An Explanation of Some Exotic Continued Fractions Found by Brillhart

H. M. STARK*

Department of Mathematics, M.I.T., Cambridge, Massachusetts, U.S.A. and The University of Michigan, Ann Arbor, Michigan, U.S.A.

1. Introduction

Late in 1964, John Brillhart embarked upon some extended computations of the continued fraction expansions of cubic irrationalities. Assisted by Michael Morrison, he was hoping that some kind of pattern would emerge—any such pattern would of course be of tremendous value if it could be proved to exist. No pattern was found but something equally unexpected occurred. The real root of the equation

$$x^3 - 8x - 10 = 0 \tag{1}$$

was found to have the continued fraction expansion:

$$x = [3, 3, 7, 4, 2, 30, 1, 8, 3, 1, 1, 1, 9, 2, 2, 1, 3, 22986, 2, 1, 32, 8, 2, 1, 8, 55, 1, 5, 2, 28, 1, 5, 1, 1501790, \ldots].$$

Altogether 8 partial quotients over 10000 were found: if we write $x = [a_0, a_1, a_2, \ldots]$ then

$$a_{17} = 22986, \quad a_{33} = 1501790, \quad a_{59} = 35657, \quad a_{81} = 49405,$$
$$a_{103} = 53460, \quad a_{121} = 16467250, \quad a_{139} = 48120, \quad a_{161} = 325927.$$

This was brought to my attention by D. H. Lehmer who noted that the discriminant of (1) is $-4 \cdot 163$ and asked if the amazingly large partial quotients found were related to the fact that the class-number of $Q(\sqrt{-163})$ is one.

A. O. L. Atkin later brought to my attention a fact which greatly helped

* This research was supported in part by the N.S.F. under contract GP–13630

towards an affirmative answer to Lehmer's question. He noted that if we translate (1) by setting $x + 2 = f$, then we get the new equation

$$f^3 - 6f^2 + 4f - 2 = 0. \tag{2}$$

This equation may be found on p. 725 of Weber (1908). Its occurrence there relates to the quadratic field $Q(\sqrt{-163})$ and its form is due to the fact that the class-number of this field is one. Let us define the Schläfli modular function f by

$$f(z) = q^{-1/48} \prod_{n=1}^{\infty} (1 + q^{n-1/2}), \qquad q = e^{2\pi i z}, \qquad \operatorname{Im} z > 0. \tag{3}$$

Then $f(\sqrt{-163})$ is the real root of (2)!

2. Modular Functions and Quadratic Fields

Define $\gamma_2(z)$ by the equation

$$f(2z + 3)^{24} + \gamma_2(z) f(2z + 3)^{16} - 256 = 0 \tag{4}$$

and set

$$j(z) = \gamma_2(z)^3. \tag{5}$$

The function $j(z)$ is regular inside the upper halfplane and invariant under the full modular group, i.e.,

$$j\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) = j(z) \text{ if } \alpha\delta - \beta\gamma = 1 \text{ and } \alpha, \beta, \gamma, \delta \text{ are integers;}$$

this property and the first two terms of the expansion

$$j(z) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots$$

completely determine $j(z)$.

Let $d < 0$ be the discriminant of the complex quadratic field $Q(\sqrt{d})$ and let $h(d)$ be its class-number. If 1 and ω form an integral basis of $Q(\sqrt{d})$ then the importance of $j(z)$ is illustrated by the fact that $j(\omega)$ generates the maximal unramified abelian extension of $Q(\sqrt{d})$ when it is adjoined to $Q(\sqrt{d})$. In fact $j(\omega)$ is an algebraic integer of degree exactly $h(d)$. For our purposes here, we will assume for the rest of this paper that

$$|d| \equiv 3 (\text{mod } 8) \text{ and } 3 \nmid d. \tag{6}$$

In this case, we may take $\omega = \dfrac{-3 + \sqrt{d}}{2}$. But now in fact $j(\omega)$ is a perfect cube; $\gamma_2(\omega)$ is also an algebraic integer of degree $h(d)$ (Weber, 1908).

