a scheme of degree $(q^2 + q + 1)(q + 1)$ and rank 6.

Proof. For each i, the relation s_i is symmetric if and only if $*$ induces an automorphism of the diagram in Fig. 2.4 corresponding to s_i. This proves statement (1) for $i \neq 3, 4$. The rest follows from the fact that $*$ maps the diagram of s_3 to that of s_4 and vice versa.

To prove statement (2), let $(p, l), (p', l') \in \Omega$. It is easily seen that

$$\begin{aligned} ((p,l),(p',l')) \in s_3 \quad &\Leftrightarrow \quad p \neq p',\ l \neq l' \quad \text{and} \quad (p', l) \notin \mathcal{I},\ (p, l') \in \mathcal{I} \\ &\Leftrightarrow \quad ((p,l),(p,l')) \in s_1 \quad \text{and} \quad ((p,l'),(p',l')) \in s_2 \\ &\Leftrightarrow \quad ((p,l),(p',l')) \in s_1 \cdot s_2. \end{aligned}$$

It follows that $s_3 = s_1 \cdot s_2$. By statement (1), this implies that

$$s_4 = s_3^* = (s_1 \cdot s_2)^* = s_2^* \cdot s_1^* = s_2 \cdot s_1.$$

To prove the remaining equalities, we argue as before to get

$$\begin{aligned} ((p,l),(p',l')) \in s_5 \quad &\Leftrightarrow \quad ((p,l),(p',l'')) \in s_3 \quad \text{and} \quad ((p',l''),(p',l')) \in s_1 \\ &\Leftrightarrow \quad ((p,l),(p',l')) \in s_3 \cdot s_1, \end{aligned}$$

where $l'' = pp'$. Since $s_3 = s_1 \cdot s_2$, we obtain

$$s_5 = s_3 \cdot s_1 = s_1 \cdot s_2 \cdot s_1.$$

This last equality in statement (2) can be proved similarly.

To prove statement (3), we note that (Ω, S) is indeed a rainbow: the condition (CC1) holds as $s_0 = 1_{\Omega}$ and the condition (CC2) follows from statement (1). It suffices to verify the condition (CC3), which follows from the lemma below (the calculation of $|\Omega|$ follows from the facts that in any projective plane of order q, there are q^2+q+1 points and each point belongs to $q+1$ distinct lines).

LEMMA 2.7.46. *For any $i,j,k\in\{0,\dots,5\}$, the number $|\alpha s_i\cap\beta s_j^*|$ is a polynomial in q that does not depend on the pair $(\alpha,\beta)\in s_k$.*

Proof. Let $i,j,k\in\{0,\dots,5\}$ and $(\alpha,\beta)\in s_k$. Set

$$c_{ij}^k := c_{ij}^k(\alpha,\beta)=|\alpha s_i\cap\beta s_j^*|.$$

Denote $c_{ii^*}^0$ by n_i. Then by an easy computation, one can get that

$$n_i=\begin{cases}1 & \text{if} \quad k=0,\\ q & \text{if} \quad k=1,2,\\ q^2 & \text{if} \quad k=3,4,\\ q^3 & \text{if} \quad k=5.\end{cases}$$

In what follows, we compute $c_{ij}^k\neq 0$ with $i,j,k\in\{1,\dots,5\}$. By statement (2) and its proof, we have:

$$s_1\cdot s_2=s_3,\qquad s_2\cdot s_1=s_4,\qquad s_3\cdot s_1=s_5,$$
$$s_1\cdot s_4=s_5,\qquad s_2\cdot s_3=s_5,\qquad s_4\cdot s_2=s_5.$$

In each of these cases, $s_i\cdot s_j=s_k$ implies $n_in_j=n_k$ (see the formula for n_i). Thus,

$$c_{12}^3=c_{21}^4=c_{31}^5=c_{14}^5=c_{23}^5=c_{42}^5=1.$$

Since both $s_0\cup s_1$ and $s_0\cup s_2$ are equivalence relations on Ω, we have,

$$c_{11}^1=c_{22}^2=q-1.$$

Let $A_i := A_{s_i}, i=0,1,\dots,5$. Then from what we got above, it follows that each of the products

$$A_1A_2,\ A_2A_1,\ A_3A_1,\ A_1A_4,\ A_2A_3,\ A_4A_2,\ A_1A_1,\ A_2A_2$$

is a linear combination of $A_0,\dots,A_5$. Because of this, each of the products below is also a combination of this type:

$$A_1A_3=A_1^2A_2=qA_2+(q-1)A_1A_2,\qquad A_1A_5=A_1^2A_4=qA_4+(q-1)A_1A_4,$$
$$A_2A_4=A_2^2A_1=qA_1+(q-1)A_2A_1,\qquad A_2A_5=A_2^2A_3=qA_3+(q-1)A_2A_3,$$
$$A_3A_2=A_1A_2^2=qA_1+(q-1)A_1A_2,\qquad A_3A_4=A_1A_2^2A_1=qA_1^2+(q-1)A_1A_4,$$
$$A_4A_1=A_2A_1^2=qA_2+(q-1)A_2A_1,\qquad A_4A_3=A_2A_1^2A_2=qA_2^2+(q-1)A_2A_3,$$
$$A_4^2=A_2^2A_3=qA_3+(q-1)A_2A_3,\qquad A_5A_1=A_3A_1^2=qA_3+(q-1)A_3A_1,$$
$$A_5A_2=A_4A_2^2=qA_4+(q-1)A_4A_2.$$

Then each of the rest products is also a combination of the same type:

$$A_3^2=A_1(A_2A_1A_2)=A_1A_5,\qquad A_3A_5=A_1A_2^2A_3=qA_1A_3+(q-1)A_1A_5,$$
$$A_4A_5=A_2A_1^2A_4=qA_2A_4+(q-1)A_4^2,\qquad A_5A_3=A_4^2A_2=qA_3A_2+(q-1)A_5A_2,$$
$$A_5A_4=A_1A_4^2=qA_1A_3+(q-1)A_1A_5,\qquad A_5^2=A_1A_5A_3=qA_4A_3+(q-1)A_1(A_4A_3).$$

Now the computation that we did shows that each c_{ij}^k is a polynomial of q. We are done. □

2.7.47. Any scheme algebraically isomorphic to the scheme associated with a projective (respectively, affine) plane, is associated with a projective (respectively, affine) plane of the same order.

Proof. Let $\mathcal{X}$ be the coherent configuration associated with a projective plane $\mathcal{P}$ and φ an algebraic isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$. Then $\mathcal{X}'$ has two fibers (Corollary 2.3.24):

$$P' := P^{\varphi} \quad \text{and} \quad L' := L^{\varphi},$$

where P and L are fibers of $\mathcal{X}$. Set

$$\Omega' := P' \cup L' \quad \text{and} \quad s'_i := \varphi(s_i), \quad i = 1, \ldots, 8.$$

Since the algebraic isomorphism φ preserves intersection numbers, we obtain

$$c_{s'_5 s'_6}^{s'_3} = c_{s_5 s_6}^{s_3} = 1 \quad \text{and} \quad c_{s'_6 s'_5}^{s'_4} = c_{s_6 s_5}^{s_4} = 1.$$

Let us define an incidence relation on Ω', where the set of points, the set of lines, and the incidence relation are respectively P', L', and s'_5. Then the above formulas imply respectively that the axioms (P1) and (P2) are satisfied.

Let the order of $\mathcal{P}$ be $q \geq 2$. Our next goal is to show that (P', L') satisfies the axiom (P3). Fix a point p'_1. Since the number of lines incident to p'_1 equals $n_{s'_5} = q+1 \geq 3$, one can find three distinct lines l'_1, l'_2 and l'_3 incident to p'_1. Because

$$|L'| = q^2 + q + 1 > q + 1,$$

there exists a line l'_4 not incident to p'_1. Denote by q'_i the point which is incident to l'_4 and l'_i, $i = 1, 2, 3$. Now choose a point p'_2 which is incident to l'_2 but different from p'_1 and q'_2 (here we use the fact that $n_{s'_6} = q + 1 \geq 3$). Then the four points p'_1, q'_1, q'_3 and p'_2 satisfy the axiom (P3) obviously.

Now let $\mathcal{X} = (\Omega, S)$ be a scheme of a finite affine plane $\mathcal{A}$ of order q. And let φ be an algebraic isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$. Since $\mathcal{X}$ is symmetric of degree q^2, $\mathcal{X}'$ is symmetric of degree q^2.

Set

$$\Omega' := \Omega^{\varphi} \quad \text{and} \quad s' := \varphi(s), \quad s \in S.$$

Since φ preserves intersection numbers, formula (2.5.5)a implies that the nonzero intersection numbers $c_{r's'}^{t'}$ with $1_{\Omega'} \notin \{r', s'\}$ are as follows:

$$c_{r's'}^{t'} = \begin{cases} q-1 & \text{if} \quad r' = s' \quad \text{and} \quad t' = 1_{\Omega'}, \\ q-2 & \text{if} \quad r' = s' = t', \\ 1 & \text{if} \quad r' \neq s' \neq t' \neq r'. \end{cases}$$

Observe that for any irreflexive basis relation s',

$$s' \cdot s' = s' \cup 1_{\Omega'}. \tag{2.7.30}$$

This yields that $e_{s'} := 1_{\Omega'} \cup s'$ is a parabolic of $\mathcal{X}'$. For any $\alpha' \in \Omega'$, the class of $e_{s'}$ that contains α' is denoted by $l_{s',\alpha'}$. Then, we have

$$l_{s',\alpha'} = \{\alpha'\} \cup \alpha' s' \quad \text{and} \quad \Omega'/e_{s'} = \{l_{s',\alpha'} : \alpha' \in \Omega'\}. \tag{2.7.31}$$

Claim: Let $s', t' \in S'$ and $\alpha', \beta' \in \Omega'$. Then $|l_{s',\alpha'} \cap l_{t',\beta'}| = 1$ whenever $s' \neq t'$.

Proof. Set $r' := r(\alpha', \beta')$. If $r' = s'$ (respectively, $r' = t'$), then $l_{s',\alpha'} \cap l_{t',\beta'} = \{\beta'\}$ (respectively, $l_{s',\alpha'} \cap l_{t',\beta'} = \{\alpha'\}$) by the first formula in (2.7.31). To prove the claim, we may assume that

$$s' \neq r' \neq t'.$$

Then $c_{s't'}^{r'} = 1$ (see the formulas for intersection numbers). Therefore,

$$l_{s',\alpha'} \cap l_{t',\beta'} = \{\gamma'\},$$

for a uniquely determined point γ', as required. □

Now we define an incidence structure with point set Ω', line set

$$\mathcal{L}' = \{l_{s',\alpha'} : s' \in S'^{\#}, \alpha' \in \Omega'\},$$

and the incidence relation given by inclusion. Our next goal is to prove that this incidence structure is an affine plane.

Let α' and β' be distinct points. Then

$$\alpha', \beta' \in l_{s',\alpha'},$$

where $s' = r(\alpha', \beta')$. Let $l_{t',\gamma'}$ be another line containing α' and β'. By the second formula in (2.7.31), we have $s' \subseteq e_{t'}$. Thus, $s' = t'$. It follows that

$$l_{t',\gamma'} = l_{s',\alpha'}.$$

This yields that $l_{s',\alpha'}$ is the unique line containing α' and β'. Thus, the axiom (AP1) holds.

Let β' be a point and $l_{s',\alpha'}$ a line such that $\beta' \notin l_{s',\alpha'}$. It follows that $l_{s',\alpha'} \neq l_{s',\beta'}$. Then, $l_{s',\alpha'} \cap l_{s',\beta'} = \varnothing$ by the definition, i.e., the line $l_{s',\beta'}$ is parallel to the line $l_{s',\alpha'}$. For any $t' \neq s'$, the line $l_{t',\beta'}$ intersects the line $l_{s',\alpha'}$ by the claim. Thus, the line $l_{s',\beta'}$ is the unique line parallel to the line $l_{s',\alpha'}$. This proves the axiom (AP2).

To prove the axiom (AP3), let $l' \in \mathcal{L}'$ be a line. Then

$$2 \leq q = |l'| < |\Omega'| = q^2.$$

Therefore, there exist distinct points $\alpha', \beta' \in l'$ and a point $\gamma' \notin l'$. Since l' is the unique line containing α' and β' (axiom (AP1)), the three points α', β', and γ' statify the axiom (AP3).

Denote the constructed affine plane by $\mathcal{A}'$. Then the irreflexive basis relations of $\mathcal{X}'$ are in one-to-one corrspondence with the parallel classes of $\mathcal{A}'$: $s' \in S'^{\#}$ corresponds to the parallel class $\{l_{s',\alpha'} : \alpha' \in \Omega'\}$. It follows that $\mathcal{X}'$ is the scheme of the affine plane $\mathcal{A}'$. We are done. □

2.7.48. Among the affine schemes, there exist

(1) schurian schemes, which are not separable,
(2) normal Cayley schemes, which are not schurian.

Proof. To prove statement (1), take two nonisomorphic affine planes of order q, one of which is the Galois plane (there exist infinitely many q such that there at least two such planes and the smallest q equals 9; see the table on page 11 in[1]). The schemes of these planes are algebraically isomorphic (Theorem 2.5.8) but not combinatorially isomorphic. Thus, each of these schemes is not separable. The required example is given by the scheme of the Galois plane (this scheme is schurian by Theorem 2.5.7).

To prove statement (2), take a non-Galois translation plane $\mathcal{A}$ (by definition, a translation plane is an affine plane whose automrophism group has a regular subgroup acting on the points). Let $\mathcal{X}$ be the scheme of $\mathcal{A}$. Then $\mathcal{X}$ is not schurian.

[1]G.E. Moorhouse, *Incidence Geometry*, University of Wyoming, Math 5700 course notes, 2017.

Moreover, by Theorem 2.3.15 in[2], $\mathcal{X}$ is a normal Cayley scheme over the regular subgroup of $\mathrm{Aut}(\mathcal{A})$. □

2.7.49. In any (n,k,λ)-design, the number r of blocks containing a point does not depend on the choice of this point. Moreover,

$$nr = bk \quad \text{and} \quad \lambda(n-1) = r(k-1),$$

where b is the number of blocks.

Proof. To prove the statement, we may assume without loss of generality that $\lambda > 0$. Let $\alpha \in \Omega$ and $B_1, \ldots, B_r$ be all the elements in $\mathfrak{B}$ that contains α. Then

$$kr = \sum_{i=1}^{r} |B_i| = (n-1)\lambda + r.$$

Indeed, we can count the sum $\sum_{i=1}^{r} |B_i|$ in two different ways: the first one uses the fact that $|B_i| = k$ for each i, whereas the second one is obtained from the fact that any $\beta \neq \alpha$ is counted λ times and α is counted r times. This proves the second equality in question and shows that the number r does not depend on the choice of α. The first equality follows by counting the sum of the cardinalites of all blocks in two different ways. □

2.7.50. A design $\mathfrak{D}$ is said to be *symmetric* if the number of blocks is equal to the number of points. The following three statements are equivalent:

(1) $\mathfrak{D}$ is symmetric,
(2) any two distinct blocks of $\mathfrak{D}$ have the same number of common points,
(3) $\mathfrak{D}$ is a coherent design, the corresponding coherent configuration of which has type $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$.

Proof. Let $\mathfrak{D}$ be an (n,k,λ)-design on Δ, b the number of blocks, and r the number defined in Exercise 2.7.49. Without loss of generality, we may assume that $n > 1$ and $b > 1$. If $r = \lambda$, then $k = n$ by Exercise (2.7.49) and we are done. In what follows, $\lambda \neq r$.

Let A be the $\{0,1\}$-matrix with rows and columns indexed respectively by Δ and $\mathfrak{B}$ such that

$$A_{\alpha,B} = 1 \quad \Leftrightarrow \quad \alpha \in B.$$

By formula (2.5.7),

$$J_\Delta A = kJ_{\Delta,\mathfrak{B}} \quad \text{and} \quad AJ_{\mathfrak{B}} = rJ_{\Delta,\mathfrak{B}}. \tag{2.7.32}$$

And by formula (2.5.9),

$$AA^T = \lambda(J_\Delta - I_\Delta) + rI_\Delta. \tag{2.7.33}$$

This matrix is nonsingular as $r \neq \lambda$. Thus,

$$n = \mathrm{rk}(AA^T) = \mathrm{rk}(A) \leq b. \tag{2.7.34}$$

[2] M. Biliotti, V. Jha, and N. L. Johnson, *Foundations of Tanslation Planes*, Pure and Applied Mathematics, A Program of Monographs, Textbooks, and Lecture Notes, No. 243, New York, Basel, 2001

(3) ⇒ (2) Let $\mathcal{X}$ be the coherent configuration associated with $\mathfrak{D}$. In the notation of Proposition 2.5.11, each block of $\mathfrak{D}$ has the form $\alpha s, \alpha \in \Gamma$, where $\Gamma = \Omega_-(s)$. The assumption on $\mathfrak{D}$ implies that for any $\alpha, \beta \in \Gamma$,

$$\alpha s \neq \beta s \quad \Leftrightarrow \quad (\alpha, \beta) \in t,$$

where t is the unique irreflexive basis relation in S_Γ. Hence, the cardinality of the intersection of any two distinct blocks is $c_{ss^*}^t$.

(2) ⇒ (1) Assume that any two distinct blocks of $\mathfrak{D}$ have exactly l common points, for a fixed integer $l > 0$. Then the complementary design $\mathfrak{D}'$ is a (b, r, l)-design $\mathfrak{D}'$ on $\mathfrak{B}$ with blocks

$$\{B \in \mathfrak{B} : \alpha \in B\}, \quad \alpha \in \Delta.$$

Applying formula (2.7.34) to the designs $\mathfrak{D}$ and $\mathfrak{D}'$, we obtain respectively $n \leq b$ and $b \leq n$. It follows that $n = b$, as required.

(1) ⇒ (3) Assume $n = b$. Then $r = k$ by Exercise 2.7.49. Moreover, $J_{\Delta,\mathfrak{B}}$ and A are square matrices. Since AA^T is nonsingular and $\mathrm{rk}(A) = \mathrm{rk}(AA^T)$, it follows that A is nonsigurlar. By the second equality in formula (2.7.32),

$$A^{-1} J_{\Delta,\mathfrak{B}} = \frac{1}{r} J_{\mathfrak{B}}.$$

By the first equality in that formula, we obtain

$$A^{-1}(J_\Delta A) = A^{-1}(k J_{\Delta,\mathfrak{B}}) = \frac{k}{r} J_{\mathfrak{B}}.$$

These two formulas together with formula (2.7.33) yield that

$$\begin{aligned} A^T A &= A^{-1}(AA^T)A \\ &= A^{-1}(\lambda J_\Delta + (r-\lambda) I_\Delta) A \\ &= \lambda A^{-1}(J_\Delta A) + (r-\lambda) I_{\mathfrak{B}} \\ &= \frac{\lambda k}{r} J_{\mathfrak{B}} + (r-\lambda) I_{\mathfrak{B}} \\ &= \lambda J_{\mathfrak{B}} + (r-\lambda) I_{\mathfrak{B}}. \end{aligned} \tag{2.7.35}$$

Let $\Omega = \Delta \cup \mathfrak{B}$. From now on, it is convenient to consider the linear spaces Mat_Δ, $\mathrm{Mat}_{\mathfrak{B}}$, $\mathrm{Mat}_{\Delta,\mathfrak{B}}$, and $\mathrm{Mat}_{\mathfrak{B},\Delta}$ as subspaces of Mat_Ω via the natural injections from Δ and $\mathfrak{B}$ to Ω.

Define a rainbow $\mathcal{X} = (\Omega, S)$ with S the partition of Ω^2 into the relations belonging to the union of the following sets:

$$S_\Delta = S(\mathcal{T}_\Delta), \quad S_{\mathfrak{B}} = S(\mathcal{T}_{\mathfrak{B}}), \quad S_{\Delta,\mathfrak{B}} = \{s, s'\}, \quad S_{\mathfrak{B},\Delta} = S^*_{\Delta,\mathfrak{B}},$$

where the adjacency matrix of s is equal to A and $s' = (\Delta \times \mathfrak{B}) \backslash s$. By formulas (2.7.32), (2.7.33), and (2.7.35), it follows that that $\mathrm{Adj}(\mathcal{X})$ is closed with respect to matrix multiplication. Thus, $\mathcal{X}$ is a coherent configuration of type $\left[\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right]$. Since $\mathfrak{B} = \mathfrak{B}_s$, the design $\mathfrak{D}$ is coherent. We are done. □

2.7.51. [**?**] Define a system of linked designs to be a collection $\{\Omega_1, \ldots, \Omega_m\}$ of sets ($m \geq 3$) and an incidence relation $I_{ij} \subseteq \Omega_i \times \Omega_j$ for all distinct i and j, such that for all distinct i, j, and k,

(LD1) the pair $(\Omega_i, \{\alpha I_{ji} : \alpha \in \Omega_j\})$ is a symmetric design,

(LD2) the number of elements in Ω_k incident with both $\alpha \in \Omega_i$ and $\beta \in \Omega_j$ depends only on whether or not $(\alpha, \beta) \in I_{ij}$.

