A COMBINATORIAL DERIVATION OF THE CORRELATION FUNCTION OF THE FIELD OPERATOR IN THE FREE FERMION LIMIT OF THE XXZ HEISENBERG CHAIN.

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ABSTRACT

We discuss the connection between quantum Heisenberg XXZ spin chain in the limiting case of zero anisotropy ($\Delta \to 0$) and some aspects of enumerative combinatorics and the theory of partitions. The representation of the Bethe wave functions via the Schur functions allows to apply the theory of symmetric functions to calculation of the thermal correlation functions as well as of the form-factors. The determinantal expressions of the form-factors and of the thermal correlation functions are obtained. We provide a combinatorial interpretation of the correlation functions in terms of nests of the self-avoiding lattice paths. The interpretation proposed is in turn related to enumeration of the boxed plane partitions. The asymptotical behavior of the thermal correlation functions is studied in the limit of small temperature provided that the characteristic parameters of the system are large enough. The leading asymptotics of the correlators are found to be proportional to the squared numbers of boxed plane partitions.

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I. XXZ HEISENBERG MODEL AND ITS FREE FERMION LIMIT $\Delta = 0$

• • • Quantum XXZ Heisenberg model describes a chain of spins $\frac{1}{2}$. Its Hamiltonian in absence of magnetic field takes the form:

$$\hat{H}_{XXZ} = -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-} + \frac{\Delta}{2}(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1)),$$

where $\Delta \in \mathbb{R}$ is the anisotropy, and M+1 is the number of sites. The local spin operators $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ and σ_k^z, dependent on the lattice argument k, are defined as $(M+1)$-fold tensor products:

$$\sigma_k^{\#} = \sigma^0 \otimes \cdots \otimes \sigma^0 \otimes \underbrace{\sigma^{\#}}_{k} \otimes \sigma^0 \otimes \cdots \otimes \sigma^0 ,$$

where σ^0 is 2×2 unit matrix, and $\sigma^{\#}$ at kth site is a Pauli matrix, $\sigma^{\#} \in \mathfrak{su}(2)$ ($\#$ is either x, y, z or $\pm$). The commutation rules are:

$$[\,\sigma_k^+, \sigma_l^-\,] = \delta_{k,l}\,\sigma_l^z\,, \quad [\,\sigma_k^z, \sigma_l^{\pm}\,] = \pm 2\,\delta_{k,l}\,\sigma_l^{\pm}\,.$$

We introduce spin "up" and spin "down" states, $|\uparrow\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The Pauli operators act on $|\uparrow\rangle$ and $|\downarrow\rangle$ as follows:

$$\sigma^- \,|\uparrow\rangle = |\downarrow\rangle, \qquad \sigma^- \,|\downarrow\rangle = 0\,, \qquad \sigma^- = \begin{pmatrix} 00 \\ 10 \end{pmatrix}\,,$$

$$\sigma^+ \,|\uparrow\rangle = 0, \qquad \sigma^+ \,|\downarrow\rangle = |\uparrow\rangle\,, \qquad \sigma^+ = \begin{pmatrix} 01 \\ 00 \end{pmatrix}\,.$$

The lattice spin operators defined above act over the state-space $\mathfrak{H}_{M+1} = \bigotimes_{k=0}^{M} \mathfrak{h}_k$ given by the product of $M+1$ copies of linear spaces $\mathfrak{h}_k \equiv \mathbb{C}^2$. The state-space $\mathfrak{H}_{M+1}$ is spanned over the state-vectors $\bigotimes_{k=0}^{M} |s\rangle_k$, where $s = \uparrow, \downarrow$.

The periodic boundary conditions $\sigma^{\#}_{k+(M+1)} = \sigma^{\#}_k$ are imposed.

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Let the sites with spin "down" states are labeled by decreasing coordinates $M \geq \mu_1 > \mu_2 > \ \ldots \ > \mu_N \geq 0$, which constitute strict partition $\boldsymbol{\mu} = (\mu_1, \mu_2, \ \ldots \ , \mu_N)$.
In the free-fermion limit, we define N-excitation state-vectors $|\Psi_N(\mathbf{u})\rangle$:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}^2) \left(\prod_{k=1}^{N} \sigma^-_{\mu_k} \right) |\Uparrow\rangle \,,$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ is $\boldsymbol{\lambda} = \boldsymbol{\mu} - \boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N = (N-1, N-2, \ldots, 1, 0)$. Besides, $\mathcal{M} \equiv M + 1 - N \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$.