Let

$$j = j\left(\frac{-3 + \sqrt{d}}{2}\right), \qquad \gamma = \gamma_2\left(\frac{-3 + \sqrt{d}}{2}\right), \qquad f = f(\sqrt{d}).$$

We see from (4) that

$$f^{24} + \gamma f^{16} - 256 = 0. \tag{7}$$

Thus f^8 is the root of a cubic equation over $Q(\gamma)\,[= Q(j)]$ and hence is an algebraic integer of degree $\leqslant 3h(d)$. In fact, with the restriction (6), f^8 is an algebraic integer of degree exactly $3h(d)$. But in fact we can do better. Weber (1908) conjectured that f is itself an algebraic integer of degree $3h(d)$. He verified this conjecture in many numerical cases, including $d = -163$, but the conjecture itself has only recently been proved by Birch (1968) (Weber did prove that f^2 has degree $3h(d)$).

For notational convenience we write the (unique) cubic equation for f^k over $Q(j)$ as

$$f^{3k} + B_k f^{2k} + A_k f^k - 2^k = 0 \tag{8}$$

where k is a positive integer. Here A_k and B_k are integers in $Q(j)$. When we deal with $d = -163$, we get $B_1 = -6$, $A_1 = 4$ which yields (2).

3. Our Basic Goal and some Numerical Tests Thereof

It is now clear that there is a relation between (1) and $Q(\sqrt{-163})$. Churchhouse and Muir (1969) have recently made some extended computations which related the successive large partial quotients of $f(\sqrt{-163})$ with the successive factors in (3). They also have a very nice discussion of what it means for several partial quotients of $f(\sqrt{-163})$ to be "larger than expected". Our aim is different from theirs. If indeed the theory of modular functions is responsible for the large partial quotients found, then we should be able to find modular functions that converge to the numerator and denominator of the corresponding convergents at $\frac{1}{2}(-3 + \sqrt{-163})$ and the ratio of these modular functions as series in z should give exceptionally good approximations to $f(2z + 3)$. Our goal is to find such functions.

If indeed this is possible then we would expect $f(\sqrt{d})$ to have some excellent approximations for other d satisfying (6) and $h(d) = 1$. There are four other fields which satisfy these conditions. Computations then reveal the following:

For $d = -67$, $f^3 - 2f^2 - 2f - 2 = 0$,
$f = [2, 1, 11, 2, 3, 1, 23, 2, 3, 1, 1337, 2, 8, 3, 2, 1, 7, 4, 2, 2, 87431, \ldots]$.

For $d = -43$, $f^3 - 2f^2 - 2 = 0$,
$f = [2, 2, 1, 3, 1, 1, 1, 1, 2, 1, 5, 456, 1, 30, 1, 3, 4, 29866, \ldots]$.

For $d = -19$, $f^3 - 2f - 2 = 0$,
$f = [1, 1, 3, 2, 1, 95, 2, 1, 1, 2, 1, 127, \ldots]$.

For $d = -11$, $f^3 - 2f^2 + 2f - 2 = 0$,
$f = [1, 1, 1, 5, 4, 2, 305, \ldots]$.

Note the tendency of the large a_n's to drift forwards and also to decrease. This seems a persistent enough pattern to support our goal. The modular functions that we get should converge to the numerator and denominator of the exceptional approximations in these cases also.

Next let us ask what turns out to be the key question. Why should just f have large partial quotients? Why not f^2 or f^4 or f^8, for example? In fact these other numbers also have large partial quotients. Consider the case of $d = -163$. We repeat some of the details of f for ease in comparison.

f is a root of $x^3 - 6x^2 + 4x - 2 = 0$, $a_0 = 5$,
$a_{17} = 22986$, $a_{33} = 1501790$, $a_{59} = 35657$,
$a_{81} = 49405$, $a_{103} = 53460$, $a_{121} = 16467250$.

f^2 is a root of $x^3 - 28x^2 - 8x - 4 = 0$, $a_0 = 28$,
$a_{11} = 126425$, $a_{31} = 8259853$, $a_{49} = 1620$,
$a_{77} = 271730$, $a_{99} = 294038$, $a_{121} = 90569882$.

f^4 is a root of $x^3 - 800x^2 - 160x - 16 = 0$, $a_0 = 800$,
$a_4 = 3202800$, $a_{12} = 209249628$, $a_{32} = 41061$,
$a_{58} = 19068$, $a_{78} = 20634$, $a_{100} = 6355781$.

f^8 is a root of $x^3 - 640320x^2 - 256 = 0$, $a_0 = 640320$,
$a_1 = 1601600400$, $a_3 = 2135467200$, $a_7 = 20533337$,
$a_{19} = 9535605$, $a_{35} = 10318433$, $a_{55} = 3178287878$.

