# SOME OPEN PROBLEMS FOR COHERENT CONFIGURATIONS

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## 1. Introduction

Coherent configurations were introduced independently by D. Higman [42] as a tool to study permutation groups, and by B. Weisfeiler and A. Leman [69] in connection with the graph isomorphism problem. In subsequent twenty years, except for a seminal Babai's paper [2] on an upper bound for the order of uniprimitive permutation groups, the main attention was paid to association schemes [6] and distance-regular graphs [11]. By the end of 90s, the theory of coherent configurations received an impetus related to a deeper study of the homogeneous case [33,74] and new applications to the graph isomorphism problem. The theory, developed in the mid-2000s, is presented in a special issue of the European Journal of Combinatorics [64]. In spite of several important results obtained in recent years (including Babai's algorithm testing graph isomorphism in quasipolynomial time [3]), it should be recognized that the general theory of coherent configuration is far from being complete.

This text presents some general problems in the theory of coherent configurations. The list of these problems is not exhaustive and mainly reflects the tastes of the author, who became acquainted with coherent configurations, researching the graph isomorphism problem. In particular, it does not include many interesting problems, arising in the structural theory and representation theory; two surveys concerning these topics can be found in [76] and [34], respectively. However, we believe that a solution of each of presented problems can lead to a better understanding of coherent configurations and further development of the theory.

We do not give a sufficient motivation for the presented problems, focusing mainly on the context in which they arise. We divided them into seven groups, each of which is allocated to a separate section. Sections 2–7 contain the problems concerning, respectively, exponentiation of a coherent configuration by a group, primitive coherent configurations, combinatorial bases, equivalenced coherent configurations, the Klin conjecture, and the separability number. In Section 8, we present a number of specific problems that are not related to each other. Unfortunately, at present there is no common system of notation and concepts in the theory of coherent configurations. Therefore, for the sake of convenience, we have collected the relevant material in Section 9.

**Notation.** In what follows, $\Omega$ always denotes a finite set of cardinality $n$. The symmetric group of $\Omega$ is denoted by $\mathrm{Sym}(\Omega)$, and also $\mathrm{Sym}(n)$ if $\Omega = \{1, \ldots, n\}$.

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## 2. Exponentiation

Let $m$ be a positive integer, $G \leq \mathrm{Sym}(m)$, and $\mathcal{X}$ a coherent configuration. Then the group $G$ acts as an isomorphism group of the direct sum or tensor product of $m$ copies of $\mathcal{X}$. Denoting this coherent configuration by $\mathcal{X}^m$ in both cases, one can see that the algebraic fusion $(\mathcal{X}^m)^G$ is canonically isomorphic, respectively, to the *wreath product* $\mathcal{X} \wr G$ [31, Subsecion 3.3] and exponentiation $\mathcal{X} \uparrow G$ of the configuration $\mathcal{X}$ by the group $G$ [23, Section 3]. In the former case, we also have $\mathcal{X} \wr G = \mathcal{X} \wr \mathcal{Y}$, where $\mathcal{Y}$ is the coherent configuration associated with $G$.

**Question 2.1.** *How to define the exponentiation $\mathcal{X} \uparrow \mathcal{Y}$ of the coherent configurations $\mathcal{X}$ and $\mathcal{Y}$ so that if $\mathcal{Y}$ is associated with a group $G$, then $\mathcal{X} \uparrow \mathcal{Y} = \mathcal{X} \uparrow G$?*

It is easily seen that if $\mathcal{X}$ is of degree $n$, then $\mathcal{X} \uparrow G$ has a fusion isomorphic to the Hamming scheme

$$H(n,m) = \mathcal{T}_n \uparrow \mathrm{Sym}(m).$$

On the other hand, it is well known (see e.g. [11, Theorem 9,2,1]) that

$$\mathrm{Aut}(H(n,m)) = \mathrm{Sym}(n) \uparrow \mathrm{Sym}(m),$$

where the right-hand side denotes the wreath product of $\mathrm{Sym}(n)$ by $\mathrm{Sym}(m)$ in the product action. Thus, if we need to use the O'Nan-Scott theorem [73, Theorem 2.4] to analyze the automorphism group of a coherent configuration, then we should be able to solve the following computational problem.

**Problem 2.2.** *Given a primitive coherent configuration $\mathcal{X}$, test whether it has a fusion isomorphic to $H(m,n)$ with $m > 1$ and $n > 1$, and if so, then find this fusion and an explicit isomorphism from it onto $H(m,n)$.*

This problem can be solved in polynomial time in $n$ whenever $\mathcal{X}$ is associated with a group of odd order: by [62], one can efficiently find the group $\mathrm{Aut}(\mathcal{X})$ and then apply algorithm from [24, Subsection 5.1]. In fact, we do not even know any efficient algorithm to test whether a given primitive permutation group is a subgroup of the wreath product in product action of two smaller groups.

It is a folklore that $\mathrm{Aut}(\mathcal{X} \wr \mathcal{Y}) = \mathrm{Aut}(\mathcal{X}) \wr \mathrm{Aut}(\mathcal{Y})$. In the case of exponentiation, it was proved in [23, Section 3] that $\mathcal{X} \uparrow G$ is schurian (respectively, primitive) if $\mathcal{X}$ is schurian (resp. $G$ is transitive and $\mathcal{X}$ is primitive and non-regular). Moreover, it was observed there that

$$\mathrm{Aut}(\mathcal{X}) \uparrow G \leq \mathrm{Aut}(\mathcal{X} \uparrow G) \leq \mathrm{Aut}(\mathcal{X}) \uparrow G^{(1)},$$

where $G^{(1)}$ is largest subgroup of $\mathrm{Sym}(m)$ that has the same orbits as $G$ (both inclusions are attained for $\mathcal{X} = \mathcal{T}_2$ and for $G = \mathrm{Alt}(5) \leq \mathrm{Sym}(6)$ and $G = \mathrm{Alt}(4)$, respectively). In particular, this shows that the group $\mathrm{Aut}(\mathcal{X} \uparrow G)$ preserves the product decomposition $\Omega^m$.

**Problem 2.3.** *Find the automorphism group of $\mathcal{X} \uparrow G$ in terms of* $\mathrm{Aut}(\mathcal{X})$ *and* $G$.