Then every such system defines a coherent configuration $\mathcal{X}$ on the union of the Ω_i, such that

(1) $F = \{\Omega_1, \ldots, \Omega_m\}$,
(2) $\mathcal{X}_{\Omega_i} = \mathcal{T}_{\Omega_i}$ for all i,
(3) $S_{\Omega_i,\Omega_j} = \{I_{ij}, I'_{ij}\}$ for all $i \ne j$, where $I'_{ij} = (\Omega_i \times \Omega_j) \setminus I_{ij}$.

Proof. By condition (LD1), the implication (1) $\Rightarrow$ (3) in Exericse 2.7.50 yields that for any $i \ne j$, the pair $(\Omega_i \cup \Omega_j, S_{ij})$ with

$$(S_{ij})_{\Omega_i} = \mathcal{T}_{\Omega_i}, \quad (S_{ij})_{\Omega_i,\Omega_j} = \{I_{ij}, I'_{ij}\}, \quad (S_{ij})_{\Omega_j} = \mathcal{T}_{\Omega_j}$$

is a coherent configuration. For each i, let I_{ii} be the irreflexive basis relation in $\mathcal{T}_{\Omega_i}$ and $I'_{ii} = 1_{\Omega_i}$. It is straightforward that

$$(\Omega, S) := (\Omega, \{I_{ij}, I'_{ij} : 1 \le i, j \le m\})$$

is a rainbow, where $\Omega = \bigcup_i \Omega_i$. To complete the proof, it suffices to show that (Ω, S) is a coherent configuration.

Let $1 \le i, j \le m$, $r \in S_{ik}$, and $s \in S_{kj}$. Assume that $r \cdot s \ne \varnothing$. Then we need to verify that the number $|\alpha r \cap \beta s^*|$ does not depend on the choice of $(\alpha, \beta) \in t$, where $t = I_{ij}$; the case $t = I'_{ij}$ is proved similarly.

First, suppose that $i \ne k \ne j \ne i$. If $r = I_{ik}$ and $s = I_{kj}$, then the required statement follows from the condition (LD2). If $r = I'_{ik}$ and $s = I_{kj}$, then

$$\beta s^* = (\alpha I_{ik} \cap \beta I^*_{kj}) \cup (\alpha r \cap \beta s^*)$$

is a disjoint union. Since $|\beta I^*_{kj}|$ is the k-parameter of the symmetric design in the condition (LD1) and $|\alpha I_{ik} \cap \beta I^*_{kj}|$ is constant, we are done. The same argument works if $r = I_{ik}$ and $s = I'_{kj}$. Finally, if $r = I'_{ik}$ and $s = I'_{kj}$, then the required statement follows from the facts that the decomposition

$$\beta s^* = (\alpha I_{ik} \cap \beta s^*) \cup (\alpha r \cap \beta s^*)$$

is disjoint and that the numbers $|\beta s^*|$ and $|\alpha I_{ik} \cap \beta s^*|$ are constants.

Second, suppose that $\{i, j, k\} = \{a, b\}$ with $1 \le a, b \le m$. Then $|\alpha r \cap \beta s^*|$ is an intersection number of the coherent configuration $(\Omega_a \cup \Omega_b, S_{ab})$ and we are done. □

2.7.52. The mapping (2.6.1) is a closure operator in the class of all rainbows $\mathcal{X}$ on Ω, i.e., the following statements hold:

(1) $\mathcal{X} \le \mathrm{WL}(\mathcal{X})$,
(2) if $\mathcal{X} \le \mathcal{X}'$, then $\mathrm{WL}(\mathcal{X}) \le \mathrm{WL}(\mathcal{X}')$,
(3) $\mathrm{WL}(\mathrm{WL}(\mathcal{X})) = \mathrm{WL}(\mathcal{X})$.

Proof. By the definition of coherent closures, one can see that

$$\mathcal{Y} \in \mathfrak{T}(\Omega, S(\mathcal{X})) \quad \Rightarrow \quad \mathcal{X} \le \mathcal{Y} \quad \text{and} \quad \mathrm{WL}(\mathcal{X}) \le \mathcal{Y}. \tag{2.7.36}$$

Setting $\mathcal{X} := \mathcal{X}$ and $\mathcal{Y} := \mathrm{WL}(\mathcal{X})$ in formula (2.7.36), we get statement (1).

By statement (1), we have

$$\mathcal{X} \le \mathcal{X}' \le \mathrm{WL}(\mathcal{X}').$$

This implies that $\mathrm{WL}(\mathcal{X}')$ belongs to $\mathfrak{T}(\Omega, S(\mathcal{X}))$. Thus, statement (2) follows from formula (2.7.36) for $\mathcal{X} = \mathcal{X}$ and $\mathcal{Y} = \mathrm{WL}(\mathcal{X}')$.

Setting $\mathcal{X} := \mathrm{WL}(\mathcal{X})$ and $\mathcal{Y} := \mathrm{WL}(\mathcal{X})$ in formula (2.7.36), we have

$$\mathrm{WL}(\mathrm{WL}(\mathcal{X})) \le \mathrm{WL}(\mathcal{X}).$$

The reverse inclusion in statement (3) follows by statement (1). $\square$

2.7.53. Let S and T be sets of binary relations on Ω. Assume that $S^{\cup} \subseteq T^{\cup}$. Then $\mathrm{WL}(S) \leq \mathrm{WL}(T)$.

Proof. Denote $\mathrm{WL}(T)$ by $\mathcal{X}$. By the assumption and the definition of coherent closure,

$$S \subseteq S^{\cup} \subseteq T^{\cup} \subseteq S(\mathcal{X})^{\cup}.$$

This implies that $\mathcal{X} \in \mathfrak{T}(\Omega, S)$. Since any coherent configuration in $\mathfrak{T}(\Omega, S)$ is a fission of $\mathrm{WL}(S)$, we are done. $\square$

2.7.54. Let T be a set of binary relations on Ω. Denote by S the partition of Ω^2 such that (α, β) and (α', β') belong to the same class if and only if

$$\forall t \in \{1_\Omega\} \cup T \cup T^* : \qquad (\alpha, \beta) \in t \;\Leftrightarrow\; (\alpha', \beta') \in t.$$

Then (Ω, S) is a rainbow and $\mathrm{WL}(T) = \mathrm{WL}(S)$.

Proof. Let $U = \{1_\Omega\} \cup T \cup T^*$. We claim that

$$U \subseteq S^{\cup}. \tag{2.7.37}$$

Indeed, given $s \in S$ and $t \in U$, we have

$$s \cap t \neq \varnothing \quad \Rightarrow \quad s \subseteq t,$$

for otherwise there exist pairs $(\alpha, \beta), (\alpha', \beta') \in s$ such that $(\alpha, \beta) \in t$ but $(\alpha', \beta') \notin t$, a contradiction. Moreover, for any $s' \in S$ such that $s' \neq s$, by formula (2.7.37) and the definition of S, there exists $u' \in U$ such that

$$s' \subseteq u' \quad \text{and} \quad s \cap u' = \varnothing. \tag{2.7.38}$$

Next we prove that (Ω, S) is a rainbow. Since $1_\Omega \in U$, formula (2.7.37) implies that $1_\Omega \in S^{\cup}$. Thus, condition (CC1) holds.

To verify condition (CC2), let $s \in S$. For $(\beta, \alpha), (\beta', \alpha') \in \Omega^2$, one can see that

$$\begin{aligned}(\beta, \alpha), (\beta', \alpha') \in s^* &\iff (\alpha, \beta), (\alpha', \beta') \in s \\ &\iff (\alpha, \beta) \in t \Leftrightarrow (\alpha', \beta') \in t,\ \forall t \in U \\ &\iff (\beta, \alpha) \in t^* \Leftrightarrow (\beta', \alpha') \in t^*,\ \forall t \in U.\end{aligned}$$

Since $U^* = U$, we are done.

Finally, we show that $\mathrm{WL}(S) = \mathrm{WL}(T)$. By formula (2.7.37), we have $T^{\cup} \subseteq S^{\cup}$. By Exercise (2.7.53), this yields that

$$\mathrm{WL}(T) \leq \mathrm{WL}(S).$$

To prove the reverse inclusion, it suffices to prove the following claim.

Claim: $S \subseteq S(\mathrm{WL}(T))^{\cup}$.

Proof. Let $s \in S$. Suppose first that s is contained in $v := \bigcup_{u \in U} u$. By formula (2.7.37), there exists $u \in U$ such that $s \subseteq u$. Then for each $s' \neq s$ with $s' \subseteq u$, there exists $u' \in U$ satisfying formula (2.7.38). Let w be the union of all such u'. Then

$$w \cap s = \varnothing \quad \text{and} \quad u \backslash s \subseteq w,$$

where the second formula holds because $u \in S^{\cup}$. It follows that

$$s = u \backslash w.$$

Observe that $u, w \in U^{\cup} \subseteq S(\mathrm{WL}(T))^{\cup}$ as $\mathrm{WL}(T)$ is a rainbow. The claim follows in this case. To complete the proof, it suffices to note that if $s \nsubseteq v$, then by the definition of S and v we have $s = \Omega^2 \setminus v$. □

2.7.55. Let $\mathfrak{X} = (\Omega, D)$ be a colored graph, and let φ be an algebraic isomorphism from $\mathcal{X} = \mathrm{WL}(\mathcal{P}_{c_{\mathfrak{X}}})$ onto another coherent configuration. Define a graph $\mathfrak{X}' = \mathfrak{X}^{\varphi}$ by

$$\Omega(\mathfrak{X}') = \Omega^{\varphi} \quad \text{and} \quad D(\mathfrak{X}') = D^{\varphi}$$

with a coloring $c_{\mathfrak{X}'}$ each color class of which is of the form $(c_{\mathfrak{X}}^{-1}(i))^{\varphi}$ for some color i of $c_{\mathfrak{X}}$. Then the colored graphs $\mathfrak{X}$ and $\mathfrak{X}^{\varphi}$ are isomorphic if and only if φ is induced by an isomorphism.

Proof. Set

$$T := \mathcal{P}_{c_{\mathfrak{X}}}, \quad T' := \mathcal{P}_{c_{\mathfrak{X}'}}, \quad \text{and} \quad \mathcal{X}' := \mathcal{X}^{\varphi}.$$

To prove the necessity, let $f \in \mathrm{Iso}(\mathfrak{X}, \mathfrak{X}')$ be a (colored graph) isomorphism. Then for any color class $t \in T^{\natural}$,

$$t^f = t^{\varphi}. \tag{2.7.39}$$

Without loss of generality, we may assume that (Ω, T) is a rainbow. Indeed, if $\mathfrak{X}$ is not a complete colored graph, then $\mathfrak{X}$ is replaced by a complete colored graph with an additional color class. Now T is a partition of Ω^2 satisfying the condition (CC1). If $T \neq T^*$, then T is replaced by $\{t \cap s^* : t, s \in T, t \cap s^* \neq \varnothing\}$. After these replacements, f and φ still satisfy formula (2.7.39).

To complete the proof of the necessity, it suffices to verify formula (2.7.39) for all $t \in S$, where $S = S(\mathcal{X})$. By Lemma 2.6.3, it suffices to verify this formula for all $t = w_k(r, s, t)$, where $w_k(r, s, t)$ is defined on page 99. However, one can easily see that

$$w_k(r, s, t)^f = w_k(r^f, s^f, t^f) = w_k(r^{\varphi}, s^{\varphi}, t^{\varphi}) = w_k(r, s, t)^{\varphi}.$$

We are done.

To prove the sufficiency, suppose that $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$. Since $T \subseteq S^{\cup}$, it follows that

$$\varphi(t) = t^f, \quad \forall t \in T.$$

This yields that $f \in \mathrm{Iso}(T, T')$. □

2.7.56. Let $\mathfrak{X}$ be an undirected cycle on n vertices. Then $\mathrm{WL}(\mathfrak{X}) = \mathrm{Inv}(D_{2n})$.

Proof. Without loss of generality, we may assume that $\Omega = \{1, \ldots, n\}$. Then,

$$S(\mathcal{X}) = \{s_0 = 1_{\Omega}, s_1, \ldots, s_d\},$$

where $\mathcal{X} = \mathrm{Inv}(D_{2n})$, $d = \lfloor \frac{n}{2} \rfloor$, and $s_1, \ldots, s_d$ are defined as in Exercise (2.7.33). In particular, we may assume that the arc set of $\mathfrak{X}$ equals s_1. Then

$$\mathrm{WL}(\mathfrak{X}) = \mathrm{WL}(\{s_1\}) \leq \mathcal{X}.$$

To prove the converse inclusion, note that s_0 and s_1 are relations of $\mathrm{WL}(\mathfrak{X})$. Using formula (2.7.18) and induction on $i = 0, \ldots, d$, one can see that each s_i is a relation of $\mathrm{WL}(\mathfrak{X})$, i.e., $S(\mathcal{X}) \subseteq S(\mathrm{WL}(\mathfrak{X}))^{\cup}$. It follows that

$$\mathcal{X} \leq \mathrm{WL}(\mathfrak{X}).$$

We are done. □

2.7.57. Let $\mathfrak{X}$ be a vertex-disjoint union of two connected graphs $\mathfrak{X}_1$ and $\mathfrak{X}_2$ on Ω_1 and Ω_2, respectively. Assume that $\Delta \in F(\mathrm{WL}(\mathfrak{X}))$ is such that

$$|\Delta \cap \Omega_1| \neq |\Delta \cap \Omega_2|.$$

Then the graphs $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are not isomorphic.

Proof. Suppose the conclusion is false. Then there exists $f \in \mathrm{Iso}(\mathfrak{X}_1, \mathfrak{X}_2)$. Let $\Omega = \Omega_1 \cup \Omega_2$ and $\tilde{f} \in \mathrm{Sym}(\Omega)$ be such that

$$\alpha^{\tilde{f}} = \begin{cases} \alpha^f & \text{if} \quad \alpha \in \Omega_1, \\ \alpha^{f^{-1}} & \text{if} \quad \alpha \in \Omega_2. \end{cases}$$

Then obviously $\tilde{f} \in \mathrm{Aut}(\mathfrak{X})$. By formula (2.6.3), we have $\tilde{f} \in \mathrm{Aut}(\mathrm{WL}(\mathcal{X}))$. This implies that $\Delta^{\tilde{f}} = \Delta$ for any $\Delta \in F(\mathrm{WL}(\mathcal{X}))$. Since $\Omega_1^{\tilde{f}} = \Omega_2$, it follows that

$$(\Delta \cap \Omega_1)^{\tilde{f}} = \Delta^{\tilde{f}} \cap \Omega_1^{\tilde{f}} = \Delta \cap \Omega_2.$$

Thus, $|\Delta \cap \Omega_2| = |\Delta \cap \Omega_1|$ for all Δ, a contradiction. □

2.7.58. Let $\mathfrak{X}$ be a graph and φ an algebraic isomorphism from $\mathrm{WL}(\mathfrak{X})$ to another coherent configuration. Then

(1) if $s_d(\mathfrak{X})$ is the relation on $\Omega(\mathfrak{X})$ consisting of all pairs of vertices at distance d in $\mathfrak{X}$, then $s_d(\mathfrak{X})^\varphi = s_d(\mathfrak{X}^\varphi)$,
(2) if the graph $\mathfrak{X}$ is distance-regular, then the graph $\mathfrak{X}^\varphi$ is also distance-regular and $\mathrm{IA}(\mathfrak{X}) = \mathrm{IA}(\mathfrak{X}^\varphi)$.

Proof. To prove statement (1), by Exercise (2.7.55) we have $\varphi(A) = A'$, where A and A' are respectively the adjacency matrices of $\mathfrak{X}$ and $\mathfrak{X}^\varphi$. Thus, applying the notation in the proof of Theorem 2.6.7, for each $i \geq 0$,

$$\varphi(A_i) = \varphi(\sum_{j=0}^{i} A^j) = \sum_{j=0}^{i} A'^j = A'_i.$$

Thus, one can see that

$$(s_i)^\varphi = (s_f(A_i)^\varphi = s_f(A'_i) = s'_i,$$

where s_f is defined on the end of page 102. In particular,

$$s_d(\mathfrak{X})^\varphi = (s_d \setminus s_{d-1})^\varphi = s_d(\mathfrak{X}^\varphi),$$

as required.

To prove statement (2), observe that if $\mathfrak{X}$ is distance-regular, then

$$S(\mathrm{WL}(\mathfrak{X})) = \{s_i : i = 0, \ldots, d\},$$

where d is the diameter of $\mathfrak{X}$. By statement (1),

$$S(\mathrm{WL}(\mathfrak{X}^\varphi)) = \{s'_i : i = 0, \ldots, d\}.$$

This implies that $\mathfrak{X}^\varphi$ is distance-regular. Since φ preserves intersection numbers, $\mathrm{IA}(\mathfrak{X}) = \mathrm{IA}(\mathfrak{X}^\varphi)$. □

2.7.59. Let $\mathfrak{X}$ be a connected but not 2-connected undirected graph[3] with at least 3 vertices. Then the coherent configuration of $\mathfrak{X}$ is not homogeneous.

[3]An undirected graph is said to be k-connected if no two of its vertices are separated by fewer than k other vertices.

Proof. Since $\mathfrak{X}$ is not 2-connected, there exists a vertex α such that $\mathfrak{X}'$ is not connected, where $\mathfrak{X}'$ is the subgraph of $\mathfrak{X}$ by removing the vertex α. Set $\Omega := \Omega(\mathfrak{X})$ and $\Omega' := \Omega(\mathfrak{X}')$.

Since $|\Omega'| \geq 2$, the integer

$$d := \max\{d(\alpha, \beta) : \beta \in \Omega'\}.$$

is positive, where here and below the distances are taken in the graph $\mathfrak{X}$. Choose $\beta \in \Omega'$ such that $d(\alpha, \beta) = d$. Since $\mathfrak{X}'$ is not connected, there exists a vertex $\gamma \in \Omega'$ such that β and γ belong to distinct connected components of $\mathfrak{X}'$. Since $\mathfrak{X}$ is connected and $\mathfrak{X}'$ is not connected, any path from β to γ in $\mathfrak{X}$ passes through α. This implies that

$$d' = d(\beta, \gamma) = d(\alpha, \beta) + d(\alpha, \gamma) > d.$$

Denote by $s_{d'}$ the relation on Ω "to be at distance d'". Then $s_{d'}$ is a nonempty relation of $\mathcal{X} := \mathrm{WL}(\mathfrak{X})$ (statement (2) of Theorem 2.6.7). This implies that $\Omega(s_{d'})$ is a homogeneity set of $\mathcal{X}$. On the other hand, by the definitions of d and d', $(\alpha, \delta) \notin s_{d'}$ for all $\delta \in \Omega$. So $\alpha \notin \Omega(s_{d'})$ and hence $\Omega(s_{d'}) \neq \Omega$. Thus, $\mathcal{X}$ is not homogeneous, as required. □

2.7.60. Let $\mathcal{X}$ be an antisymmetric scheme of rank 3, and $S = \{s_0, s_1, s_2\}$, where $s_0 = 1_\Omega$. Then the graphs associated with s_1 and s_2 are doubly regular tournaments, $n_{s_1} = n_{s_2} = (n-1)/2$, and the intersection numbers of $\mathcal{X}$ are determined from the formulas

$$c_{11}^0 = 0, \quad c_{12}^0 = \frac{n-1}{2}, \quad c_{11}^1 = c_{12}^1 = c_{12}^2 = \frac{n-3}{4}, \quad c_{11}^2 = \frac{n+1}{4}, \tag{2.7.40}$$

where $c_{ij}^k = c_{s_i s_j}^{s_k}$ for all i, j, k. In particular, $n = 3 \pmod 4$.

