Twenty One Low-dimensional Problems

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Problem 1 (M. Shapiro) *A rational periodic map.* [1]

This problem (in the original setting) is **solved** by M. Vsemirnov (PostScript file available). Other solutions are announced by M.Shapiro–A.Vainshtein–M.Gekhtman and S. Evdokimov.

Prove that the rational mapping of the plane into itself defined as

$$f_a : (x, y) \mapsto \Big(\frac{1}{y}, \frac{x(1 + ay + y^2)}{1 + y}\Big)$$

is periodic only for $a = 2$.

If $a = 2$, then we get a mapping $f(x, y) = (1/y, x(1+y))$ that has period 5, i.e. $f^5 = \mathrm{id}$. *Explain the reason why.* For example, note that the positive quadrant $Q = \{x > 0, y > 0\}$ is stable under f and f has exactly one fixed point in Q. Is it true that there is a diffeomorphism $\varphi : Q \to D$, where D is the open unit disk, such that $\varphi \circ f = r_5 \circ \varphi$, where r_5 is rotation through 72°.

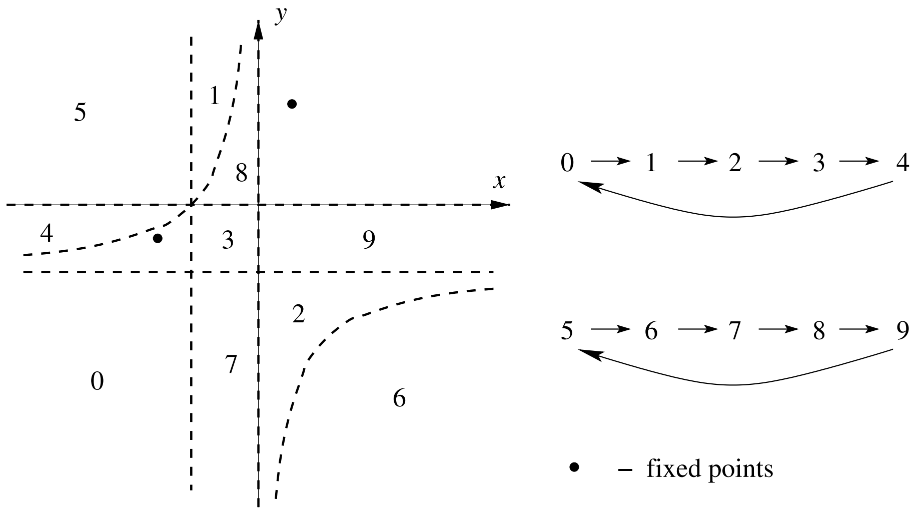


Figure 1: Periodic rational map $f : \mathbb{R}^2 \to \mathbb{R}^2$

Is it true in general that a periodic mapping of a disk with one fixed point is equivalent to a rotation?

[1] Last revision note: the problem is related to cluster algebras and the pentagon property of the dilogarithm.

Problem 2 (B. Shapiro) *Roots of a complex polynomial and its derivative and doubly stochastic matrices.* [2]

(A) Let P be a complex polynomial of degree n and $A = (a_1, ..., a_n)$ the vector of its roots (in arbitrary order). We construct a new vector $B = (b_1, ..., b_{n-1}, \beta)$, where b_1, ..., b_{n-1} are the roots of the derivative P' and $\beta = \sum a_i/n = \sum b_j/(n-1)$. Since the roots of the derivative lie inside the convex hull of initial roots a_i, there is a row-stochastic matrix M with last row $1/n, 1/n, ..., 1/n$ such that

$$B = MA,$$

where A and B are understood as column vectors. A row-stochastic matrix consists of non-negative numbers the sum of which in every row is 1.

If $n > 3$, then the choice of such matrix M is not unique. The question is whether it is possible to choose it among doubly stochastic matrices (such that the sums in every column are also 1).

(B) Problem (A) has the following generalization to an arbitrary dimension $k \geq 2$ ($k = 2$ corresponds to the case of point sets in the plane $\mathbb{C}$ considered above).

