On degreewise finite Eilenberg–Mac Lane spectra

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AI†-generated research note

15 October 2026

Abstract

We construct degreewise finite simplicial models of integral Eilenberg–Mac Lane spaces using integer-wall arrangements and finite orbit-incidence categories. A canonical lift gives an explicit primitive integral cocycle. Compatible pointed-set operators form a very special Γ-space and a degreewise finite prespectrum representing $H\mathbb{Z}$, with the Ω-spectrum condition in its tail. Maps into the models classify lattice torsors decorated by periodic covector flags; a common refinement relates these flags to Goncharov hypersimplices.

Untwisted Postnikov extensions with torsion invariants preserve degreewise finiteness. We give explicit square and prime-three examples, a tower through all primes, and finite encodings of classical cochain operations and their homotopies. A common toric incidence presentation supports different Postnikov towers through changes of the first face map. A general semialgebraic replacement theorem also yields models of BG for every compact Lie group. Applied to polarized abelian varieties, it gives finite Heisenberg deloopings with infinite-order Postnikov invariant.

Finiteness imposes exact restrictions: simplicial self-maps act eventually periodically on each homology group, excluding simplicial doubling and ordinary addition on finite integral models.

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†The constructions belong to the author, developed and fixed in a dialog with AI.

Matrix symmetries instead use coherent finite spans; finite matrix subgroups admit strict actions after replacement. Separately, geometric reordering and infinite-dimensional manifold recognition identify the original toric realizations homeomorphically with their affine topological group models. The resulting group law is explicit for the circle arrangement. The homotopy-theoretic machinery and cochain operations are classical; the emphasis is their explicit finite presentation, its geometry, and its limitations.

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1 Purpose, conventions, and main results

The motivating example is the simplicial set **SC** of circular orders, whose realization is $K(\mathbb{Z}, 2)$ [51]. Its finiteness suggests two questions. Can one construct comparably finite models of higher integral Eilenberg–Mac Lane spaces? Can those models retain enough cochain information to build nontrivial Postnikov extensions? This note gives constructions addressing both questions and explains how classical cochain operations act on the models. The resulting models need not be small, and no computational complexity bound is asserted.

There is also a general existence route: finite semialgebraic carrier replacement applies to compact simplicial tori and to nerves of compact Lie groups (Corollary 7.16). The toric construction adds explicit compatible integral cocycles, pointed-set operators, and the covector interpretation. Its orbit-incidence categories retain multiple incidence branches, so it uses the same carrier-comparison principle with additional structure; it is not identified with the face-poset construction of that theorem. No assertion of historical priority for bare existence is needed. Section 25.7 records a more

specific literature comparison: classical first-face twisting already gives common degreewise finite presentations with finite coefficients, while the possible contribution here concerns the explicit integral carrier geometry and its compatible structures.

The linear part has a precise strengthening. Refinement localization organizes all integer matrices coherently (Theorem 9.8). A diagram of lattices with finitely many objects admits a strict diagram of degreewise finite models exactly when each hom-set has finite matrix image (Theorem 9.9). The canonical cocycles and their higher comparisons have explicit finite integer alphabets; the latter can be evaluated from one endpoint and one carry (Section 9.6).

This master note retains four related research strands. The finite models, spectrum, and cochain operations form the first; the topology of their realizations forms the second. Polarized abelian varieties and Heisenberg deloopings form the third, and the Goncharov–matroid comparison forms the fourth. The separate companion papers develop these strands with their own definitions and proof routes. The master retains the broader comparisons and verification record.

Definition 1.1. A simplicial set X is *degreewise finite* if X_n is a finite set for every $n \geq 0$. A degreewise finite model of a space B is such an X together with a weak equivalence $|X| \simeq B$.

There may be infinitely many nondegenerate simplices altogether. Thus "finite" in this note does not mean a finite CW complex. For each fixed r, the skeleton $\mathrm{sk}_r X$ is generated by finitely many nondegenerate simplices, and its realization is a finite CW complex. This does not say that $X = \mathrm{sk}_r X$ for any fixed r. The toric models below have nondegenerate simplices in arbitrarily high degrees. The models are generally not Kan complexes. This distinction is essential: in a degreewise finite Kan complex every homotopy group is finite, since its elements have representatives among finitely many simplices of the corresponding degree.

The title refers to a degreewise finite simplicial prespectrum whose geometric realization represents $H\mathbb{Z}$. Its addition is encoded simplicially by finite refinement diagrams and their coherences. An ordinary continuous abelian group law on each original one-input realization is obtained by the geometric homeomorphism below. The suspension structure maps of the prespectrum are nevertheless ordinary simplicial maps: the qualification about crossed or twisted data concerns addition, not every map in the construction. The Ω-spectrum condition holds in the tail; an unshifted Ω-spectrum for $H\mathbb{Z}$ cannot have a degreewise finite zeroth space (Section 21).

The main conclusions are as follows.

1. For $k \geq 1$, a finite orbit-incidence category in each degree gives a simplicial set $D_0^{(k)}$ with $|D_0^{(k)}| \simeq K(\mathbb{Z}, k+1)$. A formula reading one integer from a lifted flag gives a primitive cocycle $c_{k+1} \in Z^{k+1}(D_0^{(k)}; \mathbb{Z})$. A bounded integer carry gives explicit compatible cochain homotopies for its additivity and a homotopy-coherent comparison of the entire pointed-set diagram with the standard integral cocycle diagram. Section 5 identifies the exact structures represented on a simplicial base: lattice torsors with compatible periodic covector flags. On finite bases, every integral class has such a representative after sufficiently many subdivisions. Section 6 gives a minor-compatible refinement of Goncharov hypersimplices by the addition arrangements, and characteristic maps from integral flag cocycles with explicit transport of homotopies. Section 7 compares this construction with the elliptic geometry of Arnoux–Marin. Divisor addition explains the additive Hopf class on their projective tower. More directly, cyclic differences and adjacent merges give the ordered root tori a simplicial structure. A finite signed-wall refinement yields a very special Γ-model naturally weakly equivalent to $X_0^{(1)}$ (Theorem 7.7). The projective symmetric quotient and the natural root–weight isogeny require separate comparisons; neither is silently identified with this simplicial equivalence. For an arbitrary complex abelian variety A, the ordered merge diagram is the classical bar construction BA; its signed refinement gives a finite model of $K(H_1(A;\mathbb{Z}),2)$. The complex structure and polarization are additional data (Section 7.4). The Poincaré line gives the exact polarization correction for a merge; the theorem of the cube makes these corrections bilinear and coherent. An external four-cocycle records the alternating lattice pairing on two inputs, with an explicit cup-one nullcochain on the diagonal and finite carry comparisons under changes of marking. Its integral two-stage Postnikov space is the Betti delooping of an algebraic Heisenberg group stack. A compact nerve built from triangle phases, followed by a face-compatible triangulation and free degeneracies, gives a degreewise finite model even though this invariant has infinite order. Its nonzero Whitehead pairing prevents an infinite-loop enhancement when the polarization form is nonzero. We describe its principal Heisenberg 2-bundles by local line data and compute the quotient by automorphisms of the underlying torus bundles. The algebraic Heisenberg action on perfect complexes then relates the same biextension to Fourier–Mukai duality and theta groups. The exchange $(x, y) \mapsto (y, -x)$ becomes a monoidal symmetry after tensoring the central line with $B_L(x, y)^{-1}$. Adding edge frames represents it by a span

of degreewise finite models, whose two legs are weak equivalences. A further finite replacement realizes the original Fourier action as a strict C_4-action, with a one-vertex model available (Corollary 7.43). More generally, an explicit correction line for every matrix in $\mathrm{GL}_2(\mathbb{Z})$ gives a monoidal symmetry, acting on the central band by its determinant. We compute its composition defect for a nonsymmetric line bundle and remove it by integral powers of $[-1]^*L \otimes L^{-1}$. This gives a coherent weak action on the algebraic group stack. All matrix words admit simultaneous finite simplicial models with strictly commuting word-face maps. These define a homotopy coherent action after weak equivalences are inverted (Corollary 7.41). Every individual model is degreewise finite; the family indexed by all words is infinite. This distinction is necessary. Finiteness of the set of n-simplices forces any simplicial self-map to act eventually periodically on H_n. Consequently the infinite-order shear on $H_2(P_E)$ cannot be a simplicial self-map of any degreewise finite model of this space. The same argument excludes a simplicial self-map representing multiplication by 2 on a degreewise finite $K(\mathbb{Z}, r)$-model, and hence excludes an ordinary simplicial addition on it (Section 7.19). For a finite matrix subgroup, the word construction instead has finitely many summands in every degree. Pulling back the universal group covering gives a strict free action; collapsing an invariant contractible subcomplex gives a one-vertex model with the same coherent action and pointing (Theorem 7.42). These comparisons concern homotopy coherent actions, including homotopy fixed points, without asserting equivalences of ordinary fixed-point spaces. Together with the homology obstruction they give an exact criterion: a subgroup of $\mathrm{GL}_2(\mathbb{Z})$ admits its faithful standard slot action on a degreewise finite model precisely when it is finite (Corollary 7.45). The derived modular action is classical; we record the precise shift and kernel-sign comparison with Mukai, Polishchuk, and Beauville in Section 7.16. We also compute the exact information retained: the Whitehead pairing determines the polarization type and is the pullback of the hyperbolic form in Orlov's derived-equivalence criterion. The group of homotopy self-equivalence classes consists exactly of the integral isometries and anti-isometries of this pairing. Two non-isogenous CM elliptic curves give the same marked Postnikov space but inequivalent derived categories. A compatible complex structure and an Appell–Humbert semicharacter specify the missing geometric data. The Jacobian comparison extends the projective tower and its divisor addition to every pointed smooth projective curve. Section 8 gives degreewise finite symmetric flag models with explicit reordering, induced

reorientation, and integer correction formulas. A two-triangle circle product exhibits the geometric mechanism at one torus level. A consequence of the classical crossed-group classification also sets a precise limit: a degreewise finite crossed simplicial group has finite homotopy groups in degrees at least two, so the higher integral models require a broader structure. Section 9 supplies a strict composition law for maps with refinement data. Finite families of integer coefficient vectors extend it simultaneously to all simplicial degrees: addition, inversion, duplication and every integer matrix have explicit finite simplicial span representatives. Their refinement comparisons prove the abelian identities in the homotopy category. Proposition 8.10 shows that the current carrier fold is not open for any $k \geq 1$, so even an arbitrary product homeomorphism cannot turn it into a topological group law. A construction with exact multiplication would have to change the fold or its presentation. Theorem 8.17 also excludes strictly natural ordinary row triangulations with finitely many vertices at the circle level when $k \geq 2$. Section 10 gives the positive geometric replacement. A single convex formula supplies every higher comparison between carrier maps and affine maps. The transported affine endpoint has exact products and addition after changing the vertex operators on the row polyhedra. Inspired by Semidetnov's operadic organization [72], a finite operad of subset families records the refinement choices. Its product with the standard Barratt–Eccles operad is E_∞ and has an explicit count in every arity and nerve degree. It organizes operation roofs, rather than a strict unital ordinary simplicial action on $D_0^{(k)}$. Section 11 obtains an actual homeomorphism first after a universal degreewise finite enlargement, and then for the original one-input model. The explicit contractible factor $C = \prod_{n\geq 2} \Delta[n]/\Lambda^n_0$ has realization $\mathbb{R}^\infty$. We prove that the affine torus realization is an $\mathbb{R}^\infty$-manifold and apply Sakai's classical classification to obtain $|D_0^{(k)} \times C| \cong |A_\bullet^{(k)}|$. Thus the enlarged finite model realizes a genuine topological abelian group. Heisey's form of the classification theorem [37] lets these homeomorphisms preserve the canonical integral classes. They transport all integer matrices with strict composition, in the homotopy classes of the earlier finite roofs (Theorem 11.6). Using a tail of this product postpones the enlargement beyond any prescribed finite skeleton. For the original diagonal, Proposition 11.9 constructs arbitrarily high-dimensional transverse disks over each open simplex. Degeneracies already agree with their affine counterparts; face attachments require the further argument below. Every lifted row face-map fibre is contractible, including fibres over nontrivial

barycentric flags; the quotient row maps are quasifibrations with the expected torus fibres. Theorem 11.16 identifies the stable nerve over one fixed carrier with $\mathbb{R}^\infty$. Corollary 11.18 then provides a connectivity bound tending to infinity for $k \geq 2$ even after every proper row degeneracy is removed. The crossed action in Proposition 8.7 gives a further, global argument: repeated small interval shuffles fill the actual links away from the canonical basepoint. Thus Theorem 11.27 proves that $|D_0^{(k)}| \setminus \{*\}$ is an $\mathbb{R}^\infty$-manifold. With an anchor retained, the exact join in Theorem 11.21 gives $|\operatorname{Dec}_0 Q| \cong \mathbb{R}^\infty$ for $k \geq 2$. Proposition 11.22 identifies the actual zero fibre of anchor deletion with the preceding model. Theorem 11.31 also identifies the total space of the universal lattice torsor with $\mathbb{R}^\infty$. The basepoint link is an $\mathbb{R}^\infty$-manifold, and its contractibility is equivalent to the desired homeomorphism (Corollary 11.30). Theorem 11.33 proves local control of anchor deletion over every open set. A fixed nonzero tag moves the entire zero fibre off itself, proving contractibility of the link and hence the original homeomorphism for $k \geq 2$ (Theorem 11.36). Proposition 11.38 handles $k = 1$ by an explicit formula depending on the realization weights. Corollary 11.39 therefore gives a continuous abelian law on every original one-input realization, preserving the integral marking. A finite formula in higher degrees and compatibility with the given multi-input carrier folds remain separate issues.

2. If X is degreewise finite and $\kappa \in H^{t+1}(|X|;\mathbb{Z})$ has finite order, $t \geq 1$, then $\operatorname{hofib}(\kappa : |X| \to K(\mathbb{Z}, t+1))$ has an explicit degreewise finite model. The construction includes a cochain trivializing the pulled-back class. Coefficients are constant throughout.

3. In particular, for the fundamental class $u \in H^3(K(\mathbb{Z},3);\mathbb{Z})$, we construct a finite model of $\operatorname{hofib}(u^2)$ with its actual nonzero k-invariant, not only its homotopy groups. Those groups are $\pi_3 = \mathbb{Z}$ and $\pi_5 = \mathbb{Z}$. The next invariant $\beta_3\mathcal{P}^1\rho_3 u$ has exact order three on this space and adds $\pi_7 = \mathbb{Z}$. More generally, there is a degreewise finite model with $\pi_3 = \mathbb{Z}$ and $\pi_{2p+1} = \mathbb{Z}$ for every prime p, no other positive homotopy groups, and a nonzero order-p invariant at each added stage.

4. For finite abelian A and degreewise finite X, the mapping space into $\mathcal{K}(A,t)$ is degreewise finite and Kan, with homotopy groups $H^{t-i}(X;A)$. Finite-coefficient operations compose in a simplicial category of such mapping spaces. Each known universal relation gives a natural homotopy by finite linear algebra, and the inherited cochain operad action retains its specified coherences. Postcomposing with a strict Bockstein

map transports these families to finite integral targets. For Sq^2 on degree-three inputs, short carry corrections give the additive homotopy and its associator on the actual finite resolved products. Sections 22.2–22.4 give local generators and finite recursions for the higher diagonals, including an all-degree binary comparison proof.

5. The arrangements also form a very special Γ-space. Its associated spectrum is $\Sigma^2 H\mathbb{Z}$; reindexing supplies a degreewise finite prespectrum for $H\mathbb{Z}$, with the Ω-spectrum condition in its tail.

Sections 3–23 contain the main argument. The appendices retain comparisons with circular orders, an interval bar construction, and the earlier binary cocycle candidate. None of the unresolved assertions about that candidate is used in the main results.

1.1 Cochains and their simplicial models

Write $[n] = \{0, \dots, n\}$ with its usual order and let $\Delta[n]$ be the standard simplicial n-simplex. For an abelian group A, $C^r(\Delta[n]; A)$ denotes normalized cochains: functions on the strictly increasing $(r+1)$-tuples of vertices, extended by zero to tuples with repeated vertices. The differential is

$$(\delta a)(i_0 \dots i_{r+1}) = \sum_{j=0}^{r+1} (-1)^j a(i_0 \dots \widehat{i_j} \dots i_{r+1}).$$

Its kernel is $Z^r(\Delta[n]; A)$. Monotone vertex maps act by pullback, so these groups form simplicial abelian groups as n varies. For $r \geq 1$ the standard cocycle model is

$$\mathcal{K}(A, r)_n = Z^r(\Delta[n]; A), \qquad |\mathcal{K}(A, r)| \simeq K(A, r).$$

We distinguish this standard model, denoted $\mathcal{K}$, from the finite toric models $D_0^{(k)}$. The former is infinite in degree r when $A = \mathbb{Z}$.

A normalized cocycle $z \in Z^r(X; A)$ determines a simplicial map

$$\widehat{z} : X \longrightarrow \mathcal{K}(A, r), \qquad \widehat{z}(x) = x^* z.$$

Similarly an arbitrary cochain determines a map to $C^r(\Delta[\bullet]; A)$. These are the standard cochain–map correspondences [43, Sections 23–24]. We often write $z(x)$ for $x^* z$, a cochain on the whole simplex, not just one numerical evaluation.

For integral cochains of degrees p and q, the cup product is

$$(a \smile b)(i_0 \dots i_{p+q}) = a(i_0 \dots i_p) b(i_p \dots i_{p+q}).$$

Reduction modulo an integer $m \geq 2$ is denoted ρ_m or an overline. All homotopy fibres are taken over the zero basepoint. We put $\binom{n}{r} = 0$ when $n < r$.

2 A finite cochain resolution

The finite toric models constructed below need not satisfy the Kan horn-filling condition. Consequently an ordinary pullback of maps between them need not be a homotopy pullback. For operations that factor through a finite coefficient group, there is a useful way around this problem: keep the finite toric inputs and perform the pullback against a finite Kan resolution of the intermediate target.

Use the conventions of Section 1.1.

Fix an integer $t \geq 1$ and put

$$\mathcal{K}(A,t)_n = Z^t(\Delta[n]; A), \qquad \mathcal{E}(A,t)_n = C^{t-1}(\Delta[n]; A).$$

Monotone vertex maps act by pullback, with zero assigned whenever a face becomes degenerate. Thus these are simplicial abelian groups, and coboundary defines a simplicial homomorphism.

Lemma 2.1. *The map*

$$\delta : \mathcal{E}(A,t) \longrightarrow \mathcal{K}(A,t)$$

is a surjective Kan fibration with contractible total space and kernel $\mathcal{K}(A, t-1)$, interpreted as the constant group A when $t = 1$. If $A = \mathbb{Z}/m$, where $m \geq 2$, both its source and target are degreewise finite. Their sizes are

$$|\mathcal{E}(\mathbb{Z}/m,t)_n| = m^{\binom{n+1}{t}}, \qquad |\mathcal{K}(\mathbb{Z}/m,t)_n| = m^{\binom{n}{t}}.$$

Proof. Every positive-degree cocycle on a simplex has a cone primitive, so δ is surjective in every degree. More explicitly, for a t-cocycle z, its primitive takes the value $z(0, i_0, \ldots, i_{t-1})$ on a face avoiding vertex zero and takes the value zero on a face containing vertex zero. Simplicial abelian groups are Kan complexes, and a degreewise surjective homomorphism between them is a Kan fibration [43].

The total space is contractible. One way to see this is to prepend a new vertex and extend a cochain by zero on every face containing that vertex. This supplies an extra degeneracy contracting $\mathcal{E}(A,t)$ to zero. Equivalently,

its normalized chain complex has one copy of A in degrees $t-1$ and t, with differential an isomorphism. Its kernel under δ is precisely the cocycle group in degree $t-1$.

There are $\binom{n+1}{t}$ nondegenerate $(t-1)$-faces of $\Delta[n]$. A t-cocycle is freely specified by its $\binom{n}{t}$ anchor values on faces containing vertex zero. These observations give the two cardinalities. No field assumption on $\mathbb{Z}/m$ is used. □

The lemma is useful even when the input is not Kan. For any map $a : W \to \mathcal{K}(\mathbb{Z}/m, t)$, the ordinary pullback

$$W \times_{\mathcal{K}(\mathbb{Z}/m,t)} \mathcal{E}(\mathbb{Z}/m, t)$$

computes $\mathrm{hofib}(a)$. Indeed, the displayed resolution replaces the basepoint of the Kan target by a contractible Kan complex mapping to it by a fibration. To check independence of the fibrancy of W, replace W by a Kan complex over the target. Right properness of simplicial sets says that pulling this weak equivalence back along the fibration preserves it. Thus the finite input itself can be used in the formula.

3 Finite categories from integer walls

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and use normalized cochains. For $k \geq 1$ set

$$A_n^{(k)} = Z^k(\Delta[n]; \mathbb{T}) \cong \mathbb{T}^{\binom{n}{k}}.$$

Anchor coordinates are $\theta_{0,I}$ for k-element subsets $I \subseteq \{1, \dots, n\}$. Every face evaluation θ_F is an integral linear combination of these coordinates. For a finite pointed set S, write $S^\circ = S \setminus \{*\}$ and

$$A_n^{(k)}(S) = (A_n^{(k)})^{S^\circ}, \qquad d = |S^\circ| \binom{n}{k}.$$

On its anchor cover $\mathbb{R}^d$, use precisely the walls

$$\theta_{U,F} := \sum_{s \in U} \theta_F^s \in \mathbb{Z}, \qquad \varnothing \neq U \subseteq S^\circ. \tag{1}$$

The face index F runs through all nondegenerate k-faces. Write $\widetilde{\mathcal{P}}_n^{(k)}(S)$ for the face poset of this periodic arrangement and $L = \mathbb{Z}^d$ for its translation lattice.

Proposition 3.1. *There is a finite category*

$$C_n^{(k)}(S) = \widetilde{\mathcal{P}}_n^{(k)}(S)/L$$

whose objects are face orbits and whose arrows are translation orbits of incidences $F \le G$. It satisfies

$$N_q C_n^{(k)}(S) = (N_q \widetilde{\mathcal{P}}_n^{(k)}(S))/L, \qquad |NC_n^{(k)}(S)| \cong \mathbb{T}^d.$$

The displayed homeomorphism is a levelwise geometric one; naturality under carrier functors is not asserted for it.

Proof. The anchor walls occur in (1). Each face closure is a bounded convex polytope with width at most one in every anchor coordinate. Therefore the lattice acts freely on faces. To compose two orbit incidences, translate their middle lifts to agree. The translation is unique, so composition is well defined and associative. Nonidentity arrows strictly raise dimension.

Only finitely many face orbits meet a compact fundamental cube. Fixing a representative G of a target orbit, every incoming arrow has a unique representative ending at G. These are exactly the nonempty faces of $\overline{G}$, so there are finitely many arrows. The same unique-lift argument works for an entire composable chain once its final lift is chosen. This proves the nerve formula. Equivariant barycentric subdivision of the cover polytopes gives $|N\widetilde{\mathcal{P}}| \cong \mathbb{R}^d$; quotienting gives the last assertion. □

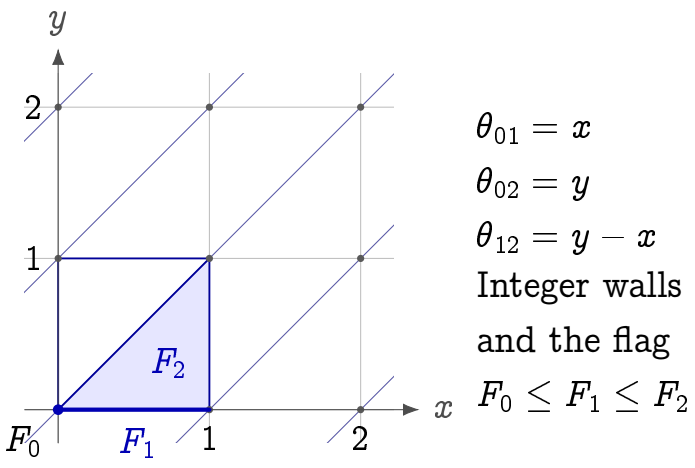


Figure 1: The lifted arrangement for one input, $k = 1$, and vertex degree two. The square is an anchor fundamental domain; the dots are lattice points. The shaded open triangle is $F_2 = \{0 < y < x < 1\}$, its lower open edge is F_1, and its initial vertex is F_0. All three families of integer walls are visible.