Even f^{24} gets in on the act. For f^{24} we find

$$a_0 = 262537412640768767\ (a_1 = 1), \quad a_2 = 1335334333499, \ldots.$$

Furthermore we can relate exceptionally good approximations to f^k with exceptionally good approximations to f^{2k}. This is because we may rewrite (8) as

$$f^{2k} = \frac{-A_k f^k + 2^k}{f^k + B_k} \tag{9}$$

The determinant of this linear fractional transformation (we will call it D_k) is

$$D_k = -(A_k B_k + 2^k)$$

and is generally not unity. Thus the continued fraction expansions of f^k and f^{2k} are not eventually the same but they are related. In fact since the expansions of f^k and $1/(f^k + B_k)$ are the same after the first term of the former and first two terms of the latter, and since

$$f^{2k} = -A_k - D_k \cdot \frac{1}{f^k + B_k},$$

we see that if p_n/q_n is a convergent of f^k and

$$a_{n+1} > 2|D_k|/g^2$$

where

$$g = (-A_k p_n + 2^k q_n, p_n + B_k q_n),$$

then

$$\frac{-A_k p_n + 2^k q_n}{p_n + B_k q_n}$$

is a convergent of f^{2k} (say the Nth) and a_{N+1} for f^{2k} is given by

$$a_{N+1} \approx \frac{a_{n+1} g^2}{|D_k|}.$$

For $d = -163$, we have $|D_1| = 22$, $|D_2| = 228$, $|D_4| = 128016$. Thus we see from the list given earlier that all of the convergents of f^k corresponding to the a_{n+1}'s listed ($k = 1, 2, 4$), except possibly those of f^4 corresponding to a_{32}, a_{58}, a_{78} yield convergents of f^{2k} and in fact the value of g is large enough in the three doubtful cases to enable them to work also. In fact the six values of a_{n+1} listed for f correspond to the six values given for f^2 which correspond to the six values given for f^4 which correspond to the six values given for f^8. This means that if we can find modular functions that converge to the numerators and denominators of the exceptional approximations to f^8, then we can do the same thing for f by using (9) inverted three times:

$$f^k = \frac{-B_k f^{2k} + 2^k}{f^{2k} + A_k} \tag{10}$$

It is interesting to note that for $d = -163$, the approximation p_{16}/q_{16} of f corresponds to the approximation $p_0/q_0 = a_0(= -\gamma)$ of f^8. If we were to pass to f^{24} by

$$f^{24} = \frac{256 f^8}{f^8 + \gamma},$$

then we would completely lose this first spectacular approximation (or rather, it corresponds to $\infty = p_{-1}/q_{-1}$) while p_{32}/q_{32} of f moves up to the integer $p_1/q_1 = a_0 + 1 (= -j + 768)$ approximation to f^{24}.

Let me close this section with an acknowledgment. With the exception of Brillhart's expansion given after (1), all the computer calculations shown above were performed by Mr. Richard Schroeppel at M.I.T. (the accuracy was insufficient to trace numerically the last two of Brillhart's eight large partial quotients for f forward to f^8). In addition, Mr. Schroeppel provided me with equation (9) along with the calculations; he used (9) to find the continued fraction expansion of f^{2k} from that of f^k as a check on his calculations.

TABLE I. The continued fraction expansion of $f(\sqrt{d})^k$ for certain values of d and k. Shown are the first few values of a_n in $f(\sqrt{d})^k = [a_0, a_1, a_2, \dots]$.