Definition 1. Given an unordered n-tuple V of vectors (points) $(v_1, ..., v_n)$ in $\mathbb{R}^k$ let us denote by $\mathrm{Pol}(V)$ the polytope in $\mathrm{Mat}(n, k)$ obtained as follows. (Here $\mathrm{Mat}(n, k)$ is the space of real $n \times k$-matrices and we assume $k \leq n$). For each ordering $i_1, ..., i_n$ of the given vectors we get a matrix in $\mathrm{Mat}(n, k)$ with the rows $v_{i_1}, ..., v_{i_n}$. Take the convex hull of all these $n!$ matrices and call it $\mathrm{Pol}(V)$.

Definition 2. An unordered n-tuple V of vectors in $\mathbb{R}^k$ is said to be wider than another such n-tuple W if $\mathrm{Pol}(W)$ is contained in $\mathrm{Pol}(V)$.

Definition 3. Given an n-tuple V of unordered vectors in $\mathbb{R}^k$ we call its *derived n-tuple* $\mathrm{Der}(V)$ the following thing. Let us place unit electric charges at each point v_1, ..., v_n and consider their common electrostatic potential

$$F(x) = \sum_{i=1}^{n} \frac{1}{|x - v_i|^{k-2}}.$$

This function has $n - 1$ critical points (counted with multiplicities). Add the meanvalue $\sum_i v_i/n$ to this $(n-1)$-tuple and you get $\mathrm{Der}(V)$.

Conjecture. Any finite point set V is wider than its derivative $\mathrm{Der}(V)$.

For $k = 1$ the proof is easy. The case $k = 2$ coincides with Problem A above since the zeros of the electrostatic field coincide with the zeros of the derivative of the polynomial in 1 complex variable whose roots are given points on $\mathbb{C}$.

Problem 3 (O. Östlund) *Untangling plane curves without second Reidemeister move.*

[2] Last revision note: Problem (A) is solved by S. Malamud (`arXiv:math.CV/0304158`). The previously announced solution by S. Evdokimov and A. Vakulenko turned out tobe incorrect because they used a lemma of B. Shapiro found to be wrong.

Let Ω_1, Ω_2, Ω_3 be Reidemeister moves considered in the class of generic immersed plane curves (plane curves that may have only transversal double points as singularities):

Ω_1 (cusp move) consists in addition/deletion of a small loop;

Ω_2 (self-tangency move) is passing through a non-generic curve with a point self-tangency;

Ω_3 (triple point move) is passing through a non-generic curve that has a triple point.

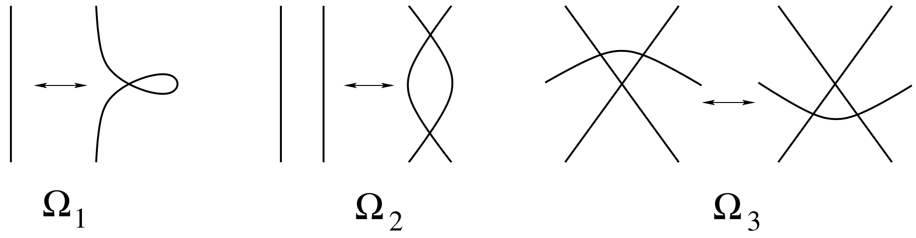


Figure 2: Reidemeister moves for curves

It is evident that every curve can be untangled (taken into the standard circle) by a sequence of Ω_1, Ω_2, Ω_3 moves (and smooth isotopies of the plane that do not influence singular points).

Conjecture. Every plane curve can be untangled using only Ω_1 and Ω_3 moves.