Why the arrows matter. The integer-wall circle has one vertex v, one open edge e, and two different arrows $v \rightrightarrows e$. Its nerve realizes a circle. Replacing this category by the orbit *poset* loses one endpoint incidence and produces an interval. Toric face categories with this incidence information are standard; see [14, Definition 7, Proposition 1.2, Lemma 3.3]. Our use concerns the cell structure of the real torus itself, not the complement of a complex toric arrangement.

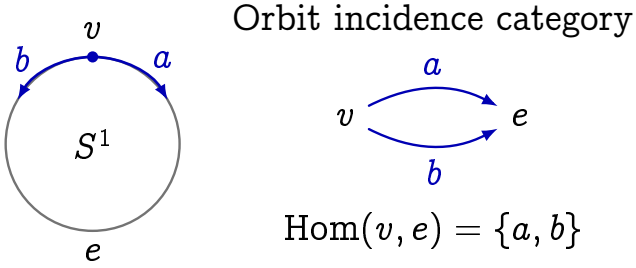


Figure 2: One vertex and one open edge still have two endpoint incidences. The two endpoint branches at v become the distinct arrows $a, b : v \rightrightarrows e$. Their nerve retains the circle. Replacing the orbit incidence category by the two-element orbit poset identifies these arrows and leaves an interval.

3.1 Simplicial and pointed-set operators

Monotone vertex pullbacks and pointed-set fibre sums lift to integer linear maps. Every target character pulls back to a source character or zero. Thus an open source face maps into one unique open target face. Its carrier defines a functor on cover face posets and then on the quotient categories. Carriers compose strictly: the image of a nonempty face lies in the open carrier obtained in two stages, so this is also the unique carrier of the composite. The simplicial and pointed-set directions commute.

Here $N_q\mathcal{C}$ denotes the nerve: its elements are strings of q composable arrows, including identities. The nerve and vertex directions commute, giving a bisimplicial set. Its diagonal provides the finite models used throughout the note:

Definition 3.2. For each finite pointed set S, define

$$X_0^{(k)}(S)_n = N_n\mathcal{C}_n^{(k)}(S), \qquad D_0^{(k)} = X_0^{(k)}(1_+).$$

A face acts simultaneously in the nerve and vertex directions; the same convention applies to degeneracies. The subscript 0 records the cut at zero modulo one, or equivalently at the integer walls on the real cover. The basepoint is the constant flag at the zero cell. In particular $X_0^{(k)}(0_+) = *$.

Proposition 3.1 makes every set in this definition finite. The next section identifies the homotopy type and gives its fundamental integral cocycle. No binary-cocycle conjecture is needed for this identification.

4 Canonical integral generators

The face categories carry an explicit integral cocycle representing their fundamental class. The construction also gives a direct proof of the homotopy type, without choosing barycentres naturally.

Fix $k \geq 1$, put $D = D_0^{(k)}$, and write

$$V_n = Z^k(\Delta[n]; \mathbb{R}), \qquad L_n = Z^k(\Delta[n]; \mathbb{Z}).$$

The anchor coordinates identify these with $\mathbb{R}^{\binom{n}{k}}$ and $\mathbb{Z}^{\binom{n}{k}}$, respectively. Retain the lifted face posets and form the simplicial set

$$\widetilde{D}_n = N_n \widetilde{\mathcal{P}}_n^{(k)}(1_+).$$

The quotient map $p : \widetilde{D} \to D$ is a torsor for L in each simplicial degree. Indeed, an orbit flag lifts uniquely after choosing any one of its lifted cells. In particular, the fibre over the degenerate basepoint is the simplicial abelian group L.

Integer parts of open cells. For an open lifted cell F and a face character θ_J, the integer part of θ_J is constant on F: its values either equal an integer or lie between two consecutive integers. Denote this integer by $\lfloor \theta_J |_F \rfloor$. This notation always uses the *open* cell. An endpoint in its closure may have a different integer part.

Every orbit cell has a unique lift whose anchor coordinates lie in $[0, 1)$. Call it the canonical lift. A monotone vertex map preserving 0 sends each target anchor character to a source anchor character or to zero. Its carrier therefore sends canonical open-cell lifts to canonical open-cell lifts, even if the image has lower dimension.

Theorem 4.1. *For a $(k+1)$-simplex $x = (C_0 \to C_1 \to \cdots \to C_{k+1})$ of D, lift the flag by making its first cell canonical, and denote its first two lifted cells by $F_0 \leq F_1$. Then*

$$c_{k+1}(x) = \left\lfloor \theta_{1,\dots,k+1} |_{F_1} \right\rfloor \tag{2}$$

defines a normalized integral cocycle. Its associated simplicial map to the standard cocycle model

$$\chi_c : D \longrightarrow \mathcal{K}(\mathbb{Z}, k+1), \qquad \mathcal{K}(\mathbb{Z}, k+1)_n = Z^{k+1}(\Delta[n]; \mathbb{Z}),$$

is a weak equivalence. In particular c_{k+1} is a primitive generator of $H^{k+1}(|D|;\mathbb{Z})$.

Proof. Define an integral k-cochain on the lifted simplicial set by

$$a(F_0 \leq \cdots \leq F_k) = \left\lfloor \theta_{0,\ldots,k}|_{F_0} \right\rfloor.$$

It is normalized: on a degenerate k-simplex the full face character pulls back to zero. This checks every degeneracy, including s_0.

Translation by $\lambda \in L_k$ gives

$$a(\lambda + e) = a(e) + \lambda(0, \ldots, k).$$

Consequently, for $e \in \widetilde{D}_{k+1}$ and $\lambda \in L_{k+1}$,

$$(\delta a)(\lambda + e) - (\delta a)(e) = \sum_{i=0}^{k+1} (-1)^i \lambda(0, \ldots, \widehat{i}, \ldots, k+1) = 0.$$

Thus δa descends to a unique cochain c on D, with

$$p^* c = \delta a. \tag{3}$$

Surjectivity of p in every degree implies $\delta c = 0$. The cochain c is normalized as well: a degenerate simplex of D has a degenerate lift, on which δa vanishes.

To compute $c(x)$, use the flag whose first cell is canonical. For $i > 0$, the diagonal face d_i retains the first nerve object and restricts along a vertex map preserving 0. Its contribution to δa is therefore zero. The face d_0 starts at the restriction of F_1, and its full character is $\theta_{1,\ldots,k+1}$. This proves (2), including its sign. Notice that the specified incidence $C_0 \to C_1$ determines the lift of F_1; the two cell orbits alone need not determine the answer.

It remains to prove primitivity. Let

$$\mathcal{E}(\mathbb{Z}, k+1)_n = C^k(\Delta[n]; \mathbb{Z}).$$

This is a contractible simplicial abelian group, and

$$\delta : \mathcal{E}(\mathbb{Z}, k+1) \longrightarrow \mathcal{K}(\mathbb{Z}, k+1)$$

is a surjective Kan fibration with kernel L. For example, the normalized complex of $\mathcal{E}(\mathbb{Z}, k+1)$ has two nonzero terms, both $\mathbb{Z}$, in degrees $k+1$ and k, with differential the identity.

Evaluation of a on all faces defines a simplicial map $\widehat{a} : \widetilde{D} \to \mathcal{E}(\mathbb{Z}, k+1)$. Equation (3) gives a commutative square

$$\begin{array}{ccc} \widetilde{D} & \xrightarrow{\widehat{a}} & \mathcal{E}(\mathbb{Z}, k+1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \delta} \\ D & \xrightarrow{\chi_c} & \mathcal{K}(\mathbb{Z}, k+1). \end{array}$$

This square is a pullback. Over any $x \in D_n$, both its proposed fibres are torsors for L_n, and

$$\widehat{a}(\lambda + e) = \lambda + \widehat{a}(e).$$

Hence $\widehat{a}$ gives a bijection between those fibres. In particular p is a Kan fibration; no Kan condition on D has been assumed.

Finally, $\widetilde{D}$ is contractible. For each fixed n, $|N\widetilde{\mathcal{P}}_n^{(k)}(1_+)|$ is the barycentric subdivision of V_n, hence contractible. The diagonal theorem for bisimplicial sets therefore makes $\widetilde{D} \to *$ a weak equivalence. In the displayed square both total spaces are contractible and the map on the fibre L is the identity. Realization takes these Kan fibrations to Serre fibrations [63] and preserves their fibre squares. Their homotopy exact sequences show that χ_c is a weak equivalence. This proves the assertion about the integral generator. □

Finite arity and the Segal maps. The same argument applies coordinatewise to any finite pointed set S, with coefficients $\mathbb{Z}^{S^\circ}$ and lattice $Z^k(\Delta[\bullet]; \mathbb{Z}^{S^\circ})$. The lifted total simplicial set is again contractible. The based-set maps act naturally on these covers and lattices by summing over fibres. In particular, the joint coordinate projection compares the lifted cover of $X_0^{(k)}(S)$ with the product of the lifted one-input covers. It is the identity on their identified lattice fibres, and both total spaces are contractible. Its base map

$$X_0^{(k)}(S) \longrightarrow \left(D_0^{(k)}\right)^{S^\circ}$$

is therefore a weak equivalence. This proves the Segal property directly and identifies the induced maps on π_{k+1} with the prescribed sums. Together with the strict pointed-set functoriality and connectedness of $|D_0^{(k)}|$, it proves that $X_0^{(k)}$ is very special. The argument requires no natural choice of barycentric subdivision homeomorphisms.

Theorem 4.2. *For every $k \geq 1$, the pointed-set diagram $X_0^{(k)}$ is degreewise finite and very special, and*

$$|X_0^{(k)}(S)| \simeq K(\mathbb{Z}^{S^\circ}, k+1).$$

In particular $|D_0^{(k)}| \simeq K(\mathbb{Z}, k+1)$*, with the explicit primitive cocycle of Theorem 4.1.*

Proof. Finiteness follows from Proposition 3.1. The preceding pullback proof and its coordinatewise version give the asserted homotopy types and the Segal equivalences. The one-input realization is connected, so its component monoid is the trivial group. This is the very-special condition, recalled in Section 21. □

5 What the models represent: periodic covector flags

The finite models have an interpretation in terms of specific realizable affine oriented matroids. On a fixed simplicial base, their maps classify structured torsors decorated by covector flags. After passing to homotopy, the decoration leaves the usual integral cohomology class. We give both statements precisely; they have different meanings of classification.

5.1 The periodic affine oriented matroids

Fix $k \geq 1$ and write $D = D_0^{(k)}$. In degree q put

$$V_q = Z^k(\Delta[q]; \mathbb{R}), \qquad L_q = Z^k(\Delta[q]; \mathbb{Z}), \qquad E_q = \binom{[q]}{k+1} \times \mathbb{Z}.$$

Here $\binom{[q]}{k+1}$ denotes the set of $(k+1)$-element subsets, written in increasing order. The walls are $\theta_J(x) = m$, for $(J, m) \in E_q$. For $x \in V_q$ define the sign vector

$$X_x(J, m) = \operatorname{sgn}(\theta_J(x) - m). \tag{4}$$

Two points have the same sign vector exactly when they lie in the same open face of the arrangement. The covector order is

$$X \leq Y \quad \Longleftrightarrow \quad X(J, m) \in \{0, Y(J, m)\} \quad \text{for every } (J, m).$$

It is precisely the face-incidence order used to construct $\widetilde{\mathcal{P}}_q^{(k)}(1_+)$. Thus the face poset can be read entirely from sign vectors, including its lower-dimensional collision strata.

There are finitely many wall directions and their levels are integers. The arrangement is locally finite, and its faces have bounded closures because

the anchor-coordinate walls already bound them. In the terminology of Delucchi–Knauer [15, Sections 1.2 and 5], this is a realizable finitary affine oriented matroid with a sliding lattice action. On sign vectors the action is explicitly

$$(\lambda X)(J, m) = X(J, m - \theta_J(\lambda)), \qquad \lambda \in L_q.$$

The lattice acts freely on faces. Its quotient must be taken as the incidence category $\mathcal{C}_q^{(k)}(1_+)$, retaining different incidence branches, rather than just the poset of face orbits. For example, a circle with one vertex and one open edge has two distinct arrows from the vertex to the edge. Identifying these arrows would lose the circle's homotopy type. The terminology above concerns this realizable periodic arrangement; we need no general axiomatization of toric oriented matroids.

Proposition 5.1. *The linear matroid of the wall normals $\theta_J \in V_q^*$ is the simplicial boundary matroid represented by the columns of*

$$\partial_k : C_k(\Delta[q]; \mathbb{R}) \longrightarrow C_{k-1}(\Delta[q]; \mathbb{R}).$$

Its rank is $\binom{q}{k}$. With the increasing orientations of the faces, this identification also preserves the oriented matroid.

Proof. Since $k \geq 1$, the map $\delta : C^{k-1}(\Delta[q]; \mathbb{R}) \to V_q$ is surjective. Its dual is injective and sends each θ_J to the vector $\partial_k[J]$, because

$$\theta_J(\delta b) = b(\partial_k[J]).$$

It therefore preserves all linear dependencies and their signs. The anchor coordinates give $\dim V_q = \binom{q}{k}$. □

For $k = 1$, write $\theta_{ij} = t_j - t_i$ with $t_0 = 0$. The normal matroid is the graphic matroid of the complete graph on $q + 1$ vertices. Modulo the integer lattice, the maximal chambers describe circular orders of $q+1$ distinct labelled points; lower faces allow collisions. For $k = 2$, the normals are the triangle columns of ∂_2. The tetrahedral boundary relations generate the space of dependencies, but need not be all its minimal dependencies. The higher models thus replace the graphic normal configuration by simplicial boundary configurations.

5.2 Exact classification on a simplicial base

The following is a definition of the structures represented here, not a claim that arbitrary oriented-matroid bundles have these classifying spaces. Write $L = K(\mathbb{Z}, k)$ and retain the lifted flag space $p : \widetilde{D} \to D$ of Section 4.

For a monotone map $\alpha : [r] \to [q]$, cocycle pullback $\alpha^* : V_q \to V_r$ sends an open face into a unique open face, its carrier, denoted $\rho_\alpha F$. These carriers preserve incidence and compose strictly. Indeed, every target face character pulls back to a source face character or to zero. They also satisfy

$$\rho_\alpha(F + \lambda) = \rho_\alpha F + \alpha^*\lambda.$$

Definition 5.2. A *periodic covector-flag system* on a simplicial set B assigns to every $\sigma \in B_q$ a simultaneous L_q-orbit of weak flags of lifted faces,

$$\mathcal{F}(\sigma) = [F_0 \leq \cdots \leq F_q].$$

For every $\alpha : [r] \to [q]$ this assignment must satisfy

$$\mathcal{F}(\alpha^*\sigma) = [\rho_\alpha F_{\alpha(0)} \leq \cdots \leq \rho_\alpha F_{\alpha(r)}]. \tag{5}$$

Repeated entries are allowed. The common lifted flag records the incidence branches as well as the face orbits.

An equivalent description uses structured torsors, making the bundle interpretation explicit. Over each $\sigma \in B_q$ take an L_q-torsor T_σ: a nonempty set with a free transitive L_q-action. For $t \in T_\sigma$ specify a weak flag $F_0(t) \leq \cdots \leq F_q(t)$, subject to

$$F_i(t + \lambda) = F_i(t) + \lambda.$$

For every $\alpha : [r] \to [q]$ give a restriction $R_\alpha : T_\sigma \to T_{\alpha^*\sigma}$, functorial in α, such that

$$\begin{aligned} R_\alpha(t + \lambda) &= R_\alpha(t) + \alpha^*\lambda, \\ F_j(R_\alpha t) &= \rho_\alpha F_{\alpha(j)}(t) \qquad (0 \leq j \leq r). \end{aligned}$$

In simplicial language, these data are a degreewise principal L-torsor $P \to B$ together with an L-equivariant simplicial map $\Phi : P \to \widetilde{D}$.

Theorem 5.3. *For every simplicial set B there are natural bijections*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{sSet}}(B, D) &\cong \{\textit{periodic covector-flag systems on } B\} \\ &\cong \{(P \to B, \Phi : P \to \widetilde{D})\}/\cong . \end{aligned}$$

The isomorphisms in the last line are L-equivariant maps over B preserving the decoration Φ.

Proof. A chain in the orbit-incidence category lifts uniquely after choosing one of its lifted cells. Thus D_q is exactly the set of simultaneous lattice orbits of the displayed flags. The simplicial-map identities are (5), proving the first bijection.

Given $f : B \to D$, form $P = B \times_D \widetilde{D}$ with its projection to $\widetilde{D}$. Conversely, an equivariant Φ descends to a simplicial map $f : B = P/L \to D$. The resulting map

$$P \longrightarrow B \times_D \widetilde{D}, \qquad t \longmapsto (\pi(t), \Phi(t))$$

is an equivariant map between nonempty torsors over each simplex, and hence an isomorphism. These constructions are inverse. □

This also gives a presheaf formulation. On the simplex category Δ/B, the groups $\mathcal{L}_B(\sigma) = L_{\dim \sigma}$ form a presheaf under cocycle pullback. The sets T_σ form a torsor presheaf for $\mathcal{L}_B$, with the specified covector-flag decoration. This is a precise version of the proposed simplicial gluing data. It is generally not a locally constant local system: the ranks vary with the simplex dimension, and the restriction homomorphisms need not be isomorphisms. No choice of a sheaf topology is needed for the statement.

5.3 The underlying higher bundle and its class

Set $G_k = |K(\mathbb{Z}, k)|$, a topological abelian group of homotopy type $K(\mathbb{Z}, k)$. The realization

$$|p| : |\widetilde{D}| \longrightarrow |D|$$

is a numerable locally trivial principal G_k-bundle. One way to check the point-set assertion is to view the simplicial sets as simplicial discrete spaces. Each level of p is a trivial principal L_q-bundle, and the base is proper because its degeneracies are injections of discrete spaces. The required levelwise fibrancy over a point is automatic. The realization theorem of Roberts–Stevenson [66, Theorem 2] then applies. It also applies to every pullback $P \to B$ above; realization identifies it with the pullback of $|p|$ along $|f|$.

The total space $|\widetilde{D}|$ is contractible by the proof of Theorem 4.1. Thus this is a universal principal G_k-bundle, and the characteristic class of a decorated bundle classified by f is

$$f^*[c_{k+1}] \in H^{k+1}(|B|; \mathbb{Z}).$$

The identity $p^* c_{k+1} = \delta a$ supplies its explicit universal cochain trivialization. For $k = 1$ these bundles have the same homotopy classification as circle bundles. For $k = 2$ they have the H^3 classification of $U(1)$-gerbes; on smooth

manifolds this is the stable classification of bundle gerbes by the Dixmier–Douady class [55, 66]. These are comparisons of classification theories: G_1 has the homotopy type of a circle, and is not being identified with that particular topological group.

The finite object here is the base of the universal bundle, degree by degree. Its lattice cover $\widetilde{D}$ is not degreewise finite. We have not constructed minimally triangulated higher-dimensional fibres. Likewise, although the $k = 1$ chambers recover circular-order data, the full model $D_0^{(1)}$ includes collision faces and flags. The theorem above does not identify it with **SC** or with the literal minimal circle-bundle classification of [51].

Proposition 5.4. *The derived classification is*

$$[B, D]_{\mathrm{Ho(sSet)}} \cong H^{k+1}(|B|; \mathbb{Z}).$$

If B has finitely many nondegenerate simplices, every such class is represented by a periodic covector-flag system on $\mathrm{Sd}^r B$ for some $r \geq 0$.

Proof. The weak equivalence χ_c proves the first assertion. For the second, use the fibrant replacement $D \to \mathrm{Ex}^\infty D$. A representing map $B \to \mathrm{Ex}^\infty D$ factors through $\mathrm{Ex}^r D$ because B is finite. Subdivision adjunction gives a map $\mathrm{Sd}^r B \to D$; the last-vertex weak equivalence $\mathrm{Sd}^r B \to B$ identifies its characteristic class with the prescribed one. These are the standard subdivision and extension constructions [42]. □

This does not assert that strict flag systems on the original B, modulo a naive simplicial homotopy, give every cohomology class. For finite B there are only finitely many maps $B \to D$, whereas a finite simplicial sphere can have $H^{k+1}(|B|; \mathbb{Z}) \cong \mathbb{Z}$. The amount of subdivision may depend on the class. If a precise equivalence relation on refined systems is desired, use adjunction to regard them as maps $B \to \mathrm{Ex}^r D$, and identify two when their images are simplicially homotopic in a common later extension stage. Finiteness of $B \times \Delta[1]$ makes this the derived classification above. This convention does not assume that subdivision preserves products.

The next section compares these periodic covector flags with the hypersimplicial Goncharov flags of the UOM bi-Grassmannian. It separates a direct polyhedral refinement from the characteristic maps that require a specified integral cocycle.

6 Goncharov flags: hypersimplicial refinement and characteristic maps

There are two precise connections with the UOM bi-Grassmannian. First, the arrangements used for crossed addition give a canonical subdivision of its hypersimplices, compatible with every minor face. Second, integral cocycles evaluated on Goncharov flags give characteristic maps to the finite Eilenberg–Mac Lane models. These maps transport specified cochain homotopies and stabilization corrections. Neither construction identifies the two flag spaces.

6.1 The oriented-matroid Goncharov flags

We retain the term used in the companion UOM notes [77, section on configurations and flags]. Fix $N \geq 1$. The hypersimplicial decomposition of

$$\Sigma_N^m = \{t \in \mathbb{R}_{\geq 0}^{m+1} : \textstyle\sum_i t_i = N\}$$

is obtained by cutting at integer coordinate values. A full-dimensional cube slice has the form

$$(a + [0,1]^{m+1}) \cap \Sigma_N^m \cong \Delta(r, m+1), \qquad r = N - \sum_i a_i,$$

where $a \in \mathbb{Z}_{\geq 0}^{m+1}$, $0 < r < m+1$, and $\Delta(r, M) = \{u \in [0,1]^M : \sum_i u_i = r\}$. An *OM Goncharov flag* assigns to each such slice a labelled uniform oriented matroid of rank r on its $m+1$ coordinate labels; uniform means that every r-element subset is a basis. On a shared face the induced minors agree, with the prescribed label and determinant-line identifications. In particular, a contraction face of one cube chart agrees with a deletion face of the adjacent chart.

Taking the oriented sum of the labelled hypersimplices defines the flag chain map. More precisely, let X_{U} be the coarse UOM cell space: it has one vertex $*$, and its positive dimensional cells are labelled by uniform oriented matroids of rank $1 \leq r < m+1$ on $m+1$ labels, with characteristic polytope $\Delta(r, m+1)$. Positive dimensional faces attach by the specified minor and label identifications. Every genuine vertex of a characteristic polytope attaches to $*$, with its face occurrence counted once. Set

$$C_*^{\mathrm{U}} = C_*^{\mathrm{cell}}(X_{\mathrm{U}}; \mathbb{Z}), \qquad C_0^{\mathrm{U}} = \mathbb{Z}[*].$$

Let $\mathcal{F}_N$ be the semisimplicial set of compatible flag labellings, with faces obtained by restriction to the boundary simplices; in degree zero ($m =$

0) use the single vertex label. Adjoin formal degeneracies when working with simplicial sets, retaining all original semisimplices as nondegenerate simplices. Then

$$\Pi : N_*(\mathcal{F}_N) \longrightarrow C_*^{\mathrm{U}}, \qquad \Pi(\mathcal{F}) = \sum_Q \epsilon_Q [M_Q], \qquad \partial_{\mathrm{U}}\Pi = \Pi\partial. \tag{6}$$

Here the sum runs over top-dimensional hypersimplices of the flag and ϵ_Q compares their characteristic orientations with the orientation of the source simplex. Internal facets occur twice with opposite orientations and the same induced minor, so they cancel. Boundary facets give the restricted flags. Degenerate chains are sent to zero. This argument uses the actual boundary faces of a hypersimplex: an endpoint is counted once, and a collapsed coordinate face is not declared to be an extra facet.

For real decorated linear flags, the quotient construction of Goncharov produces such labels by taking oriented matroids of the resulting vector configurations [33, Sections 4.1–4.2]. The combinatorial flag object retains the minor compatibilities. It does not retain all the linear or period data of the decorated flags. Keeping parent matroids and ordered normal histories is additional structure, used in the corrected geometric constructions of [77].