Proof. By the assumption on $\mathcal{X}$, we have

$$s_1^* = s_2 \quad \text{and} \quad s_1 \cup s_2 = \Omega^2 \setminus 1_\Omega.$$

Then the graphs associated with s_1 and s_2 are tournaments. Moreover, $c_{11}^0 = 0$ by the first equality and statement (1) in Exercise (2.7.6). Since $\mathcal{X}$ is a scheme,

$$n_{s_1} = n_{s_2} \quad \text{and} \quad \sum_{i=0}^{2} n_{s_i} = n.$$

As $n_{s_0} = 1$, we obtain

$$n_{s_1} = n_{s_2} = (n-1)/2 = c_{12}^0. \tag{2.7.41}$$

By formula (2.1.14),

$$n_{s_1} c_{11}^1 = n_{s_2} c_{21}^2 = n_{s_2} c_{12}^2.$$

According to formula (2.7.41), we obtain

$$c_{11}^1 = c_{21}^2 = c_{12}^2.$$

Applying formula (2.1.3), these numbers are equals to

$$c_{22}^2 = c_{21}^1 = c_{12}^1.$$

In particular,

$$c_{11}^1 = c_{12}^1.$$

By formula (2.1.7),

$$\sum_{i=0}^{2} c_{1i}^{1} = n_{s_1} = \frac{n-1}{2}.$$

Together with the obvious fact that $c_{10}^1 = 1$, we have

$$c_{11}^1 = c_{12}^1 = \frac{n-3}{4} = c_{12}^2.$$

Finally,

$$c_{10}^2 = 0,\, c_{12}^2 = \frac{n-3}{4}, \quad \sum_{i=0}^{2} c_{1i}^2 = n_1 \quad \Rightarrow \quad c_{11}^2 = \frac{n+1}{4},$$

where the first equality on the left-hand side is obvious and the third one follows from formula (2.1.7).

Since $c_{12}^1 = c_{12}^2 = \frac{n-3}{4}$, any two distinct points have $\frac{n-3}{4}$ common neighbors in the graph $\mathfrak{X}_1$ associated with s_1. It follows that $\mathfrak{X}_1$ is a doubly regular tournament. The same statement is true for the graph associated with s_2. □

2.7.61. [**?**] An antisymmetric scheme of rank 3 is schurian if and only if each irreflexive basis graph is isomorphic to a Paley tournament.

Proof. Let $\mathcal{X}$ be an antisymmetric scheme of rank 3 and s an irreflexive basis relation of $\mathcal{X}$. Then obviously, $\mathrm{WL}(s) = \mathcal{X}$ where $\mathrm{WL}(\{s\})$ is denoted by $\mathrm{WL}(s)$.

To prove the sufficiency, suppose the basis graph of s is isomorphic to a Paley tournament, i.e., a basis graph of an irreflexive basis relation t of the scheme $\mathrm{Cyc}(F, M)$, where $\mathbb{F} = \mathbb{F}_q$ with $q = 3 \pmod 4$ and M is the subgroup of $\mathbb{F}^\times$ of index 2. Now let f be the corresponding isomorphism, i.e., $s^f = t$. By formula (2.6.3), we obtain

$$f \in \mathrm{Iso}(s,t) \subseteq \mathrm{Iso}(\mathrm{WL}(s), \mathrm{WL}(t)) = \mathrm{Iso}(\mathcal{X}, \mathrm{Cyc}(\mathbb{F}, M)).$$

In other words, $\mathcal{X}$ is isomorphic to the schurian scheme $\mathrm{Cyc}(\mathbb{F}, M)$. We are done.

To prove the sufficiency, let $\mathcal{X}$ be schurian. Then the group $\mathrm{Aut}(\mathcal{X})$ acts transitively on the points and arcs of the basis graph of s. Since $\mathcal{X}$ is a tournament (Exercise (2.7.60)), we are done by the main result of J.L.Berggren in [12], which says that if the automorphism group of a tournament is transitive on the points and arcs then the tournament is isomorphic to a Paley tournament. □

2.7.62. [**?**] The following statements hold:

(1) any affine scheme is amorphic.
(2) the degree of any amorphic scheme of rank at least 4 is a square.

Proof. To prove statement (1), suppose that (Ω, S) is an affine scheme. Denote by Ψ the stabilizer of 1_Ω in $\mathrm{Sym}(S)$. According to formula 2.5.5, one can see that for any $f \in \Psi$ and any $r, s, t \in S$,

$$c_{rs}^t = c_{r^f s^f}^{t^f}.$$

Thus, f is an algebraic automorphism of $\mathcal{X}$. It follows that $\Psi \le \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$. Since obviously $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}) \le \Psi$, we have

$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}) = \Psi. \tag{2.7.42}$$

For any partition Π of S, set $\Phi_\Pi := \prod_{\pi\in\Pi}\mathrm{Sym}(\pi)$. Now if Π contains $\{1_\Omega\}$, then $\Phi_\Pi \le \Psi$. By formula (2.7.42) and Lemma 2.3.26, (Ω, S^{Φ_Π}) is a scheme. Since

$$S_\Pi = S^{\Phi_\Pi},$$

we are done.

To prove statement (2), let $\mathcal{X}$ be an amorphic scheme of rank at least 4 and degree n. By Theorem 3.3 in[4], for any irreflexive basis relations r, s, t with $r \ne s \ne t \ne r$, we have

$$c_{rs}^t = \frac{n_r n_s}{(\sqrt{n}+\epsilon)^2}, \tag{2.7.43}$$

where $\epsilon \in \{-1, 1\}$. Since c_{rs}^t is a positive integer, formula (2.7.43) implies that $\sqrt{n}$ is an integer. Thus, n is a square. □

2.7.63. [**?**] A finite affine plane is Desarguesian if and only if the corresponding scheme satisfies the 4-condition.

Proof. To prove the necessity, let $\mathcal{A}$ be a Desarguesian finite affine plane and $\mathcal{X}$ the scheme of $\mathcal{A}$. Then by the Veblen-Young theorem, $\mathcal{A}$ is an affine Galois plane. Hence, the scheme $\mathcal{X}$ is schurian (Theorem 2.5.7), i.e., $\mathcal{X} = \mathrm{Inv}(K, \Omega)$, where $K = \mathrm{Aut}(\mathcal{X})$ and Ω is the point set of $\mathcal{A}$. This implies that every basis relation of $\mathcal{X}$ is a K-orbit. Let $r_{ij} \in S$, $1 \le i, j \le 4$ and

$$\Lambda = \{\alpha \in \Omega^4 : r(\alpha_i, \alpha_j) \in r_{ij},\ 1 \le i, j \le 4\}.$$

Any quadruple $\gamma \in \Lambda$ can be treated as a 4-vertex colored subgraph $\mathfrak{X}_{\{\gamma_1,\ldots,\gamma_4\}}$, where $\mathfrak{X}$ is a colored graph associated with the rainbow $\mathcal{X}$ with respect to a fixed standard coloring. Let $(\alpha_1, \alpha_2) \in r_{12}$. Choose an arbitrary pair $(\beta_1, \beta_2) \in r_{12}$. Set

$$\Omega_{\alpha_1,\alpha_2} := \{\gamma \in \Lambda : (\gamma_1, \gamma_2) = (\alpha_1, \alpha_2)\}$$

and

$$\Omega_{\beta_1,\beta_2} := \{\gamma \in \Lambda : (\gamma_1, \gamma_2) = (\beta_1, \beta_2)\}.$$

Since r_{12} is a K-orbit, there exists $k \in K$ such that $(\alpha_1, \alpha_2)^k = (\beta_1, \beta_2)$. Then, k induces a bijection between $\Omega_{\alpha_1,\alpha_2}$ and Ω_{β_1,β_2}. Hence, $|\Omega_{\alpha_1,\alpha_2}| = |\Omega_{\beta_1,\beta_2}|$. This implies that the number of 4-vertex colored subgraphs of a given type with respect to a pair (α_1, α_2) does not depend on the choice of the pair in r_{12}. Thus, $\mathcal{X}$ satisfies the 4-condition, as required.

To prove the sufficiency, suppose that $\mathcal{X}$ satisfies the 4-condition. We have to verify that $\mathcal{A}$ is Desarguesian, i.e., given three lines containing a common point δ and given points α, α' lying on the first line, β, β' lying on the second line, and γ, γ' lying on the third line,

$$\alpha\gamma \parallel \alpha'\gamma' \quad \text{and} \quad \beta\gamma \parallel \beta'\gamma' \quad \Rightarrow \quad \alpha\beta \parallel \alpha'\beta'.$$

Without loss of generality, we may assume that the seven points $\alpha, \ldots, \delta$ are pairwise distinct. Since δ, γ, and γ' lie on the same line, we have

$$r(\delta, \gamma) = r(\delta, \gamma').$$

By the assumption, there exist two points α'', β'' and $f \in \mathrm{Iso}(\mathfrak{X}_{\{\delta,\gamma,\alpha,\beta\}}, \mathfrak{X}_{\{\delta,\gamma',\alpha'',\beta''\}})$ such that

$$(\delta, \gamma, \alpha, \beta)^f = (\delta, \gamma', \alpha'', \beta'')$$

[4]I.N. Ponomarenko and A. Rahnamai Barghi, On Amorphic C-Algebras, Journal of Mathematical Sciences, **145**(2007), No. 3, 4981-4988.

and

$$r(\gamma,\alpha)=r(\gamma',\alpha'') \quad \text{and} \quad r(\gamma,\beta)=r(\gamma',\beta'').$$

In pariticular, this means that

$$\gamma\alpha \parallel \gamma'\alpha'' \quad \text{and} \quad \gamma\beta \parallel \gamma'\beta''.$$

Taking into account that $\gamma\alpha \parallel \gamma'\alpha'$ and $\gamma\beta \parallel \gamma'\beta'$, by axiom (AP2) in the definition of an affine plane we have

$$\gamma'\alpha''=\gamma'\alpha' \quad \text{and} \quad \gamma'\beta''=\gamma'\beta'.$$

The fact that $\mathfrak{X}_{\{\delta,\gamma,\alpha,\beta\}}$ and $\mathfrak{X}_{\{\delta,\gamma',\alpha'',\beta''\}}$ are isomorphic as colored subgraphs yields that

$$r(\delta,\alpha)=r(\delta,\alpha''), \quad r(\delta,\beta)=r(\delta,\beta''), \quad \text{and} \quad \alpha\beta \parallel \alpha''\beta''. \tag{2.7.44}$$

By the first two equalities,

$$\alpha''\in\delta\alpha=\delta\alpha' \quad \text{and} \quad \beta''\in\delta\beta=\delta\beta'.$$

We conclude that

$$\alpha''=\delta\alpha'\cap\gamma'\alpha''=\delta\alpha'\cap\gamma'\alpha'=\alpha' \quad \text{and} \quad \beta''=\delta\beta'\cap\gamma'\beta''=\delta\beta'\cap\gamma'\beta'=\beta'.$$

In view of the third equality in formula (2.7.44), one can see that $\alpha\beta \parallel \alpha'\beta'$, as required. □

2.7.64. [**?, ?**] For a group G, denote by $\mathcal{X}_G$ the scheme of the strongly regular graphs $\mathfrak{X}_G$ defined by formula (2.6.11). Then

(1) $\mathrm{Aut}(\mathcal{X}_G)\cong((G\times G)\,\mathrm{Aut}(G))\,\mathrm{Sym}(3)$ whenever $|G|\geq 5$,
(2) $\mathcal{X}_G$ is schurian if and only if it satisfies the 4-condition,
(3) $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are algebraically isomorphic if and only if $|G|=|G'|$,
(4) $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are isomorphic if and only if G and G' are isomorphic.

Proof. Set $\Omega:=G\times G$ and $n:=|G|$.

(1) For any (g,h) belongs to the group $G\times G$,

$$\tau_{g,h}: \quad \Omega \quad \to \quad \Omega, \quad (\alpha_1,\alpha_2) \quad \mapsto \quad (g\alpha_1,\alpha_2 h),$$

is a permutation on Ω. It is easy to see that $\{\tau_{g,h}:(g,h)\in G\times G\}$ is a subgroup, denoted by H, of $\mathrm{Aut}(\mathfrak{X}_G)$ which is isomrophic to $G\times G$. Morover, $\mathrm{Aut}(G)$ has a natural faithful action on Ω as follows:

$$\Omega \quad \to \quad \Omega, \quad (\alpha_1,\alpha_2) \quad \mapsto \quad (\alpha_1^\sigma,\alpha_2^\sigma), \quad \sigma\in\mathrm{Aut}(G).$$

This action produces a subgroup, denoted by K, of $\mathrm{Aut}(\mathfrak{X}_G)$ which is isomorphic to $\mathrm{Aut}(G)$. Since obviously K normalizes H, $\mathrm{Aut}(\mathfrak{X}_G)$ has a subgroup HK isomorphic to $(G\times G)\,\mathrm{Aut}(G)$. In addition, the following two involutionary perumtations on Ω

$$\varphi_1:(\alpha_1,\alpha_2)\mapsto(\alpha_2^{-1},\alpha_1^{-1}) \quad \text{and} \quad \varphi_2:(\alpha_1,\alpha_2)\mapsto(\alpha_1^{-1},\alpha_1\alpha_2)$$

belong to $\mathrm{Aut}(\mathfrak{X}_G)$. Note that $\langle\varphi_1,\varphi_2\rangle$ is a subgroup, denoted by L, of $\mathrm{Aut}(\mathfrak{X}_G)$ isomorphic to $\mathrm{Sym}(3)$.

Since L normalizes HK, we conclude that $\mathrm{Aut}(\mathfrak{X}_G)$ has a subgroup HKL isomorphic to $((G\times G)\,\mathrm{Aut}(G))\,\mathrm{Sym}(3)$.

In fact, the subgroup HKL coincides with $\mathrm{Aut}(\mathfrak{X}_G)=\mathrm{Aut}(\mathcal{X}_G)$. This follows from Theorem 2.7 in [34], which tells us that

$$\mathrm{Aut}((\mathfrak{X}_G)^f)\cong((G\times G)\,\mathrm{Aut}(G))\,\mathrm{Sym}(3),$$

where $f:\Omega\to\Omega,(\alpha_1,\alpha_2)\mapsto(\alpha_1^{-1},\alpha_2)$ is a bijection of Ω.

(2) As in the first part of the proof of Exercise 2.7.63, one can see that if $\mathcal{X}_G$ is schurian then $\mathcal{X}_G$ satisfies the 4-condition. (When $\mathcal{X}_G$ is schurian, it is easily seen that $\mathrm{Aut}(G)$ is transitive on $G^\#$. This happens if and only if G is an elementary abelian p-group). Conversely, suppose $\mathcal{X}$ satisfies the 4-condition. Then by the proof of Theorem 3.1 in [34], G is isomorphic to an elementary abelian 2-group or cyclic group of order 5. By Theorem 2.10 in [34], $(\mathrm{Aut}(\mathcal{X}_G),\Omega^2)$ is primitive permutation group of rank 3. In particular, this implies that $\mathcal{X}_G$ is schurian.

(3) By Proposition 2.6.16, $\mathfrak{X}_G$ is a strongly regular graph with parameters $(n^2,3n-3,n,6)$. Hence, if $|G|=|G'|$, then $\mathrm{IA}(\mathfrak{X}_G)=\mathrm{IA}(\mathfrak{X}_{G'})$. This implies that $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are algebraically isomorphic (statement (3) of Theorem 2.6.11). Conversely, if $\varphi\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_G,\mathcal{X}_{G'})$, then $|G'|=|G^\varphi|=|G|$ (statement (2) of Proposition 2.3.22).

(4) Set $|G|=n$. The sufficiency is straightforward, since any group isomorphism $f\in\mathrm{Iso}(G,G')$ belongs to $\mathrm{Iso}(\mathfrak{X}_G,\mathfrak{X}_{G'})\subseteq\mathrm{Iso}(\mathcal{X}_G,\mathcal{X}_{G'})$ (formula (2.6.3)).

To prove the necessity, let $f\in\mathrm{Iso}(\mathcal{X}_G,\mathcal{X}_{G'})$. Then $\mathcal{X}_G$ and $\mathcal{X}_{G'}$ are algebraically isomorphic. Hence, $|G'|=n$ (statement (3)). Set

$$S(\mathcal{X}_G)=\{1_\Omega,s_G,t_G\}\quad\text{and}\quad S(\mathcal{X}_{G'})=\{1_\Omega,s_{G'},t_{G'}\}.$$

If $n=1$ or 5, then G and G' are obviously isomorphic. Otherwise,

$$n_{s_{G'}}=n_{s_G}=3n-3,\quad n_{t_G}=n_{t_{G'}}=n^2-3n+2.$$

Consequently, $s_G^f=s_{G'}$. Therefore, $f\in\mathrm{Iso}(\mathfrak{X}_G,\mathfrak{X}_{G'})$. Then we are done by Moorhouse's theorem.[5] □

2.7.65. A complete colored n-vertex graph $\mathfrak{X}$ satisfies the t-vertex condition for $t=3$ (respectively, for $t=n$) if and only if the color classes of $\mathfrak{X}$ form a coherent configuration (respectively, a schurian coherent configuration).

Proof. Let $t=3$. Assume first that the color classes of $\mathfrak{X}$ form a coherent configuration. Then the number of 3-vertex colored subgraphs $\mathfrak{X}_{\{\alpha,\beta,\gamma\}}$ of a given type with respect to the pair (α,β) is equal to c_{rs}^u, where $u=r(\alpha,\beta)$, and $r=r(\alpha,\gamma)$ and $s=r(\gamma,\beta)$ are fixed. Hence, the colored graph satisfies 3-vertex condition.

Conversely, let us verify that $S=\mathcal{P}_\mathfrak{X}$ satisfies the conditions (CC1), (CC2) and (CC3) (S is a partition because $\mathfrak{X}$ is a complete colored graph).

Observe that the definition of colored graphs implies the condition (CC1).

Let $r\in S$ and $(\alpha,\beta)\in r$. Assume that $(\alpha,\alpha)\in s$ and $(\beta,\alpha)\in u$. By the assumption, the number of 3-vertex colored subgraphs

$$\mathfrak{X}_{\{\alpha,\alpha,\beta\}},\quad(\alpha,\alpha)\in s,\quad(\beta,\alpha)\in u$$

does not depend on the choice of the pair $(\alpha,\beta)\in r$. This implies that $r^*\subseteq u$. Similarly, $u\subseteq r^*$. It follows that $r^*=u\in S$. Thus, the condition (CC2) holds.

Let $r,s,u\in S$. For each $(\alpha,\beta)\in u$, denote by $c(\alpha,\beta;r,s)$ the number of 3-vertex colored subgraphs $\mathfrak{X}_{\{\alpha,\beta,\gamma\}}$, where $(\alpha,\gamma)\in r$, $(\gamma,\beta)\in s$ and the other colors are obvious. By the 3-vertex condition, $c(\alpha,\beta;r,s)$ does not depend on the choice of the pair $(\alpha,\beta)\in u$. Since

$$c(\alpha,\beta;r,s)=|\alpha r\cap\beta s^*|,$$

[5] Proposition 4.1 in: G.E. Moorhouse, *Bruck Nets, Codes, and Characters of Loops*, Designs, Codes and Cryptography 1 (**1991**), 7-29.

the condition (CC3) holds.

Let $t = n$. Then $\mathfrak{X}$ is the only t-vertex subgraph of $\mathfrak{X}$. It follows that $\mathfrak{X}$ satisfies the n-vertex condition $\iff$ for any $s \in \mathcal{P}_{c_{\mathfrak{X}}}$ and any pairs $(\alpha, \beta), (\alpha', \beta) \in s$, there exists $f \in \mathrm{Aut}(\mathfrak{X})$ such that $(\alpha, \beta)^f = (\alpha', \beta')$ $\iff$ any $s \in \mathcal{P}_{c_{\mathfrak{X}}}$ is an orbit of $\mathrm{Aut}(\mathfrak{X})$ $\iff$ $\mathcal{P}_{c_{\mathfrak{X}}} = \mathrm{Orb}(\mathrm{Aut}(\mathfrak{X}), \Omega(\mathfrak{X})^2)$ $\iff$ the classes of $\mathfrak{X}$ form a schurian coherent configuration. $\square$

3.7. Exercises

In what follows, unless otherwise stated, $\mathcal{X}$ is a coherent configuration on Ω and $S = S(\mathcal{X})$, $F = F(\mathcal{X})$, and $E = E(\mathcal{X})$. The notations $\mathcal{X}'$ and Ω', S', F', and E' have the same meaning. The number m denotes a positive integer and $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$, $\overline{\mathcal{X}} = \overline{\mathcal{X}}^{(m)}$, etc.

3.7.1. Let $\mathcal{X}$ be a fusion of an affine scheme of degree q^2. Then

(1) for each $s \in S^{\#}$, $n_s = a_s(q-1)$ for some integer $a_s \geq 1$,
(2) $\mathcal{X}$ is primitive if and only if $a_s \geq 2$ for all $s \in S^{\#}$.

Proof. In $\mathcal{X}$, every $s \in S^{\#}$ is a union of some irreflexive basis relation of the affine scheme where each of them has valency $q-1$. Statement (1) follows.

For (2), if there exists some $s \in S^{\#}$ such that

$$a_s = 1.$$

By the property (2.5.5) of the intersection numbers in the affine scheme, we see that

$$\{1_\Omega, s\}$$

is a parabolic. Since $\mathcal{X}$ has degree q^2 and $n_s = q-1$, this parobolic is not equal to Ω^2. Thus, $\mathcal{X}$ is not primitve in this case.