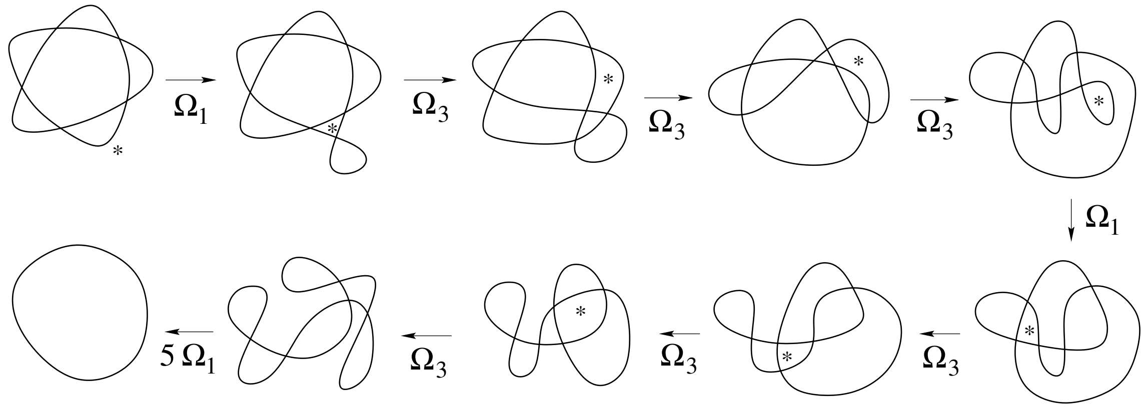


Figure 3: Untangling without Ω_2 moves (place of move shown with an asterisk)

Problem 4 (S. Tabachnikov) *Closed curve in a foliated domain.* [3]

Consider a topologically trivial domain D in the plane foliated by straightline segments. Let C be a closed immersed curve in D. *Conjecture*: there are two points of C on the same leaf with parallel tangent lines.

This conjecture is proved in some particular cases: (1) when the lines are all parallel or pass through one point, (2) when the winding number of C is non-zero. Is it true in general?

[3]This problem is solved. One of the known solutions is due to A. Malyutin who solved it in real time during the seminar.

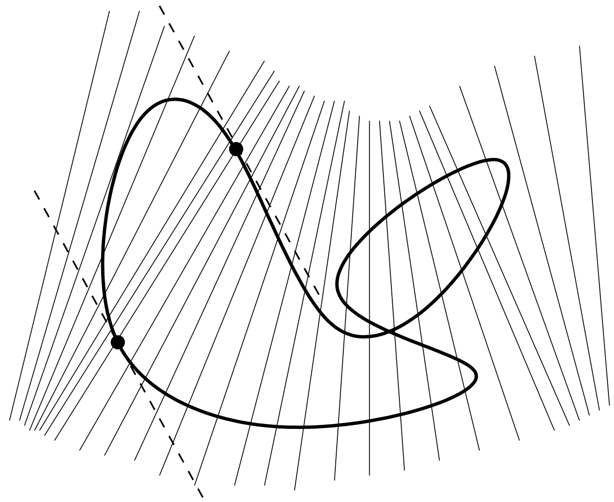


Figure 4: Parallel tangents at two points on one leaf

Problem 5 (V. I. Arnold, A. Ortiz) *Betti numbers of parabolic sets.*

Let $f(x,y)$ be a real polynomial in two variables. Denote by $P(f)$ the set of parabolic points on the surface $\{z = f(x,y)\}$, i.e. the zero set of the Hessian $H[f] = f_{xx}f_{yy} - f_{xy}^2$. Determine the maximal number of compact connected components of the set $P(f)$ for all polynomials f of given degree n.

This problem can be viewed as a specialization of the classical *oval counting problem* for polynomials representable in the form of a Hessian.

The first case when the answer is unknown is $n = 4$. Then $m = \deg H[f] = \deg f = 4$, and the Harnack inequality ensures that $b_0(P(f)) \leq (m-1)(m-2)/2 + 1 = 4$. There is a well-known construction of a polynomial ($uv - \varepsilon$, where $u = 0$, $v = 0$ are equations of ellipses that intersect in 4 points and ε is a small number) for which this estimate is attained. It is not known if it can be attained for polynomials of the form $H[f]$.

Problem 6 (V. I. Arnold) *Caustics of periodic functions.*

Let $g : S^1 \to \mathbb{R}$ be a smooth function and u, v two real parameters. The plane curve

$$C_g = \{(u,v) \mid \text{ function } G_{u,v}(\varphi) = g(\varphi) + u\cos\varphi + v\sin\varphi \text{ is not Morse}\}$$

is called the *caustic* of the function g. The condition that $G_{u,v}(\varphi)$ is not Morse means that there exists a value $\varphi \in S^1$ such that $G'_{u,v}(\varphi) = G''_{u,v}(\varphi) = 0$ (derivatives over φ).