	0	1	2	3	4	5	6	7	8	9
0	1	1	3	2	1	95	2	1	1	2
10	1	127	2	2	32	1	4	1	35	1
20	1	7	3	1	1	5	9	7	4	1

$f(\sqrt{-19})$

	0	1	2	3	4	5	6	7	8	9
0	2	2	1	3	1	1	1	1	2	1
10	5	456	1	30	1	3	4	29866	2	5
20	1	3	1	7	1	3	2	5	2	2

$f(\sqrt{-43})$

	0	1	2	3	4	5	6	7	8	9
0	2	1	11	2	3	1	23	2	3	1
10	1337	2	8	3	2	1	7	4	2	2
20	87431	1	5	1	1	1	9	130	2075	1
30	158	1	5	2	1	1	4	1	22	4
40	7	1	2	1	1	2	1	3	1	1
50	3	7	1	1	8	122	1	4	15	2
60	3	1	4	1	15	1	1	7	1	1
70	1	9	1	1	4	1	2	1	4	1
80	608	155	1	3	1	2	5	1	5	1
90	5	5	17	1	7	8	3	14	3	1

$f(\sqrt{-67})$

	0	1	2	3	4	5	6	7	8	9
0	5	3	7	4	2	30	1	8	3	1
10	1	1	9	2	2	1	3	22986	2	1
20	32	8	2	1	8	55	1	5	2	28
30	1	5	1	1501790	1	2	1	7	6	1
40	1	5	2	1	6	2	2	1	2	1
50	1	3	1	3	1	2	4	3	1	35657
60	1	17	2	15	1	1	2	1	1	5
70	3	2	1	1	7	2	1	7	1	3
80	25	49405	1	1	3	1	1	4	1	2
90	15	1	2	83	1	162	2	1	1	1
100	2	2	1	53460	1	6	4	3	4	13
110	5	15	6	1	4	1	4	1	1	2
120	1	16467250	1	3	1	7	2	6	1	95

$f(\sqrt{-163})$

	0	1	2	3	4	5	6	7	8	9
0	28	3	2	9	3	18	1	6	14	13
10	2	126425	1	5	1	3	11	8	1	2
20	5	2	2	1	4	9	1	4	2	2
30	1	8259853	12	3	1	2	2	5	3	2
40	1	7	2	11	5	22	1	3	3	1620
50	1	1	5	3	5	1	1	2	2	1
60	1	1	3	1	2	1	30	2	1	3
70	3	1	4	2	1	1	4	271730	1	1
80	2	3	2	6	2	1	3	7	1	2
90	1	20	1	6	2	8	2	1	2	294038
100	4	5	1	2	2	1	1	1	1	5
110	1	5	1	3	1	2	2	3	2	4
120	2	90569882								

$f(\sqrt{-163})^2$

	0	1	2	3	4	5	6	7	8	9
0	800	5	1600	5	3202800	4	2000	2	1	1
10	570	1	209249628	2	1	1	20	5	2	1
20	1	1	2	4	5	2	1	2	1	1
30	4	8	41061	2	2	1	2	1	2	11
40	2	1	1	10	4	1	2	2	2	4
50	1	2	1	5	1	2	2	6	19068	1
60	2	17	3	22	1	12	2	1	3	4
70	1	4	1	3	1	1	5	38	20634	3
80	1	4	3	1	1	4	2	12	1	3
90	1	4	1	1	1	4	2	4	3	7
100	6355781									

$f(\sqrt{-163})^4$

	0	1	2	3	4	5	6	7	8	9
0	640320	1601600400	320160	2135467200	261949	10	1	20533337	1	1
10	4	3	1	3	1369	2	2	14	3	9535605
20	1	3	2	1	2	1	5	1	585	3
30	1	15	2	1	1	10318433	2	1	5	3
40	1	2	1	3	1	1	10	1	1	2
50	2	1	1	3	1	3178287878				