The parameters $\mathbf{u} = (u_1, u_2, \ldots, u_N)$ and $\mathbf{u}^2 \equiv (u_1^2, u_2^2, \ldots, u_N^2)$ correspond to arbitrary complex numbers. The state $|\Uparrow\rangle$ is the fully polarized one with all spins "up": $|\Uparrow\rangle \equiv \bigotimes_{n=0}^{M} |\uparrow\rangle_n$.

The coefficients of the state-vector $|\Psi_N(\mathbf{u})\rangle$ are given by the *Schur functions*:

$$
\begin{aligned}
S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_N) &\equiv \frac{\det(x_j^{\lambda_k+N-k})_{1\le j,k\le N}}{\det(x_j^{N-k})_{1\le j,k\le N}} \\
&= \det(x_j^{\lambda_k+N-k})_{1\le j,k\le N} \prod_{1\le n<l\le N} (x_l - x_n)^{-1} \,.
\end{aligned}
$$

In the *free-fermion limit,* $\Delta \to 0$ the Hamiltonian is:

$$\hat{H}_{XX} \equiv -\frac{1}{2}\sum_{k=0}^{M}(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-}) .$$

The states $|\Psi_N(\mathbf{u})\rangle$ are the eigen-states,

$$\hat{H}_{XX} \, |\Psi_N(\mathbf{u}_N)\rangle = E_N \, |\Psi_N(\mathbf{u}_N)\rangle \, ,$$

with eigen-values $E_N \equiv E_N^{XX}(I_1, I_2, \dots, I_N) = -\sum_{l=1}^{N} \cos\left(\frac{2\pi I_l}{M+1}\right)$, if and only if u_l $(1 \le l \le N)$ satisfy the *Bethe equations*:

$$u_j^{2(M+1)} = (-1)^{N-1} \, , \qquad u_j^2 = e^{i\frac{2\pi}{M+1}I_j} \, , \qquad 1 \le j \le N \, .$$

where I_j are integers or half-integers: $M \ge I_1 > I_2 > \cdots > I_N \ge 0$.

The scalar products of the state-vectors of both limits are calculated by means of the Binet–Cauchy formula:

$$\mathcal{P}_{L/n}(\mathbf{y},\mathbf{x}) \equiv \sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}} S_{\boldsymbol{\lambda}}(x_1^2,\dots,x_N^2)S_{\boldsymbol{\lambda}}(y_1^2,\dots,y_N^2)$$

$$= \frac{\left(\prod\limits_{l=1}^{N} y_l^n x_l^n\right)\det(T_{jk})_{1\le j,k\le N}}{\prod_{1\le k<j\le N}\left(y_j^2-y_k^2\right)\prod_{1\le m<l\le N}\left(x_l^2-x_m^2\right)},$$

where summation $\sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}}$ is over non-strict partitions $\boldsymbol{\lambda}$: $L\ge\lambda_1\ge\lambda_2\ge\dots\ge\lambda_N\ge n$, $n\ge0$. The entries T_{jk} take the form:

$$T_{kj} = \frac{1-(x_k y_j)^{N+L-n}}{1-x_k y_j}.$$

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II. FORM-FACTORS

• • • Central task is to calculate *survival probability of domain wall*:

$$\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}, n, \beta) \equiv \frac{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid \bar{\mathsf{F}}^{+}_{n}\, e^{-\beta\mathcal{H}}\, \bar{\mathsf{F}}_{n} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}})\rangle}{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid e^{-\beta\mathcal{H}} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}})\rangle}\,, \quad \bar{\mathsf{F}}_{n} \equiv \prod_{j=0}^{n-1} \sigma_j^{-},$$

where $\bar{\mathsf{F}}_n$ is the field operator, and $\beta \in \mathbb{C}$.