In fact, the exponentiation can be defined in slightly general situation. Namely, suppose we are given a family $\mathcal{F} = \{\mathcal{X}_i\}_{i=1}^m$ of pairwise algebraically isomorphic coherent configurations. Then any group $G \leq \mathrm{Sym}(m)$ acts on the product

$$\mathcal{Y} = \mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_m$$

as a group of algebraic isomorphisms. The algebraic fusion $\mathcal{Y}^G$ is called the *exponentiation* of the family $\mathcal{F}$ by the group $G$, and denoted by $\mathcal{F} \uparrow G$.[1] This enables us to construct algebraically isomorphic exponentiations, which are not isomorphic. For example, let $m = 2$, $\mathcal{X}_1 = H(4,2)$ and $\mathcal{X}_2$ the coherent configuration associated with the Shrikhande graph [11, p.104]. Then $\mathcal{F} \uparrow \mathrm{Sym}(2)$ is the coherent configuration associated with a Doob graph, which is a distance-regular graph with the same parameters as the Hamming graph associated with $H(4,4)$. Thus, the coherent configurations $\mathcal{F} \uparrow \mathrm{Sym}(2)$ and $H(4,4)$ are algebraically isomorphic. They are not isomorphic, because no Doob graph is isomorphic to a Hamming graph.

**Question 2.4.** *Is it true that any coherent configuration algebraically isomorphic to $\mathcal{F} \uparrow G$ is isomorphic to $\mathcal{F}' \uparrow G'$ for some family $\mathcal{F}'$ and a group $G'$ such that $|\mathcal{F}'| = |\mathcal{F}|$ and $G' \leq \mathrm{Sym}(m)$?*

## 3. Primitive coherent configurations

By Sims's conjecture proved in [15], the maximal subdegree of a primitive group is bounded from above by a function of the minimal subdegree. Since the subdegrees of a transitive group are equal to the valences of the coherent configuration associated with this group, the following conjecture is true in the schurian case.

**Conjecture 3.1.** (L. Babai) *The maximal valency $n_{max}$ of a primitive coherent configuration $\mathcal{X}$ is bounded from above by a function of the minimal valency $n_{min}$.*

The conjecture is true if the degree of $\mathcal{X}$ is a prime [36] or $n_{min} \leq 2$ [70, pp.71-72]. Even for $n_{min} = 3$, the validity of the conjecture is not known (some partial results on primitive coherent configurations with $n_{min} = 3$ can be found in [7] and [41]). In this connection, it is also worth mentioning result [19, Theorem 3] stating that

$$n_{min} \leq 2^{cm}$$

for some constant $c > 0$, where $m$ is the minimal multiplicity of a nonprincipal irreducible representation of the adjacency algebra of $\mathcal{X}$ in the standard representation.

Only a few is known on *antisymmetric* primitive coherent configurations $\mathcal{X}$, i.e. those in which $s \neq s^*$ for all nonreflexive $s \in S$. It is easily seen that in this case, $\mathrm{Aut}(\mathcal{X})$ is a group of odd order. Therefore, in the schurian case, from the Feit-Thompson theorem, it follows that the socle of $\mathrm{Aut}(\mathcal{X})$ is an elementary abelian regular group. This implies that $\mathcal{X}$ is isomorphic to a Cayley scheme over this group, and, hence, the coherent configurations $\mathcal{X}$ is commutative.

**Question 3.2.** *What is the minimal positive integer $m = m(n)$ for which every $m$-closed primitive antisymmetric coherent configuration $\mathcal{X}$ of degree $n$ is commutative?*

One can see that $m$ can be taken of order $n^{1/3}$ up to a polylogarithmic factor. Indeed, in this case, from [68, Theorem 4], it follows that $\mathcal{X}$ is symmetric or $b(\mathcal{X}) \leq m$; however, in the latter case, the $(m+1)$-closedness of $\mathcal{X}$ implies the schurity [22, Theorem 4.8]. It should be remarked that at present, we know not so many examples of antisymmetric primitive coherent configurations: except for schurian ones, there are several constructions producing rank 3 configurations

[1] In fact, this was the original definition of the exponentiation in [23, Section 3].

(e.g. [57] and [32, Theorem 2.6.6]), by means of which one can get configurations of larger ranks via the exponentiation by a transitive group of odd order (Section 2).

The following question is taken from [61, p.11] and goes back to H. Wielandt: is it true that if a point stabilizer of a primitive permutation group $G$ has a nontrivial subgroup with $k$ non-singleton orbits, then $k+1$ is greater or equal than the rank of $G$? We note that the orbits of any subgroup of $G$ forms an *equitable partition* of the coherent configuration associated with $G$ (the defining property of an equitable partition a coherent configuration $\mathcal{X}$ is that the adjacency matrix of the corresponding equivalence relation on the points of $\mathcal{X}$ commutes with the adjacency matrix of each basis relation). Thus, a combinatorial analog of the Wielandt question can be formulated as follows.

**Question 3.3.** *Let $\Pi$ be an equitable partition of a primitive coherent configuration of rank $r$ that has $1 \le k < |\Pi|$ non-singleton classes. Is it true that $k+1 \ge r$?*

For $k = 1$, the answer is "yes" by [45, Theorem 3.1]. Since the fibers of any fission of $\mathcal{X}$ forms an equitable partition, the latter result shows also that the only primitive fusion of the wreath product $\mathcal{X} \wr \mathcal{Y}$ is of rank 2 provided that the degrees $n$ and $m$ of the configurations $\mathcal{X}$ and $\mathcal{Y}$ are greater than one: in this case, as the fission, one can take the direct sum of $\mathcal{X}$ and the complete configurations on $nm-n$ points.

**Question 3.4.** *For which coherent configurations, the only primitive fusion of it is of rank 2?*

We do not think that the coherent configurations in Question 3.4 can be completely characterized. Instead, we are interested in enough general sufficient conditions providing the corresponding property. An example of such a condition is closely related with a concept of $B$-group [71, Section 25], by which we mean a group $G$ such that any primitive group containing a regular subgroup isomorphic to $G$, is 2-transitive. Any abelian group of composite order having a cyclic Sylow subgroup is a B-group [71, Theorem 25.4]. A careful analysis of the proof shows that, in fact, the only primitive Cayley scheme over $G$ is of rank 2. Thus, every Cayley scheme of rank at least 3 over the group $G$ gives the coherent configuration in Question 3.4.

In [11, p.68], a conjecture on a possibe structure of a primitive symmetric Cayley scheme over an abelian group was formulated. A counterexample was found by A. E. Brouwer in [9]. We believe that a modified conjecture below could be true.[2] In what follows, under an *LS-scheme*, we mean a coherent configuration, all the basis graphs of which are strongly regular and have the same type: Latin square or negative Latin square.

**Conjecture 3.5.** *Let $\mathcal{X}$ be a nontrivial primitive Cayley scheme over an abelian group. Then one of the following statements holds:*

(1) *$\mathcal{X}$ has a fusion isomorphic to the exponentiation of smaller schemes,*
(2) *$\mathcal{X}$ is isomorphic to a Cayley scheme over the direct product of elementary abelian groups,*
(3) *$\mathcal{X}$ is an LS-scheme.*

[2]This statement appeared after discussions with M. Muzychuk and Š.Miklavič.