$f(\sqrt{-163})^8$

4. The Achievement of our Goal

The A_k and B_k are actually modular functions of z. However when expanded in series of fractional powers of $q = e^{2\pi i z}$, the coefficients are not always rational. This tends to be a nuisance. When we deal with f^8, we find that the coefficients of $B_8(z) = \gamma_2(z)$ are rational ($A_8(z) = 0$). Using $q = e^{2\pi i z}$, the expansion of $\gamma_2(z)$ begins,

$$\gamma_2(z) = q^{-1/3}(1 + 248q + 4124q^2 + 34752q^3 + \dots). \tag{11}$$

The expansion of $f(2z+3)^8$ begins

$$f(2z+3)^8 = q^{-1/3}(-1 + 8q - 28q^2 + 64q^3 - \dots). \tag{12}$$

Our object is to express $f(2z+3)^8$ as the continued fraction

$$f(2z+3)^8 = [a_0, a_1, a_2, \dots]$$

where the a_j are polynomials in $\gamma = \gamma_2(z)$ chosen so as to eliminate the negative (and zero) powers of q at each stage.

We set $\alpha_0 = f(2z+3)^8$ and find α_{n+1} recursively from α_n and a_n by the usual continued fraction rule

$$\alpha_{n+1} = \frac{1}{\alpha_n - a_n}.$$

Comparing (11) and (12), we see that

$$a_0 = -\gamma$$

and thus

$$\alpha_1 = \frac{1}{f(2z+3)^8 + \gamma_2(z)}$$
$$= \tfrac{1}{256} f(2z+3)^{16}$$
$$= \tfrac{1}{256} q^{-2/3}(1 - 16q + 120q^2 - 576q^3 + \dots)$$

where it is convenient to use (1). From

$$\gamma_2(z)^2 = q^{-2/3}(1 + 496q + 69752q^2 + 2115008q^3 + \dots) \tag{13}$$

we see that we should take

$$a_1 = \tfrac{1}{256}\gamma^2$$

and now

$$\alpha_2 = \frac{1}{\alpha_1 - a_1} = -\tfrac{1}{2} q^{-1/3}(1 - 136q + 14364q^2 + \dots).$$

Thus we take

$$a_2 = -\tfrac{1}{2}\gamma$$

and hence

$$\alpha_3 = \frac{1}{\alpha_2 - a_2} = \tfrac{1}{192} q^{-2/3}(1 + \tfrac{80}{3} q + \dots).$$

Now

$$a_3 = \tfrac{1}{192}\gamma^2$$

which yields

$$\alpha_4 = \frac{1}{\alpha_3 - a_3} = -\tfrac{9}{22} q^{-1/3}(1 + \dots).$$

We now take

$$a_4 = -\tfrac{9}{22}\gamma$$

and at this point we have run out of accuracy. However, it is clear how we would proceed if we were willing to start with more terms in (11) and (12). It is also clear that a_5 will be at least of degree 2 in γ.

Let us assemble the convergents p_n/q_n which we find from the usual recursion relations. In order to get rid of needless powers of 2, we will find these in terms of

$$\Gamma = -\frac{\gamma}{8}.$$

We then find

$$p_0(\Gamma) = 8\Gamma, \quad q_0(\Gamma) = 1,$$
$$p_1(\Gamma) = 2\Gamma^3 + 1, \quad q_1(\Gamma) = \tfrac{1}{4}\Gamma^2,$$
$$p_2(\Gamma) = 8\Gamma^4 + 12\Gamma, \quad q_2(\Gamma) = \Gamma^3 + 1,$$
$$p_3(\Gamma) = \tfrac{8}{3}\Gamma^6 + 6\Gamma^3 + 1, \quad q_3(\Gamma) = \tfrac{1}{3}\Gamma^5 + \tfrac{7}{12}\Gamma^2$$
$$p_4(\Gamma) = \tfrac{96}{11}\Gamma^7 + \tfrac{304}{11}\Gamma^4 + \tfrac{168}{11}\Gamma, \quad q_4(\Gamma) = \tfrac{12}{11}\Gamma^6 + \tfrac{32}{11}\Gamma^3 + 1.$$

When we set

$$\gamma = \gamma_2\left(\frac{-3+\sqrt{d}}{2}\right),$$

we will get rational p_n and q_n if $h(d) = 1$, but not necessarily integral p_n and q_n. We can expect that the factor that we must multiply through by in order to get integers will increase with n; the approximation to f^8 compared to q_n^2 is worsened by this denominator squared and thus only finitely many of the p_n/q_n should be convergents for the ordinary continued fraction expansion of $f(\sqrt{d})^8$. The number of really good approximations to $f(\sqrt{d})$ that we get will also be for this reason larger with larger $|\gamma|$ which in turn corresponds to larger $|d|$. In the five cases $d = -11, -19, -43, -67, -163$, we find that $8|\gamma$ which helps things very much. We find that $\Gamma = 4, 12, 120, 635, 80040$ respectively.