The eigen-state $|\,\Psi_N(\boldsymbol{\theta})\rangle \equiv |\,\Psi_N(e^{i\boldsymbol{\theta}/2})\rangle$ corresponds to N-particle Bethe solution $\mathbf{u}^2 = e^{i\boldsymbol{\theta}}$. The notation $\boldsymbol{\theta}^{\,\mathrm{g}} = (\theta_1^{\,\mathrm{g}}, \theta_2^{\,\mathrm{g}}, \dots, \theta_N^{\,\mathrm{g}})$ indicates that the eigen-state $|\,\Psi_N(\boldsymbol{\theta}^{\mathrm{g}})\rangle \equiv |\,\Psi_N(e^{i\boldsymbol{\theta}^{\,\mathrm{g}}/2})\rangle$ is calculated on ground-state solution. As well, $\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n} = (\theta_1^{\,\mathrm{g}}, \theta_2^{\,\mathrm{g}}, \dots, \theta_{N-n}^{\,\mathrm{g}})$ corresponds to $(N-n)$-particle solution of the Bethe equation for the ground state:

$$\theta_j^{\,\mathrm{g}} = \frac{2\pi}{M+1}\left(\frac{N-n+1}{2} - j\right), \quad 1 \le j \le N-n\,.$$

Besides, $\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}, 0, \beta) = 1$.

••• Consider the form-factor of the field operator $\bar{\mathsf{F}}_n$:

$$\langle \Psi_N(\mathbf{v}) \mid \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\mathbf{u}) \rangle \,.$$

Let us define an auxiliary operator $\mathsf{D}^n(\mathbf{u})$ which acts on an expectation $\langle \cdot \rangle_{\mathbf{u}}$ considered as function of $\mathbf{u}$ as follows:

$$\mathsf{D}^n(\mathbf{u})\,\langle \cdot \rangle_{\mathbf{u}} \;\equiv\; \mathsf{D}_{u_{N-n+1},u_{N-n+2},\dots,u_N}\;\left(\frac{\mathcal{V}_N(\mathbf{u}_N^2)}{\mathcal{V}_{N-n}(\mathbf{u}_{N-n}^2)} \times \langle \cdot \rangle_{\mathbf{u}} \right)\,,$$

where $\mathsf{D}_{u_{N-n+1},u_{N-n+2},\dots,u_N} \;\equiv\; \mathsf{D}^{n-1}_{u_{N-n+1}} \circ \mathsf{D}^{n-2}_{u_{N-n+2}} \circ \dots \circ \mathsf{D}^0_{u_N}\,,$

$$\mathsf{D}^j_{u_{N-j}} \;\equiv\; \lim_{u_{N-j}^2 \to 0} \frac{1}{j\,!}\,\frac{d^j}{d(u_{N-j}^2)^j}\,, \qquad 0 \le j \le n-1\,.$$

Now we are ready to formulate the following

Proposition 1 *The action of* $\mathsf{D}^n(\mathbf{u})$ *on* $\langle \Psi(\mathbf{v}_N) \mid \Psi(\mathbf{u}_N) \rangle$ *gives the form-factor of the field operator* $\bar{\mathsf{F}}_n$:

$$\langle \Psi(\mathbf{v}_N) \mid \bar{\mathsf{F}}_n \mid \Psi(\mathbf{u}_{N-n}) \rangle \;=\; \mathsf{D}^n(\mathbf{u}) \langle \Psi(\mathbf{v}_N) \mid \Psi(\mathbf{u}_N) \rangle\,.$$

Proposition 1 enables us to obtain two summation rules for the products of the Schur functions:

Proposition 2 *The following sums of products of the Schur functions take place*:

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}}(\mathbf{u}_{N-n}^2) = \left(\prod_{l=1}^{N-n} u_l^{-2n}\right) \frac{\det(\bar{T}_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{u}_{N-n}^2)\mathcal{V}(\mathbf{v}_N^{-2})},$$

$$\sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^{N-n}\}} S_{\boldsymbol{\lambda}}(\mathbf{v}_{N-n}^{-2}) S_{\hat{\boldsymbol{\lambda}}}(\mathbf{u}_N^2) = \left(\prod_{l=1}^{N-n} v_l^{2n}\right) \frac{\det(\tilde{T}_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{v}_{N-n}^{-2})\mathcal{V}(\mathbf{u}_N^2)},$$

where the entries of the matrices $(\bar{T}_{kj})_{1\leq k,j\leq N}$ *and* $(\tilde{T}_{kj})_{1\leq k,j\leq N}$ *are*:

$$\begin{aligned} \bar{T}_{kj} &= T^{\mathrm{o}}_{kj}\,, & 1 \leq k \leq N-n, \quad & 1 \leq j \leq N\,, \\ \bar{T}_{kj} &= v_j^{-2(N-k)}\,, & N-n+1 \leq k \leq N, \quad & 1 \leq j \leq N\,, \end{aligned}$$

and

$$\begin{aligned} \tilde{T}_{kj} &= T^{\mathrm{o}}_{kj}\,, & 1 \leq k \leq N, \quad & 1 \leq j \leq N-n\,, \\ \tilde{T}_{kj} &= u_j^{2(N-k)}\,, & 1 \leq k \leq N, \quad & N-n+1 \leq j \leq N\,. \end{aligned}$$

Here we use the notation:

$$T^{\mathrm{o}}_{kj} \equiv \frac{1-(u_k^2/v_j^2)^{M+1}}{1-u_k^2/v_j^2}\,.$$

III. CORRELATION FUNCTIONS

Let consider the following ratio of two averages at arbitrary values of parameters:

$$\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}, n, \beta) \;\equiv\; \frac{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid \bar{\mathsf{F}}^{+}_{n}\, e^{-\beta\mathcal{H}}\, \bar{\mathsf{F}}_{n} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \rangle}{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid e^{-\beta\mathcal{H}} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \rangle}\,, \quad \bar{\mathsf{F}}_{n} \;\equiv\; \prod_{j=0}^{n-1} \sigma_j^{-}\,,$$

$$\langle \Psi_{N-n}(\mathbf{v}) \mid \bar{\mathsf{F}}^{+}_{n}\, e^{-\beta\mathcal{H}}\, \bar{\mathsf{F}}_{n} \mid \Psi_{N-n}(\mathbf{u}) \rangle \;=$$

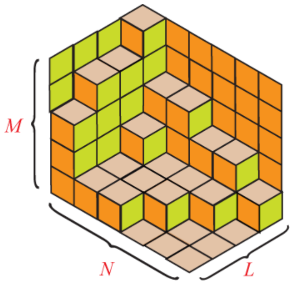
$$=\; \mathsf{D}^{n}(\mathbf{u})\, \mathsf{D}^{n}(\mathbf{v}^{-1}) \langle \Psi_{N}(\mathbf{v}) \mid e^{-\beta\mathcal{H}} \mid \Psi_{N}(\mathbf{u}) \rangle \;=\; \frac{1}{\mathcal{V}_{N-n}(\mathbf{u}^2)\mathcal{V}_{N-n}(\mathbf{v}^{-2})}$$

$$\times\, \mathsf{D}_{v^{-1}_{N-n+1}, v^{-1}_{N-n+2}, \dots, v^{-1}_{N}} \mathsf{D}_{u_{N-n+1}, u_{N-n+2}, \dots, u_{N}} \det \left(\sum_{k,l=0}^{M} F_{k;\,l}(\beta) \frac{u_i^{2l}}{v_j^{2k}} \right)_{1\le i,j\le N}$$

IV. q-BINOMIAL DETERMINANTS AND GENERATING FUNCTIONS OF PLANE PARTITIONS

• • • The correlators in question are related to the generating functions of *boxed plane partitions* and *self-avoiding lattice walks*.

An array $(\pi_{i,j})_{i,j\geq 1}$ of non-negative integers that are non-increasing as functions of both i and j $(i,j=1,2,\dots)$ is called boxed plane partition π. Plane partition is represented by cubes arranged as stacks with coordinates (i,j) and with height equal to $\pi_{i,j}$. The plane partition is contained inside a box $\mathcal{B}(L,N,M)$ provided that $i\leq L$, $j\leq N$ and $\pi_{i,j}\leq M$ for all cubes of the plane partition.