6.2 A common subdivision, compatible with every minor face

For M inputs consider the periodic arrangement

$$\mathcal{A}_M = \left\{ \sum_{i \in I} t_i = j : \varnothing \neq I \subseteq [M],\ j \in \mathbb{Z} \right\}. \tag{7}$$

For every $k \geq 1$, this is exactly the lifted arrangement underlying the crossed addition model $X_0^{(k)}(S)$ in vertex degree $q = k$, when $|S^\circ| = M$. Indeed, $Z^k(\Delta[k]; \mathbb{R}) = \mathbb{R}$ and each input has one coordinate. Thus this connection concerns the multi-input models, where the subset-sum walls encode addition.

Proposition 6.1. *Restriction of $\mathcal{A}_M$ to each hypersimplex $\Delta(r, M)$ gives a canonical finite polytopal subdivision. These subdivisions commute with coordinate permutations, integer translations, and all coordinate-face restrictions. In particular they respect the deletion and contraction faces of a hypersimplicial Goncharov flag.*

Proof. The bounding walls $t_i = 0, 1$ and the affine span $\sum_i t_i = r$ belong to $\mathcal{A}_M$. Hence the hypersimplex is a union of arrangement faces. For a fixed I, only the levels $0 \leq j \leq |I|$ can meet it; the subdivision is therefore finite.

On a coordinate face $t_e = \varepsilon$, with $\varepsilon \in \{0, 1\}$, the wall for (I, j) restricts to

$$\sum_{i \in I \setminus \{e\}} t_i = j - \varepsilon \mathbf{1}_{e \in I}.$$

After ignoring constant equations and repetitions, the nonconstant walls are exactly $\mathcal{A}_{M-1}$. Deletion gives $\Delta(r, M-1)$; contraction gives $\Delta(r-1, M-1)$. More generally, fixing $t_C = 1$ and $t_D = 0$ gives the same prescribed subdivision on

$$\Delta\big(r - |C|, [M] \setminus (C \cup D)\big).$$

This formula is independent of the order of restrictions. Translation $t = a + u$, $a \in \mathbb{Z}^M$, changes a wall level by the integer $\sum_{i \in I} a_i$ and preserves the arrangement. Permutations preserve the set of subsets I. □

Equivalently, intersect the single arrangement $\mathcal{A}_{m+1}$ with the whole simplex Σ_N^m. Its restriction refines all the integer cube slices at once. Thus the refinements agree on their common walls. The same construction subdivides a compatible UOM-labelled hypersimplicial complex while keeping each original UOM label on its carrier. The new interior cuts do not assert additional UOM minor relations.

For an oriented coarse cell Q, let

$$s_{\mathcal{A}}[Q] = \sum_{\substack{F \subseteq Q \text{ fine cell} \\ \dim F = \dim Q}} [F],$$

with the orientations induced from Q. Cancellation of interior facets proves

$$\partial s_{\mathcal{A}} = s_{\mathcal{A}} \partial. \tag{8}$$

The restriction calculation in the proposition makes this subdivision compatible with the original minor-face incidence signs. Each closed hypersimplex still realizes the same ball; its refined covector flags give the barycentric subdivision of this finer cell structure. The facewise subdivision also descends through the coordinate-face and integer-translation identifications used here, retaining all face occurrences.

The octahedron and the carry walls. For $\Delta(2, 4)$ the distinct interior walls are just

$$t_1 + t_2 = 1, \qquad t_1 + t_3 = 1, \qquad t_1 + t_4 = 1.$$

Complementary pairs describe the same walls. With $u_i = t_1 + t_{i+1} - 1$, $1 \le i \le 3$, the octahedron becomes

$$|u_1| + |u_2| + |u_3| \le 1,$$

and the cuts become $u_i = 0$. There are eight tetrahedra: join the centre $(1/2, 1/2, 1/2, 1/2)$ to each boundary triangle. The boundary is unchanged, and the numbers of cells are $(7, 18, 20, 8)$ in dimensions zero through three. These are exactly the walls across which the local pairwise integer carries

$$\lfloor t_i + t_j \rfloor - \lfloor t_i \rfloor - \lfloor t_j \rfloor$$

change in the open cube. The hypersimplicial geometry and the carry geometry therefore meet in a concrete local construction.

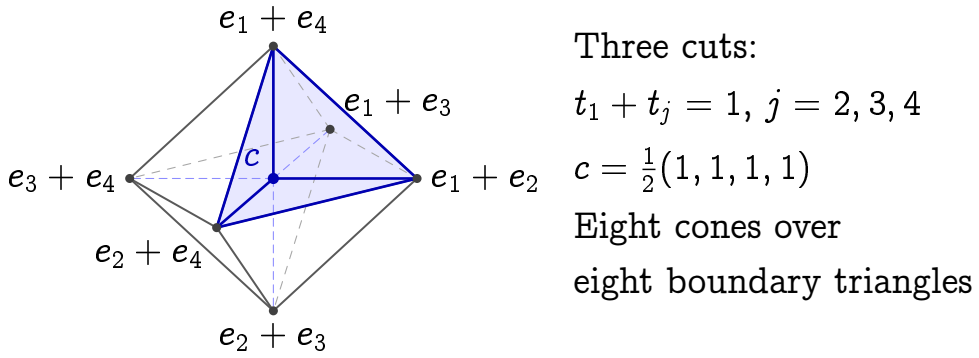


Figure 3: The subset-sum subdivision of the octahedron $\Delta(2, 4)$, shown in an oblique projection. The three cutting planes divide it into eight tetrahedra by joining the centre to each original boundary triangle. One tetrahedron is emphasized in blue. Its three edges from c lie inside the octahedron; the boundary triangulation is unchanged.

The precise level of comparison. If $P_{\mathcal{A}}(Q)$ is the refined face poset of one lifted hypersimplex, inclusion and quotient give

$$P_{\mathcal{A}}(Q) \longrightarrow \widetilde{\mathcal{P}}_k^{(k)}(S) \longrightarrow C_k^{(k)}(S).$$

Taking nerves gives a map to the fixed-$q = k$ row $N_\bullet C_k^{(k)}(S)$ of the bisimplicial construction. It is not yet a map into its diagonal $X_0^{(k)}(S)_q = N_q C_q^{(k)}(S)$: that would require compatibility in the varying vertex degree as well. Nor does it directly give a map to the one-input model $D_0^{(k)}$. The lattice quotient can identify the faces $t_e = 0$ and $t_e = 1$; their ranks and deletion or contraction labels remain extra data. The next construction supplies a characteristic map after specifying an integral cocycle.

6.3 From a Goncharov cocycle to periodic covector flags

Let F be a simplicial parameter space of compatible flags, for example a finite simplicial test diagram in $\mathcal{F}_N$ with its formal degeneracies, or a resolution

retaining specified histories. Assume its flag sum gives the chain map $\Pi : N_*(F) \to C^{\mathrm{U}}_*$ above. Fix $n \geq 2$ and an integral cocycle

$$\omega : C^{\mathrm{U}}_n \longrightarrow \mathbb{Z}, \qquad \omega \partial_{\mathrm{U}} = 0.$$

Then $z = \Pi^* \omega$ is a normalized n-cocycle on F, with the local formula

$$z(\mathcal{F}) = \sum_{Q} \epsilon_Q \omega(M_Q). \tag{9}$$

Put $D = D_0^{(n-1)}$, $c = c_n$, and $K = K(\mathbb{Z}, n)$.

Theorem 6.2. *The evaluated cocycle determines a canonical homotopy class*

$$[f_\omega] = [\chi_c]^{-1} [\widehat{z}] \in [F, D]_{\mathrm{Ho}(\mathrm{sSet})}.$$

There is an explicit natural comparison span

$$F \xleftarrow{u} \mathcal{R}(z) \xrightarrow{v} D, \qquad u \text{ a weak equivalence}, \tag{10}$$

and a normalized cochain $b \in C^{n-1}(\mathcal{R}(z); \mathbb{Z})$ such that $\delta b = u^* z - v^* c$. If F has finitely many nondegenerate simplices, its class also has a strict representative $\mathrm{Sd}^r F \to D$ for some r; equivalently, it has a periodic covector-flag representative after subdivision.

Proof. Closedness of z follows from (6). The canonical generator gives the weak equivalence $\chi_c : D \to K$, proving the first assertion. To make the comparison explicit, set

$$E_q = C^{n-1}(\Delta[q]; \mathbb{Z}), \qquad L_q = Z^{n-1}(\Delta[q]; \mathbb{Z})$$

and define

$$\mathcal{R}(z)_q = \{(\sigma, d, h) \in F_q \times D_q \times E_q : \delta h = \sigma^* z - d^* c\}. \tag{11}$$

Pullback of cochains gives all simplicial operators. The last coordinate is a cochain-valued simplicial map; evaluation in degree $n-1$ gives b, with the stated coboundary identity.

For the weak equivalence, form the principal L-torsor $P_z = F \times_K E$, where $E \to K$ is δ. The equivariant map $\widehat{a} : \widetilde{D} \to E$ from Section 4 identifies

$$\mathcal{R}(z) \cong (P_z \times \widetilde{D})/L, \qquad [(\sigma, w), e] \longmapsto (\sigma, p(e), w - \widehat{a}(e)),$$

where L translates both factors simultaneously. After realization this is the bundle associated to the numerable principal $|L|$-bundle $|P_z| \to |F|$, with fibre $|\widetilde{D}|$. The realization statement follows as in Section 5, using [66]. The fibre is contractible, so u is a weak equivalence. The subdivision assertion is Proposition 5.4. $\square$

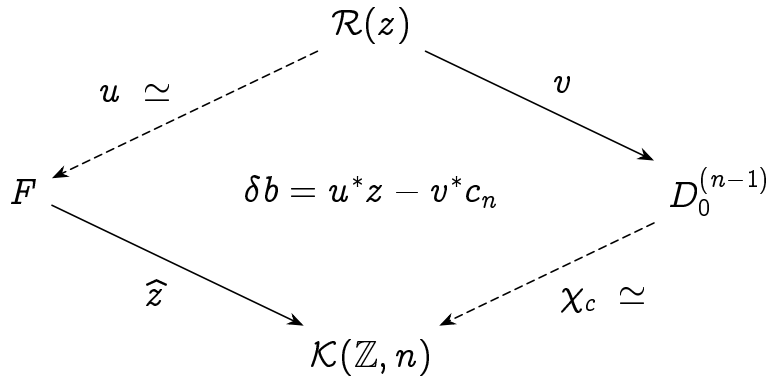


Figure 4: The Goncharov characteristic bridge, for an integral cocycle $z = \Pi^* \omega$. The middle equation records a specified homotopy between the two routes to the cocycle model. Dashed arrows are weak equivalences. The canonical comparison space $\mathcal{R}(z)$ generally has infinitely many simplices in a degree; a finite source F instead admits a strict periodic-flag representative on a sufficiently subdivided source.

The span is canonical in the given cocycle data, but its middle space is generally infinite in each sufficiently large degree: in degree $n-1$, the coordinate h already ranges over $\mathbb{Z}$. The finite comparison for a finite F is the subdivision span, whose strict representative need not be canonical. There is no uniform subdivision bound asserted here. A rational cocycle alone gives a class with rational coefficients; a specified integral refinement is needed for this theorem.

6.4 Homotopies, lower-row contractions, and stabilization

The characteristic construction retains the homotopies supplied by flags and their histories. If two flag chain maps satisfy

$$\Pi_1 - \Pi_0 = \partial_{\mathrm{U}} H + H \partial,$$

then evaluating the fixed closed ω gives

$$z_1 - z_0 = \delta b_{01}, \qquad b_{01} = \omega H. \tag{12}$$

With t the zero-cochain on $\Delta[1]$ taking values $0, 1$ at its endpoints, the cocycle

$$\mathrm{pr}_F^* z_0 + \delta \left(\mathrm{pr}_F^* b_{01} \smile \mathrm{pr}_{\Delta[1]}^* t \right)$$

gives the corresponding simplicial homotopy of maps into K. If a second chain homotopy satisfies $H_{12} - H_{02} + H_{01} = \partial_{\mathrm{U}} J - J \partial$, then

$$b_{12} - b_{02} + b_{01} = \delta(-\omega J).$$

Thus the next coherence also has an explicit cochain filler. In particular, any supplied contraction $\Pi = \partial_{\mathrm{U}} H + H\partial$ gives the primitive $z = \delta(\omega H)$ and a nullhomotopy of the characteristic map. This applies to lower-row contractions when they have actually been constructed; it does not prove their existence.

There is a useful strict formulation on the comparison spans. Suppose $f : F \to F'$ and $h \in C^{n-1}(F; \mathbb{Z})$ satisfy

$$f^* z' - z = \delta h.$$

Then there is a simplicial map over f and the identity of D,

$$\mathcal{R}(z) \longrightarrow \mathcal{R}(z'), \qquad (\sigma, d, a) \longmapsto (f\sigma, d, a + \sigma^* h). \tag{13}$$

The defining equation follows by taking δ. Corrections compose by $h + f^* h'$, so the construction preserves specified compositions strictly.

Corollary 6.3. *Let $s_N : F_N \to F_{N+1}$ be a tower of simplicial sets. Suppose it carries normalized integral cocycles z_N of degree n and specified integral cochains h_N of degree $n - 1$ with*

$$s_N^* z_{N+1} - z_N = \delta h_N.$$

Its comparison spans form a strict tower over D, and define a characteristic map

$$\operatorname{hocolim}_N F_N \longrightarrow D$$

in the homotopy category.

Proof. Use (13) for each structure map. The projections to D commute strictly. The projections to F_N are objectwise weak equivalences, hence induce a weak equivalence on homotopy colimits. This gives the asserted map. □

This is compatibility of characteristic maps with stabilization. It is not a stable equivalence between the UOM object and an Eilenberg–Mac Lane spectrum. The cochain corrections are part of the input, including their orientation conventions. The ordered-history constructions in [77] are a source of the geometric homotopies to which this criterion can be applied; discarding those histories is not justified by the minor identities alone.

6.5 A characteristic-class anchor and the remaining comparison

For an actual oriented circle bundle, e^2 is an integral degree-four class. Consequently its classifying data give a map to $D_0^{(3)}$, and finite test diagrams give strict periodic covector flags after subdivision. For an unoriented rank-two bundle, the Euler class has orientation coefficients and its square is again untwisted. On an oriented real two-plane bundle this square is p_1. This is a concrete starting point for the rank-two covector-sphere bundles in the UOM program; an extension to other coarse or resolved bases requires the appropriate bundle or integral cocycle there. The combinatorial Pontryagin-class program of Gelfand–MacPherson [31] provides a classical comparison target for these constructions.

Remark 6.4 (Scope of the bigrassmannian connection). The local subdivision theorem and the characteristic-map theorem establish different parts of a comparison. What remains is to construct a map using the UOM labels and ordered normal histories that agrees with this local subdivision in every simplicial direction and realizes the desired integral characteristic cocycle. The corrected UOM notes do not grant an automatic sphere bundle to bare compatible hypersimplicial labels. Nor does acyclicity of an augmented full hypersimplex imply acyclicity of a truncated flag complex. Those distinctions must be retained when comparing lower rows and stable objects.

The connection with broader theories has an established setting: Goncharov uses bigrassmannian and period complexes to construct characteristic classes in rational Deligne cohomology [33]; hypersimplicial flag constructions also enter the gauge-theoretic work of Dimofte–Gabella–Goncharov [17]. Our results supply common polyhedral data and an ordinary integral-cohomology target once an integral cocycle is specified. They do not yet construct a regulator, an algebraic K-theory comparison, or a physical invariant. Determinant magnitudes, periods, and integral normalizations require their own comparison maps.

7 Elliptic divisors and the Arnoux–Marin comparison

7.1 Elliptic divisors and stable addition

There is a geometric comparison with the elliptic quotient used by Arnoux–Marin [1, §§12–14]. It explains addition on the projective tower, but its

finiteness is stagewise. Fix a complex elliptic curve E with origin 0, and put

$$L = \mathcal{O}_E([0]), \qquad F_N = |N[0]| \quad (N \geq 1),$$
$$A_{N-1} = \{(z_1, \ldots, z_N) \in E^N : \textstyle\sum_i z_i = 0\}.$$

Here the last vertical bars denote a complete linear system. The Abel–Jacobi criterion and Riemann–Roch give

$$A_{N-1}/\mathfrak{S}_N \cong F_N = \mathbb{P}H^0(E, L^N) \cong \mathbb{CP}^{N-1}.$$

Proposition 7.1. *The ordinarily permutation-equivariant line*

$$\bigotimes_{i=1}^{N} \mathrm{pr}_i^* L \Big|_{A_{N-1}}$$

descends to a line $\mathcal{M}_N \cong \mathcal{O}_{F_N}(1)$. Writing s_0 for the canonical section of L, stabilization and multiplication are

$$i_N : F_N \longrightarrow F_{N+1}, \quad D \longmapsto D + [0],$$
$$\mu_{N,M} : F_N \times F_M \longrightarrow F_{N+M}, \quad (D, D') \longmapsto D + D'.$$

On sections these are multiplication by s_0 and multiplication of two sections, respectively. Thus i_N is a projective hyperplane inclusion, and

$$i_N^* \mathcal{M}_{N+1} \cong \mathcal{M}_N \otimes L|_0, \qquad \mu_{N,M}^* \mathcal{M}_{N+M} \cong \mathcal{M}_N \boxtimes \mathcal{M}_M.$$

In particular, with $h_N = c_1(\mathcal{O}_{F_N}(1))$,

$$\mu_{N,M}^* h_{N+M} = \mathrm{pr}_1^* h_N + \mathrm{pr}_2^* h_M.$$

Proof. A stabilizer only permutes tensor factors at coincident points; it acts trivially on their one-dimensional tensor product. The equivariant line therefore descends. For $N \geq 2$, the product of the canonical sections descends with zero divisor the hyperplane of divisors containing 0. For $N > 2$ its generic zero is simple. For $N = 2$, the cover is $z \mapsto \{z, -z\}$: the product vanishes to order two at $z = 0$, while a local quotient coordinate is z^2, so its descended zero is again simple. This identifies the descended line with $\mathcal{O}(1)$; the $N = 1$ assertion is an isomorphism of lines over a point. Appending 0 adds the constant factor $L|_0$, and concatenating two tuples tensors their norm lines. These identifications descend to the displayed bundle formulas. Equivalently, multiplication of nonzero sections identifies the pullback of the target tautological line with the tensor product of the two source tautological lines; dualizing gives the same $\mathcal{O}(1)$ formula. □

A choice of a nonzero vector in $L|_0$ makes the stabilization identifications compatible. This vector is not $s_0(0)$, which vanishes. Alternatively, the lines $\mathcal{M}_N \otimes (L|_0)^{-\otimes N}$ are canonically compatible with both stabilization and multiplication.

Proposition 7.2. *In compactly generated spaces the colimit*

$$M = \operatorname*{colim}_{N \geq 1} (F_N, i_N) \cong \mathbb{CP}^\infty$$

is a strict commutative topological monoid under divisor addition, with unit the stable class of $[0]$*. Its multiplication represents the usual addition on* $K(\mathbb{Z}, 2)$*. Consequently*

$$\mathcal{G}(S) = M^{S \setminus \{*\}}$$

with pointed-set fibre sums is a very special topological Γ*-space. Its deloopings have homotopy types* $K(\mathbb{Z}, r+2)$*, and its spectrum is* $\Sigma^2 H\mathbb{Z}$*. These conclusions alone provide no degreewise finite simplicial* Γ*-model. In particular, the direct colimit of the refined Arnoux–Marin flag triangulations has infinitely many vertices.*

Proof. The section inclusions identify the tower with the usual hyperplane tower. Give its colimit the weak CW topology and form products in compactly generated spaces. Products commute with these colimits, so the compatible continuous maps $\mu_{N,M}$ induce a continuous multiplication on M. Stabilizing either input stabilizes the output; hence the operation is well defined, associative and commutative, and adding $[0]$ is the identity in the colimit. Its pullback on the integral generator is the sum of the two generators, which characterizes addition on $K(\mathbb{Z}, 2)$. The Segal maps of $\mathcal{G}$ are product isomorphisms and M is connected. Segal's delooping theorem therefore gives the asserted spaces and spectrum [71, Proposition 1.4]; see also [70, pp. 331–332] for the pointed-set convention.

For the finiteness assertion, choose $N \geq 2$ and a nonzero $z \in E[N]$. The divisor $N[z]$ is an original arrangement vertex, and remains a vertex after adding the coordinate walls and passing to barycentric flags. Indeed, a lift of its tuple has all coordinate differences integral in each real factor, so it is a zero-dimensional root-arrangement cell. Every divisor in the stabilized image of a lower stage contains 0, whereas $N[z]$ does not. It is therefore a new vertex at stage N, and the strict flag inclusions preserve it thereafter. There are thus infinitely many vertices in the colimit. $\square$

Finite projective stages, a finite number of CW cells in each dimension, and degreewise finite simplicial sets are distinct conditions. Section 7.3 supplies degreewise finite data with coherent Segal comparisons by a different organization of the ordered real root tori. Its finite diagonal is not the direct colimit of the projective triangulations.

7.2 The root–weight bridge to the elliptic arrangements

Put $N = q + 1$ and define

$$V_N = \{x \in \mathbb{R}^N : \textstyle\sum_i x_i = 0\}, \qquad \pi_N(x) = x - \tfrac{1}{N}\Big(\textstyle\sum_i x_i\Big)\mathbf{1},$$

$$\Lambda_N^{\mathrm{rt}} = V_N \cap \mathbb{Z}^N, \qquad \Lambda_N^{\mathrm{wt}} = \pi_N(\mathbb{Z}^N).$$

The labels distinguish these lattices from the carrier spaces Q_q used elsewhere. The anchored circle-configuration row is

$$(\mathbb{R}/\mathbb{Z})^N/(\mathbb{R}/\mathbb{Z})_{\mathrm{diag}} \cong V_N/\Lambda_N^{\mathrm{wt}}.$$

The corresponding real factor in the AM construction is instead $V_N/\Lambda_N^{\mathrm{rt}}$. They have the same arrangement upstairs, with walls $x_i - x_j \in \mathbb{Z}$, but different lattices.

Proposition 7.3 (The finite cover and its elliptic analogue). *There is a canonical exact sequence*

$$0 \longrightarrow \Lambda_N^{\mathrm{rt}} \longrightarrow \Lambda_N^{\mathrm{wt}} \longrightarrow \mathbb{Z}/N\mathbb{Z} \longrightarrow 0.$$

Thus the identity upstairs induces a degree-N cover of real tori. For an elliptic curve E, put

$$K_N = \ker\big(\Sigma : E^N \longrightarrow E\big), \qquad D_N = E^N/E_{\mathrm{diag}}.$$

The inclusion followed by the quotient gives an isogeny

$$p_N : K_N \longrightarrow D_N, \qquad \ker p_N = \{(a, \dots, a) : a \in E[N]\}.$$

It is a free $E[N]$-quotient and has degree N^2.

Proof. The class of $\pi_N(a)$ modulo Λ_N^{rt} is determined by $\sum_i a_i$ modulo N. It vanishes exactly when this sum is divisible by N, and every residue occurs. For the elliptic assertion, every diagonal-translation class has a representative of sum zero: choose b with $Nb = \sum_i z_i$ and translate (z_i) by $-b$. Two such representatives differ by a diagonal N-torsion point. Multiplication by N on E has N^2 kernel points. □

Choose oriented real coordinates $E \cong (\mathbb{R}/\mathbb{Z})^2$. Then $K_N \cong (V_N \times V_N)/(\Lambda_N^{\mathrm{rt}} \times \Lambda_N^{\mathrm{rt}})$, while D_N uses the weight lattice in both factors. The common root walls therefore give a precise degreewise bridge between the ordered elliptic AM arrangements and two copies of the circle-configuration arrangement. This statement precedes quotienting by the symmetric group.

For $\alpha : [m] \to [q]$, put $M = m + 1$ and let $r_\alpha(x)_j = x_{\alpha(j)}$. Coordinate restriction, including repetition, induces a strict map $D_N \to D_M$. Upstairs its centered expression is

$$C_\alpha = \pi_M r_\alpha : V_N \longrightarrow V_M.$$

These maps compose strictly and satisfy $C_\alpha \Lambda_N^{\mathrm{wt}} \subseteq \Lambda_M^{\mathrm{wt}}$: subtraction of a constant before restriction has no effect after the final projection.