Conjecture 3.5 is true when the scheme $\mathcal{X}$ is schurian [52, Theorem 1.1]. It should also be remarked that in each of these three cases, there is a scheme that does not satisfy the conditions of the two other cases. To see this, let $\mathcal{X}$ be the coherent configuration of the Schrikhande graph [11, p.104]. Then we are in case (3) and not in case (1). Since, the only abelian regular subgroup of the group $\mathrm{Aut}(\mathcal{X})$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_4$, we are also not in case (2). An easy argument shows that $\mathcal{X} \uparrow \mathbb{Z}_3$ belongs to case (1) and does not belong other cases. Finally, any strongly regular graph on $4000 = 5^3 2^5$ points [9] gives an example belonging only to case (2).

## 4. Combinatorial base of a coherent configuration

Let $\mathcal{X}$ be the coherent configuration associated with a permutation group $G$. Then, obviously,

$$b(G) \le b(\mathcal{X}),$$

where $b(G)$ is the base number of $G$. This inequality can be strict, because $\mathcal{X}$ is also associated with the group $G^{(2)} = \mathrm{Aut}(\mathcal{X})$,[3] whereas, in general, $b(G) \ne b(G^{(2)})$. On the other hand, the equality $b(\mathrm{Aut}(\mathcal{X})) = b(\mathcal{X})$ holds for cyclotomic schemes [26, Theorem 1.2], for schurian primitive antisymmetric coherent configurations that are not a fission of a Hamming schemes [62, Theorem 9.1] and the coherent configurations associated with some finite simple permutation groups [63]).

**Question 4.1.** *How big can $b(\mathcal{X})$ be in comparison with $b(\mathrm{Aut}(\mathcal{X}))$, where $\mathcal{X}$ is a (primitive) schurian coherent configuration $\mathcal{X}$.*

Clearly, $b(\mathrm{Aut}(\mathcal{X})) = 0$ if and only if $b(\mathcal{X}) = 0$. A pure combinatorial characterization of coherent configurations of groups $G$ with $b(G) = 1$ [26, Theorem 9.3] shows that $b(\mathrm{Aut}(\mathcal{X})) = 1$ if and only if $b(\mathcal{X}) = 1$ in the schurian case. No answer is known when $b(\mathrm{Aut}(\mathcal{X})) = 2$ (some partial results are discussed in the end of this section).

In [23, Corollary 4.8], it was proved that $b(\mathcal{X}) \le n_{av}$ for all nontrivial primitive coherent configurations $\mathcal{X}$, where $n_{av}$ is the average valency of $\mathcal{X}$. If the Babai conjecture 3.1 was true, then this would imply that $b(\mathcal{X})$ is bounded from above by a function on $n_{min}$. Again, nothing is known even for $n_{min} = 3$.

**Conjecture 4.2.** *The base number of a primitive coherent configuration $\mathcal{X}$ is bounded from above by a function of the minimal valency $n_{min}$.*

By [67, Theorem 1.3], all primitive groups of odd order have a base of size at most 3. This result was slightly generalized in [62, Theorem 9.1], where it was proved that $b(\mathcal{X}) \le 3$ for any schurian primitive antisymmetric coherent configuration $\mathcal{X}$. If we drop the schurity condition, then the upper bound $b(\mathcal{X}) \le \log n$ is unknown even under an additional assumption that $\mathcal{X}$ is $(\log n)$-closed. This suggests the following question.

**Question 4.3.** *What is the minimal positive integer $m$ for which $b(\mathcal{X}) \le 3$ for every $m$-closed primitive antisymmetric coherent configuration $\mathcal{X}$?*

From [18, Section 5] (see also [1]), it follows that if $m$ in Question 4.3 is a constant, then the problem of finding a nontrivial factor of a univariable polynomial over a finite field can be solved efficiently assuming the generalized Riemann hypothesis.

[3]According to [72], one can define the 2-closure $G^{(2)}$ of a permutation group $G$ as the automorphism group of the coherent configuration associated with $G$.

General upper bounds for the base number of a nontrivial primitive coherent configuration $\mathcal{X}$ were given in [2, Theorem 2.4] in terms of $n$, $\mathrm{rk}(\mathcal{X})$, and $n_{max}$, and in [23, Corollary 4.8] in terms of $n_{av}$ (see above). We observe that if $\mathcal{X}$ is equivalenced of valency $k = \log n$, then these results give the upper bounds $cn$ and $\log n$, respectively; the situation is completely reversed for $k = cn$. The primitivity assumption is essential in both results. On the opposite side, it was proved in [63] that $b(\mathcal{X}) \leq 2$ for a homogeneous coherent configuration $\mathcal{X}$, whenever

$$2c(n_{max} - 1) < n,$$

where $c$ is the maximum number $c(s) = \sum_{r\in S} c_{rr^*}^s$ with nonreflexive $s \in S$.

**Problem 4.4.** *Given an integer $k \geq 3$, find a nontrivial sufficient condition in terms of intersection numbers of a homogeneous coherent configuration $\mathcal{X}$ to guarantee the fulfillment of the inequality $b(\mathcal{X}) \leq k$.*

From [22, Subsection 7.6], it follows that $b(\mathcal{X}) \leq \log\log q$, where $\mathcal{X}$ is the coherent configuration of rank 4 associated with a projective plane of order $q$. One can see that in this case, $b(\mathcal{X}) \geq k$ for some $k \geq 4$, only if every $k$ points of the projective plane form a proper subplane whenever among these points there is a quadrangle. Thus, the base of $\mathcal{X}$ is large only if the plane has sufficiently many different subplanes. It should be remarked that such a plane can not be Desarguesian (in fact, in the Desarguesian case, $b(\mathcal{X}) \leq 5$).