Conversely, assume

$$a_s \geq 2, \quad \forall s \in S^{\#}.$$

Choose arbitrarily $s \in S^{\#}$. Set

$$s = t_1 \cup \ldots \cup t_m$$

where each t_i is a basis relation of the affine scheme and $m \geq 2$. Then by (2.5.5), every irreflexive basis relation of the affine scheme t satisfies

$$t \in t_1 t_2.$$

Thus, in $\mathcal{X}$

$$\langle s \rangle = \Omega^2.$$

It follows that $\mathcal{X}$ is primitive. □

3.7.2. A coherent configuration of a disconnected graph is either non-homogeneous or imprimitive.

Proof. Let $\mathfrak{X}$ be a disconnected graph and $\mathcal{X} = \mathrm{WL}(\mathfrak{X})$. Suppose that $\mathcal{X}$ is homogeneous, it suffices to show that $\mathcal{X}$ is nonprimitive. By Proposition 2.6.8, the equivalence relation $e_{con}(\mathfrak{X})$ determined by the connected components of $\mathfrak{X}$ belongs to E. Since $\mathfrak{X}$ is disconnected,

$$e_{con}(\mathfrak{X}) \neq \Omega^2.$$

If $e_{con}(\mathfrak{X}) = 1_\Omega$, then $\mathcal{X} = \mathcal{D}_\Omega$. Since we are assuming that $\mathcal{X}$ is homogeneous, this is a contradiction. We conclude that $\mathcal{X}$ is nonprimitive, as wanted. □

3.7.3. Let $\mathcal{X}$ be a primitive scheme. Then given $s \in S^{\#}$, there exists a positive integer m such that $s^m = S$, where s^m.

3.7.4. Let $\mathcal{X}$ and $\mathcal{X}'$ be algebraically isomorphic coherent configurations. Then $\mathcal{X}$ is primitive (respectively, imprimitive) if and only if $\mathcal{X}'$ is primitive (respectively, imprimitive).

Proof. If φ is an algebraic isomorphism from $\mathcal{X}$ to $\mathcal{X}'$, then it induces a bijection from $E(\mathcal{X})$ to $E(\mathcal{X}')$ by Exercise (2.7.30). Thus,

$$E(\mathcal{X}) = \{1_\Omega, \Omega^2\} \quad \Leftrightarrow \quad E(\mathcal{X}') = \{1_{\Omega'}, \Omega'^2\}.$$

If follows that $\mathcal{X}$ is primitive if and only if $\mathcal{X}'$ is primitive. □

3.7.5. [**?**, Theorem 4.2.1] Let $\mathcal{X}$ be the scheme of a distance-regular graph of diameter d (and $n_{s_1} > 2$). Then $\mathcal{X}$ is imprimitive only if s_2(the graph $\mathfrak{X}$) is a bipartite graph or s_d is the disjoint union of cliques (here, s_2 and s_d are defined by formula (**??**)).

Proof. Set $c_{ij}^k = c_{s_i s_j}^{s_k}$ for $s_i, s_j, s_k \in S$.

Suppose $\mathcal{X}$ is imprimitive. As the graph $\mathfrak{X}$ is connected, $\langle s_1 \rangle = \Omega^2$. Since the set of parabolics $E = \{\langle s \rangle : s \in S^\cup\}$ (Corollary 2.1.20), there exists $i > 1$ such that $\langle s_i \rangle \neq \Omega^2$. Among all such i, choose the smallest one, denoted by i. We claim that

$$c_{ii}^j = 0, \quad j < i \tag{3.7.1}$$

Otherwise, $s_j \subseteq s_i^2$. This implies that

$$\Omega^2 = \langle s_j \rangle \subseteq \langle s_i \rangle,$$

a contradiction. The claim follows. In particular, if $d = 2$, then $i = 2$. Then formula (3.7.1) implies that any conncected component of the graph s_2 is complete, i.e. the graph s_2 is the disjoint union of cliques.

Now we assume that $d > 2$. If $i = 2$, our goal is to prove that $\mathfrak{X}$ is a bipartite graph. To this end, it suffices to show that $\mathfrak{X}$ does not have odd cycle (Proposition 1.6.1 in[6]). We first show that there is no 3-cylce in $\mathfrak{X}$, i.e.,

$$c_{11}^1 = 0. \tag{3.7.2}$$

Suppose on the contrary that $c_{11}^1 \neq 0$. Since $d > 2$, there exist a path

$$\gamma_0 - \gamma_1 - \gamma_2 - \gamma_3$$

in $\mathfrak{X}$ of length 3. Since $c_{11}^1 \neq 0$, there exists a point

$$\gamma \in \gamma_0 s_1 \cap \gamma_1 s_1.$$

Then $d(\gamma, \gamma_2) \leq 2$. If $d(\gamma, \gamma_2) = 1$, then $d(\gamma, \gamma_3) = 2$ (note that $d(\gamma, \gamma_3) > 1$). It follows that, $(gamm, \gamma_1) \in s_1$ and

$$\gamma_3 \in \gamma s_2 \cap \gamma_1 s_2.$$

This implies that $c_{22}^1 \neq 0$, a contradiction to formula (3.7.1). Similarly, if $d(\gamma, \gamma_2) = 2$, then

$$\gamma_2 \in \gamma s_2 \cap \gamma_0 s_2,$$

a contradiction. Formula (3.7.2) follows. This implies that there is no 3-cycle in $\mathfrak{X}$. Now suppose on the contrary that $\mathfrak{X}$ has an odd cycle

$$\beta_0 - \beta_1 - \cdots - \beta_{2m} - \beta_0$$

for some integer $m > 1$. By formula (3.7.2),

$$(\beta_0, \beta_2),\ (\beta_2, \beta_4),\ \ldots,\ (\beta_{2m-2}, \beta_{2m}),\ (\beta_{2m}, \beta_1) \in s_2.$$

[6] R. Diestel, *Graph Theory(Electronic Edition)*, Heidelberg, New York: Springer-Verlag, 2005

This implies that $s_1 \in s_2^{m+1}$. Hence,

$$\Omega^2 = \langle s_1 \rangle \subseteq \langle s_2 \rangle,$$

a contradiction. We are done.

Now assume $i > 2$. If $2 < i < d$, choose a path of length d in $\mathfrak{X}$ as follows

$$\gamma_0 - \gamma_1 \cdots \gamma_i - \gamma_{i+1} \cdots \gamma_d.$$

Since $n_{s_1} > 2$, there exists $\delta \in \gamma_{i+1}s_1$ such that $\gamma_i \neq \delta \neq \gamma_{i+2}$ (if $d = i+1$, we choose such δ satisfying $\delta \neq \gamma_d$ and γ_{i+2} is a point in $\gamma_d s_1$ different from δ). Then

$$d(\gamma_0, \delta) \in \{i + l : l = 0, 1, 2\}.$$

Then for each $l \in \{0, 1, 2\}$, note that $j := d(\delta, \gamma_{i+l}) \leq 2$. For each l, by the triple

$$(\gamma_l, \gamma_{i+l}, \delta)$$

one can see that $c_{i,i}^j \neq 0$. Since $i > 2$, this is a contradiction to formula (3.7.1). Thus, we have $i = d$. Since for any $j < d$, $c_{dd}^j \neq 0$, we see that the graph s_d is the disjoint union of cliques. □

3.7.6. Let e be a parabolic of $\mathcal{X}$ with indecomposable components e_i, $i \in I$, and π_e the mapping (**??**). Then

$$F(\mathcal{X}_{\Omega/e}) = \{\Omega(\pi_e(e_i)) : \ i \in I\}.$$

In particular, $\mathcal{X}_{\Omega/e}$ is homogeneous if and only if e is indecomposable.

Proof. By Exercise (2.7.10), for any $\Delta \in F(\mathcal{X})$,

$$e \cdot 1_\Delta \cdot e$$

is an indecoposable component of e. Conversely, if e_i is an indecomposable component of e, choose α and $\Delta \in F(\mathcal{X})$ such that

$$\alpha \in \Delta \quad \text{and} \quad (\alpha, \alpha) \in e_i.$$

In particular,

$$e_i \cap e \cdot 1_\Delta \cdot e \neq \varnothing.$$

Since both of them are indecoposable components of e, we deduce that

$$e_i = e \cdot 1_\Delta \cdot e.$$

For any

$$\bar{\Delta} \in F(\mathcal{X}_{\Omega/e})$$

by Theorem 3.1.10, there exists $\Delta \in F(\mathcal{X})$ such that

$$1_{\bar{\Delta}} = \pi_e(1_\Delta) = \pi_e(e \cdot 1_\Delta \cdot e).$$

Thus,

$$\bar{\Delta} = \Omega(\pi_e(e_i),$$

where $e_i = e \cdot 1_\Delta \cdot e$ is an indecoposable component of e. Since by theorem 3.1.10,

$$\Delta \quad \mapsto \quad \bar{\Delta}$$

is surjective. The proof is complete. □

3.7.7. Let e be the parabolic of $\mathcal{X}$ such that $\Omega/e = F$. Then $\mathcal{X}_{\Omega/e} = \mathcal{D}_F$.

Proof. By the assumption,

$$e = \bigcup_{\Delta \in F} \Delta \times \Delta.$$

For any $s \in S$, suppose

$$s \in S_{\Delta,\Gamma},$$

where $\Delta, \Gamma \in F$. It follows that

$$s_{\Omega/e} = (\Delta, \Gamma).$$

Since

$$S(\mathcal{X}_{\Omega/e}) = \{s_{\Omega/e} : \quad s \in S\},$$

the assertion then holds. □

3.7.8. Let $\mathcal{X} \leq \mathcal{X}'$ and $e \in E$. Then $\mathcal{X}_{\Omega/e} \leq \mathcal{X}'_{\Omega/e}$.

Proof. Note that

$$e \in S(\mathcal{X})^{\cup} \subseteq S(\mathcal{X}')^{\cup}.$$

This implies that $e \in E(\mathcal{X}')$. For any

$$s_{\Omega/e} \in S(\mathcal{X}_{\Omega/e})$$

where $s \in S(\mathcal{X})$, set

$$s = s_1 \cup \cdots \cup s_m, \quad s_i \in S(\mathcal{X}').$$

Then obviously,

$$s_{\Omega/e} = \cup_{i=1}^m (s_i)_{\Omega/e} \in S(\mathcal{X}'_{\Omega/e}).$$

The proof is complete. □

3.7.9. Let $e_0, e_1 \in E$ be such that $e_0 \subseteq e_1$. Then

(1) the quotient of $\mathcal{X}_{\Omega/e_0}$ modulo $\pi_{e_0}(e_1)$ is canonically isomorphic to $\mathcal{X}_{\Omega/e_1}$,
(2) for any $\Delta \in \Omega/e_1$, the quotient of $\mathcal{X}_\Delta$ modulo $(e_0)_\Delta$ is canonically isomorphic to the restriction of $\mathcal{X}_{\Omega/e_0}$ to $\pi_{e_0}(\Delta)$.

Proof. Let

$$\bar{\Omega}, \quad \bar{s} \quad \text{and} \quad \bar{S}$$

respectively denote Ω/e_0, $\pi_{e_0}(s)$ for any $s \in S^{\cup}$ and

$$\{\bar{s} : s \in S\}.$$

By Theorem 3.1.10, $\bar{e}_1$ is a parabolic of $\mathcal{X}_{\Omega/e}$. For each $s \in S$,

$$\bar{s} = \{(\Delta, \Gamma) : \quad \Delta, \Gamma \in \Omega/e_0 \quad \text{and} \quad s \cap \Delta \times \Gamma \neq \varnothing\}.$$

Since $e_0 \subseteq e_1$, for each $\Delta \in \Omega/e_0$ there exists a uniquely determined $\Delta' \in \Omega/e_1$ such that

$$\Delta \subseteq \Delta'.$$

For each $\Delta \in \Omega/e_1$, denote

$$\bar{\Delta} = \{\Delta' : \quad \Delta' \subseteq \Delta, \quad \Delta' \in \Omega/e_0\}. \tag{3.7.3}$$

Thus,

$$\bar{e}_1 = \bigcup_{\Delta \in \Omega/e_1} \bar{\Delta} \times \bar{\Delta}.$$

Observe that

$$\bar{s}_{\bar{\Omega}/\bar{e}_1} = \{(\bar{\Delta}, \bar{\Gamma}) : \bar{\Delta}, \bar{\Gamma} \in \bar{\Omega}/\bar{e}_1, \quad \bar{s} \cap \bar{\Delta} \times \bar{\Gamma} \neq \varnothing\}.$$

Furthermore,

$$(\bar\Delta, \bar\Gamma) \in \bar s_{\bar\Omega/\bar e_1}$$

if and only if there exist $\Delta', \Gamma' \in \Omega/e_0$ such that

$$\Delta' \subseteq \Delta, \quad \Gamma' \subseteq \Gamma \quad \text{and} \quad s \cap \Delta' \times \Gamma' \neq \varnothing$$

if and only if

$$s \cap \Delta' \times \Gamma' \neq \varnothing$$

if and only if

$$(\Delta, \Gamma) \in s_{\Omega/e_1}.$$

Thus,

$$\bar s_{\bar\Omega/\bar e_1} \mapsto s_{\Omega/e_1}$$

establishes the required canonical isomorphism in (1).

For (2), note that

$$(e_0)_\Delta = \bigcup_{\Delta' \in \bar\Delta} \Delta' \times \Delta',$$

where $\bar\Delta$ is defined in (3.7.3). Let

$$r_\Delta \in S(\mathcal{X}_\Delta).$$

Then

$$(r_\Delta)_{\Delta/(e_0)_\Delta} = \{(\Delta', \Gamma') : \quad \Delta', \Gamma' \in \bar\Delta, \quad r \cap \Delta' \times \Gamma' \neq \varnothing\}.$$

In addition,

$$(r_{\Omega/e_0})_{\bar\Delta} = \{(\Delta', \Gamma') : \quad \Delta', \Gamma' \in \bar\Delta, \quad r \cap \Delta' \times \Gamma' \neq \varnothing\}.$$

Hence,

$$(r_\Delta)_{\Delta/(e_0)_\Delta} \mapsto (r_{\Omega/e_0})_{\bar\Delta}$$

produces the cannocial isomorphism in (2). □

3.7.10. Let $\mathcal{X}$ be a semiregular coherent configuration, and let e be the union of all relations in a system of distinct representative of $\{S_{\Delta,\Gamma}\}_{\Delta,\Gamma\in F}$ given in statement (3) of Exercise 2.7.13. Then

(1) e is an indecomposable parabolic of $\mathcal{X}$,
(2) given $\Delta \in F$ and $\Gamma \in \Omega/e$, we have $\Delta \cap \Gamma = \{\alpha_{\Delta,\Gamma}\}$ for some point $\alpha_{\Delta,\Gamma}$,
(3) for any $\Delta \in F$, the mapping $f : \Omega/e \to \Delta$, $\Gamma \mapsto \alpha_{\Delta,\Gamma}$ is a bijection,
(4) $f \in \mathrm{Iso}(\mathcal{X}_{\Omega/e}, \mathcal{X}_\Delta)$.

Proof. Denote

$$F = \{\Delta_i : \quad 1 \le i \le m\}.$$

Set

$$S_{ij} = S_{\Delta_i,\Delta_j}.$$

Recall that we choose

$$t_{1i} \in S_{1i}, \quad 1 \le i \le m$$

with $t_{11} = 1_{\Delta_1}$. Then

$$\{t_{ij} : t_{ij} = t_{1i}^* \cdot t_{1j}, 1 \le i, j \le m\}$$

is the system of distinct representative. Also,

$$e = \bigcup_{1 \le i,j \le m} t_{ij}.$$

Observe that, for each i,
$$t_{ii} = t_{1i}^* t_{1i} = 1_{\Delta_i} \subseteq e.$$
This implies that e is reflexive. In addition,
$$t_{ij}^* = t_{1j}^* t_{1i} = t_{ji} \subseteq e,$$
which yields that e is symmetric. Since we have proved in Exercise 2.7.13 that e is closed with respect to composition of relations, e is transitive. We conclude that e is a parabolic of $\mathcal{X}$.

By the construction of e, it is easily seen that
$$e = e \cdot 1_{\Delta_1} \cdot e,$$
which is an indecomposable component of e by Exercise 2.7.10. In particular, e is indecomposable and (1) is proved.

For each $\Gamma \in \Omega/e$, choose $\alpha \in \Gamma$. There exists $1 \leq i \leq m$ satisfying
$$\alpha \in \Delta_i$$
Then
$$\Gamma = \alpha e = \bigcup_{1 \leq l,j \leq m} \alpha t_{lj} = \bigcup_{j=1}^{m} \alpha t_{ij}.$$
Then, for each $\Delta_k \in F$,
$$\Gamma \cap \Delta_k = \alpha t_{ik} := \alpha_{\Delta_k, \Gamma}.$$
Thus, statement (2) is valid.

For (3), since
$$\alpha_{\Delta,\Gamma} = \Delta \cap \Gamma,$$
we deduce that f is a well-defined injection. Furthermore, for any $\alpha \in \Delta$, then
$$\Gamma e \cap \Delta = \{\alpha\},$$
where
$$\Gamma = \alpha e \in \Omega/e.$$
It follows that f is a surjection. We are done.

To prove (4), fix $1 \leq i \leq m$. For any
$$1 \leq j, k \leq m \quad \text{and} \quad s \in S_{jk},$$
it is trivial to see that
$$e \cdot s \cdot e = e \cdot (t_{ij} \cdot s \cdot t_{ki}) \cdot e.$$
Denote
$$s' = t_{ij} \cdot s \cdot t_{ki}.$$
Using the notation in (3.1.3),
$$s^e = s'^e.$$
This implies that
$$s_{\Omega/e} = s'_{\Omega/e}.$$
We conclude that for each fixed i,
$$S(\mathcal{X}_{\Omega/e}) = \{s_{\Omega/e} : \quad s \in S_{ii}\}.$$
Also, by (2), we have
$$\Omega/e = \{\alpha_i e : \quad \alpha_i \in \Delta_i\}.$$
Moreover, for any $s \in S_{ii}$,
$$(\alpha_i e,\, \alpha_i' e) \in s_{\Omega/e} \quad \Leftrightarrow \quad s \cap \alpha_i e \times \alpha_i' e \neq \varnothing \quad \Leftrightarrow \quad (\alpha_i,\, \alpha_i') \in s.$$

It follows that, for any $s \in S_{ii}$

$$s_{\Omega/e} = \{(\alpha_i e, \alpha_i' e) : \quad (\alpha_i, \alpha_i') \in s\}.$$

Now suppose

$$f : \Omega/e \to \Delta_i, \quad \alpha_i e \mapsto \alpha_i e \cap \Delta_i$$

is defined as in (3). Then

$$f(\alpha_i e) = \alpha_i.$$

Thus,

$$(s_{\Omega/e})^f = \{(\alpha_i e^f, \alpha_i' e^f) : \quad (\alpha_i, \alpha_i') \in s\} = \{(\alpha_i, \alpha_i') : \quad (\alpha_i, \alpha_i') \in s\} = s_{\Delta_i}.$$

The proof for (4) is complete. □

3.7.11. A scheme is schurian if and only if it is isomorphic to the quotient of a regular scheme.

Proof. The sufficiency holds, since every regular scheme is schurian by Theorem 2.2.8 and any quotient of a schurian coherent configuration is also schurian by Corollary 3.1.15.

To prove the necessity, assume that $\mathcal{X}$ is a schurian scheme. Then

$$\mathcal{X} = \mathrm{Inv}(K, \Omega),$$

where $K = \mathrm{Aut}(\mathcal{X})$. Since $\mathcal{X}$ is homogeneous, K is transitive on Ω. Without loss of generality, we assume that

$$\Omega = \{Hk : \quad k \in K\}$$

for a subgroup H of K and K acts on Ω by right multiplication. Also, for each $s \in S(\mathcal{X})$, there exists $k_s \in K$ such that

(3.7.4) $$s = \mathrm{Orb}(K, (H, Hk_s)) = \{(Hk, Hk_s k) : k \in K\}.$$

Thus, by formula (2.2.4)

(3.7.5) $$(Hx, Hy) \in s \quad \Leftrightarrow \quad (H, Hyx^{-1}) \in s \quad \Leftrightarrow \quad Hyx^{-1} \subseteq Hk_s H.$$

Let

$$\mathcal{X}' = \mathrm{Inv}(K_{right}, K).$$

Then $\mathcal{X}'$ is a regular scheme on $\Gamma := K$ and

$$S(\mathcal{X}') = \{s_k : k \in K\}$$

where

$$s_k = \{(\alpha, k^{-1}\alpha) : \alpha \in K)\}.$$

Observe that

$$e = \bigcup_{h \in H} s_h$$

is a parabolic of $\mathcal{X}'$. And for any $\alpha \in K$,

$$\alpha e = H\alpha.$$

Observe that, for $H\alpha, H\beta \in \Gamma/e$

(3.7.6) $$(H\alpha, H\beta) \in (s_k)_{\Gamma/e} \quad \Leftrightarrow \quad H\beta \cap k^{-1}H\alpha \neq \varnothing \quad \Leftrightarrow \quad H\beta\alpha^{-1} \subseteq Hk^{-1}H.$$

By (3.7.5) and (3.7.6), we deduce that the bijection

$$f : \Omega \to \Gamma/e, \quad H\alpha \mapsto H\alpha$$

satisfies

$$s^f = (s_{k_s^{-1}})_{\Gamma/e}$$

where k_s is as in(3.7.4). We are done. □

3.7.12. Let $\Delta \subseteq \Omega$. Then

(1) $\mathrm{WL}(\mathrm{Inv}(K), 1_\Delta) \le \mathrm{Inv}(K_{\{\Delta\}})$ for any $K \le \mathrm{Sym}(\Omega)$,
(2) $\mathrm{Aut}(\mathrm{WL}(\mathcal{X}, 1_\Delta)) = \mathrm{Aut}(\mathcal{X})_{\{\Delta\}}$.