Proposition 7.4 (Failure of centered descent to AM roots). *The centered maps do not in general descend from K_N to K_M. In particular, the maps p_N cannot be made into a simplicial map with these prescribed restriction maps on the D_N. This failure cannot be removed by subdivision or by replacing a prescribed composite $K_N \to D_M$ by a homotopic map.*

Proof. Take $N = 3$, $M = 2$, and delete the first coordinate. The root lattice vector $(1, -1, 0)$ is sent to

$$C_\alpha(1, -1, 0) = (-\tfrac{1}{2}, \tfrac{1}{2}) \notin \Lambda_2^{\mathrm{rt}}.$$

Consequently the induced homomorphism on fundamental groups of $K_3 \to D_2$ does not land in the subgroup defining the cover p_2. The covering-space lifting criterion forbids a lift, and this subgroup condition is invariant under homotopy; the fundamental groups here are abelian. The same calculation in either real factor proves the elliptic assertion. □

This is an obstruction to the specified centered bridge, not to every possible simplicial structure on the abstract tori $K_N \cong E^{N-1}$. Clearing the denominator of one map gives the genuine homomorphism

$$(F_\alpha z)_j = M z_{\alpha(j)} - \sum_{a=0}^{m} z_{\alpha(a)}.$$

In the category with morphisms tensored with $\mathbb{Q}$, the maps $M^{-1}F_\alpha = p_M^{-1} D(\alpha) p_N$ do satisfy the simplicial identities. The integral F_α themselves need not do so: for $\beta : [\ell] \to [m]$ one has $F_\beta F_\alpha = [M] F_{\alpha\circ\beta}$, and $F_{\mathrm{id}_{[q]}} = [N]$.

This rational statement discards the finite lattice indices and is not an integral equivalence or a choice of lifts.

There is also an exact integral correspondence description. For $\alpha : [m] \to [q]$, define

$$R_\alpha = \{(x, y) \in K_N \times K_M : p_M(y) = D(\alpha) p_N(x)\}.$$

The projection $R_\alpha \to K_N$ is a principal $E[M]$-cover. For $\beta : [\ell] \to [m]$, forgetting the middle point gives

$$R_\alpha \times_{K_M} R_\beta \longrightarrow R_{\alpha \circ \beta}.$$

Each fiber is exactly the set of lifts of the specified point of D_M, hence is a free transitive $E[M]$-set. Therefore

$$(R_\alpha \times_{K_M} R_\beta)/E[M] \cong R_{\alpha \circ \beta}.$$

For longer strings, quotienting by the product of the internal deck groups is independent of the order of quotienting. These identifications are coherently associative. They describe the strict diagram $D_\bullet$ through its finite covering atlases; unquotiented span composition is not that diagram. The covers are fibrations, so the indicated ordinary pullbacks also compute homotopy pullbacks. None of this makes p_N a weak equivalence: for $N > 1$ its map on fundamental groups has cokernel $(\mathbb{Z}/N\mathbb{Z})^2$.

Two distinctions limit the application. First, the extra AM tower cuts $x_i \in \mathbb{Z}$ are not the common root-wall family. Their weight-lattice translates give the finite refinement $x_i \in N^{-1}\mathbb{Z}$ in a fixed degree. This produces a common periodic refinement, but establishes no simplicial compatibility for the extra cuts. Second, symmetric quotients and the AM inclusions that append zero form the projective tower; labelled deletion is not a map of unordered divisors. The root–weight calculation alone identifies neither that tower nor its primitive Hopf line with a realization of these correspondences. Finally, this is a $k = 1$ arrangement statement in one or two real factors. The higher-cocycle rows have rank $\binom{q}{k}$; taking an elliptic product does not turn the root arrangement into the $k = 2$ cocycle arrangement.

7.3 A merge refinement of the ordered root tori

There is a strict simplicial organization of the ordered root tori using adjacent sums. It is distinct from centered deletion and from the finite isogenies of Section 7.2. Put $N = q + 1$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and

$$V_N = \{a \in \mathbb{R}^N : \textstyle\sum_i a_i = 0\},$$

$$\Lambda_N = V_N \cap \mathbb{Z}^N, \qquad K_N(\mathbb{T}) = \ker(\Sigma : \mathbb{T}^N \to \mathbb{T}).$$

Cyclic differences give an isomorphism of compact groups

$$\delta_q : \mathbb{T}^N/\mathbb{T}_{\mathrm{diag}} \xrightarrow{\cong} K_N(\mathbb{T}), \qquad a_0 = y_0 - y_q, \quad a_i = y_i - y_{i-1}\ (1 \le i \le q).$$

Its inverse chooses $y_0 = 0$ and $y_i = \sum_{j=1}^{i} a_j$. The same formulas identify the real anchor cover and its integer lattice with V_N and Λ_N. In particular $K_N(\mathbb{T}) = V_N/\Lambda_N$ here, with the root lattice exactly. This isomorphism is not inverse to the root–weight covering: already for $N = 2$, following δ_1 by that covering doubles the phase difference $y_1 - y_0$. Write this integral cover isomorphism in the reverse direction as

$$\Psi_q : V_N \longrightarrow Z^1(\Delta[q]; \mathbb{R}), \qquad (\Psi_q a)_{ij} = \sum_{h=i+1}^{j} a_h \quad (i < j).$$

Transporting deletion and repetition of the y coordinates gives

$$d_i(a_0, \dots, a_q) = \begin{cases} (a_0, \dots, a_i + a_{i+1}, \dots, a_q), & 0 \le i < q, \\ (a_q + a_0, a_1, \dots, a_{q-1}), & i = q, \end{cases}$$

while s_i inserts a zero immediately after a_i. These are integral linear maps and satisfy the simplicial identities, since the original deletion and repetition maps do. In particular s_q is the AM operation that appends zero. For a finite pointed set S, use $V_N(S) = V_N^{S^\circ}$ and $\Lambda_N(S) = \Lambda_N^{S^\circ}$. A pointed map $f : S \to T$ acts by $b_i^t = \sum_{f(s)=t} a_i^s$ for $t \neq *$. These fibre sums commute with all the simplicial operators.

Definition 7.5 (Signed refinement). On $V_N(S)$ take every nonconstant integer-wall family

$$\sum_{s \in S^\circ} \sum_{i=0}^{q} \epsilon_{s,i} a_i^s \in \mathbb{Z}, \qquad \epsilon_{s,i} \in \{-1, 0, 1\}.$$

Discard forms identically zero on $V_N(S)$ and identify duplicate hyperplanes. Let $\widetilde{\mathcal{P}}_q^{\pm}(S)$ be its open-face poset, ordered by incidence, and let $\mathcal{C}_q^{\pm}(S) = \widetilde{\mathcal{P}}_q^{\pm}(S)/\Lambda_N(S)$ be the orbit incidence category, retaining every incidence branch. When $q = 0$ or $S = 0_+$ the space is zero and the category is terminal. Constant flags at the zero cell give compatible basepoints in every degree.

Proposition 7.6. *These categories form a strict functor $\Delta^{\mathrm{op}} \times \mathrm{Fin}_* \to \mathrm{Cat}$. Every category is finite, and its nerve realizes to the root torus $V_N(S)/\Lambda_N(S)$. The signed arrangement contains the AM root and coordinate walls, and the pullback under Ψ_q of every original $k = 1$ subset-sum wall.*

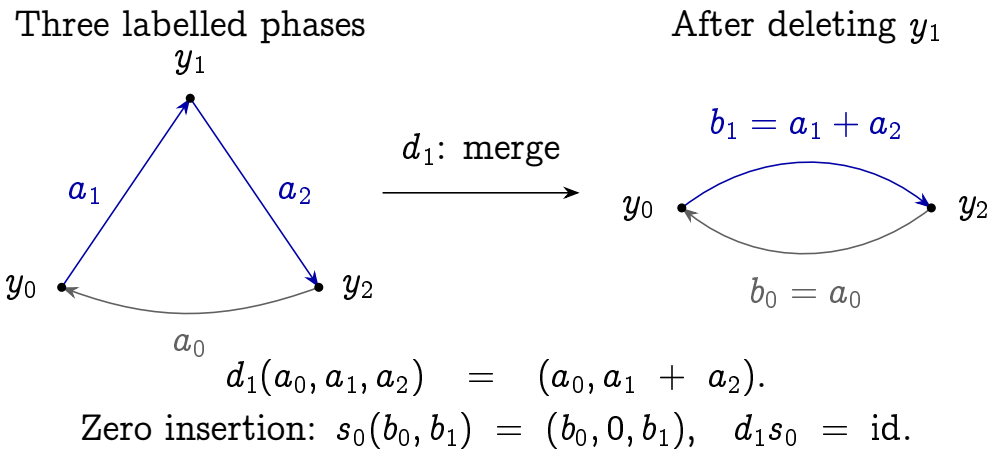


Figure 5: Cyclic differences and adjacent merge, drawn schematically. The edge from y_i to y_j has value $y_j - y_i$; the cyclic sum is zero. Deleting y_1 adds its two incident differences. Repeating y_0 inserts zero and gives the section s_0. The drawing records labels and group-valued differences; it imposes no circular order or positivity of arc lengths.

Proof. A merge copies the coefficient of the merged target coordinate to both source coordinates; zero insertion deletes a coefficient. A fibre sum copies the target input's coefficient block to each input in its fibre, and assigns zero to discarded inputs. Thus pullback preserves the coefficient set $\{-1, 0, 1\}$, or produces a constant equation. Every open source face therefore maps into one open target face. Its carrier is order-preserving and lattice-equivariant. Carriers compose strictly: the image of a source face is a nonempty subset of its intermediate open carrier, on all of which the next target sign vector is constant. The linear operators commute, so their carriers commute as well.

There are finitely many normal directions and the arrangement is locally finite. It includes each coordinate family $a_i^s \in \mathbb{Z}$; hence face closures are bounded convex polytopes. A compact lattice fundamental domain meets finitely many faces, and each closed face has finitely many subfaces. No nonzero lattice translation fixes a bounded face. Consequently orbit incidences compose by uniquely matching their middle lifts, and there are finitely many objects and arrows. Simultaneous lifts of flags identify the nerve quotient with the quotient of the barycentric subdivision upstairs. Its realization is the stated torus.

The coefficient choices $a_i^s - a_j^s$ and a_i^s give the AM walls. Finally, an old wall pulls back to

$$\sum_{s \in U} (\Psi_q a^s)_{ij} = \sum_{s \in U} \sum_{h=i+1}^{j} a_h^s \in \mathbb{Z},$$

whose coefficients belong to $\{0,1\}$. □

Theorem 7.7 (A finite root-arrangement Gamma model). *Set $X^{\pm}(S)_n = N_n\mathcal{C}_n^{\pm}(S)$. There is a natural map of degreewise finite simplicial pointed-set diagrams*

$$R : X^{\pm} \longrightarrow X_0^{(1)}$$

which is a weak equivalence at every finite pointed set. Consequently $X^{\pm}$ is very special and its associated spectrum is equivalent to $\Sigma^2 H\mathbb{Z}$. In particular its one-input diagonal is a degreewise finite $K(\mathbb{Z},2)$ model, with primitive class R^*c_2.

Proof. Under Ψ_q, the signed arrangement refines the original one. Send each signed face to the unique original open face containing it. This carrier functor respects all incidence branches and lattice translations. It is strictly natural for both types of operators: either composite is the unique original target open face containing the actual linear image of the fine source face. Here the original operators themselves carry each original open face into one open face. Thus the functors give a natural map of bisimplicial sets and hence the displayed diagonal map.

Fix q, S. Choose arithmetic vertex means as barycentres upstairs. The fine and coarse barycentric realizations are homeomorphic to the same vector space through Ψ_q. For a fine flag, its fine geometric image and the geometric image of its coarse carrier flag both lie in the closed coarse carrier of its largest face. The straight segment between these two maps is continuous and lattice-equivariant. It descends to a homotopy on the quotient torus between the geometric refinement map and the identity. The row refinement map is therefore a homotopy equivalence. No naturality of these chosen barycentric homeomorphisms or of the row homotopies is needed: the carrier map itself is natural. The diagonal weak-equivalence theorem for bisimplicial sets now proves that $R(S)$ is a weak equivalence for every S.

Finite categories give finite nerve sets in every degree, so the diagonal is degreewise finite. The natural comparison with $X_0^{(1)}$ carries the Segal diagrams to each other, proving the Segal equivalences by two-out-of-three. Its one-input realization is connected by Theorem 4.2, hence the component monoid is a group. The established spectrum identification for $X_0^{(1)}$ and the natural equivalence of very special Gamma models give the last assertion. □

The enlargement is already necessary from $q = 2$ to $q = 1$: under $d_0(a_0, a_1, a_2) = (a_0 + a_1, a_2)$ the target root wall $b_0 - b_1 \in \mathbb{Z}$ pulls back

to $a_0 + a_1 - a_2 \in \mathbb{Z}$. On V_3 this is $-2a_2 \in \mathbb{Z}$, which is not an original root-wall family and is not supplied by the integer coordinate cuts. The signed refinement includes it. Restricting the signed arrangement to an appended zero recovers exactly the previous signed arrangement.

This construction concerns labelled roots and monotone vertex maps; Fin_* above denotes the input category. It does not assert compatibility with arbitrary vertex reordering. Although the walls are invariant under root-coordinate permutations, the labelled merge faces do not descend to unordered divisors. For $E \cong \mathbb{T}^2$ the same integral formulas work in both real factors and give $K(\mathbb{Z}^2, 2)$ before symmetric quotient. For instance, $(x, y, -x - y)$ and $(x, -x - y, y)$ give the same unordered elliptic divisor, whereas d_0 gives the pairs $\{x + y, -x - y\}$ and $\{-y, y\}$, generically distinct. Thus no identification of the symmetric quotient tower, its primitive Hopf line, or its divisor multiplication with the one-circle Gamma model is implied. Establishing such a comparison that retains the symmetric quotient and the Hopf class is a further problem. The elementary signed refinement above is the precise finite bridge proved here.

7.4 Abelian varieties behind the root construction

The ordered root spaces have an algebraic interpretation in every dimension. Let A be a complex abelian variety of dimension g, and define

$$K_N(A) = \ker(\Sigma : A^N \to A), \qquad D_N(A) = A^N / A_{\mathrm{diag}}.$$

Both are abelian varieties isomorphic to A^{N-1}. The restriction of the quotient map gives an isogeny

$$p_N : K_N(A) \longrightarrow D_N(A), \qquad \ker p_N = A[N]_{\mathrm{diag}}, \qquad \deg p_N = N^{2g}.$$

Indeed, the kernel consists exactly of diagonal tuples with $Nt = 0$; surjectivity follows by translating a representative (y_i) by a solution of $Nt = -\sum_i y_i$. Over $\mathbb{C}$, $A[N] \cong (\mathbb{Z}/N)^{2g}$ [49, Chapters I.2 and I.7].

There is also the different algebraic isomorphism

$$\delta_q : D_{q+1}(A) \xrightarrow{\cong} K_{q+1}(A), \qquad [y] \longmapsto (y_0 - y_q, y_1 - y_0, \dots, y_q - y_{q-1}).$$

Its inverse uses $y_0 = 0$ and $y_i = \sum_{j=1}^{i} a_j$. Thus it is defined by integral group operations, without a choice of division points. In the free coordinates $(a_1, \dots, a_q)$, the merge faces drop the first coordinate, add adjacent coordinates, or drop the last coordinate; degeneracies insert zero. These are exactly the usual bar operators. Consequently $q \mapsto K_{q+1}(A)$ is a simplicial

abelian variety isomorphic to the reduced bar construction of the algebraic group A. For its analytic topology,

$$\left|K_{\bullet+1}(A)\right| = BA \simeq K(H_1(A;\mathbb{Z}),2) \simeq K(\mathbb{Z}^{2g},2).$$

The last equivalence uses a lattice basis for the underlying real torus $A \cong \mathbb{T}^{2g}$.

With that basis, use the signed arrangement on $V_{q+1}^{2g|S^\circ|}$, keeping the $2g$ coordinates grouped within each pointed-set input. Merge operators and input fibre sums again copy or delete signed coefficients. The preceding carrier-refinement proof therefore gives a degreewise finite very special Gamma model with one-input space $K(H_1(A;\mathbb{Z}),2)$ and spectrum $\Sigma^2 H(H_1(A;\mathbb{Z}))$. The classical bar object supplies the homotopy type; the signed walls supply its finite simplicial presentation. This presentation uses the real lattice and does not retain the complex moduli or a polarization of A.

The unordered quotient has a different role:

$$K_N(A)/\mathfrak{S}_N \cong \Sigma^{-1}(0) \subseteq \operatorname{Sym}^N A.$$

For an elliptic curve this is $|N[0]| \cong \mathbb{CP}^{N-1}$. For larger g it need not be projective space. Already for an abelian surface and $N=2$ it is $A/\{\pm 1\}$, with sixteen ordinary double points at the images of $A[2]$; its minimal resolution is a K3 surface, the Kummer surface [30, Section 2]. In higher dimension the symmetric-product points are effective zero-cycles, rather than the divisors in a linear system used for the elliptic identification. The labelled merge faces still do not descend to these unordered quotients. The next subsection retains the polarization through explicit line-bundle corrections for merges. A comparison with the unordered quotients that also retains their algebraic structure requires further work.

7.5 Polarizations and the line correction for a merge

The abelian variety can retain more than its underlying real torus if a line bundle is recorded as additional data. Let L be a line bundle on a complex abelian variety A, rigidified by $L|_0 \cong \mathbb{C}$. With $m : A \times A \to A$ the sum map, put

$$B_L = m^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}.$$

Thus its fibre at (x,y) is $L_{x+y} \otimes L_x^{-1} \otimes L_y^{-1}$. The rigidification trivializes its restrictions to both axes. Let $\mathcal{P}$ be the normalized Poincaré line on $A \times A^\vee$, and use the convention $\lambda_L(y) = [t_y^*L \otimes L^{-1}]$.

Proposition 7.8. *There is a canonical axis-rigidified isomorphism*

$$B_L \cong (\mathrm{id}_A \times \lambda_L)^*\mathcal{P}.$$

The line B_L is symmetric and has canonical compatible multiplication laws in each variable:

$$B_L(x+y,z) \cong B_L(x,z) \otimes B_L(y,z), \qquad B_L(x,y+z) \cong B_L(x,y) \otimes B_L(x,z).$$

In other words it is a symmetric biextension. If L is ample, λ_L is its polarization.

Proof. The theorem of the square makes λ_L a homomorphism. The normalized family $B_L|_{A\times\{y\}}$ represents $\lambda_L(y)$, so the universal property of $\mathcal{P}$ gives the displayed identification [49, Chapter I, Sections 5 and 8]. The quotient of the two sides of the first multiplication law is

$$L_{x+y+z} \otimes L_x \otimes L_y \otimes L_z \otimes L_{x+y}^{-1} \otimes L_{x+z}^{-1} \otimes L_{y+z}^{-1}.$$

The theorem of the cube trivializes it. Normalization at the origin makes the trivialization unique: every invertible regular function on A^3 is constant. Symmetry gives the second law, and the same uniqueness on higher products proves their associativity and interchange compatibility. See also [41, Propositions 3.2.1.7 and 3.2.3.1] for the canonical biextension and cubical formulations. □

On the labelled root variety $K_N(A)$ define

$$\mathcal{M}_N = \bigotimes_{j=0}^{N-1} p_j^* L \Big|_{K_N(A)}.$$

The ordinary permutation action descends this line to $K_N(A)/\mathfrak{S}_N$: stabilizers merely permute identical one-dimensional factors and act trivially on their tensor product. Appending or inserting zero pulls $\mathcal{M}_{N+1}$ back to $\mathcal{M}_N$, by the rigidification. Concatenation $c : K_N(A) \times K_M(A) \to K_{N+M}(A)$ gives

$$c^* \mathcal{M}_{N+M} \cong \mathcal{M}_N \boxtimes \mathcal{M}_M.$$

Proposition 7.9. *Let $d_i : K_N(A) \to K_{N-1}(A)$ be a merge face, $N \geq 2$, and let u_i select the two entries merged by that face, including (a_{N-1}, a_0) for the wrapping face. Then canonically*

$$d_i^* \mathcal{M}_{N-1} \cong \mathcal{M}_N \otimes u_i^* B_L. \tag{14}$$

For three successive labels the corrections satisfy

$$B_L(x,y) \otimes B_L(x+y,z) \cong B_L(y,z) \otimes B_L(x,y+z), \tag{15}$$

and these comparisons satisfy the pentagon and unit identities.

Proof. A merge replaces $L_x \otimes L_y$ by L_{x+y} and leaves every other factor unchanged, proving (14). Both sides of (15) cancel canonically to $L_{x+y+z} \otimes L_x^{-1} \otimes L_y^{-1} \otimes L_z^{-1}$. For four labels every parenthesization cancels to the same line, so the pentagon commutes. The rigidification gives the unit identities, and merges on disjoint pairs commute by tensor symmetry. These cancellation identities hold for any rigidified line; the theorem of the cube supplies the additional bilinearity asserted in Proposition 7.8. □

The correction need not have a nonvanishing global section. Writing $\alpha = c_1(L)$, one has

$$c_1(B_L) = m^*\alpha - p_1^*\alpha - p_2^*\alpha, \qquad \Delta^* c_1(B_L) = 2\alpha.$$

Here $[2]^*$ multiplies $H^2(A;\mathbb{Z})$ by four. Thus for an ample L on a positive-dimensional A, even a continuous global trivialization of B_L is impossible.

The first elliptic stage makes the obstruction visible. For $A = E$ and $L = \mathcal{O}_E([0])$, the identification $K_2(E) = \{(a,-a)\} \cong E$ gives

$$\mathcal{M}_2 = L \otimes [-1]^* L, \qquad \deg \mathcal{M}_2 = 2.$$

Both bar faces go to $K_1(E) = \{0\}$, whose norm line is trivial. Thus the norm lines do not constitute an ordinary simplicial line bundle with isomorphisms along all faces. The correction $B_L(a,-a) = \mathcal{M}_2^{-1}$ cancels this obstruction exactly. For the double cover $\pi : E \to E/\{\pm 1\} \cong \mathbb{CP}^1$, $\mathcal{M}_2 \cong \pi^*\mathcal{O}(1)$; hence this is the same norm line as the first Arnoux–Marin Hopf stage.

These formulas give coherent line data on the finite polyhedral refinements: restrict the actual bundles and the displayed isomorphisms to characteristic polytopes, retaining their boundary identifications and lattice descent. The affine maps between refined polytopes preserve these pullback formulas exactly. More generally, for such a map T put $C_T = T^*\mathcal{M}_{\text{target}} \otimes \mathcal{M}_{\text{source}}^{-1}$. For composable maps the canonical cancellation isomorphism is

$$C_{UT} \cong C_T \otimes T^* C_U.$$

It supplies all composition comparisons. Finiteness counts the polyhedral cells, not the possible values of transition functions. Triviality on individual cells does not remove the gluing data. Nor does restriction alone put an ordinary simplicial line bundle on the finite diagonal: barycentric carrier maps and actual affine maps require the coherent geometric comparisons described earlier.

Finally, the symmetric cocycle in (15) is the Picard-valued coboundary of L. This does not trivialize its line on $A \times A$ or erase the extra biextension

structure. The biextension records the precise failure of multiplicativity; Section 7.6 explains the degree of the associated topological data.

There is also a limit to the theta data retained by this package. A rigidified $Q \in \mathrm{Pic}^0(A)$ is multiplicative, so $\bigotimes_i Q_{a_i} \cong Q_{\sum_i a_i} \cong \mathbb{C}$ on $K_N(A)$. Consequently replacing L by $L \otimes Q$ changes neither $\mathcal{M}_N$ nor B_L, up to their canonical rigidified identifications. These data retain the polarization homomorphism, rather than a particular translated theta line.

7.6 Algebraic classifying stacks and the polarization pairing

Let $A/\mathbb{C}$ be an abelian variety of dimension g, and put $\Lambda = H_1(A(\mathbb{C}); \mathbb{Z})$, of rank $2g$. The schemes $\ker(\Sigma : A^{q+1} \to A)$, with adjacent merges and zero insertion, are an algebraic bar construction: forgetting $a_0 = -\sum_{i=1}^{q} a_i$ identifies them with the usual nerve A^q. The associated stack is the algebraic classifying stack $\mathcal{B}A$. Realizing the complex-point nerve instead gives

$$B_{\mathrm{top}}A \simeq K(\Lambda, 2), \qquad H^*(B_{\mathrm{top}}A; \mathbb{Z}) = \mathrm{Sym}^*(\Lambda^\vee[2]).$$

The finite carrier construction models this latter homotopy type. The polarization pairing is additional data, recorded below.