Example. The caustic of the function $g(\varphi) = \cos 2\varphi$ is the astroid $u = -4\cos^3\varphi$, $v = 4\sin^3\varphi$.

For generic (Morse) functions g caustics are fronts (smooth curves with generic singularities) that satisfy a number of specific conditions:

1. A caustic has at least 4 cusps.
2. The number of cusps is even.
3. If P_1, P_2, ..., P_{2n} are cusps, then the barycentres of the sets P_1, P_3, ..., P_{2n-1} and P_2, P_4, ..., P_{2n} coincide. In particular, if $n = 2$, they form a parallelogram.

4. The alternating length of a caustic (we change sign after each cusp) is 0.
5. From every point of the plane one can draw at least two tangents to the caustic.
6. Caustics do not have inflexion points.

Problem. Describe all curves that are caustics of periodic functions, i.e. give a necessary and sufficient condition for a front to be a caustic.

Problem 7 (V. Vassiliev) *Loops in the space of knots.*

Given the figure-eight knot (or any other non-trivial knot equivalent to its mirror image), let us join it with its mirror image by a path in the space of knots, and then consider the mirror image of this path.

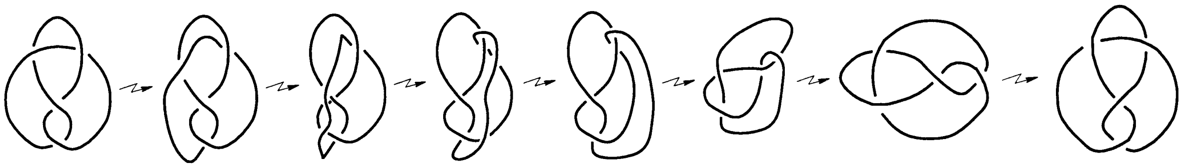


Figure 5: Deformation of figure 8 knot into its mirros image

What can be said about the homology (or homotopy) class of the obtained closed loop in the space of knots? Is it trivial?

Problem 8 (A. Skopenkov) *Plane projection of a spacial line arrangement.*

A number of lines is drawn in the plane so that each line is parallel either to the x-axis or to the y-axis. The intersection points of these lines are marked so as to show which lines should go above the other. When such a picture can be realized as a projection of a set of lines in 3-space? Does the answer depend only on combinatorial picture or also on geometry (i.e. on distances between intersection points)?

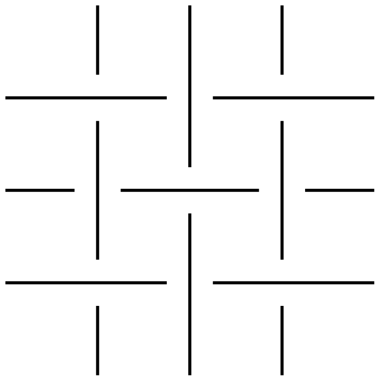


Figure 6: A configuration not realizable by straight lines

Problem 9 (D. von der Flaass) *Real sequence under constraints.*

A doubly-infinite sequence $\{x_n\}_{n\in\mathbb{Z}}$ is said to satisfy the constraints function $f(d)$ defined for all positive integers d if $|x_n - x_m| \geq f(|n-m|)$ for all $n \neq m$. For the constraints function $f(d) = 1/d$, find the minimum span of a sequence satisfying it. By the span (finite or infinite) we mean the difference of the supremum and the infimum of the sequence. The conjectured answer is $\varphi + 1$, where $\varphi = (1+\sqrt{5})/2$, is the golden ratio. This span is achieved by the sequence $x_n = n \bmod (\varphi + 1)$. For a motivation and some details, see `http://www.cdam.lse.ac.uk/Reports/Abstracts/cdam-98-12.html`