Proof. By Theorem 2.6.3,

$$\mathrm{Aut}(\mathrm{Inv}(K) \cup \{1_\Delta\})) = \mathrm{Aut}(\mathrm{Inv}(\mathrm{WL}(\mathrm{Inv}(K), 1_\Delta))).$$

Observe that

$$K_{\{\Delta\}} \le \mathrm{Aut}(\mathrm{Inv}(K) \cup \{1_\Delta\}).$$

By Galois Correspondence, it follows that

$$\mathrm{Inv}(K_{\{\Delta\}}) \ge \mathrm{Inv}(\mathrm{Aut}(\mathrm{WL}(\mathrm{Inv}(K), 1_\Delta))) \ge \mathrm{WL}(\mathrm{Inv}(K), 1_\Delta).$$

Statement (1) follows.

Note that

$$\mathcal{X} \le \mathrm{WL}(\mathcal{X}, 1_\Delta).$$

Hence,

$$\mathrm{Aut}(\mathcal{X}) \ge \mathrm{Aut}(\mathrm{WL}(\mathcal{X}, 1_\Delta)).$$

This yields that

$$\mathrm{Aut}(\mathcal{X})_{\{\Delta\}} \ge \mathrm{Aut}(\mathrm{WL}(\mathcal{X}, 1_\Delta)).$$

Now let $K = \mathrm{Aut}(\mathcal{X})$ in (1). Then we obtain

$$\mathrm{WL}(\mathcal{X}, 1_\Delta) \le \mathrm{WL}(\mathrm{Inv}(K), 1_\Delta) \le \mathrm{Inv}(K_{\{\Delta\}}).$$

It then follows that

$$K_{\{\Delta\}} \le \mathrm{Aut}(\mathrm{Inv}(K_{\{\Delta\}})) \le \mathrm{Aut}(\mathrm{WL}(\mathcal{X}, 1_\Delta)).$$

We are done. □

3.7.13. Let S be a set of binary relations on Ω, and let e be an equivalence relation on Ω. Then $\mathrm{WL}(S_{\Omega/e}) \le \mathrm{WL}(S)_{\Omega/e}$.

Proof. Since each $s \in S$ is a union of basis relations in $\mathrm{WL}(S)$, $s_{\Omega/e}$ is a union of basis relations in $\mathrm{WL}(S)_{\Omega/e}$. This implies that

$$\mathrm{WL}(S_{\Omega/e}) \le \mathrm{WL}(S)_{\Omega/e},$$

as desired. □

3.7.14. Let e be a residually thin parabolic of $\mathcal{X}$. Then

(1) $s \cdot s^* \subseteq e$ for any $s \in S$,
(2) $\mathcal{X}_e = \mathrm{WL}(\mathcal{X}, 1_\Delta)$ for any $\Delta \in \Omega/e$.

Proof. To prove statement (1), choose arbitrary pairs

$$(\alpha,\beta)\in s \quad \text{and} \quad (\beta,\gamma)\in s^*.$$

It suffices to show that $(\alpha,\gamma)\in e$. To this end, let

$$\Delta,\quad \Gamma \quad \text{and} \quad \Sigma\in\Omega/e$$

satifying

$$\alpha\in\Delta,\quad \beta\in\Gamma \quad \text{and} \quad \gamma\in\Sigma.$$

Then

$$(\Delta,\Gamma)\in s_{\Omega/e} \quad \text{and} \quad (\Gamma,\Sigma)\in s^*_{\Omega/e}.$$

By the assumption, $\mathcal{X}_{\Omega/e}$ is semiregular, which implies that

$$n_{s_{\Omega/e}}=1=n_{s^*_{\Omega/e}}.$$

Then,

$$\Delta=\Sigma \quad \Rightarrow \quad (\alpha,\gamma)\in\Delta^2\subseteq e,$$

as required.

To prove statement (2), let $\Delta\in\Omega/e$ and $\mathcal{X}'=\mathrm{WL}(\mathcal{X},1_\Delta)$. Observe that

$$1_\Delta=(1_\Omega)_{\Delta,\Delta}\in S_e^{\cup}.$$

This implies that

$$\mathcal{X}'\leq\mathcal{X}_e. \tag{3.7.7}$$

To prove the inverse inclusion, let $s_{\Delta,\Gamma}\in S_e$. Set $\Lambda:=\Omega_+(s)$. Then $\Delta\times\Lambda\in S(\mathcal{X}')^{\cup}$. Since $\mathcal{X}_e$ is semiregular, one can see that

$$s_{\Delta,\Gamma}=s\cap\Delta\times\Gamma=s\cap\Delta\times\Lambda\in S(\mathcal{X}')^{\cup}.$$

Thus, $s_{\Gamma,\Delta}=(s^*_{\Delta,\Gamma})^*\in S(\mathcal{X}')^{\cup}$ for any $s_{\Gamma,\Delta}\in S_e$. It follows that for any $t_{\Gamma,\Gamma'}\in S_e$, there exist $s_{\Gamma,\Delta}$ and $s'_{\Delta,\Gamma'}\in S_e$ such that

$$t_{\Gamma,\Gamma'}=s_{\Gamma,\Delta}s'_{\Delta,\Gamma'},$$

because $\mathcal{X}_e$ is semiregular. It follows that $t_{\Gamma,\Gamma'}\in S(\mathcal{X}')^{\cup}$. Hence, $\mathcal{X}_e\leq\mathcal{X}'$. In view of formula (3.7.7), we are done. □

3.7.15. Let $e\in E$ and $\varphi\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X},\mathcal{X}')$. Then e is residually thin in $\mathcal{X}$ if and only if $e'=\varphi(e)$ is residually thin in $\mathcal{X}'$.

Proof. It is obvious that

$$\varphi^{-1}\in\mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}',\mathcal{X}) \quad \text{and} \quad e=\varphi^{-1}(e').$$

Thus, it suffices to show e' is residually thin in $\mathcal{X}'$. By Exercise 2.7.30, e' is a parabolic of $\mathcal{X}'$. By formula (3.1.10), φ induces an algebraic isomorphism

$$\varphi_{\Omega/e}:\,S_{\Omega/e}\,\rightarrow\,S'_{\Omega'/e'}.$$

Since e is residually thin, $S_{\Omega/e}$ is semiregular. It follows that $S'_{\Omega'/e'}$ is semiregular. Thus, e' is residualy thin in $\mathcal{X}'$, as requried. □

3.7.16. The thin residue of a scheme $\mathcal{X}$ is equal to the minimal parabolic of $\mathcal{X}$ containing $s\cdot s^*$ for any $s\in S$.

Proof. Let
$$T = \bigcup_{s \in S} ss^* \quad \text{and} \quad e = \langle T \rangle.$$
Then we have the following claim.

Claim. For any $r, s \in S$, $r^* \cdot s^* \cdot s \cdot r \subseteq e$.

Proof. For any $t \in sr$, we see that $c_{sr}^t \neq 0$. Thus $c_{r^*s^*}^{t^*} \neq 0$ (formula (2.1.3). This implies that $c_{tr^*}^s \neq 0$ (formula (2.1.9). Hence, $s \in tr^*$. Then,
$$t^* \cdot s \cdot r \subseteq t^* \cdot t \cdot r^* \cdot r \subseteq e.$$
Since this true for any $t \in sr$, the claim is proved. □

Let $s \in S$. Our next goal is to prove that the following claim.

Claim. $s \cdot e \cdot s^* \subseteq e$.

Proof. By Exercise (3.7.1), it suffices to show that for any path $t_1 \cdot \cdots \cdot t_m$ with t_i or $t_i^* \in T$,
$$s \cdot (t_1 \cdot \cdots \cdot t_m) \cdot s^* \subseteq e.$$
There exist $s_1, \ldots, s_m \in S$ such that $t_i \subseteq s_i \cdot s_i^*$. It follows that
$$s \cdot (t_1 \cdot \cdots \cdot t_m) \cdot s^* \subseteq s \cdot (s_1 \cdot s_1^* \cdot \cdots \cdot s_m \cdot s_m^*) s \subseteq (s \cdot s_1 \cdot s_1^* \cdot s^*) \cdot (ss_2 s_2^* s^*) \cdot \cdots \cdot (s^* s_m s_m^* s) \subseteq e,$$
where the last containment follows from the first claim. □

Now let e' is a residually thin parabolic of $\mathcal{X}$, by statement (1) of Exercise 3.7.14, for all $s \in S$,
$$s \cdot s^* \subseteq e'.$$
This implies that $e \subseteq e'$. To complete the proof, it suffices to show that e is a residually thin parabolic,i.e. $n_{s_{\Omega/e}} = 1$ for all $s \in S$. Let $s \in S$. Suppose
$$(\Delta, \Gamma) \quad \text{and} \quad (\Delta, \Gamma') \in s_{\Omega/e},$$
for $\Delta, \Gamma, \Gamma' \in \Omega/e$. Choose $\alpha, \beta, \alpha', \beta' \in \Omega$ such that
$$(\alpha, \beta) \in s_{\Delta \times \Gamma} \quad \text{and} \quad (\alpha', \beta') \in s_{\Delta \times \Gamma'}.$$
Since $(\beta, \alpha) \in s^*$, $(\alpha, \alpha') \in e$, and $(\alpha', \beta') \in s$, one can see that
$$(\beta, \beta') \in s^* \cdot e \cdot s \subseteq e.$$
This implies that $\Gamma = \Gamma'$. We are done. □

3.7.17. [**?**] Let p be a prime. A scheme $\mathcal{X}$ is called a *p-scheme* if $|s|$ is a p-power for each $s \in S$. For such a scheme,

(1) $|\Omega|$ is a p-power,
(2) the thin radical of $\mathcal{X}$ is not equal to 1_Ω unless $|\Omega| = 1$,
(3) if $|\Omega| = p$, then $\mathcal{X}$ is regular,
(4) any quotient of $\mathcal{X}$ is a p-scheme,
(5) the thin residue of $\mathcal{X}$ is not equal to Ω^2 unless $|\Omega| = 1$.

Proof. Since $\mathcal{X}$ is a shceme, $1_\Omega \in S$. Thus,
$$|\Omega| = |1_\Omega|$$
is a p-power by the assumption. Statement (1) follows.

To prove statement (2), assume that $|\Omega| \neq 1$. For each $s \in S$, since by formula (2.1.11)
$$|s| = n_s |\Omega|$$

is a p-power, n_s is a p-power. Observe that, by formula (2.1.13)

$$|\Omega| = \sum_{s\in S} n_s,$$

which is a p-power. As $n_{1_\Omega} = 1$, there exists at one $s \in S^{\#}$ such that $n_s = 1$. This s is contained in the thin radical of $\mathcal{X}$.

To prove statement (3), note that

$$p = |\Omega| = \sum_{s\in S} n_s$$

and each n_s is a p-power. Thus, for any $s \in S$, $n_s = 1$, i.e., $\mathcal{X}$ is regular, as desired.

To prove statement (4), let $e \in E$ and $s_{\Omega/e} \in S_{\Omega/e}$. Observe that $(\Delta, \Gamma) \in s_{\Omega/e}$ if and only if $s_{\Delta\times\Gamma} \neq \varnothing$. It follows that

$$s = \bigcup_{(\Delta,\Gamma)\in s_{\Omega/e}} s_{\Delta,\Gamma},$$

is a disjoint union. However, the cardinality $|s_{\Delta,\Gamma}|$ does not depend on the choice of $(\Delta, \Gamma) \in s_{\Omega/e}$ (Proposition 2.1.18). Thus, $|s_{\Omega/e}|$ is a divisor of $|s|$. Since $|s|$ is a p-power, $|s_{\Omega/e}|$ is also a p-power. We are done.

□

3.7.18. [**?**] Any quasiregular coherent configuration $\mathcal{X}$ with all non-singleton fibers of the same prime cardinality is the direct sum of semiregular coherent configurations. In particular, $\mathcal{X}$ is schurian and separable.

Proof. By Theorem 3.2.2,

$$\mathcal{X} = \boxplus_{i=1}^{m} \mathcal{X}_{\Omega_i},$$

where $\Omega_1, \ldots, \Omega_m$ are homogeneuity sets of $\mathcal{X}$ such that

$$\Delta, \Gamma \in F \quad \text{and} \quad |S_{\Delta,\Gamma}| > 1 \quad \Leftrightarrow \quad \Delta, \Gamma \in \Omega_i, \text{ for some } 1 \le i \le m.$$

To complete the proof, it suffices to prove that for any i

$$\Delta, \Gamma \in \Omega_i \quad \text{and} \quad s \in S_{\Delta,\Gamma} \quad \Rightarrow \quad n_s = 1. \tag{3.7.8}$$

Observe that if $\Delta \in F$ with $|\Delta| = 1$, then for any $\Gamma \in F$,

$$|S_{\Delta,\Gamma}| = |S_{\Gamma,\Delta}| = 1.$$

Hence, each singleton fiber consists of some Ω_i and in this case statement (3.7.8) holds.

Now assume that Δ, Γ and s statisfy the assumption on the left-hand side in (3.7.8) with

$$|\Delta| = |\Gamma| = p,$$

where p is a prime. By formula (2.1.5)

$$|s| = n_s|\Delta| = pn_s = |s^*| = pn_{s^*}. \tag{3.7.9}$$

Since $\mathcal{X}$ is quasiregular, S_Δ can be seen as a group, with order p. Furthermore,

$$S_\Delta \times S_{\Delta,\Gamma} \quad \to \quad S_{\Delta,\Gamma}, \quad (r,\, s) \mapsto r \cdot s \tag{3.7.10}$$

defines an action of S_Δ on $S_{\Delta,\Gamma}$. We claim that there exist

$$r \neq 1_\Delta \quad \text{and} \quad s \in S_{\Delta,\Gamma}$$

such that

$$r \cdot s \neq s. \tag{3.7.11}$$

Observe for each $s \in S_{\Delta,\Gamma}$, by (3.7.9)

$$|\Delta \times \Gamma| = p^2 \quad \text{and} \quad |S_{\Delta,\Gamma}| > 1 \quad \Rightarrow \quad n_{s^*} < p.$$

Hence, there exist $\alpha, \alpha' \in \Delta$ and $\beta \in \Gamma$ such that

$$(\alpha, \beta) \in s \quad \text{and} \quad (\alpha', \beta) \notin s.$$

Then

$$r := r(\alpha', \alpha) \in S_\Delta \quad \text{and} \quad r \cdot s \neq s.$$

It follows that the orbit $\mathcal{O}$ of s under the action in (3.7.10) has size p since S_Δ has order p. Then

$$\sum_{t \in \mathcal{O}} |t| = p(pn_s) \leq |\Delta \times \Gamma| = p^2,$$

where the first equality holds as each $t \in \mathcal{O}$ is such that $n_t = n_s$. Thus,

$$\mathcal{O} = S_{\Delta,\Gamma} \quad \text{and} \quad n_t = 1, \forall\, t \in S_{\Delta,\Gamma}.$$

Hence, statement (3.7.8) follows as required. □

3.7.19. [**?**] A coherent configuration $\mathcal{X}$ is said to be *quasitrivial* if

$$\mathrm{Aut}(\mathcal{X})^\Delta = \mathrm{Sym}(\Delta) \quad \text{for all } \Delta \in F,$$

and *semitrivial* if, in addition, the group $\mathrm{Aut}(\mathcal{X})^{\Delta \cup \Gamma}$ is isomorphic to both $\mathrm{Sym}(\Delta)$ and $\mathrm{Sym}(\Gamma)$ for all $\Delta, \Gamma \in F$. Prove that every quasitrivial coherent configuration is the direct sum of semitrivial coherent configurations.

3.7.20. Any coherent configuration with all fibers of cardinality at most 3 is the direct sum of the coherent configurations isomorphic to $\mathcal{Y} \otimes \mathcal{D}_{m_\mathcal{Y}}$, where $\mathcal{Y}$ is a scheme of degree at most 3 and $m_\mathcal{Y} \geq 1$. In particular, $\mathcal{X}$ is schurian and separable.

3.7.21. Let $\mathcal{X}$ be a commutative subtensor product on $\Omega = \Omega_1 \times \Omega_2$, and let e_1 and e_2 be the parabolics of $\mathcal{X}$ defined by formula (**??**). Then

(1) for each $\Delta \in \Omega/e_1$, the mapping $\tau_\Delta : \Delta \to \Omega/e_2$, $\alpha \mapsto \alpha e_2$ is a bijection,
(2) $\tau_\Delta \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_{\Omega/e_2})$ and also $(s_\Delta)^{\tau_\Delta} = s_{\Omega/e_2}$ for all $s \in S$,
(3) if $\Gamma \in \Omega/e_1$, then $\tau_\Delta \tau_\Gamma^{-1} \in \mathrm{Iso}(\mathcal{X}_\Delta, \mathcal{X}_\Gamma, \varphi_{\Delta,\Gamma})$ (for $\varphi_{\Delta,\Gamma}$, see Example **??**).

Proof. For (1), there exists $\alpha_1 \in \Omega_1$ such that

$$\Delta = \{(\alpha_1, \alpha_2) :\ \alpha_2 \in \Omega_2\}.$$

For $\alpha = (\alpha_1, \alpha_2)$, one can see that

$$\tau_\Delta(\alpha) = \alpha e_2 = \{(\beta, \alpha_2) :\ \beta \in \Omega_1\}.$$

Observe that when α runs over Δ, α_2 will runs over Ω_2. Thus, τ_Δ is surjective. It is also straightforward that τ_Δ is injective.

□

3.7.22. Let $\mathcal{X}$ be a Cayley scheme over a group G. Then the following two statements are equivalent:

(1) $\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_k$ for some $k \geq 1$,
(2) $\mathrm{rk}(\mathcal{X}) = \mathrm{rk}(\mathcal{X}_1) \cdot \ldots \cdot \mathrm{rk}(\mathcal{X}_k)$and $G = G_1 \times \cdots \times G_k$, where G_i is an $\mathcal{X}$-group such that $\mathcal{X}_{G_i} = \mathcal{X}_i$.

Moreover, if one of these statements holds, then $\mathcal{X}$ is normal if and only if $\mathcal{X}_i$ is a normal Cayley scheme over G_i for all i.

Proof. (1) $\Rightarrow$ (2) Observe that

$$F(\mathcal{X}) = \{G\} = \{\Delta_1 \times \ldots \times \Delta_k : \Delta_i \in F(\mathcal{X}_i),\, 1 \le i \le k\}.$$

Hence, each $\Delta_i = G_i$ and G_i is a group with

$$G = G_1 \times \ldots \times G_k.$$

Obviously, $\mathcal{X}_{G_i} = \mathcal{X}_i$. Also,

$$\rho(G_i) = \bigcup_{s_i \in S(\mathcal{X}_i)} 1_{G_1} \otimes \ldots \otimes s_i \otimes \ldots \otimes 1_{G_k},$$

which is a partial parabolic of $\mathcal{X}$. Thus, each G_i is an $\mathcal{X}$-group.

(2) $\Rightarrow$ (1) For any $s \in S$, there exists $X \in \mathcal{S}(\mathfrak{A})$ such that $s = X^{\rho}$ (Theorem 2.4.17).

□

3.7.23. The extension of trivial coherent configuration $\mathcal{T}_\Omega$ with respect to the points of a set $\Delta \subseteq \Omega$, is equal to $\mathcal{D}_\Delta \boxplus \mathcal{T}_{\Omega\setminus\Delta}$.