The generating function of plane partitions inside $\mathcal{B}(L,N,P)$ is defined as formal series $Z_q(L,N,P) \equiv \sum_{\{\boldsymbol{\pi}\}} q^{|\boldsymbol{\pi}|}$ (summation over all partitions inside the box), and it takes the form:

$$Z_q(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} = \prod_{j=1}^{L}\prod_{k=1}^{N} \frac{1-q^{P+j+k-1}}{1-q^{j+k-1}} \,.$$

The limit $q \to 1$ leads to the MacMahon formula:

$$A(L,N,P) = \prod_{j=1}^{L}\prod_{k=1}^{N}\prod_{i=1}^{P} \frac{i+j+k-1}{i+j+k-2} = \prod_{j=1}^{L}\prod_{k=1}^{N} \frac{P+j+k-1}{j+k-1} \,.$$

• • • To study the asymptotical behavior of the correlation functions, we need the determinant of a block-matrix $(\bar{\mathsf{T}})_{1\le j,k\le N}$ given by entries:

$$\bar{\mathsf{T}}_{kj} = \frac{1-q^{(P+1)(j+k-1)}}{1-q^{j+k-1}}\,, \quad 1\le k\le L, \qquad 1\le j\le N\,,$$

$$\bar{\mathsf{T}}_{kj} = q^{j(N-k)}\,, \qquad L+1\le k\le N, \quad 1\le j\le N\,,$$

where P and $L \le N$ are arbitrary. The matrix $(\bar{\mathsf{T}})_{1\le j,k\le N}$ consists of two blocks of the sizes $L\times N$ and $(N-L)\times N$.

Several definitions are in order.

The q-*binomial determinant* $\binom{\mathbf{a}}{\mathbf{b}}_q$ is defined by

$$\binom{\mathbf{a}}{\mathbf{b}}_q \equiv \begin{pmatrix} a_1, & a_2, & \dots & a_S \\ b_1, & b_2, & \dots & b_S \end{pmatrix}_q \equiv \det\left(\begin{bmatrix} a_j \\ b_i \end{bmatrix}\right)_{1\le i,j\le S},$$

where $\mathbf{a}$ and $\mathbf{b}$ are ordered tuples: $0 \le a_1 < a_2 < \dots < a_S$ and $0 \le b_1 < b_2 < \dots < b_S$. The entries $\begin{bmatrix} a_j \\ b_i \end{bmatrix}$ are the q-*binomial coefficients*:

$$\begin{bmatrix} N \\ r \end{bmatrix} \equiv \frac{(1-q^N)(1-q^{N-1})\dots(1-q^{N-r+1})}{(1-q)(1-q^2)\dots(1-q^r)}, \qquad q \in \mathbb{R}.$$

The q-binomial coefficients are replaced at $q \to 1$ by the binomial coefficients $\binom{a_j}{b_i}$. The q-binomial determinant is transformed to the *binomial determinant*:

$$\binom{\mathbf{a}}{\mathbf{b}} \equiv \begin{pmatrix} a_1, & a_2, & \dots & a_S \\ b_1, & b_2, & \dots & b_S \end{pmatrix} = \det\left(\binom{a_j}{b_i}\right)_{1\le i,j\le S}.$$

The binomial determinant is positive at $b_i \le a_i$, $\forall i$.

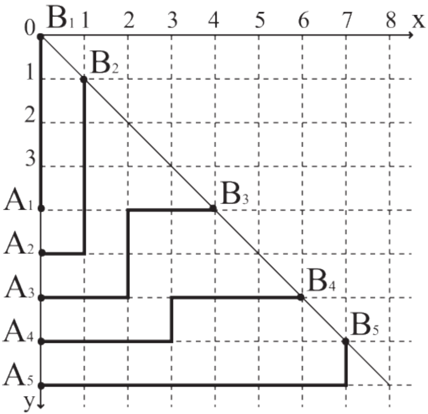


Рис.: S-Tuple $(w_1, w_2, \ldots, w_S)$ of self-avoiding walks for $S = 5$.