The mixed Chern class. Use the rigidified line L and its correction B_L from Section 7.5. Write $\eta = c_1(L) = \sum_{i<j} e_{ij} u_i u_j$ in an integral basis of $H^1(A; \mathbb{Z})$. Since $m^* u_i = u_i^{(1)} + u_i^{(2)}$,

$$c_1(B_L) = \sum_{i<j} e_{ij}(u_i^{(1)} u_j^{(2)} - u_j^{(1)} u_i^{(2)}).$$

The map $\bigwedge^2 \Lambda^\vee \to \Lambda^\vee \otimes \Lambda^\vee$ in this formula is injective. Thus a nonzero polarization class forbids an isomorphism $m^*L \cong p_1^*L \otimes p_2^*L$, already topologically.

What algebraically trivial lines provide. For $L \in \mathrm{Pic}^0(A)$, rigidification gives the unique normalized multiplicative structure; its associativity follows by comparing on A^3 and using that invertible regular functions there are constant. Removing the zero section gives a commutative extension, and the Barsotti–Weil identification is

$$\mathrm{Pic}^0(A) \cong \mathrm{Ext}^1(A, \mathbb{G}_m), \qquad 1 \to \mathbb{G}_m \to G_L \to A \to 1.$$

See [49, Chapter I, Proposition 8.4 and Propositions 9.2–9.3]. The extension defines a $\mathbb{G}_m$-gerbe $\mathcal{B}G_L \to \mathcal{B}A$: over an A-torsor its objects are lifts to G_L-torsors. It does not give an ordinary line bundle on $\mathcal{B}A$. Indeed descent

along $* \to \mathcal{B}A$ identifies $\mathrm{Pic}(\mathcal{B}A)$ with algebraic characters $A \to \mathbb{G}_m$, all trivial by properness and connectedness. The gerbe's underlying topological degree-three class vanishes, since $H^3(B_{\mathrm{top}}A;\mathbb{Z}) = 0$; its algebraic extension can nevertheless be nontrivial.

For a nonzero polarization class, the reduced coproduct above obstructs transgression in $A \to EA \to B_{\mathrm{top}}A$. Moreover $H^4(B_{\mathrm{top}}A;\mathbb{Z}) = \mathrm{Sym}^2 \Lambda^\vee$ does not identify an alternating polarization form with an intrinsic quadratic class. The following external operation records its alternating lattice pairing.

Proposition 7.10 (Finite cochains for the alternating pairing). *Choose a basis of Λ, and let C_Λ be the degreewise finite carrier model for $K(\Lambda,2)$, with normalized fundamental cocycles $c_1,\dots,c_{2g} \in Z^2(C_\Lambda;\mathbb{Z})$. For an alternating integral form $E = (e_{ij})$ on Λ, set*

$$B_E = \sum_{i<j} e_{ij}\,(\mathrm{pr}_1^*\,c_i \smile \mathrm{pr}_2^*\,c_j - \mathrm{pr}_1^*\,c_j \smile \mathrm{pr}_2^*\,c_i)\,.$$

This is an integral four-cocycle on $C_\Lambda \times C_\Lambda$. Its class is nonzero exactly when $E \neq 0$, is bilinear on the two degree-two factors, and satisfies $\tau^*[B_E] = -[B_E]$ for factor interchange τ. For the diagonal Δ there is an exact cochain identity

$$\Delta^* B_E = \delta H_E, \qquad H_E = -\sum_{i<j} e_{ij}\, c_i \smile_1 c_j.$$

Proof. Cup products of cocycles are cocycles. The Künneth summand $H^2(C_\Lambda)\otimes H^2(C_\Lambda)$ identifies $[B_E]$ with $\sum_{i<j} e_{ij}([c_i]\otimes[c_j]-[c_j]\otimes[c_i])$, proving nonvanishing and antisymmetry. Degree-two classes commute, so its diagonal class vanishes. More precisely, the convention of (82) gives $\delta(c_i \smile_1 c_j) = -(c_i \smile c_j - c_j \smile c_i)$, which proves the displayed primitive with its stated minus sign. □

These are finite explicit evaluations: on a four-simplex (x,y),

$$B_E(x,y) = \sum_{i<j} e_{ij}\big((c_i)_{012}(x)(c_j)_{234}(y) - (c_j)_{012}(x)(c_i)_{234}(y)\big),$$

and on a three-simplex x,

$$H_E(x) = \sum_{i<j} e_{ij}\big((c_i)_{013}(x)(c_j)_{123}(x) - (c_i)_{023}(x)(c_j)_{012}(x)\big).$$

The homotopy fibre P_E of the classifying map $C_\Lambda \times C_\Lambda \to K(\mathbb{Z},4)$ is consequently a two-stage Postnikov space with

$$\pi_2(P_E) = \Lambda \oplus \Lambda, \qquad \pi_3(P_E) = \mathbb{Z},$$

and no other positive homotopy groups, by the fibre exact sequence. Its k-invariant is $[B_E]$, and H_E supplies a lift of the diagonal to P_E. For $E \neq 0$ this invariant has infinite order. For a polarization, Corollary 7.19 below constructs a degreewise finite model of this fibre by means of its compact Heisenberg nerve. For a polarization, taking its alternating Riemann form as E retains the lattice pairing, not the full polarized abelian variety. The operation is defined from the lattice pairing. Comparing it with algebraic biextension data requires attention to the degree shift: B_L is symmetric under swapping its degree-one inputs, whereas $[B_E]$ is alternating on its degree-two inputs.

Changes of integral marking. The pairing has an explicit comparison under a change of lattice basis. Use the finite carrier presentations of Theorem 9.4. On their common finite refinement, that theorem gives

$$f_P^* c' = Pc + \delta b, \qquad |c| = 2, \quad |b| = 1.$$

Suppose the alternating matrices satisfy $P^{\mathrm{t}} E' P = E$. For vector cochains write $\langle u \smile v \rangle_{E'} = \sum_{i,j} E'_{ij} u_i \smile v_j$, and let subscripts x, y mean pullback from the two factors. Then the three-cochain

$$T_P = \langle b_x \smile Pc_y \rangle_{E'} + \langle Pc_x \smile b_y \rangle_{E'} + \langle b_x \smile \delta b_y \rangle_{E'}$$

satisfies the exact identity

$$(f_P \times f_P)^* B_{E'} - (r \times r)^* B_E = \delta T_P,$$

where r is the refinement map and $c = r^* c_{\mathrm{source}}$. Expand the left side using $f_P^* c' = Pc + \delta b$; the three remaining terms are the differentials of the three displayed summands, since $\delta c = 0$. Thus the integral carry comparisons already constructed in the note also transport the polarization pairing. In particular symplectic changes of marking preserve its cohomology class with an explicit finite cochain homotopy.

7.7 An algebraic Heisenberg extension

The degree shift admits an algebraic construction once both inputs are retained. We use Polishchuk's Heisenberg groupoid construction [60, Definition 2.1 and the following remark] and Ganter's classification of categorical tori [29, Construction 2.3 and Theorem 4.1]. The identification with our particular cocycle is the deduction below.

Let A and the rigidified line L be as in Proposition 7.8. Put $G = A \times A$. On $G \times G$ define the axis-rigidified line

$$\mathcal{C}_L\big((x,y),(x',y')\big) = B_L(x,y') = \mathcal{P}\big(x, \lambda_L(y')\big).$$

These are fibrewise descriptions of pullbacks of line bundles.

Proposition 7.11. *The line $\mathcal{C}_L$ defines a central extension of algebraic group stacks*

$$1 \longrightarrow B\mathbb{G}_m \longrightarrow \mathcal{H}_L \longrightarrow G \longrightarrow 1.$$

Its underlying stack over G is the trivial gerbe $G \times B\mathbb{G}_m$; its multiplication carries the additional information.

Proof. For a complex scheme S, an object of $\mathcal{H}_L(S)$ is a pair (g, M), with $g : S \to G$ and M a line bundle on S. Morphisms are line-bundle isomorphisms over the same g. Set

$$(g,M) \otimes (h,N) = (g+h,\, M \otimes N \otimes (g,h)^{*}\mathcal{C}_L).$$

The biextension laws identify

$$\mathcal{C}_L(g,h) \otimes \mathcal{C}_L(g+h,k) \cong \mathcal{C}_L(h,k) \otimes \mathcal{C}_L(g,h+k).$$

Both sides expand to the same three factors; compatibility of the biextension laws gives the pentagon. Axis rigidifications supply the unit $(0, \mathcal{O}_S)$. Inverses use $-g$ and the inverse lines, with the correction $\mathcal{C}_L(g,-g)^{-1}$. The kernel consists of $(0, M)$ and is $B\mathbb{G}_m$, centrally. This is Polishchuk's construction pulled back from the Poincare extension on $A \times A^{\vee}$ along $(x,y) \mapsto (x, \lambda_L(y))$. It works also for a nonprincipal polarization; no claim that its commutator biextension is symplectic in the isomorphism sense is needed. $\square$

Write $\Lambda = H_1(A(\mathbb{C}); \mathbb{Z})$ and identify $c_1(L)$ with its alternating integral form E on Λ. On $\Gamma = \Lambda \oplus \Lambda$ set

$$J\big((x,y),(x',y')\big) = E(x,y'), \qquad I = J + J^{\mathrm{t}}.$$

Thus

$$I\big((x,y),(x',y')\big) = E(x,y') + E(x',y),$$
$$q(x,y) = \tfrac{1}{2} I\big((x,y),(x,y)\big) = E(x,y).$$

The form I is even and symmetric, with matrix $\left(\begin{smallmatrix} 0 & E \\ -E & 0 \end{smallmatrix}\right)$. Its two coordinate sublattices are isotropic. For an ample L it is nondegenerate; it is unimodular precisely for a principal polarization.

Proposition 7.12. *Let $B_{\mathrm{top}}\mathcal{H}_L$ be the delooping of the topological categorical group obtained from $\mathcal{H}_L$ by analytification. Its central-extension class is $+[B_E]$. Consequently there is an equivalence over $BG(\mathbb{C}) \simeq K(\Gamma, 2)$*

$$B_{\mathrm{top}}\mathcal{H}_L \simeq \mathrm{hofib}\big([B_E] : K(\Gamma,2) \to K(\mathbb{Z},4)\big).$$

The comparison with the earlier P_E uses the chosen equivalence $|C_\Lambda| \simeq BA(\mathbb{C})$ in both factors.

Proof. Let a_i be the degree-one basis dual to the chosen lattice basis. Since $c_1(L) = \sum_{i<j} e_{ij} a_i \smile a_j$, subtraction of the two unmixed terms from $m^* c_1(L)$ gives

$$c_1(B_L) = \sum_{i,j} e_{ij}\, a_i \otimes a_j \quad \text{in } H^1(A) \otimes H^1(A).$$

Accordingly the mixed Chern matrix of C_L on $G \times G$ is J. Ganter's sign convention is essential: input K in her Construction 2.3 has mixed Chern matrix $-K$ (Section 3 and Section 4.1); Theorem 4.1 assigns class $I = J + J^{\mathrm{t}}$ to input $-J$. Hence our line has class q, with positive sign. The class is determined by this mixed matrix modulo skew matrices: in the bar construction, coboundaries of lines on G supply exactly the skew part. Equivalently, this is the degree-four calculation in Ganter's theorem. In degree-two coordinates u_i, v_i it reads

$$q = \sum_{i,j} e_{ij} u_i v_j = \sum_{i<j} e_{ij}(u_i v_j - u_j v_i) = [B_E].$$

Finally $B\mathbb{C}^* \simeq K(\mathbb{Z},2)$, so delooping the central extension gives fibre $B^2\mathbb{C}^* \simeq K(\mathbb{Z},3)$ over $BG(\mathbb{C})$. Its class in degree four is exactly the k-invariant just computed, proving the equivalence. □

Changing the generator of the central $\mathbb{Z}$ replaces $[B_E]$ by $-[B_E]$. Multiplication by -1 on $K(\mathbb{Z},4)$ identifies their homotopy fibres, while reversing the chosen generator of π_3. Thus the sign fixes the displayed Whitehead pairing convention; it does not change the polarization-type classification or the unmarked homotopy type.

The extra delooping is necessary: the underlying gerbe over G is trivial, but the delooped extension need not be. Thus this is not a claim of a nonzero ordinary Brauer class on G.

Proposition 7.13. *Both coordinate axes split canonically in $\mathcal{H}_L$. The diagonal $\Delta : A \to G$ has the monoidal lift*

$$s_L(x) = (\Delta x, L_x).$$

After Betti delooping, this lift agrees up to homotopy over the base with the lift defined by the cochain H_E.

Proof. On either axis $\mathcal{C}_L$ is trivial. On the diagonal its value at (x, x') is $B_L(x, x')$, and the defining formula for B_L gives

$$L_x \otimes L_{x'} \otimes B_L(x, x') \cong L_{x+x'}.$$

Rigidification and the cube theorem give the monoidal coherence. This supplies a nullhomotopy of the restricted degree-four class. So does $\delta H_E = \Delta^* B_E$. After fixing the base and classifying map comparisons, the differences between such lifts are parametrized by $H^3(BA(\mathbb{C}); \mathbb{Z}) = 0$. They are therefore homotopic as lifts. No equality of the geometric and simplicial cochains is claimed. □

This construction retains algebraic data when A, its two coordinate factors, and the biextension are retained. Relative to the displayed trivial-gerbe presentation (or with both axis splittings fixed), the diagonal splitting also records the chosen L. If $M \in \mathrm{Pic}^0(A)$ is rigidified, $B_{L \otimes M} \cong B_L$, so the extension alone is unchanged, while the specified splitting is tensored by M. Without the indicated markings, tensoring objects by the pullback of M from the first factor gives a monoidal automorphism relating these diagonal splittings; no recovery of L up to unrestricted equivalence is asserted. The Betti homotopy type retains E, not the complex moduli. Using B_L directly as a twisting line on $A \times A$ would instead give a split extension of A, since B_L is the coboundary of L; the doubled group G is essential here. For an arbitrary integral form, no algebraic enhancement is asserted without a suitable algebraic line bundle. Passing from this algebraic group stack to a degreewise finite simplicial set requires another construction. The next two subsections supply it: first a compact nerve of phases, then finite face carriers and formal degeneracies.

7.8 A compact semialgebraic nerve for the Heisenberg delooping

Retain $G = A(\mathbb{C}) \times A(\mathbb{C})$ and the multiplicative line $\mathcal{C} = \mathcal{C}_L$ of Proposition 7.11. For a complex line V, write

$$\mathrm{Ph}(V) = (V \setminus \{0\})/\mathbb{R}_{>0}.$$

This is a $U(1)$-torsor, and Ph preserves tensor products and all identities between line isomorphisms. Thus the associator and unit maps of $\mathcal{C}$ pass

to phase torsors without a choice of compatible metrics. A metric merely represents a phase by its unit vector.

Proposition 7.14. *There is a proper simplicial compact semialgebraic space $X_\bullet$ with $X_0 = *$, $X_1 = G$, and*

$$X_n \longrightarrow G^n \quad \text{a principal } U(1)^{\binom{n}{2}}\text{-bundle.}$$

Its realization is equivalent over BG to $B_{\mathrm{top}}\mathcal{H}_L$, and hence to P_E with the positive class $[B_E]$ of Proposition 7.12.

Proof. An n-simplex consists of edge labels $g_{ij} \in G$ for $i < j$, with $g_{ij}+g_{jk} = g_{ik}$, and triangle labels

$$\phi_{ijk} \in \mathrm{Ph}\big(\mathcal{C}(g_{ij}, g_{jk})^\vee\big) \qquad (i < j < k).$$

The direction is a phase of an isomorphism $T_{g_{ij}} \otimes T_{g_{jk}} \longrightarrow T_{g_{ik}}$, where $T_g = (g, \mathbb{C})$ in the displayed trivial-gerbe presentation. If a is the associator

$$a : \mathcal{C}(g,h) \otimes \mathcal{C}(g+h,k) \longrightarrow \mathcal{C}(h,k) \otimes \mathcal{C}(g,h+k),$$

impose on each tetrahedron $i < j < k < l$ the equation

$$\phi_{ijk} \otimes \phi_{ikl} = (\phi_{jkl} \otimes \phi_{ijl}) \circ a.$$

These equations concern phases of functionals on the same line. Faces restrict the labels to a subset of vertices. Degeneracies repeat a vertex, put $g_{ii} = 0$, and use the canonical unit triangle whenever two consecutive vertices coincide. Coherence of the associator and unit maps gives the simplicial identities exactly.

The n consecutive edges determine all edge labels. Choose independently the $\binom{n}{2}$ anchor triangles ϕ_{0ij}, $1 \le i < j \le n$. For $0 < i < j < k$, the tetrahedron $(0,i,j,k)$ uniquely determines ϕ_{ijk}. The pentagon ensures that all remaining tetrahedra commute. Thus X_n is the fibre product over G^n of the anchor phase bundles. In particular it is a compact manifold and a principal torus bundle; no global section of any anchor line is being asserted.

All the line bundles and coherence isomorphisms are algebraic before taking phases. In finite algebraic charts their real and imaginary parts are semialgebraic. A continuous semialgebraic Hermitian metric exists on each such line bundle over the compact base, and its unit circle realizes the phase torsor semialgebraically. Tensoring and normalizing by a positive norm are semialgebraic operations. Since this is just a presentation of the phase quotient, normalization preserves every coherence identity. No exponential uniformization is used. Consequently all simplicial operators are semialgebraic.

Each degeneracy is an embedding, since it has a face as a left inverse, and its image is closed by compactness. The finite union of degeneracy images in X_n is a closed semialgebraic subset. Compatible triangulation makes its inclusion a cofibration, proving properness of $X_\bullet$.

Here is an explicit realization comparison, including the descent that is absent from the trivial-line atlas alone. Let

$$U_n = C^1(\Delta^n; U(1)) = U(1)^{\binom{n+1}{2}},$$

where normalized edge cochains restrict under faces and have value 1 on repeated edges. Edge changes of trivialization act on X_n by

$$(u \cdot \phi)_{ijk} = u_{ik}(u_{ij}u_{jk})^{-1}\phi_{ijk}.$$

The kernel is $Z^1(\Delta^n; U(1)) \cong U(1)^n$: its cochains have $u_{ik} = u_{ij}u_{jk}$ and are determined by their consecutive edges. Thus this action is not free; its isotropy retains the line automorphisms. Form the simplicial quotient stacks $D_n = [X_n/U_n]$. An object of D_n is equivalently a collection of line bundles M_{ij} on all edges, with coherent tensorators

$$M_{ij} \otimes M_{jk} \otimes C(g_{ij}, g_{jk}) \longrightarrow M_{ik}.$$

Restriction to consecutive edges is an equivalence $D_n \simeq \mathcal{H}_L^n$: an inverse composes these edges, and the associator supplies all other tensorators. Any extension of the spine is isomorphic to this one; its automorphisms are exactly the n independent automorphisms of the spine lines. In this statement we use the topological $U(1)$ version of $\mathcal{H}_L$; passage from $\mathbb{C}^*$ to $U(1)$ does not change its Betti homotopy type. The simplicial stack $D_\bullet$ is therefore a Segal stack and the coherent bar construction of this categorical group. This is also the stacky-groupoid nerve construction of [81, Section 4.1]; compare [69, Section 3.7, Theorem 94] for the realization of smooth categorical groups.

The group $U_\bullet$ is contractible after realization. Indeed, prepending a vertex and assigning 1 to every new edge is an extra degeneracy contracting this simplicial topological group. The bisimplicial action nerve has spaces

$$Y_{n,p} = X_n \times U_n^p.$$

We realize quotient stacks by their action-groupoid nerves, so this comparison uses derived classifying spaces throughout. All spaces in this bisimplicial diagram are compact semialgebraic and its degeneracy inclusions are cofibrations, by the same argument used above. Realizing first in the n direction gives the action nerve of $|U_\bullet|$ on $|X_\bullet|$. Realization commutes with the finite products here, in compactly generated spaces [44, Corollary 11.6]. Since $|U_\bullet|$

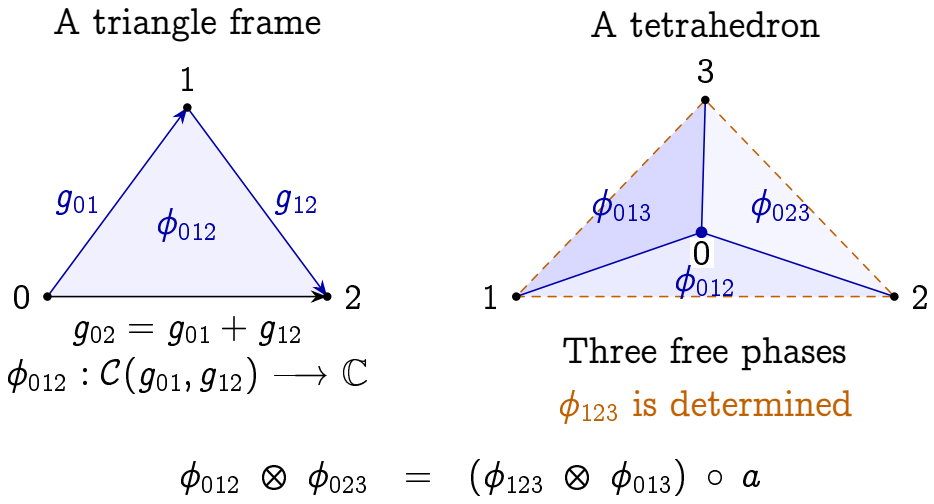


Figure 6: Triangle data and tetrahedron coherence in the compact nerve. A triangle label is the phase of a line isomorphism $\mathcal{C}(g_{01}, g_{12}) \to \mathbb{C}$, taken modulo positive real scaling. For fixed edge labels, the three faces incident to vertex 0 carry independent choices in their phase torsors. The identity below the tetrahedron, using the given associator a, uniquely determines the label on the back face (123), whose boundary is dashed. Independence is over the fixed edges; it does not choose global trivializations of the phase bundles.

is contractible, its homotopy quotient is weakly equivalent to $|X_\bullet|$. Realizing first in the other direction presents $|D_\bullet| = B_{\mathrm{top}}\mathcal{H}_L$. All comparisons are over BG. This proves the claimed equivalence. In particular, the dual in the triangle labels is dictated by the tensorator direction; it does not replace the categorical group by its inverse extension. □

The compact nerve supplies the geometric hypotheses for a finite semisimplicial carrier replacement. Such a replacement must use compatible face carriers and a realization comparison; separate finite triangulations of the spaces X_n alone are insufficient. The projection to the ordinary bar construction of G is semialgebraic. The next theorem supplies the required replacement, and Corollary 7.19 identifies both its Postnikov invariant and its obstruction to an infinite-loop structure.

7.9 Finite replacement by face carriers

Write Δ_{inj} for the category of finite nonempty ordinals and order-preserving injections, and $\|-\|$ for semisimplicial, or fat, realization. Ordinary simplicial realization is denoted by $|-|$. All realizations use compactly generated spaces. The proof combines compatible semialgebraic triangulation [57, The-

orem 2.2] with free degeneracies and homotopy invariance of fat realization [23, Section 1.1, Lemma 1.8, Theorem 2.2]. For the passage to thin realization we use [71, Definition A.4 and Proposition A.1(iv)].

Theorem 7.15. *Let $X_\bullet$ be a semisimplicial space whose levels are compact semialgebraic sets and whose face maps are continuous semialgebraic maps. There is a simplicial set Y finite in each degree and a zigzag of weak homotopy equivalences*

$$|Y| \;\simeq\; \|X_\bullet\|.$$

If $X_\bullet$ is a simplicial space and all its degeneracies are also semialgebraic, then $|Y| \simeq |X_\bullet|$. The construction requires choices of finite triangulations; it does not assert that the original degeneracies act on those triangulations or that additional operations are preserved.

Proof. Face-compatible cells. Choose a finite semialgebraic triangulation $h_0 : |K_0| \longrightarrow X_0$. Suppose triangulations have been chosen through degree $q-1$. Triangulate X_q compatibly with the finite family

$$\{d_i^{-1}(F) : 0 \le i \le q, \quad F = h_{q-1}(\operatorname{int}\sigma),\ \sigma \in K_{q-1} \text{ nonempty}\}.$$

The semialgebraic triangulation theorem provides such a triangulation $h_q : |K_q| \longrightarrow X_q$; compactness makes K_q finite. Compatibility means that every displayed subset is a union of open simplices, not that any prescribed triangulation of a subset extends without subdivision.

