A COMBINATORIAL CONJECTURE RELATED WITH COMPLEX BOUNDED SYMMETRIC DOMAINS

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1. Motivations

Let Ω be a bounded irreducible symmetric domain. To each such Ω are attached numerical invariants: the *multiplicities* a and b, and the *rank* r. These three invariants characterize the domain up to isomorphism. Hereunder is the table of all possible values, corresponding to the classification of irreducible complex bounded symmetric domains:

Type		a	b	r	g	d
$I_{m,n}$	$(1 \leq m \leq n)$	2	$n-m$	m	$m+n$	mn
II_n	$(n \geq 2)$	4	$\begin{cases} 0 & (n=2p) \\ 2 & (n=2p+1) \end{cases}$	$\left[\frac{n}{2}\right]$	$2\,(n-1)$	$\frac{n(n-1)}{2}$
III_n	$(n \geq 1)$	1	0	n	$n+1$	$\frac{n(n+1)}{2}$
IV_n	$(n > 2)$	$n-2$	0	2	n	n
V		6	4	2	12	16
VI		8	0	3	18	27

In the last two columns of this table

$$g = 2 + a(r-1) + b$$

is the *genus* of the bounded symmetric domain Ω and

$$d = \dim_{\mathbb{C}} \Omega = r + \frac{r(r-1)}{2} a + rb$$

its complex dimension.

The *Hua polynomial* χ of the bounded symmetric domain Ω is then defined by

$$\chi(s) = \prod_{j=1}^{r} \left(s + 1 + (j-1)\frac{a}{2} \right)_{1+b+(r-j)a}, \tag{1}$$

where $(s)_k$ denotes the *raising factorial*

$$(s)_k = s(s+1)\cdots(s+k-1) = \frac{\Gamma(s+k)}{\Gamma(s)}.$$

The polynomial χ is related to the *Hua integral* $\int_\Omega N(z,z)^s \omega(z)$ (where N is the *generic norm* for Ω) by

$$\int_\Omega N(z,z)^s \omega(z) = \frac{\chi(0)}{\chi(s)} \int_\Omega \omega \qquad (s > -1).$$

(see [1]).

One checks easily that $\deg \chi = d$. The expression of χ for the different types of bounded symmetric domains is as follows:

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- Type $\mathrm{I}_{m,n}$ $(1 \leq m \leq n)$: $\chi(s) = \prod_{j=1}^{m} (s+j)_n$.
- *Type* II_{2p}: $\chi(s) = \prod_{j=1}^{p} (s+2j-1)_{2p-1}$.
- *Type* II_{2p+1}: $\chi(s) = \prod_{j=1}^{p} (s+2j-1)_{2p+1}$.
- *Type* III_n: $\chi(s) = \prod_{j=1}^{n} \left(s + \frac{j+1}{2}\right)_{1+n-j}$.
- *Type* IV_n: $\chi(s) = (s+1)_{n-1} \left(s + \frac{n}{2}\right)$.
- *Type* V: $\chi(s) = (s+1)_8 (s+4)_8$.
- *Type* VI: $\chi(s) = (s+1)_9 (s+5)_9 (s+9)_9$.

Let $\mu \in \mathbb{R}$, $\mu > 0$. The following expansion

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j} \frac{(k+1)_j}{j!}. \tag{2}$$

and the associated rational function

$$F_{\chi,\mu}(t) = \sum_{j=0}^{d} c_{\mu,j} \left(\frac{1}{1-t}\right)^j \tag{3}$$

play a key role in the computation of the Bergman kernel of some Hartogs domains built over bounded symmetric domains (see [1], [2]).

2. The conjecture

Let

$$\mu_0 = \frac{g}{d+1}.$$

Conjecture. *The coefficients* $c_{\mu,j}$ *in (2):*

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{d} c_{\mu,j} \frac{(k+1)_j}{j!}.$$

are all strictly positive if and only if

$$\mu < \mu_0.$$

For $\mu = \mu_0$*, all coefficients* $c_{\mu,j}$ *in (2) are strictly positive, except* $c_{\mu,d-1} = 0$ *and except for the rank* 1 *type* $I_{1,n}$ *(where* $c_{\mu,d} = 1$ *and* $c_{\mu,j} = 0$ *for all* $j < d$*).*

We call μ_0 the *critical exponent* for Ω. The values of the critical exponent are

Type	$I_{m,n}$	II_n	III_n	IV_n	V	VI
μ_0	$\frac{m+n}{mn+1}$	$\frac{4}{n+\frac{2}{n-1}}$	$\frac{2}{n+\frac{1}{n+1}}$	$\frac{n}{n+1}$	$\frac{12}{17}$	$\frac{9}{14}$

We have always $\mu_0 < 1$, except in the rank 1 case $I_{1,n}$.

Remark 1. The conjecture has been checked with help of computer algebra software in many significant cases:

- for $\mu = \mu_0$ and the types $I_{3,3}$, IV_3, IV_4, IV_6, V, VI;
- for type V and all values of μ.

Remark 2. The coefficients $c_{\mu,j}$ are polynomial functions of μ and linear combinations of

$$\left\{\chi(k\mu) \mid k = -1, -2, \ldots, -j-1\right\}.$$

The conjecture is equivalent to the following properties:

- the least positive root of $\mu \mapsto c_{\mu,d-1}$ is μ_0;

- all positive roots of $\mu \mapsto c_{\mu,j}$ $(0 \le j < d-1)$ are greater than μ_0.