Problem 10 (S. Duzhin) *Decomposable skew functions.*

A (real) function of n (real) variables is said to be *skew-symmetric*, if it changes sign whenever any two variables are interchanged:

$$f(x_1, ..., x_i, ..., x_j, ..., x_n) = -f(x_1, ..., x_j, ..., x_i, ...x_n).$$

A skew-symmetric function $f(x_1, ..., x_n)$ is *decomposable*, if there exist functions of one variable f_1, ..., f_n such that

$$f(x_1, ..., x_n) = \det \|f_i(x_j)\|_{i,j=1}^n.$$

Theorem. In the class of analytic functions (or in any ring of functions without zero divisors) a skew-symmetric function $f(x_1, ..., x_n)$ is decomposable if and only if it satisfies the identity

$$f(x_1, x_2, ...)f(x_3, x_4, ...) - f(x_1, x_3, ...)f(x_2, x_4, ...) + f(x_1, x_4, ...)f(x_2, x_3, ...) = 0, \qquad (1)$$

where the dots mean one and the same set of $(n-2)$ variables.

Now, besides the above notion of complete decomposability, one can consider partially decomposable skew-symmetric functions. If $\lambda = (\lambda_1, ..., \lambda_k)$ is a partition of n, then by a λ-decomposable skew-symmetric function of n variables we understand the complete antisymmetrization of the product of k arbitrary functions of λ_1, ..., λ_k variables. The partition $(1, 1, ..., 1)$ gives completely decomposable functions, while the partition (n) yields the class of all skew-symmetric functions in n variables.

Problem. For a given $\lambda \vdash n$, find a criterion of λ-decomposability.

Problem 11 (S. Duzhin) *Hilbert's Sixteenth problem with separated variables.*

Hilbert's sixteenth problem concerns the number and mutual position of ovals (circular connected components) of an algebraic curve defined by the equation $\varphi(x, y) = 0$. We ask the same questions in the case of polynomials with additively separated variables: $\varphi(x, y) = f(x) - g(y)$ and, in particular, $\varphi(x, y) = f(x) - f(y)$. If the polynomial $f(x)$ is a Morse function ($f'(x)$ has no multiple roots), then the curve $\varphi(x, y) = 0$ consists of one infinite straightline component $x = y$ and a number of ovals. The combinatorics of the oval

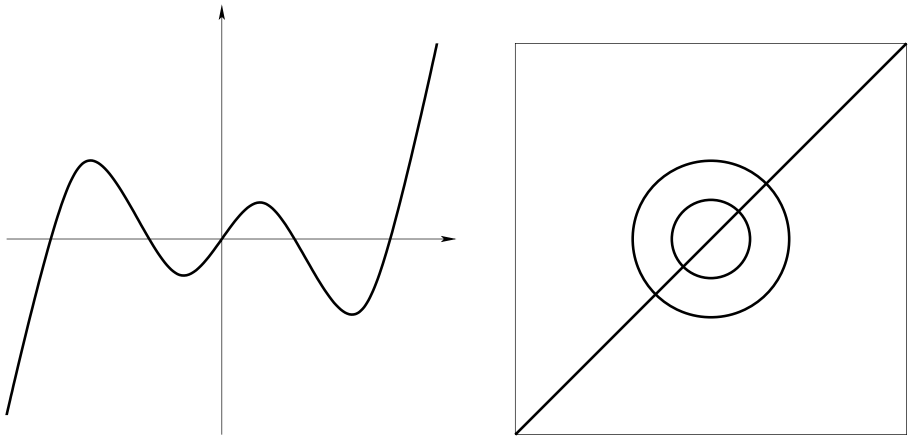


Figure 7: Graph of $f(x)$ and the curve $f(x) = f(y)$

arrangement depends only on the up-down permutation that describes the order of critical values of the polynomial.

Problem 12 (W. Known) *Big Moore Graph.*

The Big Moore Graph is defined as a regular graph of degree 57 with 3250 vertices and diameter 2. *Problem: does it exist?* In other words, is it possible to organize air traffic in a country with 3250 cities so that there are 57 air routes flying from each city and any two cities are connected either by a direct flight or by two consecutive flights with one transfer?