**Question 4.5.** *Is it true that there is an integer $b \geq 5$ such that the base number of the coherent configuration associated with a projective plane is at most $b$?*

## 5. Equivalenced coherent configurations

Following [58], a *pseudocyclic* scheme $\mathcal{X}$ can be defined as an equivalenced coherent configuration of valency $k$ such that

$$c(s) = k - 1$$

for all $s \in S^{\#}$, where $c(s)$ is defined as in Section 9.7. A natural example of a pseudocyclic scheme is given by a *Frobenius scheme*, by which we mean the coherent configuration associated with a Frobenius group in its standard permutation representation (the one point stabilizer coincides with the Frobenius complement).[4] It turns out [58, Theorem 7.4] that any pseudocyclic scheme of valency $k > 1$ and rank at least $4(k-1)k^3$ is a Frobenius scheme (if $k = 1$, then the scheme is regular). This bound was improved to $6(k-1)^2$ in [16]

**Problem 5.1.** *Given an integer $k > 1$, find the minimum number $r = r(k)$ such that any pseudocyclic scheme of valency $k > 1$ and rank $r$ is a Frobenius scheme.*

Given an affine plane of order $q$, one can construct a pseudocyclic scheme $\mathcal{X}$ on $q^2$ points of this plane: every (nonreflexive) basis graph of $\mathcal{X}$ is a disjoint union of $q$ complete graphs, the vertices of which are the points of the affine lines belonging to a given parallel class. Thus, $\mathcal{X}$ is of valency $q - 1$ and rank $q + 2$ (there are exactly $q + 1$ parallel classes). As was observed in [58, Theorem 4.1], the scheme $\mathcal{X}$

[4]Thus, any Frobenius scheme is schurian, but the automorphism group of it is not necessarily a Frobenius group.

is schurian if and only if the affine plane is Desarguesian. Since there are infinitely many non-Desarguesian affine planes, we conclude that

$$r(k) \geq k + 3$$

for infinitely many $k$'s.

Let $p$ be a prime. In [36, Theorem 3.3], it was proved that every homogeneous coherent configuration $\mathcal{X}$ of degree $p$ is a commutative pseudocyclic scheme. From the above mentioned result, it follows that $\mathcal{X}$ is schurian whenever

$$p > ck^3$$

for some constant $c > 0$. We do not believe that the exponent 3 here is best possible. In fact, we do not know any nonschurian homogeneous coherent configuration of degree $p$ and rank greater than 3.

**Question 5.2.** *What is the maximal rank of a nonschurian homogeneous coherent configuration of prime degree?*

Concerning the rank 3 case, one can find sporadic examples with $p = 19, 23, 29$ in the catalog of small association schemes [35]. The construction described in [32, Theorem 2.6.6] enables us to construct an antisymmetric scheme of degree $2n + 1$ and rank 3 from any antisymmetric scheme of degree $n$ and rank 3. Statement 3 in Theorem 2.6.6 from that paper states that the obtained scheme is non-schurian for $n \geq 7$. However, the proof of this statement was not presented there. I was able to prove this only under an additional assumption that the original scheme has intransitive automorphism group. However, using this statement and the above sporadic examples, I cannot construct any infinite family of non-schurian schemes of prime degree and rank 3.

**Question 5.3.** *Whether there exists an infinite family of non-schurian schemes of prime degree and rank* 3*?*

One can see that the coherent configuration of a permutation group is pseudocyclic (respectively, equivalenced) if and only if the group is Frobenius (respectively, 3/2-transitive). On the other hand, a recent complete characterization of 3/2-transitive groups [53] shows that if such a group has rank $r$ and subdegree $k$,[5] then it is a Frobenius group, whenever

$$r > ck^2$$

for some $c > 0$ and the degree is large enough. Thus, the following conjecture is true at least in the schurian case.

**Conjecture 5.4.** *Every equivalenced coherent configuration $\mathcal{X}$ of valency $k$ and rank $r > ck^2$ is pseudocyclic, whenever the degree of $\mathcal{X}$ is large enough.*

Conjecture 5.4 is true for $k \leq 3$. Indeed, for $k = 1$, this obvious. In [46], this was proved for $k = 3$. Finally, let $\mathcal{X}$ be an equivalenced coherent configuration of valency 2. Then $\mathcal{X}$ is commutative and no multiplicity of a nonprincipal irreducible representation of the adjacency algebra of $\mathcal{X}$ in the standard representation, equals 1 [44, Lemma 4.2]. This immediately implies that all of these multiplicities equal 2. Thus, $\mathcal{X}$ is pseudocyclic.

[5]This exactly means that the equivalenced coherent configuration associated with this group is of rank $r$ and valency $k$.

Every imprimitive 3/2-transitive group is a Frobenius group [71, Theorem 10.4]. A natural analog of this theorem for coherent configurations could be formulated as follows: every imprimitive equivalenced coherent configuration $\mathcal{X}$ is a Frobenius scheme. If this was true, then Conjecture 5.4 holds in imprimitive case. However, we can prove this statement only if $\mathcal{X}$ is 2-closed [23, Theorem 5.11]. This suggests the following weakening of Conjecture 5.4.

**Question 5.5.** *Is Conjecture 5.4 is true if the coherent configuration $\mathcal{X}$ is 2-closed?*

It is easily seen that every coherent configuration with all valences equal to one is schurian and separable. However, there exist non-schurian and non-separable quasi-thin coherent configurations, i.e., those in which $n_s \le 2$ for all $s \in S$. In homogeneous case, they were in a sense characterized in [59, Theorem 1.1]. However, this result cannot be considered as final. First, because nothing is known in non-homogeneous case. Second, even in homogeneous case it is not clear how broad is the class of non-schurian and/or non-separable such schemes. A partial result obtained in [59] states that any homogeneous coherent configuration of degree $n$ and maximal valency 2 is schurian whenever

$$n_2 > 3n/7,$$

where $n_2$ is the number of basis relations of valency 2.

**Question 5.6.** *Is it possible to find a necessary and sufficient condition for a quasi-thin coherent configuration to be schurian and/or separable?*

## 6. The Klin conjecture

Let $\mathcal{G}$ be an undirected graph. Following [40,65], we say that two subgraphs of $\mathcal{G}$ are of the *same type with respect to a pair* $(\alpha, \beta)$ *of vertices* if both contain $\alpha$ and $\beta$ and there exists an isomorphism of one onto the other mapping $\alpha$ to $\alpha$ and $\beta$ to $\beta$.

**Definition 6.1.** *The graph $\mathcal{G}$ satisfies the $t$-vertex condition for $t \ge 2$ if the number of $t$-vertex subgraphs of a given type with respect to a given pair $(\alpha, \beta)$ of vertices depends only on whether $\alpha$ and $\beta$ are equal, adjacent, or nonadjacent.*

Clearly, the graph $\mathcal{G}$ is regular (resp. strongly regular) if and only if it satisfies the $t$-vertex condition for $t = 2$ (resp. $t = 3$). Moreover, an $n$-vertex graph satisfies the $n$-vertex condition if and only if it is a rank 3 graph, i.e. the associated coherent configuration of rank 3 is schurian. A lot of examples of non-rank 3 strongly regular graphs satisfying the 4-vertex condition can be found in [49].

**Conjecture 6.2.** (M.Klin, see [32, p.74]) *There exists an integer $t_0$ such that any strongly regular graph satisfying the $t_0$-vertex condition is of rank* 3.