Proof. Denote the extension under consideration by $\mathcal{X}$. Note that in

$$\mathcal{Y} := \mathcal{D}_\Delta \boxplus \mathcal{T}_{\Omega\setminus\Delta},$$

for any $\delta \in \Delta$, $\{\delta\}$ is a fiber of $\mathcal{Y}$. Thus

$$\mathcal{X} \le \mathcal{Y}.$$

By Theorem 3.2.3, as Δ is a homogeneuity set of $\mathcal{X}$,

$$\mathcal{X} \ge \mathcal{X}_\Delta \boxplus \mathcal{X}_{\Omega\setminus\Delta}.$$

Since obviously

$$\mathcal{X}_\Delta = \mathcal{D}_\Delta \quad \text{and} \quad \mathcal{X}_{\Omega\setminus\Delta} \ge \mathcal{T}_{\Omega\setminus\Delta},$$

we obtain

$$\mathcal{X} \ge \mathcal{Y}.$$

The proof is complete. □

3.7.24. Let $\alpha \in \Omega$. Assume that for every $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, there exists $\alpha' \in \Omega'$ and $\varphi_{\alpha,\alpha'} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'})$ extending φ. Then $\mathcal{X}$ is separable if so is $\mathcal{X}_\alpha$.

Proof. Choose an arbitary $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$. Let $\varphi_{\alpha,\alpha'} \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'})$ extending φ. Assume that $\mathcal{X}_\alpha$ is separable. Then there exists an isomorphism f such that

$$f \in \mathrm{Iso}(\mathcal{X}_\alpha, \mathcal{X}'_{\alpha'}, \varphi_{\alpha,\alpha'}).$$

For any $s \in S$, denote s by $s_1 \cup \ldots \cup s_m$ with $s_i \in S(\mathcal{X}_\alpha)$. Since $\varphi_{\alpha,\alpha'}$ extending φ,

$$\varphi(s) = \varphi_{\alpha,\alpha'}(\cup_{i=1}^m s_i) = \cup_{i=1}^m \varphi_{\alpha,\alpha'}(s_i) = \cup_{i=1}^m s_i^f = s^f.$$

This implies that $f \in \mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ and hence $\mathcal{X}$ is separable.

□

3.7.25. Let $\alpha \in \Omega$ and

$$T_\alpha = \{r_{u,v} : \ r \in S, \ u, v \in S \setminus S_1\}^\natural,$$

where $r_{u,v} = r \cap (\alpha u \times \alpha v)$. Then the pair

$$\mathcal{X}_\alpha^\perp = (\alpha S_{1'}, T_\alpha)$$

with $S_{1'} = \{s \in S : \ n_s > 1\}$, is a rainbow and

$$\mathcal{X}_\alpha = \mathcal{D}_{\alpha S_1} \boxplus \mathrm{WL}(\mathcal{X}_\alpha^\perp).$$

Proof. Observe that $\Delta := \alpha S_{1'}$ is a homogeneity set of $\mathcal{X}_\alpha$ (statement (1) of Lemma 3.3.5). Our first goal is to show that $\mathcal{X}_\alpha^\perp$ is a rainbow.

For any $\beta, \gamma \in \alpha S_{1'}$, there exist $u, v \in S_{1'}$ and $r \in S$ such that

$$\beta \in \alpha u, \quad \gamma \in \alpha v, \quad \text{and} \quad (\beta, \gamma) \in r. \tag{3.7.12}$$

This implies that $(\beta, \gamma) \in r_{u,v}$. Hence, T_α is a partition of Δ^2.

For any $\beta \in \Delta$, suppose $\beta \in \alpha u$ for some $u \in S_{1'}$. Then $(\beta, \beta) \in t_{u,u}$ where $t \in S$ is the basis relation containing (β, β). We conclude that $1_\Delta \in T_\alpha^\cup$. Hence, $\mathcal{X}_\alpha^\perp$ satisfies the condtion (CC1).

Obviously, for any $r_{u,v} \in T_\alpha$, $(r_{u,v})^* = r^*_{v,u} \in T_\alpha$. Thus, $\mathcal{X}_\alpha^\perp$ satisfies the condtion (CC2).

For any $s \in S_1$, if $\alpha s \neq \varnothing$ then $\alpha s = \{\beta\}$ and $\{\beta\}$ is a fiber of $\mathcal{X}_\alpha$ (Lemma 3.3.5). It follows that $\Gamma := \alpha S_1$ is a homogeneity set of $\mathcal{X}_\alpha$ and each fiber of $(\mathcal{X}_\alpha)_\Gamma$ is a singleton set. Thus,

$$(\mathcal{X}_\alpha)_\Gamma = \mathcal{D}_\Gamma.$$

By statement (2) of Lemma 3.3.5,

$$T_\alpha \subseteq S((\mathcal{X}_\alpha)_\Delta)^\cup.$$

This yields that

$$\mathrm{WL}(\mathcal{X}_\alpha^\perp) \leq (\mathcal{X}_\alpha)_\Delta.$$

Thus,

$$\mathcal{D}_\Gamma \boxplus \mathrm{WL}(\mathcal{X}_\alpha^\perp) \leq \mathcal{D}_\Gamma \boxplus (\mathcal{X}_\alpha)_\Delta = \mathcal{X}_\alpha. \tag{3.7.13}$$

To prove the inverse inclusion, since obviously $\{\alpha\}$ is a fiber of the coherent configuration on the left-hand side in (3.7.13), it suffices to show that

$$\mathcal{X} \leq \mathcal{D}_\Gamma \boxplus \mathrm{WL}(\mathcal{X}_\alpha^\perp).$$

To this end, we claim that for each $s \in S$ and any pair in s there exists a relation of the coherent configuration on the left-hand side in (3.7.13) which is contained in s and contains this pair.

Now let $s \in S$ and $(\beta, \gamma) \in s$, if $\beta, \gamma \in \alpha S_{1'}$ then $(\beta, \gamma) \in s_{u,v}$ for some $u, v \in S_{1'}$ and $s_{u,v} \subseteq s$. If $\beta, \gamma \in \alpha S_1$, then $(\beta, \gamma) \in s$. If $\beta \in \alpha r$ and $\gamma \in \alpha t$ with $r \in S_1$ and $t \in S_{1'}$, then $(\beta, \gamma) \in r^* t$. Since r is thin, $r^* t$ is a basis relation. Hence, $s = r^* t$. Set $\Lambda := \Omega_+(s)$. Then $\Lambda = \Omega_+(t)$. It follows that

$$(\beta, \gamma) \in \{\beta\} \times \Lambda \subseteq s.$$

If $\beta \in \alpha S_{1'}$ and $\gamma \in \alpha S_1$, the claim can be proved similarly. We are done. □

3.7.26. [**?**] Any primitive scheme admitting a one point extension with exactly one non-singleton fiber, is trivial.

3.7.27. Let $\alpha \in \Omega$ and Δ a base of $\mathcal{X}_\alpha$. Then $\{\alpha\} \cup \Delta$ is a base of $\mathcal{X}$. In particular,

$$b(\mathcal{X}) \le 1 + b(\mathcal{X}_\alpha).$$

If α belongs to a minimal base Δ with rank $b(\mathcal{X})$ of $\mathcal{X}$, then the equality occures.

Proof. Denote Δ by $\{\beta, \dots, \tau\}$. Note that

$$\mathcal{X}_{\alpha,\beta,\dots,\tau} = (\mathcal{X}_\alpha)_{\beta,\dots,\tau}.$$

This is the trivial coherent configuration since Δ is a base of $\mathcal{X}_\alpha$. We are done. □

3.7.28. The class of all partly regular coherent configurations is closed with respect to taking fissions and tensor products.

Proof. Let $\mathcal{X}_1$ be a partly coherent configuration on Ω_1 and $\mathcal{X}_1'$ be a fission of $\mathcal{X}_1$. By definition of partly coherent configuration, there exists a point $\alpha_1 \in \Omega_1$ such that

$$|\alpha_1 s_1| \le 1 \quad \text{for} \quad \text{all} \quad s_1 \in S(\mathcal{X}_1).$$

Taking into account that for any $s_1' \in S(\mathcal{X}_1')$ there exists $s_1 \in S(\mathcal{X}_1)$ such that $s_1' \subseteq s_1$,

$$|\alpha_1 s_1'| \le |\alpha_1 s_1| \le 1.$$

This yields that $\mathcal{X}_1'$ is partly regular.

Suppose further that $\mathcal{X}_2$ is partly regular coherent configuration on Ω_2. Then there exists α_2 such that $|\alpha_2 s_2| \le 1$ for all $s_2 \in S(\mathcal{X}_2)$. It follows that for any $s_1 \otimes s_2 \in S(\mathcal{X}_1 \otimes \mathcal{X}_2)$, where $s_i \in S(\mathcal{X}_i)$, $i = 1, 2$,

$$|(\alpha_1, \alpha_2)(s_1 \otimes s_2)| = |\alpha_1 s_1||\alpha_2 s_2| \le 1.$$

This implies that $\mathcal{X}_1 \otimes \mathcal{X}_2$ is also partly regular, as desired.

□

3.7.29. Let Ω_1 and Ω_2 be sets. Then the only proper fusion of the wreath product $\mathcal{T}_{\Omega_1} \wr \mathcal{T}_{\Omega_2}$ is the trivial scheme $\mathcal{T}_{\Omega_1 \times \Omega_2}$.

Proof. Observe that

$$\mathrm{rk}(\mathcal{T}_{\Omega_1} \wr \mathcal{T}_{\Omega_2}) = \mathrm{rk}(\mathcal{T}_{\Omega_1}) + \mathrm{rk}(\mathcal{T}_{\Omega_2}) - 1 = 3.$$

Thus, the proper fusion of the wreath product of these two trivial schemes should have rank 2 and hence must be the trivial scheme on $\Omega_1 \times \Omega_2$. □

3.7.30. Let $\mathcal{X}$ be a scheme and $\mathcal{Y} = \mathrm{Inv}(K, \Delta)$, where Δ is a set and $K \le \mathrm{Sym}(\Delta)$ is a transitive group. Then K acts as a group of isomorphisms of the direct sum $\mathcal{X}'$ of $|\Delta|$ copies of $\mathcal{X}$, and $\mathcal{X} \wr \mathcal{Y} \cong (\mathcal{X}')^K$.

Proof. Let $\mathcal{X}$ be a scheme on Ω. For each $\delta \in \Delta$, there is a bijection f_δ : $\Omega \to \Omega_\delta$. Set $\mathcal{X}_\delta := \mathcal{X}^{f_\delta}$. For any $s \in S$, set $s_\delta := s^{f_\delta}$. Then

$$\mathcal{X}' = \boxplus_{\delta \in \Delta} \mathcal{X}_\delta.$$

We then have the following bijection

$$f : \Omega \times \Delta \quad \to \quad \bigsqcup_{\delta \in \Delta} \Omega_\delta, \quad (\alpha, \delta) \mapsto \alpha^{f_\delta}.$$

To complete the proof, it suffices to show that

$$S(\mathcal{X} \wr \mathcal{Y})^f = S((\mathcal{X}')^K).$$

Let $r \in S(\mathcal{X} \wr \mathcal{Y})$. If $r = s \otimes 1_\Delta$ where $s \in S(\mathcal{X})$, then

$$\begin{aligned} r^f =& \{((\alpha,\delta)^f, (\beta,\delta)^f) : (\alpha,\beta) \in s, \delta \in \Delta\} \\ =& \{(\alpha^{f_\delta}, \beta^{f_\delta}) : (\alpha,\beta) \in s, \delta \in \Delta\} \\ =& \bigcup_{\delta \in \Delta} s_\delta. \end{aligned}$$

Observe that for each $\delta \in \Delta$, $(\mathcal{X}')^K$ has a basis relation s_δ^K. Since K is transitive on Δ,

$$(s_\delta)^K = \bigcup_{k \in K} s_{\delta^k} = \bigcup_{\gamma \in \Delta} s_\gamma = r^f.$$

Thus, $r^f \in S((\mathcal{X}')^K)$.

If $r = \Omega^2 \otimes t$, where $t \in S(\mathcal{Y})^\#$, then $t = (\delta, \delta')^K$ for some $(\delta, \delta') \in \Delta^2$ with $\delta \neq \delta'$. In addition,

$$\begin{aligned} r^f =& \{((\alpha,\delta)^f, (\beta,\delta')^f) : \alpha,\beta \in \Omega, (\gamma,\gamma') \in t\} \\ =& \{(\alpha^{f_\gamma}, \beta^{f_{\gamma'}}) : \alpha,\beta \in \Omega, (\gamma,\gamma') \in t\} \\ =& \bigcup_{k \in K} \Omega_{\delta^k} \times \Omega_{\delta'^k} \\ =& (\Omega_\delta \times \Omega_{\delta'})^K. \end{aligned}$$

Here $\Omega_\delta \times \Omega_{\delta'}$ is a basis relation of $\mathcal{X}'$ and hence $r^f \in S((\mathcal{X}')^K)$. We conclude that f induces an injective map from $S(\mathcal{X} \wr \mathcal{Y})$ to $S((\mathcal{X}')^K)$. Since obviously this map is surjective, we are done. □

3.7.31. Let $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$ be schemes and Φ a family of the algebraic isomorphisms

$$\varphi_\alpha \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_{1\alpha}), \quad \alpha \in \Omega_2,$$

where $\mathcal{X}_{1\alpha}$ is a scheme on the set $\Omega_\alpha = \Omega_1 \times \{\alpha\}$. Define a rainbow $\mathcal{X}$ on the set $\Omega = \Omega_1 \times \Omega_2$ with $S(\mathcal{X}) = S^{(1)} \cup S^{(2)}$, where

$$S^{(1)} = \{ \bigcup_{\alpha \in \Omega_2} \varphi_\alpha(s_1) : \; s_1 \in S_1\} \quad \text{and} \quad S^{(2)} = \{ \bigcup_{(\alpha,\beta) \in s_2} \Omega_\alpha \times \Omega_\beta : \; s_2 \in S_2^\#\}.$$

Then $\mathcal{X}$ is a scheme, called the *wreath product of* $\mathcal{X}_1$ *by* $\mathcal{X}_2$ *with respect to the family* Φ; it is denoted by $\mathcal{X}_1 \wr_\Phi \mathcal{X}_2$. Moreover,

(1) the equivalence relation e with classes Ω_α, $\alpha \in \Omega_2$, is an indecomposable parabolic of $\mathcal{X}$,
(2) if for each α, the algebraic isomorphism φ_α is induced by the bijection $\beta \mapsto (\beta, \alpha)$, $\beta \in \Omega_1$, then $\mathcal{X} = \mathcal{X}_1 \wr \mathcal{X}_2$,
(3) $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$ is isomorphic to $\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1) \times \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2)$.

Proof. For each $s_1 \in S_1$ and $s_2 \in S_2^\#$, denote

$$\tilde{s}_1 := \bigcup_{\alpha \in \Omega_2} \varphi_\alpha(s_1) \quad \text{and} \quad \tilde{s}_2 := \bigcup_{(\alpha,\beta) \in s_2} \Omega_\alpha \times \Omega_\beta.$$

By (3) of Proposition 2.3.18, each algebraic isomorphism φ_α maps reflexive relations to relexive ones. Thus, $\varphi_\alpha(1_{\Omega_1}) = 1_{\Omega_\alpha}$. This implies that

$$\widetilde{1_{\Omega_1}} = 1_\Omega \in S(\mathcal{X}).$$

To prove that $\mathcal{X}$ is a scheme, since it is obviously a rainbow, it suffices to prove that condition (CC3) holds. It can be easily computed that

$$c_{\tilde{r}_1\tilde{s}_1}^{\tilde{t}_1} = c_{r_1s_1}^{t_1}, \quad r_1, s_1, t_1 \in S_1; \quad c_{\tilde{r}_2\tilde{s}_2}^{\tilde{t}_2} = |\Omega_1| c_{r_2s_2}^{t_2}, \quad r_2, s_2, t_2 \in S_2^{\#}.$$

And,

$$c_{\tilde{r}_2\tilde{r}_2^*}^{1_\Omega} = |\Omega_1| n_{r_2}, \quad r_2 \in S_2^{\#}; \quad c_{\tilde{s}_1\tilde{s}_2}^{\tilde{s}_2} = c_{\tilde{s}_2\tilde{s}_1}^{\tilde{s}_2} = n_{s_1}, \quad s_1 \in S_1, s_2 \in S_2^{\#}.$$

Other types of intersection numbers are zero.

To prove statement (1), note that e is the union of all basis relations contained in $S^{(1)}$. Hence e is a parabolic. Since $\mathcal{X}$ is a scheme, e is indecomposable by Proposition 2.1.24.

To prove statement (2), suppose each φ_α has the form in the assumption. Then,

$$\tilde{s}_1 = s_1 \otimes 1_{\Omega_2},\ s_1 \in S_1 \quad \text{and} \quad \tilde{s}_2 = \Omega_1^2 \otimes s_2,\ s_2 \in S_2^{\#}.$$

Therefore, $S(\mathcal{X}) = S(\mathcal{X}_1 \wr \mathcal{X}_2)$, as wanted.

To prove statement (3), note that there is a bijection

$$\pi :\ \Omega/e \to \Omega_2,\ \Omega_\alpha \mapsto \alpha.$$

Furthermore,

$$(\mathcal{X}_{\Omega/e})^\pi = \mathcal{X}_2.$$

Also, for any $\Omega_\alpha \in \Omega/e$,

$$\mathcal{X}_{\Omega_\alpha} = \mathcal{X}_{1\alpha}.$$

Now let $\psi \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X})$. Then ψ induces an algebraic isomorphism $\psi_{\Omega/e}$ of $\mathcal{X}_{\Omega/e}$. Obviously,

$$\varphi_2 := \pi\psi\pi^{-1} \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2).$$

Moreover, by Exercise (2.7.31), for each $\Omega_\alpha \in \Omega/e$,

$$\psi_{1\alpha} : \mathcal{X}_{\Omega_\alpha} \to \mathcal{X}_{\Omega_\alpha}, \quad s_{\Omega_\alpha} \mapsto \psi(s)_{\Omega_\alpha}$$

is an algebraic isomorphism. Fix $\alpha \in \Omega_2$. Then,

$$\psi_1 := \varphi_\alpha^{-1}\psi_{1\alpha}\varphi_\alpha \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1).$$

Therefore, we obtain the following group monomorphism

$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}) \to \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1) \times \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2), \quad \psi \mapsto (\psi_1, \psi_2). \tag{3.7.14}$$

Conversely, let $\psi_i \in \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_i), i = 1, 2$. Then

$$\psi(\tilde{s}_1) = \widetilde{\psi_1(s_1)},\ s_1 \in S_1 \quad \text{and} \quad \psi(\tilde{s}_2)) = \widetilde{\psi_2(s_2)},\ s_2 \in S_2^{\#}$$

defines an algebraic isomorphism ψ of $\mathcal{X}$. It is straightforward to see that the image of ψ with respect to the mapping (3.7.14) is (ψ_1, ψ_2). As a consequence, the mapping (3.7.14) is a group homomorphism.

□

3.7.32. Let $\mathcal{X}$ be a scheme on $\Omega_1 \times \Omega_2$, and let e be the equivalence relation with classes $\Omega_\alpha = \Omega_1 \times \{\alpha\}$, $\alpha \in \Omega_2$. Assume that e is an indecomposable parabolic of $\mathcal{X}$. Take an arbitrary $\alpha \in \Omega_2$ and set

$$\Phi = \{\varphi_{\Omega_\alpha,\Omega_\beta} :\ \beta \in \Omega_2\},$$

where $\varphi_{\Omega_\alpha,\Omega_\beta}$ is the algebraic isomorphism defined in Example 2.3.16 . Then $\mathcal{X}$ is a fission of the scheme $\mathcal{X}_1 \wr_\Phi \mathcal{X}_2$, where $\mathcal{X}_1 = \mathcal{X}_{\Omega_\alpha}$ and $\mathcal{X}_2 = \mathcal{X}_{\Omega/e}$.