The three best approximations (of the five that we have found) to $f(2z+3)^8$ are given by

$$\frac{p_0(\Gamma)}{q_0(\Gamma)}, \quad \frac{p_2(\Gamma)}{q_2(\Gamma)}, \quad \frac{11p_4(\Gamma)}{11q_4(\Gamma)},$$

where in each case the numerators and denominators are (not necessarily relatively prime) integers when $d = -11, -19, -32, -67, -163$. When we trace these back to approximations to $f(2z+3)$ by applying (10), we find that for $d = -11$, the first two already give the convergents $p_3/q_3 (a_4 = 4)$ and $p_5/q_5 (a_6 = 305)$ of $f(\sqrt{-11})$ and for $d = -19, -43, -67, -163$, all three give convergents for $f(\sqrt{d})$. For each of these last four discriminants we get the first two spectacular approximations indicated in the expansions earlier but the third becomes spectacular only as $|d|$ grows. The third corresponds to p_{14}/q_{14} for $f(\sqrt{-19})$ $(a_{15} = 1)$, to p_{26}/q_{26} for $f(\sqrt{-43})$ $(a_{27} = 5)$, to p_{27}/q_{27} for $f(\sqrt{-67})$ $(a_{28} = 2075)$, and to p_{58}/q_{58} for $f(\sqrt{-163})$ $(a_{59} = 35657)$. It would be interesting to know if there is a ninth non-spectacular convergent to $f(\sqrt{-163})$ that comes from this process.

It is certainly possible to analyze all of this further to include a discussion of what multiples of the numerators and denominators we actually end up with. The numbers A_k and B_k are related to A_{2k} and B_{2k}. If we transpose the f^{2k} and constant terms of (8) to the other side of the equation and then square both sides, we find

$$B_{2k} = 2A_k - B_k^2, \qquad A_{2k} = A_k^2 + 2^{k+1}B_k. \tag{14}$$

If we recall that $B_8 = \gamma$, $A_8 = 0$, then we find from (14) that we may set for $k = 1, 2, 4, 8$,

$$A_k = 2^k a_k, \qquad B_k = 2^{[k/2]+1} b_k, \tag{15}$$

and a_k and b_k are integers (in particular $4|\Gamma$). This conclusion is true for any discriminant d satisfying the restriction (6); we are of course then dealing with algebraic integers.

For any algebraic integer Γ, such that $2|\Gamma$, we find that

$$(p_0(\Gamma), q_0(\Gamma)) = 1, \qquad (p_2(\Gamma), q_2(\Gamma)) = 1, \qquad (11p_4(\Gamma), 11q_4(\Gamma))|11;$$

if Γ is also rational then

$$(11p_4(\Gamma), 11q_4(\Gamma)) = \begin{cases} 1 & \text{if } 11\nmid\Gamma \\ 11 & \text{if } 11|\Gamma. \end{cases}$$

In the cases of interest to us, this factor 11 occurs only for $d = -67$ but it makes a_{28} of $f(\sqrt{-67})$ about 121 times larger than it would otherwise have been.

Now when $4|\Gamma$ we see that

$$16|p_0(\Gamma), \qquad 16|p_2(\Gamma), \qquad 16|11p_4(\Gamma),$$
$$(q_0(\Gamma), 2) = (q_2(\Gamma), 2) = (11q_4(\Gamma), 2) = 1.$$

Suppose we have an approximation to f^8, $p(8)/q(8)$, where

$$16|p(8), (q(8), 2) = 1. \tag{16}$$

Then we get an approximation to f^4 given by (10),

$$\frac{p(4)}{q(4)} = \frac{-B_4 p(8) + 16q(8)}{p(8) + A_4 q(8)}.$$

Thanks to (15) and (16), we may remove a factor 16 from the numerator and denominator and take

$$p(4) = -\tfrac{1}{2}b_4 p(8) + q(8), \qquad q(4) = \tfrac{1}{16}p(8) + a_4 q(8)$$