Let us cite some results supporting the Klin conjecture. The class of strongly regular graphs with fixed minimal eigenvalue $-m$, where $m \ge 2$ is an integer, contains finitely many graphs other than Latin square graphs $\mathrm{LS}_m(n)$ and Steiner graphs $\mathrm{S}_m(n)$ [60]. The graphs with $m = 2$ were characterized in [66]; one can check that each of them is of rank 3 or does not satisfy the 4-vertex condition. Next, any $\mathrm{LS}_3(n)$- or $\mathrm{S}_3(n)$-graph satisfying the 4-vertex condition is of rank 3: for the Steiner graphs, this follows from [48], whereas for the Latin square graphs, it can be proved by the technique used in [17].

**Problem 6.3.** *Check Klin's conjecture for $LS_m(n)$- and $S_m(n)$-graphs with $m \ge 4$.*

In fact, $8 \leq t_0 \leq n$. The low bound was proved in [65], where an infinite family of non-rank 3 strongly regular graphs satisfying the 7-vertex condition was constructed. The trivial upper bound can be replaced by $n/3$ under an additional assumption. Namely, we note that Definition 6.1 can be naturally extended to directed and colored graphs. Then, one can say that a coherent configuration $\mathcal{X}$ satisfies the *t-vertex condition* if every basis graph of $\mathcal{X}$ satisfies the $t$-vertex condition [32, p.71]. In these terms, by [22, Theorem 6.4], any $m$-closed coherent configuration satisfies the $3m$-vertex condition. This implies that a strongly regular graph is of rank 3 whenever the associated coherent configuration is $n/3$-closed.

**Question 6.4.** *Is it true that there exists a function $f(t) : \mathbb{N} \to \mathbb{N}$ such that a coherent configuration satisfying the $t$-vertex condition is $f(t)$-closed?*

It was proved in [21, Theorem 1.1] that there exists $\varepsilon > 0$ such that for any sufficiently large positive integer $n$, one can find a nonschurian coherent configuration on $n$ points, which is $m$-closed for some $m \geq \lfloor \varepsilon n \rfloor$. [6] Since a strongly regular graph is of rank 3 if and only if the associated coherent configuration is schurian, this shows that an analog of the Klin conjecture for the colored graphs is not true.

**Question 6.5.** *Is it true that given an integer $r \geq 3$, for every sufficiently large $m \geq r$, there exists a homogeneous nonschurian $m$-closed coherent configuration of rank $r$?*

Note that if the answers to Questions 6.4 and 6.5 are positive, then the Klin conjecture is false.

Let $\mathcal{X}_i$ be the coherent configuration associated with a strongly regular graph $\mathcal{G}_i$, $i = 1, 2$. Then from [22, Theorem 6.1], one can deduce that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are algebraically $m$-isomorphic and $\mathcal{G}_1$ satisfies the $m$-vertex condition, then $\mathcal{G}_2$ also satisfies the $m$-vertex condition. Thus, if the Klin conjecture is true, then $\mathcal{G}_1$ is a rank 3 graph if and only if so is $\mathcal{G}_2$. This enables us to formulate a weaker conjecture.

**Conjecture 6.6.** *There exists an integer $t_1$ such that if $\mathcal{X}_1$ is a schurian coherent configuration of rank 3 and $\mathcal{X}_2$ is a coherent configuration algebraically $m$-isomorphic to $\mathcal{X}_1$, where $m \geq t_1$, then $\mathcal{X}_1$ and $\mathcal{X}_2$ are isomorphic.*

In terms of Section 7, this conjecture states that any schurian coherent configuration $\mathcal{X}$ of rank 3 is $t_1$-separable. For many of them of small degree, this is true for $t_1 = 1$, which exactly means that the corresponding graph of rank 3 is uniquely determined by its parameters (see [11]). From [22, Section 7], it also follows that for the Hamming, Johnson and Grassmann coherent configurations of rank 3, one can take $t_1 = 2$. We believe that Conjecture 6.6 can be proved with the help of the classification of rank 3 groups (see [12, Section 6]).

It seems that the schurity condition in Conjecture 6.6 is essential. This is supported by the following considerations. Take two nonisomorphic finite groups $G_1$ and $G_2$. Then the Latin square graphs associated with the multiplication tables of $G_1$ and $G_2$ are strongly regular and nonisomorphic (see [38, Section 3]). Assume, in addition, that $G_1$ and $G_2$ can not be distinguished by the "$m$-dimensional invariants"[7] (it can be expected that such groups do exist among the $p$-groups of

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[6] Such a coherent configuration can be chosen to be homogeneous [18, p.93].

[7] For instance, there exists an isomorphism preserving bijection between the $k$-generated subgroups of $G_1$ and $G_2$ with $k \leq m$.

class 2). In this situation, it seems very likely that the coherent configurations of rank 3 associated with the above graphs are algebraically $cm$-isomorphic for a constant $c > 0$.

**Question 6.7.** *Is it true that for any positive integer $m$ there exist two algebraically $m$-isomorphic coherent configurations of rank 3, which are not isomorphic?*

The answer is "yes" for $m = 1$. However, even for $m = 2$, we do not know two algebraically $m$-isomorphic coherent configurations of rank 3, which are not isomorphic.

## 7. Separability number

According to [22, Definition 4.1], a coherent configuration $\mathcal{X}$ is said to be *$m$-separable* for a positive integer $m$ if every algebraic $m$-isomorphism from $\mathcal{X}$ to another coherent configuration is induced by a combinatorial isomorphism. In other words, $\mathcal{X}$ is $m$-separable if and only if $\mathcal{X}$ is determined up to isomorphism by the $m$-dim intersection numbers (which are the ordinary intersection numbers for $m = 1$). The minimal $m$ for which $\mathcal{X}$ is $m$-separable is called the *separability number* of $\mathcal{X}$ and is denoted by $s(\mathcal{X})$; it follows from [22, Theorem 4.3] that

$$1 \leq s(\mathcal{X}) \leq \lceil n/3 \rceil.$$

We do not know whether the upper bound is tight, but there exists a positive constant $c < 1$ such that for all sufficiently large positive integer $n$, one can find a (schurian) coherent configuration of degree $n$ admitting an algebraic $m$-isomorphism with $m \geq \lfloor cn \rfloor$ which is not induced by a combinatorial isomorphism.

From [26, Theorem 1.1], it follows that every cyclotomic scheme $\mathcal{X}$ over a finite field is determined up to isomorphism by the 3-dim intersection numbers.[8] In fact, it was proved that if $\mathcal{X}$ is not trivial, then

$$s(\mathcal{X}) \leq b(\mathrm{Aut}(\mathcal{X})) \leq 3.$$

On the other hand, any homogeneous nonschurian coherent configuration of prime degree $p$ and rank 3 (see Section 5) is algebraically isomorphic to a cyclotomic scheme over $\mathbb{F}_p$. Therefore, for such a scheme, we have $s(\mathcal{X}) = 2$.