Problem 13 (S. Vorobyov `<vorobyov@csd.uu.se>`) *Completely unimodal pseudo-boolean functions.*

A pseudo-boolean function (PBF) is an injective mapping from $\{0,1\}^n \to [1...2^n]$. Call a PBF *completely unimodal* if it has a unique local minimum on each 2-dimensional face, (e.g., a square clockwise numbered 1234 is a CU PBF, but 1324 is not: 1 and 2 are two local minima).

QUESTIONS:

1) How many n-dimensional CU PBF are there? (counting or not counting symmetries, whichever is simpler, asymptotical growth is enough)

2) How could one construct large (say for $n > 10$; I do not know how large are "small dimensions" `;-)` *random* nontrivial instances of CU PBF?

Here is the first nontrivial example: it is Hamiltonian, but most importantly, it has a *saddle* — four coplanar points 1845 with two local minima 1, 4:

The following is a trivial construction:

— take two identical CU PBF of size n,

— add 2^n to each vertex of the 2nd one,

— glue them as two n-dimensional cubes to form the $n+1$-dimensional one.

Also "trivial" is a *completely absorbing* CU PBF:

— either it has dimension 1,

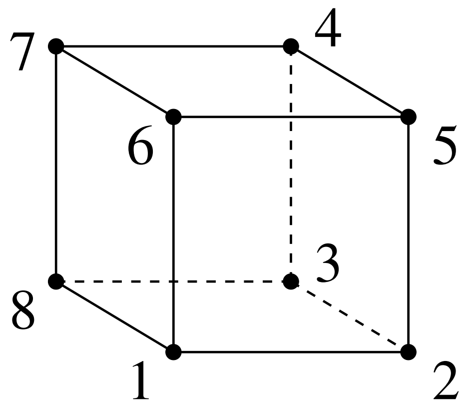


Figure 8: A non-trivial PBF

— or there is a facet F (of dimension $n-1$) such that each vertex on it is smaller than its neighbor on the opposite facet, and F itself is completely absorbing.

The trivial construction above produces completely absorbing cubes.

3) Obviously, any CU PBF has a unique global minimum. A trivial algorithm to find it is the so-called Randomized Local Search: start at any vertex and iteratively switch to its uniformly and randomly chosen better neighbor. Obviously, you do not have to take more than 2^n iterations.

It is known that for completely absorbing CU PBF the above algorithm stops in expected $O(n^2)$ time (polynomial in n).

A big open problem is: *is it polynomial or not for arbitrary CU PBFs*?

Problem 14 (Boris Shapiro <shapiro@matematik.su.se>) *Counting rational M-functions.*

Consider a real rational function f in one variable of degree n. In general position it has $2n-2$ critical points and values. Suppose they are all real (f is an M-function). Then the restriction of this mapping to the real line induces a mapping $S^1 \to S^1$ with $2n-2$ ramification points. Count the number of topological types of such mappings.

Problem 15 (V. Vassiliev <vassil@vassil.mccme.rssi.ru>) *Energy-type functionals on the space of knots.*

Prove that all (or many) energy- or thickness- type functionals on the space of knots are quite useless for the algorithmic separation of knots. More precisely: prove that in any component of the space of knots (e.g. in the component of the unknot) there are infinitely many local minima of this functional.

In the case of the "thickness" functional this can be formulated as follows. Given an empty bike balloon, we can fold and tie it in such a way that, after we start to fill it in, it will not be unfolded to the regular circle-wise form (and moreover we can do it in infinitely many different ways, so that all the resulting stable positions will be not $O(3)$-equivalent). There are obvious candidates for such foldings, but, as far as I know, the propperty was not proved for them.

Problem 16 (Arkady Skopenkov <skopenko@mccme.ru>) *Invariant of embeddings of surfaces into 4-space.*

Define an invariant of embeddings $N \to \mathbb{R}^4$ of a closed orientable 2-manifold N assuming its values in $H_1(N)$ and prove that every element of $H_1(N)$ can be realized as an invariant of some embedding.

See A. Skopenkov, Commentarii Matematici Helvetici, 72 (1997) 543–555, a new paper of the same author to appear in the same journal, and references therein.