Binomial determinant gives the number of self-avoiding walks across two-dimensional lattice. Each path w_i from a tuple $(w_1, w_2, \ldots, w_S)$ goes from $A_i = (0, a_i)$ to $B_i = (b_i, b_i)$, $1 \le i \le S$. Only steps to north and to east are allowed.

So, let us consider $(\bar{\mathsf{T}})_{1\le j,k\le N}$ given by the entries:

$$\bar{\mathsf{T}}_{kj} = \frac{1-q^{(P+1)(j+k-1)}}{1-q^{j+k-1}}\,, \quad 1\le k\le L, \qquad 1\le j\le N\,,$$
$$\bar{\mathsf{T}}_{kj} = q^{j(N-k)}\,, \qquad L+1\le k\le N, \quad 1\le j\le N\,,$$

where P and $L \le N$ are arbitrary. Now we formulate the following

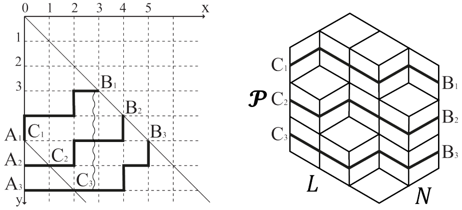
Proposition 3 *Let the matrix* $(\bar{\mathsf{T}})_{1\le j,k\le N}$, *be defined by the entries* (32) *with* $\frac{P}{2} < N < P$. *The determinant of* $(\bar{\mathsf{T}})_{1\le j,k\le N}$ *is given as:*

$$q^{-\frac{L}{2}(L-1)(N-L)}\,\frac{\det(\bar{\mathsf{T}})_{1\le j,k\le N}}{\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_L/q)}$$
$$= q^{-\frac{N}{2}(\mathcal{P}-1)\mathcal{P}}\begin{pmatrix} L+N, & L+N+1, & \dots & L+N+\mathcal{P}-1 \\ L, & L+1, & \dots & L+\mathcal{P}-1 \end{pmatrix}_q$$
$$= \prod_{k=1}^{\mathcal{P}}\prod_{j=1}^{L}\frac{1-q^{j+k+N-1}}{1-q^{j+k-1}} = Z_q(L,N,\mathcal{P})\,,$$

where $\mathcal{P}\equiv P-N+1$, *and* $Z_q(L,N,\mathcal{P})$ *is the generating function of plane partitions.*

Proposition 3 relates $\det \bar{\mathsf{T}}$ to the q-binomial determinant, which is transformed at $q \to 1$ to the binomial determinant equal, in turn, to the number of $\mathcal{P}$-tuples of lattice self-avoiding paths between the *end points* $A_l = (0, N + L + l - 1)$ and $B_l = (L + l - 1, L + l - 1)$, $1 \le l \le \mathcal{P}$.

The Figure gives appropriate picture with *end points* A_l and B_l at $\mathcal{P} = L = 3$ and $N = 2$.



The generating function $Z_q(L,N,\mathcal{P})$ gives at $q \to 1$ the number of plane partitions $A(L,N,\mathcal{P})$ inside $\mathcal{B}(L,N,\mathcal{P})$:

$$Z_q(L,N,\mathcal{P}) = \prod_{j=1}^{L}\prod_{k=1}^{N} \frac{1-q^{\mathcal{P}+j+k-1}}{1-q^{j+k-1}} \underset{q\to 1}{\longrightarrow}$$

$$\underset{q\to 1}{\longrightarrow} A(L,N,\mathcal{P}) = \det\left(\begin{pmatrix} N+L+i-1 \\ L+j-1 \end{pmatrix}\right)_{1\le i,j\le \mathcal{P}} .$$

RHS expresses the fact that number of plane partitions $A(L,N,\mathcal{P})$ is equal to the number of self-avoiding lattice paths. Just the paths constituting a $\mathcal{P}$-tuple are in bi-jection with, so-called, gradient lines corresponding to a plane partition inside $\mathcal{B}(L,N,\mathcal{P})$.