**Question 7.1.** *Does there exist a cyclotomic scheme $\mathcal{X}$ over a finite field for which $s(\mathcal{X}) = 3$?*

While solving the Schur-Klin hypothesis[9], infinitely many Cayley schemes $\mathcal{X}$ with $s(\mathcal{X}) \geq 2$ were found in [25]. It seems to be true that all these schemes are determined by the 2-dim intersection numbers. On the other hand, if the separability number of every Cayley scheme over a cyclic group was bounded from above by a constant $m \geq 2$, then the isomorphism testing of circulant graphs can be substantially simplified in comparison with the algorithms from [27] or [56].

**Question 7.2.** *Is it true that there exists a constant $m$ such that $s(\mathcal{X}) \leq m$ for every Cayley scheme $\mathcal{X}$ over a cyclic group?*

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[8] By a cyclotomic scheme over a finite field field $\mathbb{F}$, we mean the coherent configuration associated with a group $GK \leq \mathrm{Sym}(\mathbb{F})$, where $G = \mathbb{F}^+$ and $K \leq \mathbb{F}^*$.

[9] This hypothesis, which states that every Cayley scheme over a cyclic group is schurian, proved to be incorrect.

With any finite projective plane $\mathcal{P}$, one can associate two coherent configurations: homogeneous of rank 4 and non-homogeneous of rank 8 (the two fibers of it are the points and lines of $\mathcal{P}$). For both of them, the 2-extension was found in [30]. [10] It turned out that all homogeneous (resp. non-homogeneous) coherent configurations $\mathcal{X}$ of projective planes of the same order are algebraically 2-isomorphic. Thus, $s(\mathcal{X}) \geq 3$. It was also shown that the equality is attained when the plane $\mathcal{P}$ is Desarguesian. Note that the Desarguesian projective planes in the theory of planes are analogs of classical polygons in the theory of generalized polygons [55]. Each of them is associated both homogeneous and non-homogeneous coherent configuration: in the former case see [43], whereas in the latter the construction is standard.

**Conjecture 7.3.** *There exists a constant $m$ such that $s(\mathcal{X}) \leq m$ for the coherent configuration $\mathcal{X}$ of every classical generalized polygon.*

We complete the section by remarking that if the answer to Question 2.4 is "yes" for the family $\mathcal{F}$ consisting of $m$ copies of $\mathcal{X}$, then it is quite natural to ask about the separability number of the coherent configuration $\mathcal{X} \uparrow G$ in terms of $s(\mathcal{X})$.

## 8. Miscellaneous

The character table of an association scheme $\mathcal{X}$ of rank $r$ contains at most $r^2$ nonzero complex numbers, with the help of which all intersection numbers of $\mathcal{X}$ can be computed [6, Theorem 3.6]. For a complete coherent configuration of degree $n$, the rank $r$ equals $n^2$ and there are exactly $n^3 = r^{3/2}$ nonzero intersection numbers.

**Problem 8.1.** *Given a positive integer $r$, find the minimal number $m = m(r)$ such that for every coherent configuration of rank $r$, there exist $m$ nonzero intersection numbers that determine all the intersection numbers of $\mathcal{X}$.*

Every homogeneous coherent configuration of rank $r \leq 5$ is commutative [70, p.84]. The imprimitive noncommutative coherent configurations $\mathcal{X}$ of rank 6 were studied in [37] (an infinite family of examples give the coherent configurations on flags of a finite projective plane [43]). The complete description of possible array of the intersection numbers of $\mathcal{X}$ in terms of two irreducible linear characters of the its adjacency algebra can be found in [39]. However, up to now no primitive noncommutative coherent configuration of rank 6 is known. Even in the schurian case, the following problem is open.

**Problem 8.2.** *Determine (up to algebraic isomorphism) all noncommutative coherent configuration of rank* 6.

Recall that the vertex connectivity of an undirected graph is defined to be the minimum number of vertices that one has to remove in order to disconnect the graph. It was proved in [10] that the connectivity of any incomplete basis graph of the coherent configuration associated with a distance-regular graph equals the valency $k$ of the graph provided that $k > 2$. For any connected undirected graph of an arbitrary homogeneous coherent configuration $\mathcal{X}$, the equality was proved under the assumption that $\mathcal{X}$ is $k$-closed [28]. Some partial result on this problem can be found in [50]

[10] In a sense, it contains other natural coherent configurations associated with a projective plane.

**Conjecture 8.3.** (A.E.Brouwer, [8]) *The vertex connectivity of a connected graph of a homogeneous coherent configuration equals the valency of the graph.*

One of the basic computational problems in the coherent configuration theory is to construct the *coherent closure* of a set of binary relations on $\Omega$, which is the smallest coherent configuration $\mathcal{X}$ on $\Omega$ such that each input relation is a union of some basis relations of $\mathcal{X}$. The standard way to find the coherent closure, is the Weisfeiler-Leman algorithm, on each step of which more and more subtle partition of $\Omega\times\Omega$ is constructed. Clearly, the number of such steps is bounded from above by $n^2$. We do not know any example, for which this bound is attained, and it is interesting to find a tight upper bound. The running time of a practical implementation of the Weisfeiler-Leman algorithm in [4] is $O(n^7)$; at present, the best bound of the algorithm is $O(n^3\log n)$ [5, 47].

**Question 8.4.** *What is the computational complexity of finding the coherent closure?*

It seems to be difficult to find a nontrivial necessary and sufficient condition for a coherent configuration to be schurian. From the computational point of view, schurity testing is not harder than finding the automorphism group of a coherent configuration. Indeed, as a certificate, it suffices to find any permutation group, with which a given coherent configuration is associated. An efficient schurity testing is quite obvious for the class of abelian groups, because in this case the homogeneous components of the input coherent configuration are regular. In [62], a polynomial-time algorithm to test the schurity of a antisymmetic coherent configurations was proposed (this corresponds to the case of the groups of odd order).

**Problem 8.5.** *Construct an efficient algorithm to test whether a given coherent configuration $\mathcal{X}$ is schurian, and (if so) find a permutation group, with which $\mathcal{X}$ is associated.*

The problem of finding a nontrivial fixed point free automorphism of a graph is NP-complete [54] (a fixed point free element of a given transitive group can be found in polynomial time [13]). On the other hand, this problem can be solved in polynomial time if the required automorphism is a full cycle [27]. In fact, the latter paper contains also a polynomial-time for recognizing Cayley schemes over a cyclic group. This can be considered as the first step to solving the following problem.