(this enlarges the corresponding a_{n+1} by a factor of about 256). We see also that $(p(4), 2) = 1$. We now get an approximation to f^2,

$$\frac{p(2)}{q(2)} = \frac{-B_2 p(4) + 4q(4)}{p(4) + A_2 q(4)}.$$

Here we can't remove any factors of 2 and so we set

$$p(2) = -4b_2 p(4) + 4q(4), \qquad q(2) = p(4) + 4a_2 q(4).$$

Note that $(q(2), 2) = 1$ and $4|p(2)$. We are now ready to go to f,

$$\frac{p(1)}{q(1)} = \frac{-B_1 p(2) + 2q(2)}{p(2) + A_1 q(2)}.$$

Here we can save a factor 2 and so we take

$$p(1) = -b_1 p(2) + q(2), \qquad q(1) = \tfrac{1}{2}p(2) + a_1 q(2).$$

For $d = -19, -47, -67, -163$, we find that in each of the first three spectacular convergents, $(p(1), q(1))$ is divisible by 3 (but not 9). When $d = -11$, $(p(1), q(1))$ is not divisible by 3.

When we come to compare the first two spectacular convergents to $f(\sqrt{d})$, we note that the second is even more spectacular than the first. The reason is that although the same powers of 2 and 3 come out of $(p(1), q(1))$ for the first two convergents, the result is relatively prime for the first and usually not for the second. There are extra common factors

$$g = 7, 1, 7, 7, 7$$

in $p(1)$ and $q(1)$ for the second spectacular convergent with $d = -11$, $-19, -43, -67, -163$ respectively. The result is that the $a_{??+1}$ corresponding to the second spectacular convergent is about

$$\tfrac{256}{192} g^2 = \tfrac{4}{3} g^2$$

times as large as the $a_{?+1}$ corresponding to the first spectacular convergent.

5. Other Applications of our Results

While there are only five discriminants d satisfying (6) and $h(d) = 1$, there is no reason why we should restrict ourselves to these. If d satisfies (6) then we may give exceptionally good approximations to $f(\sqrt{d})$ by quotients of algebraic integers of degree $h(d)$. For example, we may find good approximations to $f(\sqrt{-427})$ (an algebraic integer of degree 6) by the quotients of two integers in $Q(\sqrt{61})$.

There is still another direction in which we may proceed. Consider the cubic equation

$$x^3 + tx^2 - 256 = 0, \tag{17}$$

where t is a negative integer, large in absolute value. Any such equation will have a unique real root and the continued fraction expansion of this root will have some partial quotients of the order of t^2. For if we let $y > \sqrt{3}$ be determined (uniquely) by

$$\gamma_2\left(\frac{-3 + iy}{2}\right) = t,$$

then the real root of (17) is $f(iy)^8$. We have even explicitly given in the last section three excellent convergents as quotients of polynomials in $\Gamma = -\frac{1}{8}t$.

When t is a large positive integer, (17) has three real roots (this is true for $t > 12$). In the last section, we have found good approximations to one of them. In fact if we determine $y > 1$ uniquely by

$$\gamma_2(iy) = t$$

then we have found good approximations to $f(3 + 2iy)^8$ which is one of the two negative roots of (17) (the one furthest removed from 0). The other two roots of (17) are of the order of $t^{-1/2}$ and hence not well approximable by polynomials in t.

Now let us consider the cubic equation

$$x^3 - 2s^2x^2 + 8sx - 16 = 0 \tag{18}$$

where s is an integer, large in absolute value. Any such equation will have a unique real root and the continued fraction expansion of this root will have some partial quotients on the order of s^5. In fact, if we determine $y > \sqrt{3}$ uniquely by

$$\gamma_2\left(\frac{-3 + iy}{2}\right) = t = 4s(4 - s^3),$$

the real root of (18) is $f(iy)^4$. We have $A_4 = 8s$, $B_4 = -2s^2$ and these numbers satisfy the relation (14) (with $k = 4$) where $B_8 = 4s(4 - s^3)$, $A_8 = 0$. Since

$$|D_4| = 16s^3 - 16,$$

and since we have approximations to $f(iy)^8$ with partial quotients of the order of at least t^2 which is of the order of s^8, we see that we get approximations to $f(iy)^4$ with partial quotients of the order of at least s^5.