• • • The form-factor of $\bar{\mathsf{F}}_n$ taken in the q-parametrization, $\mathbf{v}_N^{-2} = \mathbf{q}_N \equiv (q, q^2, \ldots, q^N)$, $\mathbf{u}_N^2 = \mathbf{q}_N/q \equiv (1, q, \ldots, q^{N-1})$, acquires the form:

$$\langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}}) \rangle =$$

$$= q^{\frac{n}{2}(N-n)(N-n-1)} \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(\mathbf{q}_N) S_{\boldsymbol{\lambda}}(\mathbf{q}_{N-n}/q) = \frac{\det \bar{\mathsf{T}}}{\mathcal{V}(\mathbf{q}_N)\mathcal{V}(\mathbf{q}_{N-n}/q)} ,$$

where $\bar{\mathsf{T}}$ is given as above with $L = N - n$ and $P = M$. We obtain:

$$\langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}}) \rangle = q^{\frac{n}{2}(N-n)(N-n-1)} Z_q(N-n, N, \mathcal{M}) .$$

The form-factor is the generating function of plane partitions inside $\mathcal{B}(N-n, N, M-N+1)$. We obtain the MacMahon formula at $q \to 1$:

$$\lim_{q \to 1} \langle \Psi(\mathbf{q}_N^{-\frac{1}{2}}) \mid \bar{\mathsf{F}}_n \mid \Psi((\mathbf{q}_{N-n}/q)^{\frac{1}{2}}) \rangle = A(N-n, N, M-N+1) .$$

IV. LOW TEMPERATURE

••• We assume that $M \gg 1$ and $1 \ll N \ll M$, and estimate the thermal correlation function *survival probability of domain wall*:

$$\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}, n, \beta) \equiv \frac{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid \bar{\mathsf{F}}_n \, e^{-\beta \mathcal{H}} \, \bar{\mathsf{F}}_n \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \rangle}{\langle \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \mid e^{-\beta \mathcal{H}} \mid \Psi_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}) \rangle}, \quad \bar{\mathsf{F}}_n \equiv \prod_{j=0}^{n-1} \sigma_j^-,$$

$$\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}, n, \beta) = $$

$$= \frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n})(M+1)^{N-n}} \sum_{\{\boldsymbol{\theta}_{N-n}\}} e^{-\beta(E_{N-n}(\boldsymbol{\theta}_{N-n}) - E_{N-n}(\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}))}$$

$$\times \left| \mathcal{V}(e^{i\boldsymbol{\theta}_{N-n}}) \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^{N-n}\}} S_{\hat{\boldsymbol{\lambda}}}(e^{-i\boldsymbol{\theta}_{N-n}}) S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}^{\,\mathrm{g}}_{N-n}}) \right|^2 .$$

If the chain is long enough while the number of quasi-particles is moderate, $N \ll M$, we replace the sums by the integrals since $\cos\theta_l^{\rm g} \simeq 1$ and $\cos\theta_l \approx 1$, $\forall l$. At large β, we approximate for $\mathcal{F}(\boldsymbol{\theta}^{\rm g}_{N-n}, n, \beta)$:

$$\mathcal{F}(\boldsymbol{\theta}^{\rm g}_{N-n}, n, \beta) \simeq \frac{A^2(N-n, N, M-N+1)}{\beta^{\frac{(N-n)^2}{2}}} \frac{\mathcal{I}_{N-n}}{\mathcal{N}^2(\boldsymbol{\theta}^{\rm g}_{N-n})},$$

where $\mathcal{I}_{N-n}$ is the *Mehta integral* ,

$$\mathcal{I}_N \equiv \frac{1}{N!} \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \cdots \int\limits_{-\infty}^{\infty} e^{-\frac{1}{2}\sum\limits_{l=1}^{N} x_l^2} \prod_{1\le k<l\le N} \left|x_k - x_l\right|^2 \frac{dx_1 dx_2 \dots dx_N}{(2\pi)^N},$$

The low temperature decay of the correlator is governed by the critical exponent $(N-n)^2/2$. The estimate demonstrates that $\mathcal{F}(\boldsymbol{\theta}^{\rm g}_{N-n}, n, \beta)$ is related to the matrix integrals of the theory of Gaussian Matrix Ensemble.