**Problem 8.6.** *Construct an efficient algorithm to test whether a given coherent configuration $\mathcal{X}$ is isomorphic to a Cayley scheme over a given abelian group $G$, and (if so) find a regular subgroup of the group* $\mathrm{Aut}(\mathcal{X})$ *that is isomorphic to $G$.*

In general, Problem 8.6 seems to be hard. A good idea is to start with the case when $G$ is the direct product of a constant number of cyclic groups. Even for $G=\mathbb{Z}_p\times\mathbb{Z}_p$, where $p$ is a prime, the problem is open. It should be remarked that in this case, the recognition problem gives, as byproduct, a polynomial-time isomorphism test for graphs isomorphic to Cayley graphs over $G$ (this follows from the fact that $\mathbb{Z}_p\times\mathbb{Z}_p$ is a CI-group, see [51]).

## 9. Coherent configurations

Throughout this section, we use the following notation. For a relation $s\subset\Omega\times\Omega$, we set $s^*=\{(\beta,\alpha):\ (\alpha,\beta)\in s\}$ and $\alpha s=\{\beta\in\Omega:\ (\alpha,\beta)\in s\}$ for all $\alpha\in\Omega$. If

$S$ is a set of relations, then $S^{\cup}$ denotes the set of all unions of the elements of $S$, and $S^* = \{s^* : s \in S\}$. The details on the concepts defined below can be found in [29].[11]

9.1. **Definitions.** By a *coherent configuration* on a finite set $\Omega$, we mean a pair $\mathcal{X} = (\Omega, S)$, where $S$ is a partition of $\Omega \times \Omega$ satisfying the following conditions:

(1) $1_\Omega = \mathrm{diag}(\Omega \times \Omega)$ belongs to $S^{\cup}$,
(2) $S^* = S$,
(3) given $r, s, t \in S$, the number $c_{rs}^t = |\alpha r \cap \beta s^*|$ does not depend on $(\alpha, \beta) \in t$.

The numbers $|\Omega|$ and $|S|$ are called the *degree* and *rank* of $\mathcal{X}$. The elements of $\Omega$ and $S$ are called the *points* and *basis relations*. The numbers $c_{rs}^t$ are called the *intersection numbers*.

9.2. **Graphs.** Any coherent configuration can be considered as a colored graph, the colored classes of which are the basis relations. By a *basis graph* (resp. *graph*) of $\mathcal{X}$, we mean any graph with vertex set $\Omega$ and arc set $s \in S$ (resp. $s \in S^{\cup}$). With every distance-regular graph, one can associate a coherent configuration the basis relations of which are the relations "to be at distance $i$", where $0 \le i \le d$ with $d$ the diameter of the graph.

9.3. **Adjacency algebra.** A $\mathbb{C}$-linear space $\mathcal{A}$ spanned by the adjacency matrices of basis graphs of the coherent configuration $\mathcal{X}$ is a matrix algebra; it is called the *adjacency algebra* of $\mathcal{X}$. We say that $\mathcal{X}$ is *commutative* if so is $\mathcal{A}$. The linear space $\mathbb{C}\Omega$ is an $\mathcal{A}$-module and the afforded linear representation of $\mathcal{A}$ is said to be *standard*. When the coherent configuration $\mathcal{X}$ is homogeneous (see Subsection 9.7 below), the matrix $(1/n)J$, where $J \in \mathcal{A}$ is the all one matrix, is a primitive central idempotent of $\mathcal{A}$; the multiplicity and degree of the afforded irreducible representation of $\mathcal{A}$ equal 1. This representation of $\mathcal{A}$ is said to be *principal*.

9.4. **Combinatorial isomorphisms and schurity.** We say that $\mathcal{X}$ is (combinatorially) *isomorphic* to a coherent configuration $\mathcal{X}' = (\Omega', S')$, if there exists a (combinatorial) *isomorphism* from $\mathcal{X}$ onto $\mathcal{X}'$, i.e. a bijection $f : \Omega \to \Omega'$ such that $s^f \in S'$ for all $s \in S$, where $s^f = \{(\alpha^f, \beta^f) : (\alpha, \beta) \in s\}$. The group of all isomorphisms of the coherent configuration $\mathcal{X}$ to itself contains a normal subgroup

$$\mathrm{Aut}(\mathcal{X}) = \{f \in \mathrm{Sym}(\Omega) : s^f = s, \ s \in S\}$$

called the *automorphism group* of $\mathcal{X}$. Conversely, if $G \le \mathrm{Sym}(\Omega)$ and $S$ is the set of orbits of the componentwise action of $G$ on $\Omega \times \Omega$, then $(\Omega, S)$ is a coherent configuration associated with $G$. A coherent configuration is called *schurian* if it is associated with some permutation group.

9.5. **Algebraic isomorphisms and separability.** We say that $\mathcal{X}$ is *algebraically isomorphic* to a coherent configuration $\mathcal{X}' = (\Omega', S')$, if there exists an *algebraic isomorphism* from $\mathcal{X}$ onto $\mathcal{X}'$, i.e. a bijection $\varphi : S \to S'$ such that

$$c_{rs}^t = c_{r's'}^{t'}, \qquad r, s, t \in S. \tag{1}$$

Each (combinatorial) isomorphism $f$ from $\mathcal{X}$ to $\mathcal{X}'$ naturally induces an algebraic isomorphism $\varphi$ between these coherent configurations defined by $s^\varphi := s^f$ for all

[11] In the present text, the terminology is slightly different: we use terms "fission" and "algebraic isomorphism" instead of "extension" and "similarity".

$s \in S$. A coherent configuration is said to be *separable* if, every algebraic isomorphism from it onto another coherent configuration is induced by a combinatorial isomorphism.

9.6. **Fibers and valences.** Any set $\Delta \subset \Omega$ for which $1_\Delta \in S$, is called a *fiber* of $\mathcal{X}$. Each fiber $\Delta$ is associated with a *homogeneous component*, which is the coherent configuration $(\Delta, S_\Delta)$, where $S_\Delta = \{s \in S : \ s \subset \Delta^2\}$. In particular, for any basis relation $s \in S$ there exist uniquely determined fibers $\Delta, \Gamma$ such that $s \subset \Delta \times \Gamma$. Set

$$n_s = c_{ss^*}^r,$$

where $r = 1_\Delta$. Then the number $|\delta s| = n_s$ does not depend on $\delta \in \Delta$ and is called the *valency* of $s$.