Remark 3. As the function $F_{\chi,\mu}$ is related to the Bergman kernel of a family of bounded (non homogeneous) domains, it is known that all derivatives $F^{(k)}_{\chi,\mu}$ $(k > 0)$ of this function are strictly positive on $[0,1[$ for all $\mu > 0$. The conjecture says more about this function when $0 < \mu < \mu_0$.

Remark 4. If the conjecture is true, it would allow to compare the Bergman metric of some Hartogs domains built over bounded symmetric domains, with the Kähler-Einstein metric of the same domains.

3. Examples

3.1. **Type V (the exceptional case of dimension 16).** The critical exponent is

$$\mu_0 = \frac{12}{17}.$$

The *Hua polynomial* χ is

$$\chi(s) = (s+1)_8(s+4)_8.$$

A computation with Maple gives

$$\chi\left(\frac{12}{17}s\right) = \left(\frac{12}{17}\right)^{16} \sum_{j=0}^{16} c_j (s+1)_j,$$

with

$$c_{16} = 1, \qquad c_{15} = 0, \qquad c_{14} = \frac{595}{12}, \qquad c_{13} = \frac{4165}{6}, \qquad c_{12} = \frac{30042145}{3456},$$
$$c_{11} = \frac{14448385}{144}, \qquad c_{10} = \frac{790269316375}{746496}, \qquad c_9 = \frac{1259425781075}{124416},$$
$$c_8 = \frac{12447571001586875}{143327232}, \qquad c_7 = \frac{2957566710311675}{4478976},$$
$$c_6 = \frac{113001256229424966725}{2579890176}, \qquad c_5 = \frac{10677213117341703625}{429981696},$$
$$c_4 = \frac{651907704485453963318125}{557256278016}, \qquad c_3 = \frac{1020948478836605654 9125}{23219011584},$$
$$c_2 = \frac{114818904611324955416375}{92876046336}, \qquad c_1 = \frac{357792528548153074626 25}{15479341056},$$
$$c_0 = \frac{33368892412222545303125}{15479341056}.$$

This shows that *all coefficients* $c_{\mu_0,j}$ $(0 \le j \le 16)$ in

$$\frac{\chi(k\mu)}{\chi(0)} = \sum_{j=0}^{16} c_{\mu,j} \frac{(k+1)_j}{j!}$$

are strictly positive, except $c_{\mu_0,15} = 0$.

3.2. **Type VI (the exceptional case of dimension 27).** The critical exponent is

$$\mu_0 = \frac{9}{14}..$$

The *Hua polynomial* χ is

$$\chi(s) = (s+1)_9(s+5)_9(s+9)_9.$$

A computation with Maple gives

$$\chi\left(\frac{9}{14}s\right) = \left(\frac{9}{14}\right)^{27} \sum_{j=0}^{27} c_j (s+1)_j,$$

with

$$c_{27} = 1, \qquad c_{26} = 0, \qquad c_{25} = \frac{2275}{9},$$
$$c_{24} = \frac{56875}{9}, \qquad c_{23} = \frac{38591735}{243},$$
$$c_{21} = 15425515970150/3^{11},$$
$$c_{20} = 37061881356500/3^{9},$$
$$c_{19} = 184328710104188650/3^{14},$$
$$c_{18} = 3564334218619774600/3^{14},$$
$$c_{17} = 584735324681177419750/3^{16},$$
$$c_{16} = 10020732894163060819750/3^{16},$$
$$c_{15} = 352001611351295587864253500/3^{23},$$
$$c_{14} = 586664566244061492395923000/3^{21},$$
$$c_{13} = 1988637252859632373297511212000/3^{26},$$
$$c_{12} = 25672251717038124392289396301000/3^{26},$$
$$c_{11} = 8233663487061605972803486331644375/3^{29},$$
$$c_{10} = 89384793443821000370862374382625000/3^{29},$$
$$c_9 = 1923754293102540042201539198326959366875/3^{36},$$
$$c_8 = 209778908005712588859591649123533801875/3^{32},$$
$$c_7 = 399192552377373476318550395682751432975625/3^{37},$$
$$c_6 = 2728484170046421839052459199725228012518750/3^{37},$$
$$c_5 = 47840351197962492631409316902739852226831250/3^{38},$$
$$c_4 = 232468257762517753158460641861539626710125000/3^{38},$$
$$c_3 = 73037107041363504672642146434776735686797778125/3^{42},$$
$$c_2 = 23557400955895564936769134062033297681918662500/3^{40},$$
$$c_1 = 409456797752799914624225389536137199476376953125/3^{42},$$
$$c_0 = 394594700340674453245747775040231797415576953125/3^{42}.$$

This shows again that *all coefficients* $c_{\mu_0,j}$ $(0 \leq j \leq 27)$ in

$$F_{\chi,\mu_0}(t) = \sum_{j=0}^{27} c_{\mu_0,j} \left(\frac{1}{1-t}\right)^j$$

are strictly positive, except $c_{\mu_0,26} = 0$.

References

[1] Yin Weiping, Lu Keping, Roos Guy, New classes of domains with explicit Bergman kernel, *Science in China* Ser. A Mathematics, **47**(2004), 352–371.

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[3] Wang An, Yin Weiping, Zhang Liyou, Roos Guy, The Kähler–Einstein metric for some Hartogs domains over bounded symmetric domains, Preprint, 2004.

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