An example of such an equation is given by $s = 60$ (the value of y is then transcendental and thus certainly not connected with quadratic fields). We then have the equation

$$x^3 - 7200x^2 + 480x - 16 = 0.$$

Since $2|x$, we may set $x = 2X$ and the equation for X is then,

$$X^3 - 3600X^2 + 120X - 2 = 0.$$

The people at Atlas very kindly furnished the continued fraction expansion of the real root of this equation. It is given in Table II. Besides the expected very large partial quotients we note some others in the neighborhood of 10000. These do not come from the expansion of $f(iy)^8$. We do get partial quotients for $f(iy)^4$ on the order of s from $f(iy)^8$ but the partial quotients on the order of 10000 for $f(iy)^4$ originate with $f(iy)^4$ and come from expanding $f(iy)^4$ in a continued fraction with partial quotients being polynomials in s.

TABLE II. The continued fraction expansion of the real root of $X^3 - 3600X^2 + 120X - 2 = 0$. Shown are the values of a_n in $X = [a_0, a_1, a_2, \ldots]$.

	0	1	2	3	4	5	6	7	8	9
0	3599	1	28	1	7198	1	29	388787400	23	1
10	8998	1	13	1	10284	1	2	25400776804	1	1
20	3	4	93	3	1	2	11	1	9	1
30	99	1	3	1	3	9	1	603118914	1	1
40	2	24	1	1	3	2	1	1	2	2
50	1	1	26	1	8	1	18	1	2	2
60	1	2	1	1	3	9	3	2	1	2314761
70	6	1	2	5	5	61	1	1	4	1
80	1	5	1	22	1	4	2	1	1	1
90	9	2	1	1	2	1	2	2	1	1
100	12	1709319								

In view of all this, why did Brillhart come up with a cubic related to a quadratic field? The answer is that the magnitude of the coefficients involved in his search made sure that any spectacular approximations to $f(iy)^k$ covered by our discussion here that he might find would come with $k = 1$ (and conceivably 2). While there are infinitely many $y \geqslant \sqrt{3}$ such that $f(iy)^8$ is the root of a cubic equation of the form

$$x^3 + tx^2 - 256 = 0$$

with t an integer, and still infinitely many such y with the additional restriction that $f(iy)^4$ should generate a cubic extension of the rationals, there are only six values of y satisfying all this and having $f(iy)^2$ generate a cubic extension of the rationals. This is because the recursion relations (14) are very restrictive ($k = 2, 4$). They give a set of Diophantine equations with only six solutions (the corresponding y being $\sqrt{3}$, $\sqrt{11}$, $\sqrt{19}$, $\sqrt{43}$, $\sqrt{67}$, $\sqrt{163}$).

This is in fact the Heegner (1952) approach to proving that there are only nine values of d with $h(d) = 1$. [See also Birch (1968), Deuring (1968) and Stark (1969)].

We close by mentioning one more aspect of all this. Suppose $X = X(s)$, $Y = Y(s)$ are polynomials in s (with complex coefficients) of degrees n and $n - 2$ respectively. How small can we make the degree of

$$F(X, Y) = X^3 - 2s^2X^2Y + 8sXY^2 - 16Y^3?$$

Since there are $2n$ unknown coefficients we would expect that the degree of $F(X, Y)$ (in s) would be at least $n + 1$; if it were any smaller, we would have $2n$ homogeneous equations in the $2n$ coefficients and would expect the $2n$ unknowns to be all zero. In fact our expectations are wrong. For infinitely many n, there exist polynomials X and Y (with integral coefficients) such that the degree of $F(X, Y)$ in s is less than or equal to $n - 3$. For example, if

$$X(s) = 2s^6 - 8s^3 + 4, \qquad Y(s) = s^4 - 2s$$

then

$$F(X, Y) = -64(s^3 - 1).$$

I hope to say more about this in the future.

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