Proof. Denote $\mathcal{X}_1 \wr_\Phi \mathcal{X}_2$ by $\mathcal{X}'$. For $s \in S$, if $s \subseteq e$, then

$$s_1 = \bigcup_{\beta \in \Omega_2} (s_1)_{\Omega_\beta} = \bigcup_{\beta \in \Omega_2} \varphi_{\Omega_\alpha, \Omega_\beta}(s_1) \in S(\mathcal{X}').$$

If $s \nsubseteq e$, then $s_{\Omega_\alpha} = \varnothing$ for each $\alpha \in \Omega_2$. It follows that

$$s \subseteq \bigcup_{(\alpha,\beta) \in s} \Omega_\alpha \times \Omega_\beta \in S(\mathcal{X}').$$

We conclude that $\mathcal{X}$ is a fission of $\mathcal{X}'$. □

3.7.33. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations on Ω_1 and Ω_2, respectively, and let $\square$ denote $\boxplus$ or $\otimes$ or $\wr$; in the latter case, $\mathcal{X}_1$ and $\mathcal{X}_2$ are schemes. Then

(1) for any $\varphi_1 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1, \mathcal{X}_1')$ and $\varphi_2 \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_2, \mathcal{X}_2')$, there exists a unique

$$\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_1 \square \mathcal{X}_2, \mathcal{X}_1' \square \mathcal{X}_2')$$

such that $\varphi_{\Omega_1} = \varphi_1$ and $\varphi_{\Omega_2} = \varphi_2$,

(2) the inclusion

$$\mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1 \square \mathcal{X}_2) \geq \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_1) \times \mathrm{Aut}_{\mathrm{alg}}(\mathcal{X}_2)$$

holds with equality attained if $\mathcal{X}_1$ and $\mathcal{X}_2$ are not algebraically isomorphic,[7]

(3) for any $e_1 \in E(\mathcal{X}_1)$,

$$(\mathcal{X}_1 \square \mathcal{X}_2)_{\Omega/e} = (\mathcal{X}_1)_{\Omega_1/e_1} \square \mathcal{X}_2,$$

where $e = e_1$ if $\square = \boxplus$, and $e = e_1 \otimes 1_{\Omega_2}$ otherwise.

Proof. For statement (1), since the algebraic isomorphisms φ_i, induce bijections from $F(\mathcal{X}_i)$ to $F(\mathcal{X}_i')$, $i = 1, 2$, we see that

$$s_i \mapsto \varphi(s_i), s_i \in S(\mathcal{X}_i) \quad \text{and} \quad \Delta_1 \times \Delta_2 \mapsto \Delta_1^{\varphi_1} \times \Delta_2^{\varphi_2}$$

generates an algebraic isomorphism φ from $\mathcal{X}_1 \boxplus \mathcal{X}_2$ to $\mathcal{X}_1' \boxplus \mathcal{X}_2'$ such that $\varphi_{\Omega_i} = \varphi_i$, $i = 1, 2$.

Next,

$$s_1 \otimes s_2 \mapsto \varphi_1(s_1) \otimes \varphi_2(s_2), \quad s_1 \otimes s_2 \in S(\mathcal{X}_1 \otimes \mathcal{X}_2)$$

produces an algebraic isomorphism φ from $\mathcal{X}_1 \otimes \mathcal{X}_2$.

□

3.7.34. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be coherent configurations. Then

(1) $b(\mathcal{X}_1 \boxplus \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2)$,

(2) $b(\mathcal{X}_1 \otimes \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2) - 1$ unless $\min\{b(\mathcal{X}_1), b(\mathcal{X}_2)\} = 0$; in the latter case, $b(\mathcal{X}_1 \otimes \mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2)$,

(3) if $\mathcal{X}_1$ and $\mathcal{X}_2$ are schemes, then $b(\mathcal{X}_1 \wr \mathcal{X}_2) = |\Omega_2|\, b(\mathcal{X}_1)$.

Proof. To prove statement (1), note that if Δ_i is a subset of Ω_i, $i = 1, 2$, then

$$\mathrm{WL}(\mathcal{X}_1 \boxplus \mathcal{X}_2,\, \{1_\alpha : \alpha \in \Delta_1 \cup \Delta_2\})_{\Omega_i} = \mathrm{WL}(\mathcal{X}_i,\, \{1_\alpha : \alpha \in \Delta_i\}). \tag{3.7.15}$$

If assume further that Δ_i is a base of $\mathcal{X}_i$, then the right-hand side equals $\mathcal{D}_{\Omega_i}$. Thus, in this case

$$\mathrm{WL}(\mathcal{X}_1 \boxplus \mathcal{X}_2,\, \{1_\alpha : \alpha \in \Delta_1 \cup \Delta_2\}) \geq \mathcal{D}_{\Omega_1} \boxplus \mathcal{D}_{\Omega_2} = \mathcal{D}_{\Omega_1 \cup \Omega_2}.$$

[7]For $\square = \wr$, this condition is superfluous.

This yields that $\Delta_1 \cup \Delta_2$ is a base of $\mathcal{X}_1 \boxplus \mathcal{X}_2$. Therefore,

$$b(\mathcal{X}_1 \boxplus \mathcal{X}_2) \leq b(\mathcal{X}_1) + b(\mathcal{X}_2). \tag{3.7.16}$$

Now let $\Delta = \Delta_1 \cup \Delta_2$ be a base for $\mathcal{X}_1 \boxplus \mathcal{X}_2$, where $\Delta_i \subseteq \Omega_i$, $i = 1, 2$. By formula (3.7.15), we see that

$$\mathrm{WL}(\mathcal{X}_i, \{1_\alpha : \alpha \in \Delta_i\}) = \mathcal{D}_{\Omega_i}, \quad i = 1, 2$$

since the left-hand side is the discrete coherent configuration in this case. It follows that Δ_i is a base of $\mathcal{X}_i$. Hence, the inequality of converse direction in (3.7.16) holds.

To prove statement (2), denote $\Omega := \Omega_1 \times \Omega_2$ and $\mathcal{X} := \mathcal{X}_1 \otimes \mathcal{X}_2$. For $\alpha \in \Omega$, the first and the second coordinates of α are denoted by α_1 and α_2. Let

$$e_1 = \{(\alpha, \beta) \in \Omega^2 : \alpha_1 = \beta_1\} \quad \text{and} \quad e_2 = \{(\alpha, \beta) \in \Omega^2 : \alpha_2 = \beta_2\}.$$

Also, let

$$f_1 : \Omega/e_1 \to \Omega_1,\ \{\alpha_1\} \times \Omega_2 \mapsto \alpha_1 \quad \text{and} \quad f_2 : \Omega/e_2 \to \Omega_2,\ \Omega_1 \times \{\alpha_2\} \mapsto \alpha_2.$$

Observe that e_1 and e_2 are parabolics of $\mathcal{X}$ and that

$$(\mathcal{X}_{\Omega/e_1})^{f_1} = \mathcal{X}_1 \quad \text{and} \quad (\mathcal{X}_{\Omega/e_2})^{f_2} = \mathcal{X}_2.$$

For $(\alpha_1, \alpha_2) \in \Omega$, we claim that

$$\mathcal{X}_{(\alpha_1, \alpha_2)} = (\mathcal{X}_1)_{\alpha_1} \otimes (\mathcal{X}_2)_{\alpha_2}. \tag{3.7.17}$$

On one hand, by definition, it is easily seen that the left-hand side is contained in the right-hand side. On the other hand, observe that

$$((\mathcal{X}_{(\alpha_1, \alpha_2)})_{\Omega_i/e_i})^{f_i} = (\mathcal{X}_i)_{\alpha_i},\ i = 1, 2.$$

This yields that the right-hand side of (3.7.17) is contained in the left-hand side. Thus, formula (3.7.17) holds.

Assume first that $\min\{b(\mathcal{X}_1), b(\mathcal{X}_2)\} = 0$. Without loss of generality, we assume that $b(\mathcal{X}_1) = 0$. Denote $b(\mathcal{X}_1 \otimes \mathcal{X}_2)$ by m. For a fixed $\alpha_1 \in \Omega_1$, by repeated applying formula (3.7.17), we see that $\beta_1, \ldots, \beta_m$ is a base of $\mathcal{X}_2$. if and only if $(\alpha_1, \beta_1), (\alpha_1, \beta_2), \ldots, (\alpha_1, \beta_m)$ is a base of $\mathcal{X}_1 \otimes \mathcal{X}_2$. Hence, in this case,

$$b(\mathcal{X}_1 \otimes \mathcal{X}_2) = b(\mathcal{X}_2) = b(\mathcal{X}_1) + b(\mathcal{X}_2).$$

Now assume that $b_i := b(\mathcal{X}_i) \geq 1$. We will use induction on $b_1 + b_2$ to prove that

$$b := b(\mathcal{X}) = b_1 + b_2 - 1. \tag{3.7.18}$$

If $b_1 + b_2 = 2$, then neither $\mathcal{X}_1$ nor $\mathcal{X}_2$ are discrete coherent configuration. Thus, $\mathcal{X}$ is not discrete and therefore $b \geq 1$. Choose a base $\{\alpha_i\}$ of $\mathcal{X}_i$, $i = 1, 2$. By formula (3.7.17), $\{(\alpha_1, \alpha_2)\}$ is a base of $\mathcal{X}$. Hence, $b = 1$ in this case and formula (3.7.18) is proved.

Now assume that $c := b_1 + b_2 > 2$ and formula (3.7.18) is valid for cases where $b_1 + b_2 < c$. Choose a point $(\alpha_1, \alpha_2) \in \Omega$ such that α_i belongs to a minimal base of $\mathcal{X}_i$ for $i = 1, 2$. Formula (3.7.18) together with Exercise (3.7.27) show that

$$b \leq b(\mathcal{X}_{(\alpha_1, \alpha_2)}) + 1 = b((\mathcal{X}_1)_{\alpha_1} \otimes (\mathcal{X}_2)_{\alpha_2}) + 1.$$

By inductive hypothesis and Exercise (3.7.27), we have

$$b((\mathcal{X}_1)_{\alpha_1} \otimes (\mathcal{X}_2)_{\alpha_2}) = b((\mathcal{X}_1)_{\alpha_1}) + b((\mathcal{X}_2)_{\alpha_2}) - 1 = (b_1 - 1) + (b_2 - 1).$$

We deduce that $b \leq b_1 + b_2 - 1$.

□

3.7.35. [?, Corollary 5.2] Let $\mathfrak{X}$ be a graph with connected components $\mathfrak{X}_{ij}$, where $i = 1, \ldots, a$ and $j = 1, \ldots, a_i$ for each i. Assume that the indices are chosen so that the graphs $\mathfrak{X}_{ij}$ and $\mathfrak{X}_{i'j'}$ are isomorphic if and only if $i = i'$. Then

$$\mathrm{WL}(\mathfrak{X}) \cong \boxplus_{i=1}^{a} \mathrm{WL}(\mathcal{X}_{i1}) \wr \mathcal{T}_{a_i}.$$

3.7.36. The exponentiation preserves the partial orders of coherent configurations and permutation groups:

(1) if $\mathcal{Y} \le \mathcal{X}$, then $\mathcal{Y} \uparrow K \le \mathcal{X} \uparrow K$ for any K,
(2) if $L \le K$, then $\mathcal{X} \uparrow L \ge \mathcal{X} \uparrow K$ for any $\mathcal{X}$.

Proof. Assume that K and L are permutation groups on Δ. To prove statement (1), let $t \in S(\mathcal{Y} \uparrow K)$. Then there exists $\bigotimes_{\delta\in\Delta} s_\delta \in S(\mathcal{Y}^\Delta)$ such that each $s_\delta \in S(\mathcal{Y})$ and

$$t = (\bigotimes_{\delta\in\Delta} s_\delta)^K.$$

Since $\mathcal{Y} \le \mathcal{X}$, each $s_\delta \in S(\mathcal{X})^\cup$. It follows that $\bigotimes_{\delta\in\Delta} s_\delta \in S(\mathcal{X}^\Delta)^\cup$. This implies that $t \in S(\mathcal{X} \uparrow K)^\cup$. We are done.

To prove statement (2), let $t \in S(\mathcal{X} \uparrow K)$. Then

$$t = (\bigotimes_{\delta\in\Delta} s_\delta)^K = \bigcup_{k\in K} \bigotimes_{\delta\in\Delta} s_{\delta^{k^{-1}}}$$

where each $s_\delta \in S(\mathcal{Y})$. Let $K = \bigcup_{i=1}^m Lk_i$ be a disjoint union of right cosets of L in K. Then

$$t = \bigcup_{i=1}^{m} (\bigotimes_{\delta\in\Delta} s_{\delta^{k_i^{-1}}})^L \in S(\mathcal{X} \uparrow L)^\cup.$$

We are done. □

3.7.37. Let $\mathcal{X}$ be the scheme associated with the Hamming graph $H(d,q)$, where $d \ge 1$ and $q \ge 2$. Then

$$\mathcal{X} = \mathcal{T}_q \uparrow \mathrm{Sym}(d) \quad \text{and} \quad \mathrm{Aut}(\mathcal{X}) = \mathrm{Sym}(q) \uparrow \mathrm{Sym}(d).$$

Proof. Let $\Omega = \{1, \ldots, q\}^d$. From the statements about Hamming graph on page 84, $\mathcal{X}$ is a symmetric scheme of degree q^d and the ith basis relation is of the form

$$s_i = \{(\alpha, \beta) \in \Omega^2 : |\{j : \alpha_j \ne \beta_j\}| = i\},\ i = 0, \ldots, d.$$

Two basis relations of $\mathcal{T}_q$ are as follows:

$$t_1 = \{(1,1), \cdots, (q,q)\}, \quad t_2 = \{(i,j) | 1 \le i \ne j \le q\}.$$

If $t_{j_1} \otimes \ldots \otimes t_{j_d}$ is a basis relation of $(\mathcal{T}_q)^d$, where the number of the factor t_2 is i. Then, it is easily seen that the algebraic fusion with respect to the action of $\mathrm{Sym}(d)$ of this basis relation is s_i. Since any basis relation of $\mathcal{T}_q \uparrow \mathrm{Sym}(d)$ is established in this way, this first equality in question follows.

□

3.7.38. Let $\mathcal{X}$ be a Cayley scheme over G. Assume that $\mathcal{X}$ is the U/L-wreath product. Then

(1) if $\mathcal{X}' \le \mathcal{X}$, and L and U are $\mathcal{X}'$-groups, then $\mathcal{X}'$ is the U/L-wreath product,

(2) if $L' \leq L$ and $U' \geq U$ are $\mathcal{X}$-subgroups and $L' \trianglelefteq G$, then $\mathcal{X}$ is the U'/L'-wreath product,
(3) if $H \geq L$ is a normal $\mathcal{X}$-subgroup of G, then $\mathcal{X}_{G/H}$ is the HU/HL-wreath product.

Proof. To prove statement (1), let $s' \in S(\mathcal{X}')$ be such that $s' \not\subseteq e_U$. Our goal is to prove that $e_L \subseteq \mathrm{rad}(s')$. Since U is an $\mathcal{X}'$-group, $e_U \in S(\mathcal{X}')^\cup$. It follows that

$$s' \cap e_U = \varnothing. \tag{3.7.19}$$

By the assumption, $s' \in S^\cup$. Thus, $s' = \bigcup_{i=1}^m s_i$ for $s_i \in S$. In view of formula (3.7.19), we conclude that each $s_i \not\subseteq e_U$. This implies that, for each i, $e_L \subseteq \mathrm{rad}(s_i)$ since $\mathcal{X}$ is the U/L-wreath product. Then,

$$e_L \cdot s' \cdot e_L = \bigcup_{i=1}^m e_L \cdot s_i \cdot e_L = \bigcup_{i=1}^m s_i = s'.$$

In other words, $e_L \subseteq \mathrm{rad}(s')$, as required.

To prove statement (2), let $s \in S$ be such that $s \not\subseteq e_{U'}$. It follows that $s \not\subseteq e_U$ as $U \leq U'$. Hence, $e_L \subseteq \mathrm{rad}(s)$ because $\mathcal{X}$ is the U/L-wreath product. It follows that $e_{L'} \subseteq \mathrm{rad}(s)$ by the assumption that $L' \leq L$. We are done.

□

3.7.39. Let $\mathcal{X}$ be a Cayley scheme over a group $G = L \times H \times V$, where L, H, and V are $\mathcal{X}$-groups. Assume that $\mathcal{X}$ is the U/L-wreath product, where $U = HL$. Then

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{T}_L \wr \mathcal{X}_{G/L}) \cap \mathrm{Aut}(\mathcal{X}_U \wr \mathcal{T}_V).$$

3.7.40. [**?**] Let we are given

(1) primes p_1, p_2, p_3, p_4 such that $\{p_1, p_2\} \cap \{p_3, p_4\} = \varnothing$,
(2) a positive integer d dividing $\mathrm{GCD}(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1)$,
(3) an isomorphism $f_{ij} \in \mathrm{Iso}(M_i, M_j)$, $(i,j) \in \{(1,3), (2,3), (2,4), (1,4)\}$,

where for each i, we set $M_i \leq \mathrm{Aut}(C_{p_i})$ and $|M_i| = d$. Denote by $\mathcal{X}_{ij}$ the cyclotomic Cayley scheme over $C_{p_ip_j}$ that is associated with the group

$$M_{ij} = \{(x,y) \in M_i \times M_j : \ f_{ij}(x) = y\}.$$

Let us consider the generalized wreath product

$$\mathcal{X}(d) = (\mathcal{X}_{13} \wr_{p_3} \mathcal{X}_{23}) \wr_{p_1p_2} (\mathcal{X}_{14} \wr_{p_4} \mathcal{X}_{24}),$$

where the subscript at the sign $\wr$ denotes the number $|U/L|$ in the corresponding U/L-wreath product: for example, $\mathcal{X}_{13} \wr_{p_3} \mathcal{X}_{23}$ is a Cayley scheme over $C_{p_1p_2p_3}$ that is the U/L-wreath product with $|U| = p_1p_3$ and $|L| = p_1$. Then

(1) if the automorphism $f = f_{13} \circ f_{23}^{-1} \circ f_{24} \circ f_{14}^{-1}$ of the group K_1 is not trivial, then the Cayley scheme $\mathcal{X}(d)$ is not schurian,
(2) if, additionally, for some d' dividing d the automorphism f is identical on the subgroup of order d' and the factorgroup modulo it, then the scheme $\mathcal{X}(d')$ is not separable.

3.7.41. Let $\mathcal{X}$ be semiregular and $K = \mathrm{Aut}(\mathcal{X})$. Then

$$\widehat{\mathcal{X}} = \mathrm{Inv}(\widehat{K}^{(m)}). \tag{3.7.20}$$

In particular, the m-dimensional extension of any semiregular coherent configuration is also semiregular.

Proof. As $\mathcal{X}$ is semiregular, $\mathcal{X}^m$ is semiregular. Hence, as a fission of $\mathcal{X}^m$

$$\widehat{\mathcal{X}} = \mathrm{WL}(\mathcal{X}^m, 1_{\mathrm{Diag}(\Omega^m)})$$

is semiregular. In particular, $\widehat{\mathcal{X}}$ is schurian (Exercise (2.7.37)). It follows that

$$\widehat{\mathcal{X}} = \mathrm{Inv}(\mathrm{Aut}(\widehat{\mathcal{X}})) = \mathrm{Inv}(\widehat{K}^{(m)})$$

where the second equality follow from formula (3.5.4).

□

3.7.42. Let $\mathcal{X} = \mathcal{T}_\Omega$ and $K = \mathrm{Sym}(\Omega)$. Then

(1) $S(\widehat{\mathcal{X}}) = \mathrm{Orb}(K, \Omega^m)$; in particular, equality (3.7.20) holds,
(2) $\Delta_m = \{\alpha \in \Omega^m : \ |\{\alpha_1, \ldots, \alpha_m\}| = m\}$ is a homogeneity set of $\widehat{\mathcal{X}}$, (this also true for all integers j with $1 \le j \le m$)
(3) the equivalence relation $\sim$ on Δ_m defined by

$$\alpha \sim \beta \quad \Leftrightarrow \quad \{\alpha_1, \ldots, \alpha_m\} = \{\beta_1, \ldots, \beta_m\}$$

is a partial parabolic of $\widehat{\mathcal{X}}$,
(4) $\widehat{\mathcal{X}}_{\Omega_m/\sim}$ is isomorphic to the scheme of the Johnson graph $J(n, m)$.