Let P_q be the finite poset of nonempty open cells of this triangulation, with $F \le G$ when $F \subseteq \overline{G}$. Each d_i maps an open cell F into exactly one open cell, denoted $d_i^P F$. Continuity gives

$$d_i(\overline{F}) \subseteq \overline{d_i^P F}.$$

If $F \le F'$, the nonempty set $d_i(F)$ is contained both in $d_i^P F$ and in $\overline{d_i^P F'}$. The latter closed simplex is a union of whole open cells; therefore $d_i^P F \le d_i^P F'$. Thus the carrier map is order-preserving. Carriers compose: if $f(F) \subseteq G$ and $g(G) \subseteq H$ for nonempty open cells, then $(gf)(F) \subseteq H$, so its carrier is H. Thus the simplicial face identities hold strictly on the P_q, giving a functor $P : \Delta_{\mathrm{inj}}^{\mathrm{op}} \longrightarrow \mathrm{Poset}$. Only faces have been used in this induction.

A natural incidence comparison. Define a topological category $\mathcal{E}_q$ with objects (F, x) for $x \in \overline{F}$ and with a unique arrow

$$(F, x) \longrightarrow (G, x) \quad \text{when } F \le G.$$

The object and arrow spaces have their finite disjoint-union topologies. There are functors $\mathcal{E}_q \to P_q$ and $\mathcal{E}_q \to X_q$, the latter viewing X_q as a category with identity arrows only. The face functors are $(F, x) \mapsto (d_i^P F, d_i x)$, so both projections are strictly natural in the semisimplicial direction. In nerve degree p one has, with the indicated orientation,

$$N_p\mathcal{E}_q = \coprod_{F_0 \le \cdots \le F_p} \overline{F_0}.$$

Forgetting x is a homotopy equivalence to the discrete set N_pP_q, since every closed cell is a simplex. The degeneracies of both nerves identify whole components with components, hence are closed cofibrations. Fat realization preserves these levelwise weak equivalences, and the good-space fat–thin comparison therefore gives a weak equivalence

$$u_q : |N\mathcal{E}_q| \longrightarrow |NP_q|.$$

For completeness, evaluation $v_q : |N\mathcal{E}_q| \longrightarrow X_q$ is also a weak equivalence by an explicit homotopy. Let $b_q : |NP_q| \longrightarrow |K_q|$ be the barycentric-subdivision homeomorphism. A representative of a point of $|N\mathcal{E}_q|$ is $(F_0 \le \cdots \le F_p, x; t_0, \ldots, t_p)$. Put $\sigma_j = h_q^{-1}(\overline{F_j})$ and let $b(\sigma_j)$ be its barycentre. The formula

$$H_q(-, \lambda) = h_q \left((1-\lambda) h_q^{-1}(x) + \lambda \sum_{j=0}^{p} t_j b(\sigma_j) \right), \qquad 0 \le \lambda \le 1,$$

is defined in the simplex σ_p, because $h_q^{-1}(x) \in \sigma_0 \subseteq \sigma_p$. It respects zero barycentric coordinates and repeated entries, and so descends continuously to realization. It joins v_q to $h_q b_q u_q$. Hence v_q is a weak equivalence. These homotopies need not be natural; the comparison maps are. We have obtained a semisimplicially natural span of rowwise weak equivalences

$$|NP_\bullet| \xleftarrow{u} |N\mathcal{E}_\bullet| \xrightarrow{v} X_\bullet.$$

Fat realization preserves these weak equivalences.

Adjoin degeneracies before taking the diagonal. The sets N_pP_q are simplicial in p and semisimplicial in q. Freely adjoin degeneracies in the second variable, defining the bisimplicial set

$$D_{p,q} = \coprod_{\rho : [q] \twoheadrightarrow [r]} N_pP_r.$$

Its outer operators are entirely explicit. For $\alpha : [m] \to [q]$, factor $\rho\alpha$ uniquely as

$$[m] \xrightarrow{\tau} [s] \xrightarrow{\iota} [r], \qquad \tau \text{ surjective}, \quad \iota \text{ injective};$$

the ρ summand maps by $N_p(\iota^P)$ to the τ summand. The inner operators are the ordinary nerve operators. Let $Y = \operatorname{diag} D$. In particular

$$Y_n = \coprod_{\rho:[n]\twoheadrightarrow[r]} N_n P_r, \qquad |Y_n| = \sum_{r=0}^{n} \binom{n}{r} |N_n P_r| < \infty.$$

If $c_r = |P_r|$, this is at most $\sum_{r=0}^{n} \binom{n}{r} c_r^{n+1}$. The realization of a bisimplicial set agrees homeomorphically with its iterated realization: for a representable bisimplicial set this is the shuffle triangulation of a product of simplices, and both constructions preserve colimits. For any semisimplicial space A, let EA freely adjoin degeneracies. The mutually inverse continuous quotient maps

$$|EA| \longrightarrow \|A\|, \quad [(\rho, a), t] \longmapsto [a, \rho_* t], \qquad [a, t] \longmapsto [(\mathrm{id}, a), t]$$

give a homeomorphism; here $\rho_* t$ sums coordinates in each fibre of ρ. Using also that inner realization preserves coproducts gives

$$|Y| \cong \Big\| [q] \longmapsto |N P_q| \Big\|.$$

The preceding incidence span proves the first assertion.

Finally suppose $X_\bullet$ was simplicial. Each degeneracy $s_i : X_{q-1} \to X_q$ is a closed semialgebraic embedding: it has a continuous face left inverse and compact source. A finite triangulation of X_q compatible with its image makes that image a subcomplex, so the embedding is a closed cofibration. Thus $X_\bullet$ is good, and the canonical map $\|X_\bullet\| \to |X_\bullet|$ is a weak equivalence. This last triangulation is used only to prove goodness; it imposes no additional compatibility on the K_q above. □

Corollary 7.16. *For every integer $n \geq 1$ there is a degreewise finite simplicial set whose realization has homotopy type $K(\mathbb{Z}, n)$. For every compact Lie group G, including disconnected groups, there is a degreewise finite simplicial set Y_G with $|Y_G| \simeq BG$.*

Proof. For $k \geq 1$, use the compact simplicial torus

$$A_q^{(k)} = Z^k(\Delta[q]; \mathbb{R}/\mathbb{Z}) \cong (S^1)^{\binom{q}{k}}.$$

In anchor coordinates every simplicial operator is an integral linear map. On $S^1 \subset \mathbb{C}$ this is a monomial map, with negative powers replaced by complex conjugates, hence a polynomial map in real coordinates. All levels are compact real algebraic sets. Theorem 7.15 applies, and the usual topological

cocycle realization has homotopy type $K(\mathbb{Z}, k+1)$. This gives $n \geq 2$; for $n = 1$ use the simplicial circle $\Delta[1]/\partial\Delta[1]$.

A compact Lie group admits a faithful orthogonal representation whose image is real algebraic; see [12, Lemma 1.6 and Theorem 2.5]. Its nerve has compact real algebraic levels G^q. Faces multiply adjacent entries or forget an end entry, while degeneracies insert the identity. These maps are polynomial. The same theorem gives $|Y_G| \simeq |N_\bullet G| = BG$. No separate goodness hypothesis is needed: the semialgebraic degeneracies are closed embeddings and cofibrations, as proved in Theorem 7.15. □

Remark 7.17 (Scope). These are simplicial sets, without a Kan condition. The construction uses choices and does not specify compatible integral cocycles, pointed-set operations, or functoriality in G. For integral Eilenberg–Mac Lane spaces, the toric family $X_0^{(k)}(S)$ supplies compatible cocycles and pointed-set operations by its explicit subset-sum arrangements. Its addition is encoded by the Segal refinement span, not by a simplicial map $D_0^{(k)} \times D_0^{(k)} \to D_0^{(k)}$. The assertion does not include $K(\mathbb{Z}, 0)$: a degreewise finite simplicial set has finitely many vertices and hence finitely many path components.

Remark 7.18 (Scope and a failed stronger construction). The theorem supplies a degreewise finite simplicial model, without a Kan condition, whenever a compact semialgebraic simplicial presentation is available. A presentation with a noncompact vector-space object space or infinitely many lattice-indexed components does not yet satisfy its hypothesis. In particular, the theorem alone does not construct such a presentation for a prescribed non-torsion Postnikov invariant.

One cannot in general strengthen the cell induction by requiring old degeneracies to embed each old cell as a cell. Even an interval embedded as the horizontal part of a T-shaped graph cannot retain an open edge through the branch point in a regular cell decomposition of the graph. The free degeneracies in D are new formal degeneracies; their role is to preserve the fat realization, not to identify them with the original degeneracies of X.

Corollary 7.19. *Let A, L, E be as above, and put $\Gamma = \Lambda \oplus \Lambda$. There is a simplicial set P_E^{fin}, finite in every degree, and a zigzag of weak homotopy equivalences*

$$|P_E^{\mathrm{fin}}| \simeq B_{\mathrm{top}}\mathcal{H}_L \simeq P_E = \mathrm{hofib}\big([B_E] : K(\Gamma, 2) \to K(\mathbb{Z}, 4)\big).$$

Thus its only nonzero positive homotopy groups are $\pi_2 = \Gamma$ and $\pi_3 = \mathbb{Z}$, and its k-invariant is $+[B_E]$. Its Whitehead pairing is the symmetric

form

$$I\big((x,y),(x',y')\big) = E(x,y') + E(x',y) = J + J^{\mathrm{t}}.$$

Here the sign of the generator of π_3 is fixed by the usual boundary convention for the k-invariant and Whitehead product. For $E \neq 0$, this homotopy type admits no H-space structure, and in particular no infinite loop space structure.

Proof. Apply Theorem 7.15 to the compact semialgebraic simplicial space of Proposition 7.14. The realization comparison there and Proposition 7.12 identify the resulting finite model with the indicated homotopy fibre. Its homotopy groups follow from the fibre exact sequence.

To compute the Whitehead pairing, let $a, b \in \Gamma = \pi_2(P_E)$. Map $S^2 \vee S^2$ to P_E by representatives of a, b. Its projection extends to $S^2 \times S^2 \to K(\Gamma, 2)$. If s, t are the positive degree-two generators of the two spheres, the pullback of the quadratic class $q = [B_E]$ is

$$q(as + bt) = \big(q(a+b) - q(a) - q(b)\big)st = I(a,b)st,$$

since $s^2 = t^2 = 0$. The top cell of $S^2 \times S^2$ is attached by the Whitehead product of the two sphere inclusions. Its obstruction to lifting is consequently $I(a,b)$ times the chosen generator of $\pi_3(P_E)$, proving the formula. If $E \neq 0$, choose x, y' with $E(x,y') \neq 0$ and take $a = (x,0)$, $b = (0,y')$. Their Whitehead product is nonzero. In an H-space any two based sphere maps extend from their wedge to their product using multiplication, so all Whitehead products vanish. This proves the final assertions. □

For example, let A be an elliptic curve and let L have degree $d > 0$. In an oriented integral basis of $\Lambda \cong \mathbb{Z}^2$,

$$E = \begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix}, \qquad q = d(u_1 v_2 - u_2 v_1).$$

The corresponding generators have mixed Whitehead products

$$[u_1^\vee, v_2^\vee] = d\zeta, \qquad [u_2^\vee, v_1^\vee] = -d\zeta,$$

where ζ generates π_3. This is already a degreewise finite model with a nonzero, infinite-order Postnikov invariant.

The finiteness assertion concerns each simplicial degree separately; it asserts neither finitely many simplices in total nor the Kan condition. This is a model of the homotopy fibre, not a literal Kan fibration over the original finite toric base. The replacement uses triangulation choices and does

not identify this model with the earlier finite flag presentation or transport the algebraic group stack and its marked splittings to strict operations on the chosen finite simplicial set. The algebraic data remain in the geometric presentation from which it was obtained.

7.10 Torus bundles and their lifting 2-gerbe

Put $T = A(\mathbb{C})$, $G = T \times T$, and let $\Lambda = H_1(T;\mathbb{Z})$. The equivalence $P_E \simeq B_{\mathrm{top}}\mathcal{H}_L$ gives a geometric meaning to maps into the finite model. For the explicit local description below, let X be a finite simplicial complex or a paracompact smooth manifold; the homotopy classification also applies to CW spaces using numerable bundles and derived mapping spaces.

Proposition 7.20. *A map $X \to P_E$, up to homotopy, classifies a pair of principal topological T-bundles Q_1, Q_2, together with a trivialization of their $\mathcal{H}_L$ lifting $U(1)$-bundle 2-gerbe, up to the corresponding equivalences. If $a, b \in H^2(X;\Lambda)$ classify the two bundles, the obstruction to such a trivialization is*

$$E(a \smile b) = \sum_{i,j} e_{ij} a_i \smile b_j \in H^4(X;\mathbb{Z}).$$

For a fixed bundle $Q = Q_1 \times_X Q_2$, a lift includes a specified identification of its projected G-bundle with Q. When lifts exist, their equivalence classes with that identification fixed form a torsor under $H^3(X;\mathbb{Z})$.

Proof. The lifting 2-gerbe $\mathfrak{L}_Q$ is the 2-stack of local $\mathcal{H}_L$-lifts of Q, with the identification of the projection included. It is locally equivalent to the 2-stack of $U(1)$-gerbes. Its class is the pullback along the classifying map $f_Q : X \to BG$ of the extension class of $\mathcal{H}_L$. By Proposition 7.12, this is precisely $f_Q^*[B_E] = E(a \smile b)$. A trivialization is a global lift. Two trivializations differ by an ordinary $U(1)$-gerbe, giving the stated torsor. For bundle 2-gerbes see [75]; the lifting-gerbe equivalence is the standard result of [56, Theorem 4.2.7 and Corollary 4.2.8]. Their strict-model formulation applies through Ganter's equivalent smoothly separable crossed-module presentation of our categorical torus. Alternatively, the assertions follow directly from the homotopy-fibre description of P_E and the descent data below. $\square$

Choose an ordered good cover $\{U_i\}$ and transition maps

$$g_{ij} = (x_{ij}, y_{ij}) : U_{ij} \longrightarrow G, \qquad g_{ij} + g_{jk} = g_{ik}.$$

Let $C_{ijk} = (g_{ij}, g_{jk})^{*}\mathcal{C}_L$, so fibrewise $C_{ijk} = B_L(x_{ij}, y_{jk})$. A lift consists of line bundles M_{ij} on U_{ij} and isomorphisms on U_{ijk}

$$\varphi_{ijk} : M_{ij} \otimes M_{jk} \otimes C_{ijk} \longrightarrow M_{ik}.$$

Use Hermitian lines and the phase version of $\mathcal{C}_L$ in the $U(1)$ formulation, or arbitrary complex lines in the equivalent $\mathbb{C}^{*}$ formulation. On U_{ijkl} require

$$\varphi_{ikl} \circ (\varphi_{ijk} \otimes \mathrm{id}) = \varphi_{ijl} \circ (\mathrm{id} \otimes \varphi_{jkl}) \circ (\mathrm{id} \otimes a_{ijkl}),$$

after the canonical rearrangement of the M-factors, where

$$a_{ijkl} : C_{ijk} \otimes C_{ikl} \longrightarrow C_{jkl} \otimes C_{ijl}$$

is pulled back from the associator of $\mathcal{C}_L$. Repeated indices use $g_{ii} = 0$, $M_{ii} = \mathbb{C}$, and the unit maps. This is precisely the triangle and tetrahedron data of Proposition 7.14 and Figure 6, with arbitrary edge lines retained so that descent is intrinsic.

Equivalence includes refinement of the cover, changes of local bundle trivializations, and compatible isomorphisms of these data. If the g_{ij} and the identification with Q are fixed, an equivalence is given by lines K_i on U_i and maps $K_i \otimes M_{ij} \cong M'_{ij} \otimes K_j$ compatible with the φ's; isomorphisms of the K_i give the next level of equivalence. For two lifts of this fixed cocycle, the lines $D_{ij} = M'_{ij} \otimes M_{ij}^{-1}$ have multiplication $D_{ij} \otimes D_{jk} \cong D_{ik}$, since the C_{ijk} factors cancel. They are the overlap-line description of the difference gerbe.

On a good cover one may first trivialize the overlap lines and choose triangle isomorphisms. The discrepancy between the two tetrahedral composites is a $U(1)$-valued Cech 3-cocycle. The pentagon gives its cocycle identity; changing the choices changes it by a coboundary. With the coefficient orientation already fixed by the extension, this is the obstruction above under $H^3(X; \mathcal{C}^0(U(1))) \cong H^4(X; \mathbb{Z})$. This description explains why choosing one theta line on A does not automatically trivialize the lifting problem for arbitrary Q_1, Q_2.

The fixed-identification qualification matters. If only the two bundle isomorphism classes are retained, their gauge transformations must also be divided out. Since G is abelian, the components of its gauge group are $H^1(X; \Lambda) \oplus H^1(X; \Lambda)$. In the notation of Proposition 7.21, they act on the fixed-lift torsor by the translations

$$(\alpha, \beta) \longmapsto E(\alpha \smile b) + E(a \smile \beta) \in H^3(X; \mathbb{Z}).$$

Consequently the fibre of $[X, P_E] \to H^2(X; \Lambda)^2$ over an unobstructed pair is a torsor for the quotient by this image, rather than generally for the full group $H^3(X; \mathbb{Z})$.

There is also an algebraic lifting problem, with a different cohomological degree convention. For a separated complex scheme S of finite type and an algebraic $(A \times A)$-torsor Q, use the fppf topology. The same central extension has classifying map

$$B(A \times A) \longrightarrow B^3\mathbb{G}_m,$$

so its lifting $\mathbb{G}_m$-2-gerbe has obstruction in $H^3_{\mathrm{fppf}}(S, \mathbb{G}_m)$. When trivializable, its trivializations with fixed projection form a torsor, on equivalence classes, for $H^2_{\mathrm{fppf}}(S, \mathbb{G}_m)$. The displayed line-and-tensorator formulas apply algebraically on a suitable cover or hypercover; ordinary Cech cochains on one arbitrary cover need not compute these groups. This is the usual higher-gerbe descent formalism; see [7, Sections 4–6].

With $X = S^{\mathrm{an}}$, the comparison passes through holomorphic units:

$$H^3_{\mathrm{fppf}}(S, \mathbb{G}_m) \longrightarrow H^3(X, \mathcal{O}_X^*) \longrightarrow H^4(X; \mathbb{Z}).$$

It sends the algebraic obstruction to $E(a \smile b)$ in Betti degree four. The shift follows from the exponential sequence and the acyclicity of the sheaf of continuous complex-valued functions:

$$H^r(X; \mathcal{C}^0(\mathbb{C}^*)) \cong H^{r+1}(X; \mathbb{Z}) \qquad (r \geq 1).$$

This is not an identification of algebraic or holomorphic cohomology with Betti cohomology. In particular, a Betti trivialization need not lift to an algebraic trivialization. The obstruction is a 2-gerbe class, one degree above the group containing ordinary gerbe classes.

7.11 Lifts and the action of base homotopies

Let Λ be a finite-rank free abelian group, let $E : \Lambda \otimes \Lambda \to \mathbb{Z}$ be alternating, and put $\Gamma = \Lambda \oplus \Lambda$. Write

$$B = K(\Gamma, 2), \qquad Z = K(\mathbb{Z}, 4), \qquad q : B \longrightarrow Z, \qquad q^*\iota = E(u \smile v),$$

where u, v are the two universal degree-two classes. Here $E(c \smile d)$ means cup product with tensor coefficients followed by E, in the displayed order. Choose the following path convention for the homotopy fibre:

$$P_E = \{(z, \eta) : z \in B,\ \eta(0) = 0,\ \eta(1) = q(z)\}.$$

Thus a lift includes a path from zero to the obstruction map. This convention fixes the sign of the monodromy below.

Proposition 7.21. *Let T be a connected finite CW complex, and fix $(a,b) \in H^2(T;\Lambda)^2$. A class in $[T, P_E]$ projecting to (a,b) exists exactly when*

$$E(a \smile b) = 0 \quad \text{in } H^4(T;\mathbb{Z}).$$

When it exists, the fibre of

$$[T, P_E] \longrightarrow H^2(T;\Lambda)^2$$

is a torsor for the quotient group

$$\frac{H^3(T;\mathbb{Z})}{\operatorname{im} D_{a,b}}, \qquad D_{a,b} : H^1(T;\Lambda)^2 \longrightarrow H^3(T;\mathbb{Z}),$$

$$D_{a,b}(\alpha,\beta) = E(\alpha \smile b) + E(a \smile \beta).$$

Choosing one reference lift identifies this torsor with the displayed quotient. In contrast, for a fixed representative map $f : T \to B$ and homotopies projecting constantly to f, the set of lift classes is an $H^3(T;\mathbb{Z})$-torsor.

Proof. We use compactly generated mapping spaces and may choose topological abelian group models for the Eilenberg–MacLane targets. A lift of f is a path in $\operatorname{Map}(T,Z)$ from the zero map to qf. Such a path exists exactly when $[qf] = E(a \smile b)$ vanishes. If a path exists, the homotopy classes of these paths, relative to their endpoints, form a torsor for

$$\pi_1 \operatorname{Map}(T,Z) = H^3(T;\mathbb{Z}).$$

This proves the assertion for fixed f. These representability identifications follow from [35, Theorem 4.57], applied also to suspensions.

Now allow the base map to move within its component. A loop ℓ based at f in $\operatorname{Map}(T,B)$ transports a nullhomotopy η to $\eta * (q\ell)$, where $*$ means first follow η and then follow $q\ell$. The monodromy is therefore translation by the image of $[\ell]$ under

$$\pi_1(\operatorname{Map}(T,B), f) \longrightarrow \pi_1(\operatorname{Map}(T,Z), qf).$$

The source is canonically $H^1(T;\Lambda)^2$. Changing the reference nullhomotopy changes the path used to identify the target fundamental group with the group at zero, but only by conjugation. That group is abelian, so the translation homomorphism is independent of this choice.

To compute it, let $t \in H^1(S^1;\mathbb{Z})$ be the positive generator. The adjoint $T \times S^1 \to B$ of a base loop with class (α,β) has degree-two classes

$$a + \alpha \smile t, \qquad b + \beta \smile t,$$

with pullbacks suppressed and the S^1 factor written on the right. The integral Künneth decomposition gives

$$E\big((a+\alpha t)\smile(b+\beta t)\big) = E(a\smile b) \\ + \big(E(\alpha\smile b)+E(a\smile\beta)\big)t.$$

Indeed moving t past b contributes $(-1)^{1\cdot 2}=+1$, and the term involving two copies of t vanishes because $t^2=0$. Slant product with the positively oriented circle therefore gives precisely $D_{a,b}$, with the stated signs and order. Reversing the path convention for P_E would negate this translation, leaving its image unchanged.

Finally, two components of the fibre over f lie in the same component of the total mapping space exactly when they are related by such monodromy. One direction follows by transporting along a base loop. Conversely, any path between two lifts over f projects to a base loop and identifies its endpoints by that transport. Transport along paths from any other representative of (a,b) to f shows that these are all classes over the cohomology pair. Thus the required set is the orbit torsor under translations by $\operatorname{im} D_{a,b}$, proving the quotient formula. This is the mapping-space form of the fibre-orbit principle in [35, Section 4.3, Exercise 19]. □

Remark 7.22 (Example: a fixed-base torsor can collapse). Let $T=S^1\times S^2$, with generators $u\in H^1(T;\mathbb{Z})$ and $v\in H^2(T;\mathbb{Z})$, so uv generates $H^3(T;\mathbb{Z})$. Take $\Lambda=\mathbb{Z}e_1\oplus\mathbb{Z}e_2$ with $E(e_1,e_2)=d>0$, and set $a=e_1v$, $b=0$. The obstruction vanishes, since $H^4(T;\mathbb{Z})=0$. For $\alpha=0$ and $\beta=e_2u$ one obtains

$$D_{a,0}(0,e_2u)=d\,vu=d\,uv, \qquad \operatorname{im} D_{a,0}=d\mathbb{Z}\,uv.$$

Fixed-map lift classes form a $\mathbb{Z}$-torsor, while unrestricted classes over $(a,0)$ form a $\mathbb{Z}/d\mathbb{Z}$-torsor. For $d=1$ there is just one such unrestricted class. Consequently an unconditional H^3-torsor statement for the fibre of $[T,P_E]$ is false.