Problem 17 (Elena Kudryavtseva <eakudr@mech.math.msu.su>) *Self-intersections of curves on surfaces.*

Let S be a closed orientable surface and g_1, g_2 are two nontrivial elements of its fundamental group. Denote by n_i the minimal number of self-intersection points of a curve in the class g_i. Prove or disprove the following conjecture.

If g_2 belongs to the normal subgroup generated by g_1, then

(1) $n_1 \leq n_2$;

(2) if $n_1 = n_2$, then g_2 is a conjugate of g_1 or g_1^{-1}.

The author can prove this conjecture in the case when $n_1 = 0$, i.e. g_1 contains a simple curve, and also in a certain more general case.

Problem 19 (J.-O. Moussafir <msfr@ceremade.dauphine.fr>) *Empty integral simplices.*

An integral simplex in R^d is a simplex whose vertices belong to $\mathbb{Z}^d$. An integral simplex is said to be empty when its intersection with $\mathbb{Z}^d$ is exacly the set of vertices. The problem is to find an arithmetic characterization of empty integral simplices.

The problem is solved in dimension 1, 2, 3, and open for $d \geq 4$.

When the dimension is 2, a simplex is empty if and only if its area is 1/2.

It is clear that if s is the simplex, represented by a $d \times d$ matrix, then $|\det s| = 1$ is a sufficient condition. But when $d \geq 3$, there are integral empty simplices with arbitrarily big volume!

When $d = 3$. The arithmetic characterization is the following: Consider 2 integral parallel planes (i.e. planes containaing a 2-dimensionnal sublattice of $\mathbb{Z}^3$). They are said to be neigbours if there is no point of $\mathbb{Z}^3$ between them.

An integral simplex s is empty $\Leftrightarrow$ there are two integral parallel planes P_1 an P_2 such that P_1 and P_2 are neigbours P_1 contains 2 vertices of s, say a and b P_2 contains the other vertices of s, say c and d the only points of $\mathbb{Z}^3$ that lie in $[a, b]$ and $[c, d]$ are a, b, c and d.

For instance, if you consider two integral points in the plane $z = 0$, and two integral points in the plane $z = 1$, and if there is no points of $\mathbb{Z}^3$ between them, the corresponding simplex is empty. It is easy to see because there are no points of $\mathbb{Z}^3$ between $z = 0$ and $z = 1$. This example is generic. Modulo $SL(3, \mathbb{Z})$, every empty simplex is of that kind.

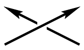
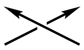
The previous characterization is arithmetic. One can express the condition in terms of GCD etc.

I have been told that H. Scarf had proven the result when $d = 3$, but I don't have his paper.

There is a thesis that deals with this question: Christian Haase, "Lattice Polytopes and Triangulations with applications to toric geometry".

Problem 20 (S. Chmutov `<chmutov@math.ohio-state.edu>`) *Casson's invariant modulo n.*

The Conway polynomial $C_{\mathcal{L}}(z)$ of a link $\mathcal{L}$ is one of the simplest invariants of links in $\mathbb{R}^3$. It is defined by two properties: it is one on the unknot and satisfies the skein relation

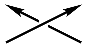
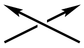
C  $(z) - C$  $(z) = z \cdot C$  (z) ,

where again we mean three links identical outside a small ball in $\mathbb{R}^3$ and which look inside the ball as shown.

If $\mathcal{L}$ is a knot then the Conway polynomial is even:

$$C_{\mathcal{L}}(z) = 1 + C_2(\mathcal{L}) \cdot z^2 + C_4(\mathcal{L}) \cdot z^4 + \ldots + C_{2n}(\mathcal{L}) \cdot z^{2n} \ .$$

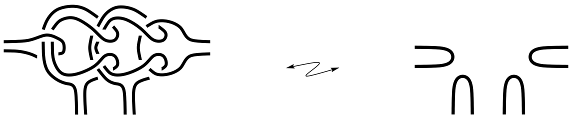
Definition. The coefficient $C_2(\mathcal{L})$ is called *Casson's knot invariant* (see [PV]). It can be defined ([Ka1]) by the initial condition $C_2(\mathit{unknot}) = 0$ and the skein relation:

$C_2($  $) - C_2($  $) = lk($  $)$,

where lk means the linking number of the two component link in the right hand side of the relation.