Proof. To prove statement (1), it suffices to show that $\widehat{\mathcal{X}} = \mathrm{Inv}(K, \Omega^m)$. It is easily seen that

$$\widehat{K}^{(m)} = (K^m)_{\mathrm{Diag}(\Omega^m)} = \mathrm{Diag}(K^m),$$

where $\mathrm{Diag}(K^m)$ is the diagonal subgroup $\{(k, \ldots, k) : k \in K\}$ of K^m. Thus, by formula (3.5.3)

$$\widehat{\mathcal{X}} = \widehat{\mathrm{Inv}(K)}^{(m)} \le \mathrm{Inv}(\mathrm{Diag}(K^m)) := \mathcal{Y}.$$

Obviously, $S(\mathcal{Y}) = \mathrm{Orb}(K, \Omega^m)$. On the other hand, by Exercise (2.7.22)for any $s \in S(\mathcal{Y})$ there exists an equivalence relation e on $\{1, \ldots, 2m\}$ such that

$$s = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : (\alpha\beta)_i = (\alpha\beta)_j \quad \Leftrightarrow \quad (i,j) \in e\}$$

By statement (1) of Theorem 3.5.7, each

$$\mathrm{Cyl}_{1_\Omega}(i,j) = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \alpha_i = \beta_j\}$$

is a relation of $S(\widehat{\mathcal{X}})$. It follows that

$$r_{ik} = \mathrm{Cyl}_{1_\Omega}(i,j)\mathrm{Cyl}_{1_\Omega}(k,j)^* = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \alpha_i = \alpha_k\}$$

is a relation of $S(\widehat{\mathcal{X}})$. Also,

$$t_{ik} = \mathrm{Cyl}_{1_\Omega}(j,i)^*\mathrm{Cyl}_{1_\Omega}(j,k) = \{(\alpha, \beta) \in \Omega^m \times \Omega^m : \beta_i = \beta_k\}$$

is a relation of $S(\widehat{\mathcal{X}})$. Let Δ be a class of e and $i \in \Delta$. For any $j \in \Delta$, define

$$u_{ij} = \begin{cases} \mathrm{Cyl}_{1_\Omega}(j, i-m) & \text{if} i > m, j \le m \\ t_{i-m,j-m} & \text{if} \quad i > m, j > m \\ \mathrm{Cyl}_{1_\Omega}(i, j-m) & \text{if} \quad i < m, j > m \\ r_{ij} & \text{if} \quad i < m, j \le m. \end{cases}$$

Set $u(\Delta) = \cap_{j \in \Delta} u_{ij}$. Then one can easily see that

$$s = \bigcap_{\Delta \in e/\{1, \ldots, 2m\}} u(\Delta).$$

This implies that $s \in S(\widehat{\mathcal{X}})^{\cup}$. Thus, $\mathcal{Y} \le \widehat{\mathcal{X}}$. Hence $\widehat{\mathcal{X}} = \mathcal{Y} = \mathrm{Orb}(K, \Omega^m)$, as required.

□

3.7.43. Let $\mathcal{X}' \ge \mathcal{X}$ and $\mathcal{Y}' \ge \mathcal{Y}$. Then

(1) $\widehat{\mathcal{X}'} \ge \widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}'} \ge \widehat{\mathcal{Y}}$,

(2) if $\psi \in \mathrm{Iso}_m(\mathcal{X}', \mathcal{Y}')$ extends $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{Y})$, then $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{Y})$ and $\widehat{\psi}$ extends $\widehat{\varphi}$.

Proof. To prove statement (1), it suffices to verify the first inclusion. Note that

$$\widehat{\mathcal{X}'} = \mathrm{WL}(\mathcal{X}'^m, 1_{\mathrm{Diag}(\Omega^m)}).$$

It follows that $\mathcal{X}^m \le \widehat{\mathcal{X}'}$ as $\mathcal{X} \le \mathcal{X}'$ and $1_{\mathrm{Diag}(\Omega^m)} \in S(\widehat{\mathcal{X}'})^{\cup}$. Thus $\widehat{\mathcal{X}'} \ge \widehat{\mathcal{X}}$, as required.

To prove statement (2), assume that $\mathcal{X}$ and $\mathcal{Y}$ are respectively coherent configurations on Ω and Δ. Observe that

$$\mathrm{Diag}(\Omega^m)^{\widehat{\psi}} = \mathrm{Diag}(\Delta^m) \tag{3.7.21}$$

and for all $s' \in S(\mathcal{X}'^m)$

$$\widehat{\psi}(s') = \psi^m(s'). \tag{3.7.22}$$

Since ψ extends φ, formula (3.7.22) shows that for all $s \in S(\mathcal{X}^m)$

$$\widehat{\psi}(s) = \varphi^m(s).$$

In view of formula (3.7.21), the restriction of $\widehat{\psi}$ to $\widehat{\mathcal{X}}$ is the m-dimensional extension of φ. We are done.

□

3.7.44. For any $\Delta \in F^{\cup}$, we have $\widehat{\mathcal{X}_\Delta} \le \widehat{\mathcal{X}}_{\Delta^m}$.

Proof. Since $1_{\mathrm{Diag}(\Omega^m)} \in S(\widehat{\mathcal{X}})^{\cup}$,

$$1_{\mathrm{Diag}(\Delta^m)} = 1_{\mathrm{Diag}(\Omega^m)} \cap \Delta^m \in S(\widehat{\mathcal{X}}_{\Delta^m}). \tag{3.7.23}$$

In addition, for any $s \in (\mathcal{X}_\Delta)^m$ one can see that

$$s \in \mathcal{X}^m_{\Delta^m} \le \widehat{\mathcal{X}}_{\Delta^m}.$$

Thus, $(\mathcal{X}_\Delta)^m \le \widehat{\mathcal{X}}_{\Delta^m}$. Together with formula (3.7.23), this shows that

$$\widehat{\mathcal{X}_\Delta} = \mathrm{WL}((\mathcal{X}_\Delta)^m, 1_{\mathrm{Diag}(\Delta^m)}) \le \widehat{\mathcal{X}}_{\Delta^m}.$$

□

3.7.45. [**?**, Lemma 6.2] Let $s \in S(\widehat{\mathcal{X}})$. Then for any indices $i, j \in \{1, \ldots, 2m\}$ the following two statements hold:

(1) $\mathrm{pr}_{i,j}(s) = \{((\alpha \cdot \beta)_i, (\alpha \cdot \beta)_j)) : (\alpha, \beta) \in s\}$ is a basis relation of $\overline{\mathcal{X}}$, where $\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m)$,

(2) if φ is an m-isomorphism from $\mathcal{X}$ to another coherent configuration, then

$$\mathrm{pr}_{i,j}(s^{\widehat{\varphi}}) = \mathrm{pr}_{i,j}(s)^{\overline{\varphi}}.$$

Proof. Let $\Omega_-(s) = \Lambda$ and $\Omega_+(s) = \Gamma$. If $1 \le i, j \le m$, then $\mathrm{pr}_{i,j}(s) = \mathrm{pr}_{i,j}(\Lambda)$. If $m < i, j \le 2m$, then $\mathrm{pr}_{i,j}(s) = \mathrm{pr}_{i-m,j-m}(\Gamma)$. In these two cases, $\mathrm{pr}_{i,j}(s)$ is a basis relation of $\overline{\mathcal{X}}$ by Theorem 3.5. 16.

Now we assume that $1 \le i \le m$ and $m < j \le 2m$. Define

$$\mathrm{Cyl}'_{1_\Omega}(i,j) = \{(\alpha, \beta) \in \Omega^{2m} : (\alpha \cdot \beta)_i = (\alpha \cdot \beta)_j\}.$$

By the proof of (3.7.42), one can see that $\mathrm{Cyl}'_{1_\Omega}(i,j)$ is a relation of $\widehat{\mathcal{X}}$. Note that

$$\mathrm{pr}_{i,j}(s)^\eta = \mathrm{pr}_{i,i}(\Lambda)^\eta \mathrm{Cyl}'_{1_\Omega}(i,j)\, \mathrm{pr}_{j,j}(\Gamma)^\eta,$$

where $\eta = \eta_m$ defined in formula (3.5.10). By Theorem 3.5.16, $\mathrm{pr}_{i,i}(\Lambda)^\eta$ and $\mathrm{pr}_{j,j}(\Lambda)^\eta$ belong to $S(\widehat{\mathcal{X}}_\Delta^{(m)})$ for $\Delta = \mathrm{Diag}(\Omega^m)$. Hence $\mathrm{pr}_{i,j}(s)^\eta$ belongs to $S(\widehat{\mathcal{X}}_\Delta^{(m)})$. This implies that $\mathrm{pr}_{i,j}(s)$ is a relation of $\overline{\mathcal{X}}$. Now if $\mathrm{pr}_{i,j}(s)$ is not a basis relation, let t be a basis relation properly contained in $\mathrm{pr}_{i,j}(s)$. Then

□

3.7.46. The mapping $\mathcal{X} \mapsto \overline{\mathcal{X}}$ is a closure operator, i.e., the following statements hold:

(1) $\mathcal{X} \le \overline{\mathcal{X}}$,
(2) if $\mathcal{X} \le \mathcal{Y}$, then $\overline{\mathcal{X}} \le \overline{\mathcal{Y}}$,
(3) $\overline{\mathcal{X}}$ is m-closed.

Proof. To prove statement (3), since $\overline{\mathcal{X}} \le \overline{\overline{\mathcal{X}}}$ (statement (1)), it suffices to prove that $\overline{\overline{\mathcal{X}}} \le \overline{\mathcal{X}}$. However,

$$\overline{\overline{\mathcal{X}}} = (\widehat{\overline{\mathcal{X}}}^{(m)})^{\eta^{-1}} \quad \text{and} \quad \overline{\mathcal{X}} = (\widehat{\mathcal{X}}^{(m)})^{\eta^{-1}},$$

where $\eta = \eta_m$. It suffices to prove that $\widehat{\overline{\mathcal{X}}}^{(m)} \le \widehat{\mathcal{X}}^{(m)}$. Since $\widehat{\overline{\mathcal{X}}}^{(m)} = \mathrm{WL}(\overline{\mathcal{X}}^m, 1_{\mathrm{Diag}(\Omega^m)})$ and $\widehat{\mathcal{X}}^{(m)} = \mathrm{WL}(\mathcal{X}^m, 1_{\mathrm{Diag}(\Omega^m)})$, it suffices to prove that

$$\overline{\mathcal{X}}^m \le \widehat{\mathcal{X}}^{(m)}.$$

For any $s_1 \otimes s_2 \otimes \cdots \otimes s_m \in S(\overline{\mathcal{X}}^m)$ with each $s_i \in S(\overline{\mathcal{X}})$,

$$s_1 \otimes s_2 \otimes \cdots \otimes s_m = \prod_{i=1}^m 1_\Omega \otimes \cdots \otimes s_i \otimes \cdots \otimes 1_\Omega$$

Thus, it suffices to prove that for each $s \in S(\overline{\mathcal{X}})$ and each $1 \le i \le m$,

$$1_\Omega \otimes \cdots \otimes s \otimes \cdots \otimes 1_\Omega \in S(\widehat{\mathcal{X}}^{(m)})^\cup. \tag{3.7.24}$$

One can see that

$$1_\Omega \otimes \cdots \otimes s \otimes \cdots \otimes 1_\Omega = \mathrm{Cyl}_{1_\Omega}(i,i) \cdot s^\eta \cdot \mathrm{Cyl}_{1_\Omega}(i,i),$$

where $\mathrm{Cyl}_{1_\Omega}(i,i) \in S(\widehat{\mathcal{X}}^{(m)})^\cup$ (Theorem 3.5.7) and $s^\eta \in S(\widehat{\mathcal{X}}^{(m)})$. Hence, formula (3.7.24) follows. We are done.

□

3.7.47. For fixed sets Ω and Ω', we define a partial order on the set of all algebraic isomorphisms $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$, where $\mathcal{X}$ and $\mathcal{X}'$ are coherent configurations on Ω and Ω', respectively. Namely, if $\psi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{Y}, \mathcal{Y}')$, then

$$\varphi \le \psi \quad \Leftrightarrow \quad \mathcal{X} \le \mathcal{Y},\ \mathcal{X}' \le \mathcal{Y}',\ \text{and } \psi \text{ extends } \varphi.$$

Then the mapping taking φ to $\mathrm{cl}(\varphi) = \overline{\varphi}$, is a closure operator, i.e., the following statements hold:

(1) $\varphi \le \mathrm{cl}(\varphi)$
(2) if $\varphi \le \psi$, then $\mathrm{cl}(\varphi) \le \mathrm{cl}(\psi)$,
(3) $\mathrm{cl}(\mathrm{cl}(\varphi)) = \mathrm{cl}(\varphi)$.

3.7.48. [**?**, Theorems 7.5 and 7.6] Let $\mathcal{X} = \mathcal{X}_1 \boxplus \cdots \boxplus \mathcal{X}_k$. Then

(1) $\overline{\mathcal{X}} = \overline{\mathcal{X}}_1 \boxplus \cdots \boxplus \overline{\mathcal{X}}_k$,
(2) $\mathcal{X}$ is m-closed if and only if so are $\mathcal{X}_1, \ldots, \mathcal{X}_k$,
(3) if $\mathcal{X}' = \mathcal{X}'_1 \boxplus \cdots \boxplus \mathcal{X}'_k$ and the algebraic isomorphism $\varphi \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}, \mathcal{X}')$ is induced by the algebraic isomorphisms $\varphi_i \in \mathrm{Iso}_{\mathrm{alg}}(\mathcal{X}_i, \mathcal{X}'_i)$, $i = 1, \ldots, k$, then $\varphi \in \mathrm{Iso}_m(\mathcal{X}, \mathcal{X}')$ if and only if $\varphi_i \in \mathrm{Iso}_m(\mathcal{X}_i, \mathcal{X}'_i)$ for all i.

3.7.49. [**?**, Corollary 5.4] Let $\mathcal{X}$ be a 2-closed scheme and $e \in E$. Assume that $e \subseteq S_1(\mathcal{X})$. Then any class $\Delta \in \Omega/e$ is a fiber of the coherent closure $\mathrm{WL}(\mathcal{X}, 1_\Delta)$.

3.7.50. [**?**, Theorem 5.9] Let $\mathcal{X}$ be a 2-closed primitive scheme. For a fixed $\alpha \in \Omega$, denote by Δ the set of all fibers $\Gamma \in F(\mathcal{X}_\alpha)$ such that the scheme $(\mathcal{X}_\alpha)_\Delta$ is imprimitive. Then

(1) if $\Delta \ne \varnothing$, then the union of all $\Gamma \in \Delta$ is a base of $\mathcal{X}$;
(2) if $\Delta = \varnothing$, then any fiber of $\mathcal{X}$ other than $\{\alpha\}$ is a base of $\mathcal{X}$.

3.7.51. [**?**] Let G be an abelian group and $\widehat{G}$ the group of all irreducible complex characters of G. For an S-ring $\mathfrak{A}$ over G, define an equivalence relation $\sim$ on $\widehat{G}$ so that

$$\xi \sim \eta \quad \Leftrightarrow \quad \xi(\underline{X}) = \eta(\underline{X}) \quad \text{for all } X \in \mathcal{S}(\mathfrak{A}).$$

Then the partition $\widehat{\mathcal{S}}$ of the group $\widehat{G}$ into the classes of $\sim$ satisfies the conditions (SR1), (SR2), and (SR3) at page **??**; in particular,

$$\widehat{\mathfrak{A}} = \mathrm{Span}\, \underline{\widehat{\mathcal{S}}}$$

is an S-ring over $\widehat{G}$. Moreover, $\mathrm{rk}(\mathfrak{A}) = \mathrm{rk}(\widehat{\mathfrak{A}})$.

Proof. Denote the principal character of G by $\widehat{e}$. Observe that, for $\chi \in \widehat{G}$

$$\chi \sim \widehat{e} \Leftrightarrow \chi(\underline{X}) = \widehat{e}(\underline{X}) = |X|, \text{ for all } X \in \mathcal{S}(\mathfrak{A}) \Leftrightarrow \chi(\underline{G}) = |G| \Leftrightarrow \chi = \widehat{e}.$$

This implies that $\{\widehat{e}\} \in \widehat{\mathcal{S}}$, i.e., the condition (SR1) holds.

For any $\xi, \eta \in \widehat{G}$ and any $g \in G$, $\xi^{-1}(g) = \xi(g^{-1})$ and $\eta^{-1}(g) = \eta(g^{-1})$. Thus,

$$\begin{aligned} \xi \sim \eta \quad &\Leftrightarrow \quad \xi(\underline{X}) = \eta(\underline{X}), \text{ for all } X \in \mathcal{S} \\ &\Leftrightarrow \quad \xi(\underline{X^{-1}}) = \eta(\underline{X^{-1}}), \text{ for all } X \in \mathcal{S} \\ &\Leftrightarrow \quad \xi^{-1}(\underline{X}) = \eta^{-1}(\underline{X}), \text{ for all } X \in \mathcal{S} \\ &\Leftrightarrow \quad \xi^{-1} \sim \eta^{-1}. \end{aligned}$$

Here the second equivalence is valid as $\mathcal{S} = \{X^{-1} : X \in \mathcal{S}\}$. It follows that the condition (SR2) holds.

Let $\widehat{X}, \widehat{Y} \in \widehat{\mathcal{S}}$. As elements in $\mathfrak{C}\widehat{G}$,

$$f := \underline{\widehat{X}} \cdot \underline{\widehat{Y}} = \sum_{\chi \in \widehat{G}} a_\chi \chi.$$

By the first orthogonality relation of character, one can see that $\chi(\underline{G}) = 0$ for any nonpincipal character $\chi \in \widehat{G}$. Then $a_\xi|G| = (\xi^{-1}f)(\underline{G})$ for any $\xi \in \widehat{G}$. Now suppose $\xi \sim \eta$. Then,

$$\begin{aligned} a_\xi|G| &= (\xi^{-1}f)(\underline{G}) \\ &= (\xi^{-1}f)(\sum_{X\in\mathcal{S}} \underline{X}) \\ &= \sum_{X\in\mathcal{S}} \xi^{-1}(\underline{X})f(\underline{X}) \\ &= \sum_{X\in\mathcal{S}} \eta^{-1}(\underline{X})f(\underline{X}) \\ &= (\eta^{-1}f)(\underline{G}) \\ &= a_\eta|G|. \end{aligned}$$

It follows that $a_\xi = a_\eta$. Thus, f is a linear combination of $\{\underline{\widehat{X}} : \widehat{X} \in \widehat{\mathcal{S}}\}$. Consequently the condition (SR3) holds.

□

3.7.52. In the conditions and notation of Exercise 3.7.51, given a group $H \le G$ denote by $H^\perp$ the group of all characters $\xi \in \widehat{G}$ such that $\ker(\xi) \ge H$. Then

(1) the mapping $\mathcal{E}(\mathfrak{A}) \to \mathcal{E}(\widehat{\mathfrak{A}})$, $H \mapsto H^\perp$ is a lattice antiisomorphism,
(2) $\widehat{\mathfrak{A}_H} = \widehat{\mathfrak{A}}_{\widehat{G}/H^\perp}$ for each $H \in \mathcal{E}(\mathfrak{A})$,
(3) $\widehat{\mathfrak{A}_{G/H}} = \widehat{\mathfrak{A}}_{H^\perp}$ for each $H \in \mathcal{E}(\mathfrak{A})$.

3.7.53. [**?**, Sec. 2.3] In the conditions and notation of Exercise 3.7.51,

(1) $\mathfrak{A} = \mathrm{Cyc}(K, G)$ for $K \le \mathrm{Aut}(G)$ if and only if $\widehat{\mathfrak{A}} = \mathrm{Cyc}(K, \widehat{G})$,
(2) $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$ if and only if $\widehat{\mathfrak{A}} = \widehat{\mathfrak{A}_1} \otimes \widehat{\mathfrak{A}_2}$,
(3) $\mathfrak{A}$ is the U/L-wreath product if and only if $\widehat{\mathfrak{A}}$ is the $L^\perp/U^\perp$-wreath product.

3.7.54. Let $\mathcal{X}$ be a coherent configuration and $\xi \in \mathrm{Irr}(\mathcal{X})$. Then

$$n_\xi \le |\,\mathrm{Supp}_{\mathcal{X}}(\xi)|\, m_\xi,$$

and the equality is simultaneously attained for all irreducible characters if and only if $\mathcal{X}$ is quasiregular.

3.7.55. [**?**, Theorem 3] There exists a constant c such that given a primitive scheme $\mathcal{X}$,

$$n_{min} \le 2^{cm_{\min}},$$

where n_{min} is the minimal valency of a nonreflexive basis relation of $\mathcal{X}$ and m_{min} is the minimal multiplicity of a nonprincipal irreducible character of $\mathcal{X}$.

3.7.56. Let $G = G_1 \times G_2 \times G_3$ be a group, where $|G_1| = |G_2| = |G_3|$. Denote by K the permutation group induced by the action of G by right multiplications on the set

$$\Omega = G/G_1 \cup G/G_2 \cup G/G_3,$$

and set $\mathcal{X} = \mathrm{Inv}(K, \Omega)$. Then

(1) $F(\mathcal{X}) = \{G/G_1, G/G_2, G/G_3\}$,

(2) $m_\xi = 1$ and $n_\xi = |\operatorname{Supp}_{\mathcal{X}}(\xi)|$ for all $\xi \in \operatorname{Irr}(\mathcal{X})$,
(3) the mapping $\xi \mapsto \operatorname{Supp}_{\mathcal{X}}(\xi)$ induces a bijection from $\operatorname{Irr}(\mathcal{X})$ onto the nonempty homogeneity sets of $\mathcal{X}$.

3.7.57. [**?**, Theorem 3.6(ii)] Let $\mathcal{X}$ be a commutative scheme of degree n, and $r, s, t \in S$. Then

$$c_{rs}^t = \frac{n_r n_s}{n} \sum_{\xi \in \operatorname{Irr}(\mathcal{X})} \frac{1}{m_\xi^2}\, \xi(r)\xi(s)\overline{\xi(t)}.$$

Bibliography

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