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Claim 1 *Let $a \in \mathbb{R}$ (or $a \in \mathbb{C}$). Define the rational map f_a of the plane (or $\mathbb{C}^2$ respectively) by*

$$f_a(x,y) = \left(\frac{1}{y}, \frac{x(1+ay+y^2)}{1+y}\right).$$

If $a \neq 2$ then f_a is not periodic.

Proof. Put $b = 2 - a$. Thus, $b \neq 0$. Let p and q be two reals which are linear independent over $\mathbb{Q}$. In particular, neither p nor q are 0's. Consider the field $K = \mathbb{Q}(p,q,b)$ and take any $z \in \mathbb{R}$ which is transcendental over K.

We start with the point (x_0, y_0) whose coordinates are rational functions in z with coefficients from K. The exact functions will be chosen later. Since z is transcendental over K, such functions behave under the action of f_a in the same way as elements of the field $K(z)$, which has transcendence degree 1 over K. In particular, we may consider *formal* Laurent series of these rational functions.

Now, let

$$x_0 = \frac{P_0(z)}{Q_0(z)}, \qquad y_0 = \frac{R_0(z)}{S_0(z)},$$

where

$$\frac{P_0(z)}{Q_0(z)} = -1 + pz + \cdots$$
$$\frac{R_0(z)}{S_0(z)} = -1 + qz + \cdots.$$

We are interested in the first two terms of Laurent series expansion of $f_a^i(x_0, y_0)$. However, to keep this information we also need next terms in the expansion of x_0 and y_0. Write down

$$\frac{P_1(z)}{Q_1(z)} = -1 + pz + rz^2 + \cdots$$
$$\frac{P_2(z)}{Q_2(z)} = -1 + qz + sz^2 + tz^3 \cdots ,$$

for some r, s and t in K. Let $(x_i, y_i) = f_a^i(x_0, y_0)$. A direct computation shows that

$$\begin{aligned}
x_1 &= -1 - qz - (q^2 + s)z^2 + \cdots , \\
y_1 &= -\frac{b}{q}z^{-1} + \frac{bpq + bq^2 + sb}{q^2} + \frac{-bpq^3 - bpqs - q^4 + bq^2r + bqt - s^2b}{q^3}z + \cdots , \\
x_2 &= -\frac{q}{b}z - \frac{qp + q^2 + s}{b}z^2 + \frac{-bqp^2 - bq^2p - bps + (1-b)q^3 - bqr - 2bqs - tb}{b^2}z^3 + \cdots , \\
y_2 &= \frac{b}{q}z^{-1} - \frac{bqp + (-b+1)q^2 + sb}{q^2} + \frac{bqps + (b+1)q^4 - bq^2r - bqt + bs^2}{q^3}z + \cdots , \\
x_3 &= \frac{q}{b}z + \frac{bqp + (-b+1)q^2 + sb}{b^2}z^2 + \cdots , \\
y_3 &= -1 - qz - \frac{(2b+1)q^2 + sb}{b}z^2 + \cdots , \\
x_4 &= -1 + qz + \cdots , \\
y_4 &= -1 + (-p + 2q)z + \cdots .
\end{aligned}$$

Fortunately, the first two coefficients in the Laurent series expansions for x_4 and y_4 do not depend on r, s and t. Hence, we can iterate our procedure and prove by induction that

$$x_{4k} = -1 + (kq - (k-1)p)z + \cdots$$
$$y_{4k} = -1 + ((k+1)q - kp)z + \cdots .$$

Since z is transcendental over K, the above evaluation shows also that the points $f_a^i(x_0, y_0)$ are well defined for any $i \geq 0$. Let

$$x_{4k} = \frac{P_k(z)}{Q_k(z)}.$$

Since p and q are linearly independent over $\mathbb{Q}$, the second terms in the (*formal*) series expansions of $\frac{P_k(z)}{Q_k(z)}$ and $\frac{P_l(z)}{Q_l(z)}$ are different, provided $k \neq l$. Hence, the difference $\frac{P_k(z)}{Q_k(z)} - \frac{P_l(z)}{Q_l(z)}$ is not zero as a rational function in the indeterminate z. Since z is transcendental over K, this implies that $\frac{P_k(z)}{Q_k(z)} - \frac{P_l(z)}{Q_l(z)} \neq 0$. Hence, f_a^4 and therefore f_a are not periodic. $\square$