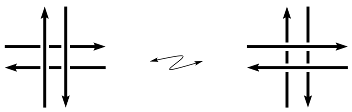
The *mod* 2 reduction of Casson's invarisnt $C_2(\mathcal{L})$ is called *Arf invariant* of a knot $\mathcal{L}$.

Recently K.Habiro [Ha] proved that *for two knots K_1 and K_2, $C_2(K_1) = C_2(K_2)$ if and only if K_1 can be obtained from K_2 by finite number of moves:*



This move looks quite complicated. It might be that if we restrict ourselves with C_2 modulo some n then the corresponding moves would be simplier. And indeed there is the following theorem by L.Kauffman.

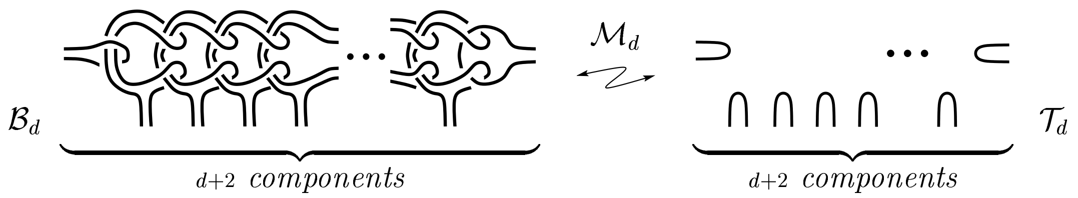
Theorem (L.Kauffman [Ka2]). *K_1 and K_2 have the same Arf invariant if and only if K_1 can be obtained from K_2 by finite number of so called pass-moves:*



Problem A (L.Kauffman): Find the moves which relate the knots with the same Casson's invariant modulo n, for $n = 3, 4, \ldots$

In fact K.Habiro found a description of the set of knots non distinguishable by Vassiliev invariants of order $\leq n$. More precisely:

Theorem (Habiro [Ha]). *Let K_1 and K_2 be two knots. Then $v(K_1) = v(K_2)$ for any Vassiliev invariant v of order $\leq d$ if and only if K_1 and K_2 are related by a finite sequence of moves $\mathcal{M}_d : \mathcal{B}_d \leftrightsquigarrow \mathcal{T}_d$:*



Problem B: Find the moves relating any two knots with the same Vassiliev invariants modulo 2 (3, 4, ..., n) up to the order d.

Problem C: Find the moves relating any two knots with the same Conway polynomial.

References

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[PV] M. Polyak, O. Viro, *On the Casson Knot Invariant*, preprint `math.GT/9903158`, to appear in Journal of Knot Theory and Its Ramifications.

Problem 21 (B. Shapiro `<shapiro@matematik.su.se>`, M. Shapiro `<mshapiro@math.kth.se>`). *Inflexion points of algebraic curves.*

This question is a special case of a general conjecture but we find it worthy to formulate it separately.

Consider a generic rational curve $g : \mathbb{C}P^1 \to \mathbb{C}P^2$ of degree n. (By generic we mean that g has only double points which are unavoidable if $n > 2$ and no other singularities.) Such a curve has exactly $3(n-2)$ distinct inflection points, see e.g. Plücker formulas. We call by inflection preimages the inverse images of inflection points on $\mathbb{C}P^1$.

Conjecture. Assume that all inflection preimages lie on $\mathbb{R}P^1$ in $\mathbb{C}P^1$. Then g can be made into a real curve by a projective transformation in $\mathbb{C}P^2$.

A part of these problems were announced on October 5, 2001 at the Moscow-Petersburg seminar on Low-Dimensional Mathematics.

Names in parenthesis refer to authors or people who communicated to me these problems. Some problems come with additions and modifications on my part.

Home page of the seminar `http://www.pdmi.ras.ru/~lowdimma`