Explicit cochain lifts and comparisons. On a simplicial model S for T, use normalized cochains. A cochain lift is a triple

$$a,b\in Z^2(S;\Lambda), \qquad h\in C^3(S;\mathbb{Z}), \qquad \delta h=E(a\smile b).$$

For example these triples describe maps to the usual cochain homotopy-fibre model: its n-simplices are such triples on $\Delta[n]$. The map $\delta: C^3(\Delta[\bullet];\mathbb{Z})\to Z^4(\Delta[\bullet];\mathbb{Z})$ is a surjective map of simplicial abelian groups with contractible source and kernel $Z^3(\Delta[\bullet];\mathbb{Z})$. Contractibility follows by adjoining an initial

vertex and extending a cochain by zero on simplices containing it. Pulling this fibration back along the cup-product map gives the stated model of P_E.

If $r, s \in C^1(S;\Lambda)$ and $a' = a + \delta r$, $b' = b + \delta s$, put

$$C_{r,s}(a,b) = E(r \smile b) + E(a \smile s) + E(r \smile \delta s).$$

The Leibniz rule gives the exact identity

$$\delta C_{r,s}(a,b) = E(a' \smile b') - E(a \smile b).$$

Consequently

$$(a,b,h) \longmapsto (a', b', h + C_{r,s}(a,b) + \delta k), \qquad k \in C^2(S;\mathbb{Z}),$$

preserves the lift equation. This is an explicit homotopy comparison, not only an equality of obstructions: on $S \times \Delta[1]$ take the vertex function χ with $\chi(0) = 0$, $\chi(1) = 1$, put $R = r \smile \chi$, $S_1 = s \smile \chi$, and use

$$A = a + \delta R, \quad B = b + \delta S_1, \quad H = h + E(R \smile b) + E(a \smile S_1) + E(R \smile \delta S_1).$$

Pullbacks are suppressed. The same identity gives $\delta H = E(A \smile B)$ and the required endpoint triples. Adding $\delta(k \smile \chi)$ to H supplies the final δk change while keeping the endpoints of the base fixed.

For signs, when r, s are closed this cylinder represents the base loop $(\alpha, \beta) = (-[r], -[s])$ in the right-t convention above, since $\delta(r \smile \chi) = -r \smile \delta\chi$. With the prism convention $\delta K + K\delta = i_1^* - i_0^*$, let w be the universal four-cocycle pulled back along a target path. The primitive $h = K(w)$ evaluates on a three-cycle z by evaluating w on $-z \times [0,1]$. Thus addition of $[h - h_0]$ as a cochain coordinate is the negative of the right-circle-coefficient path-torsor coordinate used in the proof. Both conventions give exactly the same orbit subgroup $\operatorname{im} D_{a,b}$. The displayed gauge formula supplies concrete comparisons; the mapping-space argument, rather than this formula alone, establishes the classification under all homotopies.

When a polarized abelian variety supplies the Heisenberg categorical group $\mathcal{H}_L$, the established equivalence $B_{\mathrm{top}}\mathcal{H}_L \simeq P_E$ interprets these classes as topological principal $\mathcal{H}_L$ 2-bundles. The projection retains their two underlying torus bundles. Fixing the base lift problem retains its identification with the prescribed base data; forgetting that identification allows the base homotopies responsible for $D_{a,b}$.

The degreewise finite model has the same homotopy type and therefore the same classification. Its simplicial mapping sets must be interpreted in the homotopy category: if $|S| \simeq T$ and F_E is that finite model, use

$$[S, F_E]_{\mathrm{Ho(sSet)}} = \pi_0 \operatorname{Map}_{\mathrm{sSet}}(S, RF_E),$$

where RF_E is a Kan replacement. Raw simplicial maps into the generally non-Kan F_E from one fixed triangulation do not by themselves compute this set. The Kan replacement need not be degreewise finite.

7.12 The Heisenberg action and Fourier–Mukai duality

The algebraic extension also acts on a category built from A itself. This is the translation–tensor action described by Polishchuk [60, Section 2, pp. 817–819]. We spell out its order and signs for the particular extension $\mathcal{H}_L$ used here. Write $t_x(z) = z + x$ and $\mathcal{B}_y = B_L(-, y) = \mathcal{P}(-, \lambda_L(y))$. The Poincare and biextension lines retain their rigidifications; a line over a point is not silently replaced by a chosen scalar.

Proposition 7.23. *The group stack $\mathcal{H}_L$ acts coherently on perfect complexes on A by*

$$\mathsf{T}_{(x,y;M)}(F) = M \otimes \mathcal{B}_y \otimes t_x^* F.$$

For complex points this is an action by exact autoequivalences of $D^b\operatorname{Coh}(A)$, since A is smooth and projective. With composition read from right to left, its multiplication law is

$$\mathsf{T}_{(x,y;M)} \circ \mathsf{T}_{(x',y';N)} \cong \mathsf{T}_{(x+x',y+y';\, M\otimes N\otimes B_L(x,y'))}.$$

The diagonal monoidal section of Proposition 7.13 acts by

$$\mathsf{T}_{s_L(x)}(F) \cong L^{-1} \otimes t_x^*(L \otimes F).$$

Thus tensoring by L conjugates the diagonal action to translation.

Proof. The formula has a family interpretation. For a complex scheme S, let $x, y : S \to A$, let M be a line on S, and put $\pi : A \times S \to S$. Interpret M as $\pi^* M$, t_x as translation over S, and $\mathcal{B}_y$ as the pullback of B_L along $(z, s) \mapsto (z, y(s))$. These operations preserve perfect complexes and commute with derived base change.

The biextension laws give canonical isomorphisms

$$t_x^* \mathcal{B}_{y'} \cong \mathcal{B}_{y'} \otimes \pi^* B_L(x, y'), \qquad \mathcal{B}_y \otimes \mathcal{B}_{y'} \cong \mathcal{B}_{y+y'}.$$

Substitution in the composite functor proves the multiplication law with the positive central line $B_L(x, y')$. The compatibility of the biextension laws is precisely the associativity coherence; the axis rigidifications supply the unit coherence. Isomorphisms of the parameter line M act by the corresponding natural isomorphisms; in particular its scalar automorphisms act as scalars.

Finally, the defining identity for B_L gives

$$L_x \otimes \mathcal{B}_x \cong t_x^* L \otimes L^{-1}.$$

Combining this with $s_L(x) = (x, x; L_x)$ proves the diagonal formula. Its coherence is inherited from the same line-bundle identities. In particular, L^{-1} has a coherent invariant structure for this diagonal action. □

To compare the two coordinate actions, put $\widehat{A} = A^\vee$, $P_\xi = \mathcal{P}(-, \xi)$ and $Q_x = \mathcal{P}(x, -)$. The Fourier–Mukai functor with the positive normalized kernel is

$$\Phi_{\mathcal{P}}(F) = R(p_{\widehat{A}})_*(p_A^* F \otimes \mathcal{P}).$$

It is an equivalence $D^b \operatorname{Coh}(A) \simeq D^b \operatorname{Coh}(\widehat{A})$ by Mukai's theorem [52, Theorem 2.2]. Its translation and tensor identities, with these conventions, are

$$\begin{aligned} \Phi_{\mathcal{P}} \circ t_x^* &\cong (Q_x^{-1} \otimes -) \circ \Phi_{\mathcal{P}}, \\ \Phi_{\mathcal{P}} \circ (P_\xi \otimes -) &\cong t_\xi^* \circ \Phi_{\mathcal{P}}. \end{aligned} \tag{16}$$

These are Mukai's identities [52, (3.1)]. They also follow directly from

$$\begin{aligned} (t_x \times 1)^* \mathcal{P} &\cong \mathcal{P} \otimes p_{\widehat{A}}^* Q_x, \\ (1 \times t_\xi)^* \mathcal{P} &\cong \mathcal{P} \otimes p_A^* P_\xi, \end{aligned}$$

using the projection formula and base change. In the first identity of (16), changing the integration variable from z to $z + x$ introduces $Q_{-x} = Q_x^{-1}$.

For clarity, reordering a composite into tensor-then-translation form on the dual variety retains an additional central line:

$$\Phi_{\mathcal{P}}(P_\xi \otimes t_x^* F) \cong \mathcal{P}(x, \xi)^{-1} \otimes Q_x^{-1} \otimes t_\xi^* \Phi_{\mathcal{P}}(F).$$

Indeed $t_\xi^* Q_x^{-1} \cong \mathcal{P}(x, \xi)^{-1} \otimes Q_x^{-1}$. Thus the two coordinate labels become $(\xi, -x)$ with this scalar-line correction. Taking $\xi = \lambda_L(y)$ gives the action attached to $\mathcal{H}_L$; no principal-polarization assumption is needed.

These are constructions using the algebraic variety, its Poincaré line, and its derived category. The finite Betti model retains the homotopy type with class $[B_E]$. Proposition 7.47 below proves that this homotopy type does not determine $D^b \operatorname{Coh}(A)$. Nevertheless the exchange of the two coordinate actions induces the corrected Heisenberg symmetry of Proposition 7.24, which admits a degreewise finite span in Proposition 7.25.

The classical theta group as a stabilizer. The action also contains a familiar finite subgroup of translations. Assume in this paragraph that L is ample. Put

$$K(L) = \ker(\lambda_L) = \{x \in A : t_x^* L \cong L\}.$$

Since L is ample, $K(L)$ is a finite group scheme. Over $\mathbb{C}$ it is finite and reduced. Consider pairs (x, φ), where $x \in K(L)$ and $\varphi : t_x^* L \longrightarrow L$ is an isomorphism. With the convention

$$(x, \varphi)(y, \psi) = (x + y, \varphi \circ t_x^* \psi),$$

they form the theta group $\mathcal{G}(L)$ in a translation-linearization description [53]. Our arrow convention corresponds to Mumford's pair $(z, \alpha : L \to t_z^* L)$ by $(z, \alpha) = (-x, t_{-x}^* \varphi)$. There is a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathcal{G}(L) \longrightarrow K(L) \longrightarrow 1.$$

Indeed each nonempty isomorphism fibre is a $\mathbb{G}_m$-torsor, the kernel consists of scalar automorphisms of L, and the displayed composition is associative. Equivalently, this is the stabilizer of L, together with a specified isomorphism, for the first-axis translation action above. It acts on $H^0(A, L)$ by $s \mapsto \varphi(t_x^* s)$.

For an elliptic curve and a line of degree $d > 0$, the translation subgroup is $K(L) = A[d]$, of order d^2. The central extension still contains $\mathbb{C}^*$, so it is not a finite group. This classical finiteness concerns the translation stabilizer; degreewise finiteness of P_E^{fin} concerns simplices in a model of a different, delooped object.

7.13 The Fourier exchange as a corrected symmetry

Although the derived category requires more information than the Betti homotopy type, its exchange of translation and tensoring has a precise counterpart in the Heisenberg extension itself. The correction is a line bundle, rather than a globally chosen scalar. For $g = (x, y) \in G = A \times A$, put

$$Sg = (y, -x), \qquad \mathcal{N}_g = B_L(x, y)^{-1}.$$

The symmetry of B_L here is the coherent symmetry inherited from its presentation as the coboundary of L.

Proposition 7.24. *There is a monoidal self-equivalence of algebraic group stacks*

$$\mathcal{S}_L : \mathcal{H}_L \longrightarrow \mathcal{H}_L, \qquad (x, y; M) \longmapsto (y, -x; M \otimes \mathcal{N}_{(x,y)}).$$

Its square is canonically monoidally isomorphic to simultaneous coordinate negation $(x, y; M) \mapsto (-x, -y; M)$, and its fourth power is canonically isomorphic to the identity. It acts identically on the central $B\mathbb{G}_m$. On the Betti delooping P_E, the induced homotopy equivalence acts by $(x, y) \mapsto (y, -x)$ on π_2 and by the identity on π_3.

Proof. For $g = (x, y)$ and $h = (x', y')$, bilinearity and symmetry give the canonical identity

$$\mathcal{N}_g \otimes \mathcal{N}_h \otimes \mathcal{C}_L(Sg, Sh) \cong \mathcal{C}_L(g, h) \otimes \mathcal{N}_{g+h}. \tag{17}$$

Indeed, the left side is $B_L(x, y)^{-1} \otimes B_L(x', y')^{-1} \otimes B_L(x', y)^{-1}$. Expanding $B_L(x+x', y+y')$ into its four factors gives exactly the right side. This identity is the tensorator for $\mathcal{S}_L$. For three inputs both composites expand into the same factors; the compatibility of the biextension laws and its symmetry gives the monoidal pentagon. The rigidifications give the unit map.

The two successive correction lines cancel canonically:

$$\mathcal{N}_{(x,y)} \otimes \mathcal{N}_{(y,-x)} = B_L(x, y)^{-1} \otimes B_L(y, -x)^{-1} \cong \mathbb{C}.$$

This identifies the square with simultaneous coordinate negation, including its monoidal structure, and gives the fourth-power identification. Coordinate negation fixes the central line M; it is not the categorical inverse of the object $(x, y; M)$. All corrections at $(0, 0)$ are canonically trivial, so the central band is fixed. The asserted maps on homotopy groups follow. The k-invariant is preserved, consistently with

$$q(S(x, y)) = E(y, -x) = E(x, y) = q(x, y).$$

□

When L is a principal polarization, use the isomorphism $\lambda_L : A \to A^\vee$ to define the Fourier autoequivalence $\mathfrak{F}_L = \lambda_L^* \Phi_{\mathcal{P}}$ of $D^b \operatorname{Coh}(A)$. The identities in (16) yield, with all scalar lines retained,

$$\mathfrak{F}_L \circ \mathsf{T}_{(x,y;M)} \cong \mathsf{T}_{\mathcal{S}_L(x,y;M)} \circ \mathfrak{F}_L.$$

Thus the corrected group-stack symmetry is the conjugation symmetry of this Fourier transform. The group-stack construction itself does not require a principal polarization. For a nonprincipal polarization we do not identify A with $A^\vee$ by its isogeny.

The fourth-power statement concerns this symmetry of the Heisenberg action. It does not assert a fourth-power identity for the Fourier functor itself:

Mukai's inversion formula gives $\mathfrak{F}_L^2 \cong (-1_A)^*[-g]$, and hence $\mathfrak{F}_L^4 \cong [-2g]$, where $g = \dim A$ [52, Theorem 2.2]. Shifts commute with the translation–tensor action, so they disappear under conjugation.

The line $\mathcal{N}$ need not be trivial on G. Therefore $\mathcal{S}_L$ need not preserve the trivial-line atlas used in the compact triangle nerve. The next construction retains frames of $\mathcal{N}$ as additional data and obtains a finite span instead of choosing a nonexistent global trivialization.

7.14 A finite simplicial roof for the Fourier symmetry

Let $X_\bullet$ be the compact triangle nerve of Proposition 7.14, and retain $S(x,y) = (y,-x)$ and $\mathcal{N}_g = B_L(x,y)^{-1}$ from Proposition 7.24. All line isomorphisms in this subsection are taken in phase torsors, so their automorphism group is $U(1)$.

Proposition 7.25. *There is a span of simplicial compact semialgebraic spaces*

$$X_\bullet \xleftarrow{\ p\ } Z_\bullet \xrightarrow{\ q\ } X_\bullet$$

whose two maps are weak equivalences after realization and which represents $B_{\mathrm{top}}\mathcal{S}_L$ under $|X_\bullet| \simeq P_E$. Simultaneous face-carrier replacement gives a span of simplicial sets finite in every degree,

$$Y_X \xleftarrow{\ \widehat{p}\ } Y_Z \xrightarrow{\ \widehat{q}\ } Y_X,$$

with both maps weak equivalences. Its morphism $\widehat{q}\widehat{p}^{-1}$ in the homotopy category realizes the same symmetry, acting by S on π_2 and by the identity on π_3. The assertion is a finite roof; it supplies neither a specified simplicial self-map of Y_X nor a strict C_4-action.

Proof. Adjoin frames on all edges. An element of Z_n is an element (g,ϕ) of X_n together with independent phase frames

$$u_{ij} : \mathcal{N}_{g_{ij}} \longrightarrow \mathbb{C} \qquad (0 \le i < j \le n).$$

There is no compatibility equation on these frames. Faces restrict all labels, and degeneracies repeat labels and assign the canonical unit frame to a repeated edge. In particular $Z_0 = *$, and forgetting the frames gives a simplicial map p.

Write the phase of the tensorator in (17) as

$$t_{g,h} : \mathcal{N}_g \otimes \mathcal{N}_h \otimes \mathcal{C}(Sg, Sh) \longrightarrow \mathcal{C}(g,h) \otimes \mathcal{N}_{g+h}.$$

For a triangle put

$$\widetilde{\phi}_{ijk} = (\phi_{ijk} \otimes \mathrm{id}) t_{g_{ij}, g_{jk}} : \mathcal{N}_{g_{ij}} \otimes \mathcal{N}_{g_{jk}} \otimes C(Sg_{ij}, Sg_{jk}) \longrightarrow \mathcal{N}_{g_{ik}}.$$

Define $q(g, \phi, u) = (Sg, \psi)$, where

$$\psi_{ijk} = u_{ik}\, \widetilde{\phi}_{ijk}(u_{ij}^{-1} \otimes u_{jk}^{-1} \otimes \mathrm{id}). \tag{18}$$

Monoidality of $\mathcal{S}_L$ and cancellation of the edge frames give the tetrahedron identities for ψ. The unit coherence gives the repeated-vertex identities. Thus q is simplicial.

The total space Z_n is the fibre product over X_n of the finitely many circle bundles $\mathrm{Ph}(\mathcal{N}_{g_{ij}}^{\vee})$. It is compact semialgebraic. The tensorator is algebraic before taking phases, and the same semialgebraic metric presentations used for X_n show that p, q, and every simplicial operator are semialgebraic. Every degeneracy is a closed embedding with a face left inverse. The union of its images is closed semialgebraic, so compatible triangulation proves properness of $Z_\bullet$.

Two principal actions of the same simplicial group. Put

$$U_n = C^1(\Delta^n; U(1)) = U(1)^{\binom{n+1}{2}}, \qquad (\delta v)_{ijk} = v_{ij} v_{jk} v_{ik}^{-1}.$$

Use value 1 on repeated edges. The following two actions are compatible with all simplicial operators:

$$v \cdot_p (g, \phi, u) = (g, \phi, vu),$$
$$v \cdot_q (g, \phi, u) = (g, (\delta v)\phi, vu).$$

The first action fixes p. Changing u to vu multiplies the right side of (18) by $(\delta v)_{ijk}^{-1}$, so the second action fixes q. Multiplication by δv preserves every tetrahedron equation because $\delta^2 v = 1$.

Both actions are free and transitive on the appropriate fibres. This is immediate for p. For q, its target edge labels determine the source labels uniquely by S^{-1}. Choose arbitrary frames u_{ij} on these source edges. Equation (18) then uniquely determines ϕ from the target triangles, and inverse monoidal transport ensures its tetrahedron identities. Changing the frames changes ϕ by exactly δv. Local trivializations of the finitely many frame bundles give continuous local sections. Consequently each of p_n and q_n is a locally trivial principal U_n-bundle, for its indicated action.

The realization of each principal map is an equivalence. The simplicial topological group $U_\bullet$ has a contracting extra degeneracy: prepend a new vertex and give every new edge value 1. Its zeroth face is the identity, and

its other face and degeneracy identities follow by restriction of edge labels. Therefore $|U_\bullet|$ is contractible.

For either action form the bisimplicial action nerve

$$W_{n,r} = Z_n \times U_n^r.$$

In each fixed degree n, the augmentation $|W_{n,\bullet}| \to X_n$ is a weak equivalence: it is the homotopy quotient of a locally trivial principal bundle, with contractible fibre EU_n. Local triviality over the compact semialgebraic base implies numerability. All spaces $W_{n,r}$ are compact semialgebraic and all degeneracies are cofibrations, as follows from closed semialgebraic images and compatible triangulation. Fat realization and the good-space fat–thin comparison therefore preserve these equivalences [23, Theorem 2.2].

Realizing in the n direction first instead gives the action nerve of $|U_\bullet|$ acting on $|Z_\bullet|$: realization commutes with these finite products in compactly generated spaces [44, Corollary 11.6]. Its realization is the homotopy quotient $|Z_\bullet|_{h|U_\bullet|}$. The inclusion of $|Z_\bullet|$ in this homotopy quotient is a weak equivalence, since $|U_\bullet|$ is contractible. Composing that inclusion with the augmentation gives exactly $|p|$, respectively $|q|$. The two-out-of-three property proves that both are weak equivalences.

Identify the symmetry represented by the span. Use the simplicial stack of coherent edge lines and triangle tensorators from Proposition 7.14. Applying $\mathcal{S}_L$ to the source simplex gives the edge lines $\mathcal{N}_{g_{ij}}$ and triangle tensorators $\widetilde{\phi}_{ijk}$. The frames u_{ij} identify these edge objects with the trivial edge lines of the output simplex $q(z)$. Equation (18) says exactly that these identifications intertwine every triangle tensorator. They thus give an isomorphism of coherent diagrams, compatible with restriction and repeated vertices. A natural isomorphism of these groupoid presentations induces a homotopy on their classifying spaces. Under the natural realization comparison to $B_{\mathrm{top}}\mathcal{H}_L$ this gives

$$|q| \simeq B_{\mathrm{top}}\mathcal{S}_L \circ |p|.$$

This comparison, rather than the weak-equivalence property alone, identifies the represented symmetry.

Replace the entire span by finite carriers. The construction of Theorem 7.15 extends to this two-arrow diagram as follows. Suppose finite triangulations have been chosen below degree n. First triangulate X_n compatibly with the preimages of the already chosen open cells under every face $d_i : X_n \to X_{n-1}$. Then triangulate Z_n compatibly with both the preimages of the chosen cells of Z_{n-1} under its faces and the preimages of the newly

chosen cells of X_n under p_n and q_n. This is a finite family of semialgebraic subsets, so compatible finite triangulations exist.

Let P_n^X, P_n^Z be the finite posets of their nonempty open cells. Every displayed map sends each open cell into a unique open cell. Its carrier is order-preserving: continuity maps a closed source cell into the closure of its target carrier, and a closed target simplex is a union of whole open cells. Uniqueness of the open carrier makes the actual face identities and the equations $d_i p = p d_i$, $d_i q = q d_i$ hold strictly for carriers. Hence there is a span of semisimplicial finite posets

$$P_\bullet^X \longleftarrow P_\bullet^Z \longrightarrow P_\bullet^X.$$

For each of these maps f, the incidence categories in the proof of Theorem 7.15 have the functor

$$(F, z) \longmapsto (f^P F, f(z)).$$

It commutes exactly with both forgetting the point and evaluation. Their rowwise weak-equivalence spans are therefore natural for the whole diagram, not just for the individual objects. Freely adjoining outer degeneracies and taking the diagonal now gives the asserted finite simplicial span, with

$$(Y_X)_n = \coprod_{[n]\twoheadrightarrow[r]} N_n P_r^X, \qquad (Y_Z)_n = \coprod_{[n]\twoheadrightarrow[r]} N_n P_r^Z.$$

Both sets are finite. The natural incidence comparisons and the proper fat–thin comparisons identify this span with $|X| \leftarrow |Z| \rightarrow |X|$ in the homotopy category. Thus both finite legs are weak equivalences and their roof represents $B_{\mathrm{top}}\mathcal{S}_L$. Proposition 7.24 gives its stated action on homotopy groups. □

The line correction $\mathcal{N}$ is retained through phase frames; no global trivialization is used. The triangulations need only respect faces and the two arrows. The newly adjoined degeneracies are formal, and no compatibility with fourfold iteration or its coherent comparison to the identity is imposed on the finite sets by this individual-roof construction. Section 7.20 constructs the additional word diagrams for the normalized matrix action, including its coherent composition and unit comparisons. For the original Fourier action, the corresponding finite-group construction gives a strict C_4-action on a further degreewise finite model (Corollary 7.43).

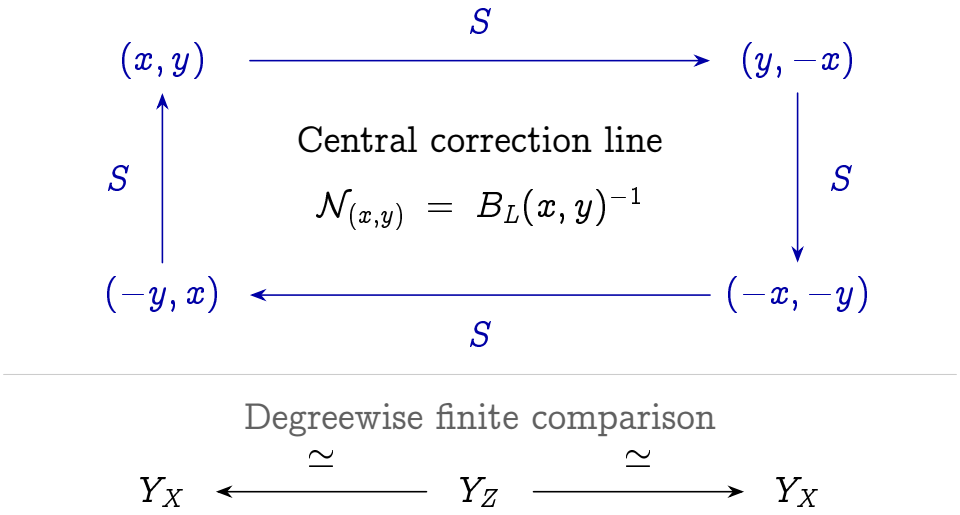


Figure 7: The four-cycle records the action on the two coordinates. At each step the group-stack symmetry retains the correction line $\mathcal{N}$, evaluated at the current pair. Both legs of the finite roof are weak equivalences. The roof represents the resulting homotopy symmetry; it does not assert a strict C_4-action on the chosen finite simplicial model Y_X.