The integral $\mathcal{I}_N$ is given as follows:

$$\mathcal{I}_N = e^{\varphi_N}, \qquad \varphi_N \equiv \sum_{k=1}^{N} \log \frac{\Gamma(k)}{(2\pi)^{1/2}} .$$

Eventually, we estimate $\frac{1}{\mathcal{N}^2(\boldsymbol{\theta}^{\mathrm{g}}_{N-n})} \simeq \left(\frac{2\pi}{M+1}\right)^{N^2} e^{2\varphi_N}$, and express the survival probability of domain wall:

$$\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \simeq A^2(N-n, N, M-N+1)\, e^{\Phi(N,M,\beta)},$$

$$\Phi(N, M, \beta) \equiv N^2 \log \frac{2\pi}{M+1} - \frac{N^2}{2} \log \beta + 3\varphi_N .$$

The asymptotics of the correlator is proportional to the squared number of plane partitions inside a box with rectangular bottom $\mathcal{B}(N-n, N, M-N+1)$.

To study the asymptotical behavior, it is convenient to express φ_N through the Barnes G-function:

$$G(z+1) \;=\; (2\pi)^{z/2} e^{\frac{-z}{2}(z+1) - \frac{\gamma}{2} z^2} \prod_{n=1}^{\infty} \Big(1+\frac{z}{n}\Big)^n e^{-z+\frac{z^2}{2n}} \,,$$

which is an integral function satisfying the relations: $G(1)=1$, $G(z+1)=\Gamma(z)G(z)$, and

$$G(n+1) \;=\; \frac{(n!)^n}{1^1\,2^2\,\ldots n^n} \;=\; \prod_{k=1}^{n} \Gamma(k)\,.$$

We obtain for φ_N and $\mathcal{I}_N$:

$$\varphi_N \;=\; \log G(N+1) - \frac{N}{2}\log 2\pi\,, \qquad \mathcal{I}_N \;=\; \frac{G(N+1)}{(2\pi)^{N/2}}\,.$$

We re-express the number of plane partitions inside $\mathcal{B}(N-n, N, M-N+1)$:

$$A(N-n, N, M-N+1) = \frac{G(N+1)\,G(N-n+1)}{G(2N-n+1)} \times \frac{G(M+2-n+N)\,G(M+2-N)}{G(M+2-n)\,G(M+2)}\,.$$

The asymptotics of $\log G(z+1)$ at $z \to \infty$ is known. For instance, it gives for φ_N at $N \gg 1$:

$$\varphi_N = \frac{N^2}{2}\log N - \frac{3N^2}{4} + \mathcal{O}(\log N)\,, \qquad N \gg 1\,.$$

Eventually, the asymptotics of survival probability $\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta)$:

$$\log \mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta) \simeq N^2 \log\left(\mathsf{A}\,\frac{N^{3/2}}{M\beta^{1/2}}\right) + \\ + 2N(N-n)\log\left(\mathsf{D}\,\frac{M-n}{2N-n}\right).$$

When M and N increase and temperature T decreases,
$T < \mathrm{const(M,N,n)}$, the correlator $\mathcal{F}(\boldsymbol{\theta}^{\mathrm{g}}_{N-n}, n, \beta)$ also decreases.

CONCLUDING REMARKS

We have discussed the N-particle thermal correlation functions of the XXZ Heisenberg model on a cyclic chain at $\Delta \to 0$. We have considered the domain wall creation operator $\bar{\mathsf{F}}_n = \prod_{j=0}^{n-1} \sigma_j^-$. The combinatorial aspects of the correlation functions of this operator are considered. The calculations are based on the symmetric functions that allows us to express the answers in the determinantal form. The representations for the form-factors are related to the generating functions of self-avoiding random walks and boxed plane partitions.

The asymptotical behavior of the thermal correlation functions in question for the operator $\bar{\mathsf{F}}_n$ is estimated for low temperatures. The asymptotical representation at low temperature demonstrates the combinatorial pre-factor and is characterized by a power law decay. Its critical exponent looks like the free energy appearing at small coupling for large-N lattice gauge theory considered by Gross, Witten (1980).

THANKS !