9.7. **Homogeneous coherent configurations.** The coherent configuration $\mathcal{X}$ is said to be *homogeneous* or *association scheme* if $1_\Omega \in S$.[12] In this case, $n_s = n_{s^*}$ for all $s \in S$. We say that a homogeneous coherent configuration $\mathcal{X}$ is *equivalenced*, if the number $k = n_s$ does not depend on the nonreflexive relation $s \in S$; in this case, $k$ is called the *valency* of $\mathcal{X}$. When $k = 1$, the coherent configuration $\mathcal{X}$ is said to be *regular* (or, in terminology of [75], thin). Another example of a homogeneous coherent configuration gives a *Cayley scheme*, which is a coherent configuration $\mathcal{X}$ the points of which are the elements of a group $G$ and such that $\mathrm{Aut}(\mathcal{X})$ contains the right regular representation of $G$.

9.8. **Primitive and imprimitive coherent configurations.** A homogeneous coherent configuration $\mathcal{X}$ is said to be *imprimitive* if the set $S^\cup$ contains an equivalence relation other than $1_\Omega$ and $\Omega^2$. Otherwise, we say that $\mathcal{X}$ is *primitive*. A homogeneous coherent configuration $\mathcal{X}$ is primitive if and only if every basis graph of $\mathcal{X}$ is strongly connected.

9.9. **Fissions and fusions.** There is a natural partial order $\geq$ on the set of all coherent configurations on the set $\Omega$. Namely, given two coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega, S')$ we set

$$\mathcal{X} \geq \mathcal{X}' \ \Leftrightarrow \ S^\cup \subset (S')^\cup.$$

In this case, we say that $\mathcal{X}'$ is the *fission* of $\mathcal{X}$ and $\mathcal{X}$ is the *fusion* of $\mathcal{X}'$. The least and the greatest elements with respect to that order are respectively the *trivial* scheme $\mathcal{T}_\Omega$ (or $\mathcal{T}_n$ if $\Omega = \{1, \ldots, n\}$) and *complete* coherent configuration. The basis relations of the first of them are $1_\Omega$ and $\Omega^2 \setminus \{1_\Omega\}$, whereas the basis relations of the second one the singletons in $\Omega^2$.

9.10. **Algebraic fusion.** Let $G$ be a group of algebraic isomorphisms of the coherent configuration $\mathcal{X}$. For $s \in S$, denote by $s^G$ the union of the basis relations $s^g$, $g \in G$, and set $S^G = \{s^G : \ s \in S\}$. Then $\mathcal{X}^G := (\Omega, S^G)$ is a coherent configuration. We say that $\mathcal{X}'$ is an *algebraic fusion* of $\mathcal{X}$ if $\mathcal{X}' = \mathcal{X}^G$ for a suitable group $G$.

9.11. **Combinatorial base.** For any point $\alpha \in \Omega$, denote by $\mathcal{X}_\alpha$ the smallest coherent configuration on $\Omega$, for which

$$1_\alpha \in S_\alpha \quad \text{and} \quad S \subset S_\alpha^\cup,$$

[12] A discussion on the term "associative scheme" can be found in [14].

where $S_\alpha$ is the set of basic relations of $\mathcal{X}_\alpha$. We say that the points $\alpha, \beta, \ldots$ forms a (combinatorial) *base* of $\mathcal{X}$ if the coherent configuration $\mathcal{X}_{\alpha,\beta,\ldots}$ is complete. The smallest size of a base is denoted by $b(\mathcal{X})$ and is called the *base number* of $\mathcal{X}$.

9.12. **Direct sum, tensor and wreath products.** In this subsection, we define coherent configurations that can be constructed in a regular way from $\mathcal{X}$ and a coherent configuration $\mathcal{X}' = (\Omega', S')$.

The *direct sum* $\mathcal{X} \boxplus \mathcal{X}'$ is defined to be the coherent configuration on the disjoint union of $\Omega$ and $\Omega'$ the basis relations of which are $s \in S$, $s' \in S'$, $\Delta \times \Delta'$, and $\Delta' \times \Delta$, where $\Delta$ and $\Delta'$ are fibers of $\mathcal{X}$ and $\mathcal{X}'$.

The *tensor product* $\mathcal{X} \otimes \mathcal{X}'$ is defined to be the coherent configuration on $\Omega \times \Omega'$ the basis relations of which are $s \otimes s' = \{((\alpha, \alpha'), (\beta, \beta')) :\ (\alpha, \beta) \in s \text{ and } (\alpha', \beta') \in s'\}$, where $s \in S$ and $s' \in S'$.

The (imprimitive) *wreath product* $\mathcal{X} \wr \mathcal{X}'$ is defined to be the coherent configuration on $\Omega \times \Omega'$ the basis relations of which are $s \otimes 1_{\Omega'}$ and $(\Omega)^2 \otimes s'$, where $s \in S$ and $s' \in S'$.

9.13. $m$**-dim intersection numbers.** For a positive integer $m$, the *$m$-extension* $\widehat{\mathcal{X}} = \widehat{\mathcal{X}}^{(m)}$ of a coherent configuration $\mathcal{X}$ is defined to be the smallest coherent configuration on $\Omega^m$, which is a fission of the $m$-fold tensor product $\mathcal{X}^m$ and for which the set $\Delta = \operatorname{diag}(\Omega^m)$ is a union of fibers. The intersection numbers of the $m$-extension are called the *$m$-dimensional numbers* of the configuration $\mathcal{X}$.

9.14. $m$**-closed coherent configurations.** By the *$m$-closure* of $\mathcal{X}$, we mean the coherent configuration

$$\bar{\mathcal{X}} = (\widehat{\mathcal{X}}_\Delta)^{f^{-1}},$$

where $\widehat{\mathcal{X}}_\Delta$ is the restriction of the $m$-extension $\widehat{\mathcal{X}}$ to the diagonal $\Delta$ and $f : \Omega \to \Delta$ is the bijection taking $\alpha$ to $(\alpha, \ldots, \alpha)$. In general, $\bar{\mathcal{X}} \geq \mathcal{X}$; we say that $\mathcal{X}$ is *$m$-closed* if $\bar{\mathcal{X}} = \mathcal{X}$.

9.15. **Algebraic** $m$**-isomorphisms.** Two coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ are said to be algebraically *$m$-isomorphic* if there exists an *algebraic $m$-isomorphism* $\varphi : \mathcal{X} \to \mathcal{X}'$. The latter means that $\varphi$ is an algebraic isomorphism having an $m$-extension, by which we mean an algebraic isomorphism $\psi : \widehat{\mathcal{X}} \to \widehat{\mathcal{X}'}$ of the $m$-extensions such that

$$\Delta^\psi = \Delta' \quad \text{and} \quad r^\psi = r^{\varphi_m}$$

for all basis relations $r$ of $\mathcal{X}^m$, where $\Delta' = \operatorname{diag}(\Omega'^m)$, and $\varphi_m : \mathcal{X}^m \to \mathcal{X}'^m$ is the algebraic isomorphism induced by $\varphi$.

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