7.15 Integral matrices and their line corrections

The Fourier exchange belongs to a larger family of algebraic symmetries. The family initially has a composition defect, which can be calculated and removed without choosing a symmetric representative of the polarization. Removing it changes the particular Fourier lift when the original L is not symmetric. Throughout this subsection, line-bundle isomorphisms are normalized using the fixed rigidification at the origin.

Write $g = (x, y)$, $h = (x', y')$, and $C(g, h) = B_L(x, y')$. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ put $\varepsilon_M = \det M \in \{1, -1\}$ and

$$\mathcal{N}_M(x, y) = L_x^{\otimes ac} \otimes L_y^{\otimes bd} \otimes B_L(x, y)^{\otimes bc}. \tag{19}$$

Negative tensor powers mean powers of the dual line. On an isomorphism $f : R \to R'$, the power -1 means the *covariant* inverse dual $(f^{-1})^\vee : R^\vee \to R'^\vee$. In particular, it inverts central scalars.

Proposition 7.26. *Each matrix $M \in \mathrm{GL}_2(\mathbb{Z})$ has a monoidal self-equivalence of algebraic group stacks*

$$\mathcal{F}_M(g; R) = \big(Mg; R^{\otimes \varepsilon_M} \otimes \mathcal{N}_M(g)\big) \quad \text{of } \mathcal{H}_L.$$

It acts by M on the two lattice summands of $\pi_2(P_E)$ and by ε_M on $\pi_3(P_E)$.

Proof. For a line N on G, write $\delta N(g,h) = N(g+h) \otimes N(g)^{-1} \otimes N(h)^{-1}$. Bilinearity of B_L gives

$$\delta \mathcal{N}_M(g,h) \cong B_L(x,x')^{ac} \otimes B_L(y,y')^{bd} \otimes B_L(x,y')^{bc} \otimes B_L(y,x')^{bc},$$
$$\mathcal{C}(Mg,Mh) \cong B_L(x,x')^{ac} \otimes B_L(y,y')^{bd} \otimes B_L(x,y')^{ad} \otimes B_L(y,x')^{bc}.$$

Since $ad - bc = \varepsilon_M$, the required tensorator is the normalized isomorphism

$$\mathcal{N}_M(g) \otimes \mathcal{N}_M(h) \otimes \mathcal{C}(Mg,Mh) \cong \mathcal{C}(g,h)^{\otimes \varepsilon_M} \otimes \mathcal{N}_M(g+h). \tag{20}$$

The biextension laws provide its coherence. Equivalently, the two pentagon composites are normalized isomorphisms of the same line bundles on the connected proper variety A^6. Their ratio is a regular unit, hence a constant, and its value at the origin is 1. The same argument handles the units. The underlying functor is an equivalence because M is invertible and tensoring or dualizing lines is an equivalence. The central band is raised to the power ε_M. This agrees with $q(M(u,v)) = \varepsilon_M q(u,v)$. □

The exact defect. Set

$$D = [-1]^* L \otimes L^{-1} \in \mathrm{Pic}^0(A).$$

This rigidified line has its normalized multiplicative structure, so $D_{x+y} \cong D_x \otimes D_y$. The cube theorem gives

$$[n]^* L \cong L^{\otimes n^2} \otimes D^{\otimes \binom{n}{2}}, \qquad B_L(x,x) \cong L_x^{\otimes 2} \otimes D_x, \tag{21}$$

for every integer n; see [49, Chapter I, Corollary 5.4]. These are isomorphisms of rigidified lines, not choices of scalar-valued multipliers.

Let $N = \left(\begin{smallmatrix} e & f \\ r & s \end{smallmatrix}\right)$. The line comparing $\mathcal{F}_M \mathcal{F}_N$ with $\mathcal{F}_{MN}$ is

$$\Delta_{M,N}(g) = \mathcal{N}_N(g)^{\otimes \varepsilon_M} \otimes \mathcal{N}_M(Ng) \otimes \mathcal{N}_{MN}(g)^{-1}.$$

An explicit calculation yields

$$\Delta_{M,N}(x,y) \cong D_x^{\otimes k_1(M,N)} \otimes D_y^{\otimes k_2(M,N)}, \tag{22}$$

where

$$k_1(M,N) = ac\binom{e}{2} + bd\binom{r}{2} + bc\, er,$$
$$k_2(M,N) = ac\binom{f}{2} + bd\binom{s}{2} + bc\, fs.$$

To see every contribution, expand

$$L_{ex+fy} \cong L_x^{e^2} \otimes L_y^{f^2} \otimes B_L(x,y)^{ef} \otimes D_x^{\binom{e}{2}} \otimes D_y^{\binom{f}{2}},$$

and similarly for L_{rx+sy}. Also

$$B_L(ex+fy, rx+sy) \cong B_L(x,x)^{er} \otimes B_L(x,y)^{es+fr} \otimes B_L(y,y)^{fs}.$$

After applying (21), all the L_x, L_y, and $B_L(x,y)$ exponents cancel against those of $\mathcal{N}_{MN}$. For example the L_x exponent is $\varepsilon_M er + ace^2 + bdr^2 + 2bc\,er = (ae+br)(ce+dr)$. The remaining D exponents are exactly (22).

An integral normalization. Use row vectors, and put

$$\beta(M) = (ac, bd), \qquad v = (1,1), \qquad \rho(M) = \frac{\beta(M) + vM - \varepsilon_M v}{2}.$$

Both components of $\rho(M)$ are integers. Each column of an invertible matrix modulo 2 is nonzero, and $pq + p + q = 1$ for each nonzero $(p,q) \in \mathbb{F}_2^2$. Thus $ac + a + c - \varepsilon_M$ and $bd + b + d - \varepsilon_M$ are even.

Remark 7.27. The parity calculation has a simple invariant formulation. The quadratic function $Q_2(s,t) = st + s + t$ on $\mathbb{F}_2^2$ is 0 at the origin and 1 at each nonzero vector. It is therefore invariant under $\mathrm{GL}_2(\mathbb{F}_2)$. Expanding $Q_2(M(s,t)) = Q_2(s,t)$ gives $\beta(M) + vM - v = 0$ modulo 2. Since $\varepsilon_M = 1$ modulo 2, this is exactly the divisibility needed to define $\rho(M)$ integrally.

Proposition 7.28. *For any rigidified ample L, the corrections*

$$\widetilde{\mathcal{N}}_M(x,y) = \mathcal{N}_M(x,y) \otimes D_x^{\otimes \rho_1(M)} \otimes D_y^{\otimes \rho_2(M)} \tag{23}$$

define a coherent weak action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathcal{H}_L$ by monoidal self-equivalences. Here "weak" means that there are specified monoidal natural isomorphisms $\widetilde{\mathcal{F}}_M \widetilde{\mathcal{F}}_N \cong \widetilde{\mathcal{F}}_{MN}$ satisfying the associativity and unit conditions. The determinant-one subgroup fixes the central band. If L is symmetric, the original corrections $\mathcal{N}_M$ already give such an action.

Proof. Multiplicativity of D leaves the tensorator calculation (20) unchanged. The preceding exponents satisfy the row-vector identity

$$2k(M,N) = \beta(MN) - \varepsilon_M \beta(N) - \beta(M)N.$$

Since $\varepsilon_{MN} = \varepsilon_M \varepsilon_N$, the definition of ρ therefore gives

$$\rho(MN) - \varepsilon_M \rho(N) - \rho(M)N = k(M, N).$$

The extra D powers in (23) cancel (22) exactly. Normalize the resulting comparison isomorphisms at the origin. They are monoidal: the two requisite composites are normalized isomorphisms of the same lines on G^2. The associativity comparison for three matrices is likewise an equality of normalized isomorphisms on G. On these connected proper varieties every regular unit is constant, so the comparisons agree. The identity matrix has trivial correction.

If L is symmetric, its normalized symmetry trivializes D. The defect vanishes and the same uniqueness argument proves coherence for the unmodified family. □

What happens to the Fourier lift. For the generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

the original corrections are $\mathcal{N}_S = B_L(x, y)^{-1}$ and $\mathcal{N}_T = L_x$. Their first defect is $\Delta_{S,T} = D_x^{-1}$. The normalization above gives

$$\widetilde{\mathcal{N}}_S = B_L(x, y)^{-1} \otimes D_x^{-1}, \qquad \widetilde{\mathcal{N}}_T = L_x \otimes D_x.$$

These two additional factors cancel in the composite $S \circ T$. Since $(ST)^3 = S^2 = -I$ and $\rho(-I) = (-1, -1)$, the raw lifts obey instead

$$(\mathcal{F}_S \mathcal{F}_T)^3(g; R) \cong \mathcal{F}_{-I}\big(g; R \otimes D_x^{-1} \otimes D_y^{-1}\big), \qquad \mathcal{F}_S^2 \cong \mathcal{F}_{-I}.$$

Thus a nontrivial D obstructs this prescribed pair of lifts from satisfying the modular relation. The normalized action exists for arbitrary L, but its S is the displayed modification of the earlier Fourier symmetry. For comparison, the reflection $\mathrm{diag}(1, -1)$ has trivial raw correction and normalized correction D_x, and inverts the central line in either case.

This determinant--1 symmetry cannot be induced by conjugation with a $\mathbb{C}$-linear derived autoequivalence: such conjugation fixes every scalar automorphism $\lambda\,\mathrm{id}$, whereas the central inversion sends λ to λ^{-1}.

One may alternatively choose $P \in \mathrm{Pic}^0(A)$ with $P^{\otimes 2} \cong D$ and replace L by $L \otimes P$. This line is symmetric and has the same polarization; the square root exists over $\mathbb{C}$ but is not canonical. The integral normalization above avoids this choice. It is a calculation with algebraic line bundles and their

coherent isomorphisms. Corollary 7.37 below rules out a strict simplicial realization of this full matrix action on any degreewise finite model; the coherent finite replacement uses word diagrams instead. No identification of every isometry of the Whitehead lattice with an algebraic symmetry is asserted.

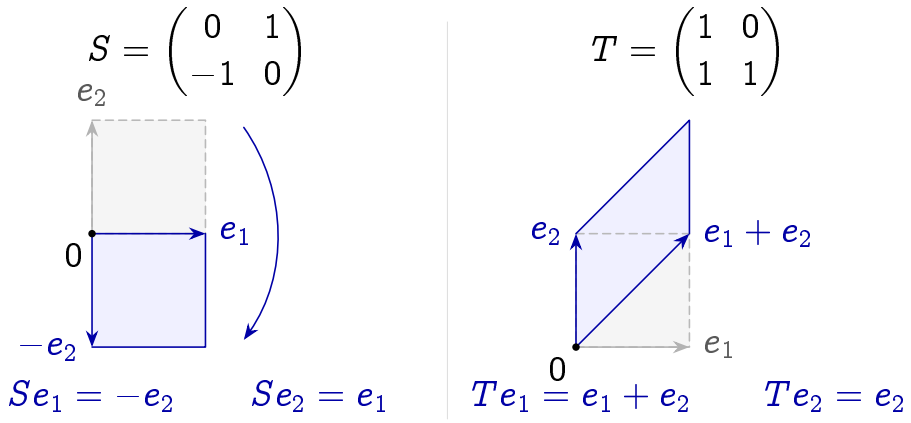


Figure 8: Clockwise rotation and shear of the auxiliary $\mathbb{Z}^2$ of slots. Gray dashed edges mark the original unit square; blue edges and arrows mark its image and the image basis. This plane is not the abelian variety. Both matrices have determinant 1 and fix the central band, but their lifts above the lattice retain correction lines. For symmetric L these are $\mathcal{N}_S(x, y) = B_L(x, y)^{-1}$ and $\mathcal{N}_T(x, y) = L_x$.

7.16 The classical modular action and its shifts

The derived counterpart of these matrix symmetries is classical. Mukai observed the modular action modulo shifts [52, Theorem 3.13 and (3.14)]; Polishchuk constructed a coherent action of its central extension [59, Theorem 3.1]. The following calculation translates that result into our positive Poincare-kernel and translation conventions.

Assume that L is a symmetric rigidified principal polarization, put $g = \dim A$, and write

$$\mathfrak{F} = \lambda_L^* \Phi_{\mathcal{P}} = \Phi_{B_L}, \qquad \mathfrak{U} = (- \otimes L^{-1}), \qquad \iota = (-1_A)^*.$$

Composition is read from right to left. The corresponding matrices are

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad (ST)^3 = S^2 = -I.$$

In particular, T is the lower triangular generator.

Proposition 7.29. *Conjugation by $\mathfrak{F}$ and $\mathfrak{U}$ induces, respectively, the corrected Heisenberg symmetries*

$$(x,y;M) \longmapsto (y,-x;M\otimes B_L(x,y)^{-1}), \qquad (x,y;M)\longmapsto (x,x+y;M\otimes L_x).$$

The unshifted derived functors satisfy natural isomorphisms

$$\mathfrak{F}^2 \cong \iota[-g], \qquad (\mathfrak{F}\mathfrak{U})^3 \cong \iota[-2g]. \tag{24}$$

Thus the matrix relation holds modulo shifts. With the shifted lift $\widetilde{\mathfrak{F}} = \mathfrak{F}[g]$, it becomes

$$(\widetilde{\mathfrak{F}}\mathfrak{U})^3 \cong \widetilde{\mathfrak{F}}^2 \cong \iota[g], \qquad \widetilde{\mathfrak{F}}^4 \cong [2g].$$

These formulas are isomorphisms of functors, not literal equalities.

Proof. The Fourier conjugation was proved in Proposition 7.24. For the other generator, the identity $L^{-1}\otimes t_x^*L \cong L_x\otimes B_L(-,x)$ gives

$$\mathfrak{U}\,\mathsf{T}_{(x,y;M)}\,\mathfrak{U}^{-1}(F) \cong M\otimes L_x\otimes B_L(-,x+y)\otimes t_x^*F.$$

This also proves the second conjugation formula without a principal-polarization or symmetry assumption on L.

To check the shifts and kernel sign, set $R = \Phi_{B_L^{-1}}$ and $V = (-\otimes L)$. Beauville uses this negative kernel and records

$$R^2 \cong (VR)^3 \cong \iota[-g] \qquad \text{[4, Sections 1.6 and 2.2].}$$

Changing the kernel sign gives $\mathfrak{F}\cong \iota R$; hence inversion gives $\mathfrak{F}\cong R^{-1}[-g]$. Since $\mathfrak{U}=V^{-1}$, we obtain

$$(\mathfrak{F}\mathfrak{U})^3 \cong (VR)^{-3}[-3g] \cong \iota[-2g].$$

Mukai's inversion formula gives $\mathfrak{F}^2\cong\iota[-g]$. The remaining identities follow by collecting shifts. □

The classical coherent lift is an action of the central extension with presentation $\langle \widetilde{S},\widetilde{T} \mid \widetilde{S}^2 = (\widetilde{S}\widetilde{T})^3\rangle$; imposing $\widetilde{S}^4=1$ gives $\mathrm{SL}_2(\mathbb{Z})$. Polishchuk's coherence theorem supplies the composition isomorphisms and their compatibility, beyond the displayed generator relations. The shifted pair uses the inverse generators of the classical lift: $\widetilde{\mathfrak{F}}\cong R^{-1}$ and $\mathfrak{U}=V^{-1}$. Shifts commute with the Heisenberg action and therefore vanish under conjugation. This explains why the algebraic group-stack symmetries obey the modular relations while these particular derived lifts retain a central shift. It gives no strict modular action on the degreewise finite simplicial model.

7.17 The polarization type retained by the Heisenberg space

Let Λ be free of rank $2g$, with $g \geq 1$, and let $E : \Lambda \otimes \Lambda \to \mathbb{Z}$ be a nondegenerate alternating form. For a polarization on a complex abelian variety this is its integral Riemann form; its positivity implies nondegeneracy [49, Chapter I, Section 2]. Put $\Gamma = \Lambda \oplus \Lambda$ and retain

$$P_E = \operatorname{hofib}\bigl(q_E : K(\Gamma, 2) \longrightarrow K(\mathbb{Z}, 4)\bigr), \qquad q_E = E(u \smile v).$$

The two copies of Λ specify the construction, but the classification below does not require a homotopy equivalence to preserve them.

Proposition 7.30. *The unmarked homotopy type of P_E determines, and is determined by, g and the polarization type*

$$d_1 \mid d_2 \mid \cdots \mid d_g, \qquad d_i > 0.$$

More precisely, its Whitehead pairing, after a choice of generator of $\pi_3(P_E) \cong \mathbb{Z}$, has matrix

$$I_E = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \quad \text{on } \pi_2(P_E) = \Gamma,$$

up to an overall sign. Its Smith normal form has each d_i repeated four times. In a suitable integral basis its symmetric bilinear form is

$$I_E \cong \bigoplus_{i=1}^{g} (d_i H \oplus d_i H), \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consequently $P_E \simeq P_{E'}$ if and only if the two polarization types, including their lengths, agree.

Proof. Since Γ is free, $K(\Gamma, 2)$ is a product of copies of $\mathbb{C}P^{\infty}$, and

$$H^4(K(\Gamma, 2); \mathbb{Z}) = \operatorname{Sym}^2(\Gamma^{\vee}).$$

Thus we can regard q_E as the integral quadratic polynomial $q_E(x, y) = E(x, y)$. Its polar form is

$$\begin{aligned} I_E\bigl((x, y), (x', y')\bigr) &= q_E(x + x', y + y') - q_E(x, y) - q_E(x', y') \\ &= E(x, y') + E(x', y). \end{aligned}$$

The displayed block matrix follows from $E^t = -E$. There is no division-by-two ambiguity here: for an integral quadratic polynomial q, its polar

form satisfies $I_q(z,z) = 2q(z)$, and the operation $q \mapsto I_q$ is injective over $\mathbb{Z}$. In coordinates it doubles the square coefficients and retains each mixed coefficient as the corresponding off-diagonal matrix entry.

To identify this form intrinsically, take $z, z' \in \pi_2(P_E)$ and use the attaching map of the top cell of $S^2 \times S^2$. It is the Whitehead product of the two inclusions of $S^2 \vee S^2$; see [35, Section 4.2, after Example 4.52]. If $a, b \in H^2(S^2 \times S^2; \mathbb{Z})$ are the two generators, the associated map to $K(\Gamma, 2)$ has class $za + z'b$. Because $a^2 = b^2 = 0$, its obstruction to a lift is

$$q_E(za + z'b) = I_E(z, z')\, a \smile b.$$

The obstruction is the coefficient of the Whitehead product in $\pi_3(P_E) = \mathbb{Z}$. Choosing the generator compatibly with the cell orientation gives $[z, z'] = I_E(z, z')$; reversing that generator changes every value by the same sign. In particular, $[z, z] = 2q_E(z)$ in this convention.

The normal form gives an integral basis $e_1, \dots, e_g, f_1, \dots, f_g$ of Λ for which

$$E(e_i, e_j) = E(f_i, f_j) = 0, \qquad E(e_i, f_j) = d_i \delta_{ij}, \qquad 0 < d_1 \mid \cdots \mid d_g.$$

The d_i are unique [78, Section 43.4.15]. For each i, order the corresponding four basis vectors of Γ as

$$(e_i^x, f_i^y, f_i^x, -e_i^y).$$

The form I_E on this block is $d_i H \oplus d_i H$; different blocks are orthogonal. Since H is unimodular, the Smith normal form of this block is $\operatorname{diag}(d_i, d_i, d_i, d_i)$. The divisibility ordering of the d_i makes the concatenation of these blocks the Smith normal form of I_E.

Any homotopy equivalence $P_E \simeq P_{E'}$ induces an integral isomorphism $F : \Gamma \to \Gamma'$ and multiplication by $\epsilon \in \{1, -1\}$ on π_3. Naturality of Whitehead products gives

$$F^t I_{E'} F = \epsilon I_E.$$

Thus $\operatorname{rank} \pi_2 = 4g$ and the Smith invariants agree, recovering every d_i, including those equal to 1. Equivalently, the Postnikov classes satisfy $F^* q_{E'} = \epsilon q_E$, since polarization of quadratic polynomials is injective.

Conversely, equality of the polarization types supplies an isomorphism $S : \Lambda \to \Lambda'$ with $S^t E' S = E$. Then $F = S \oplus S$ satisfies $F^* q_{E'} = q_E$. Representability of integral cohomology by Eilenberg–Mac Lane spaces supplies a homotopy-commutative square of the two classifying maps, with $K(F, 2)$ on the source and the identity on $K(\mathbb{Z}, 4)$ [35, Theorem 4.57 and Section 4.3]. The induced map of homotopy fibres is an isomorphism on π_2 and π_3; these are their only nonzero homotopy groups. It is therefore a homotopy equivalence of their CW models. $\square$

Remark 7.31 (Information retained and forgotten). The finite group

$$\operatorname{coker}(I_E : \Gamma \to \Gamma^\vee) \cong \bigoplus_{i=1}^{g} (\mathbb{Z}/d_i)^4$$

together with $\operatorname{rank} \pi_2$ also recovers the type. The rank is needed to count factors with $d_i = 1$. For example, the types $(1,4)$ and $(2,2)$ give respectively $(\mathbb{Z}/4)^4$ and $(\mathbb{Z}/2)^8$, hence distinct homotopy types. All principal polarizations of a fixed dimension give the same unmarked space P_E; the complex structure, a particular theta line, and a choice of the two axes are additional data. The zero-dimensional case is separate and gives $P_0 = K(\mathbb{Z}, 3)$.

7.18 Homotopy symmetries and finite matrix roofs

Assume $g \geq 1$ and retain the nondegenerate alternating form E on Λ, $\Gamma = \Lambda \oplus \Lambda$, and

$$p : P_E \longrightarrow B = K(\Gamma, 2), \qquad P_E = \operatorname{hofib}(q : B \longrightarrow K(\mathbb{Z}, 4)).$$

In particular $q = E(u \smile v)$ is nonzero. Write $\operatorname{haut}(P_E)$ for the space of homotopy self-equivalences.

Proposition 7.32. *The actions on the two nonzero homotopy groups give a group isomorphism*

$$\pi_0 \operatorname{haut}(P_E) \cong \{(F, \epsilon) \in \mathrm{GL}(\Gamma) \times \{1, -1\} : F^* q = \epsilon q\}.$$

Equivalently, with I_E the symmetric Whitehead form, the condition is $F^t I_E F = \epsilon I_E$. Based and unbased homotopy classes give the same group. For a principal polarization, an integral basis identifies I_E with $H^{\oplus 2g}$, so this is the group of integral isometries and anti-isometries of that hyperbolic form.

Proof. First check the possible ambiguity in lifting a map with domain P_E, rather than merely with domain B. The integral cohomological Serre spectral sequence of

$$K(\mathbb{Z}, 3) \longrightarrow P_E \xrightarrow{p} B$$

has trivial local coefficients because B is simply connected [73, Chapter II, Sections 8–9]. The fibre has $H^1 = H^2 = 0$ and $H^3 = \mathbb{Z}\langle z \rangle$; the base is a product of copies of $\mathbb{C}P^\infty$, so $H^1(B; \mathbb{Z}) = H^3(B; \mathbb{Z}) = 0$. The only possible differential from total degree three is

$$d_4 : E_4^{0,3} = \mathbb{Z} \longrightarrow E_4^{4,0} = H^4(B; \mathbb{Z}), \qquad z \longmapsto q.$$

This identification follows by pulling back the path fibration of $K(\mathbb{Z},4)$ along its classifying map q, with a compatible choice of sign for z. There are no earlier differentials into $(4,0)$, since the fibre rows one and two vanish. Hence

$$H^{3}(P_E;\mathbb{Z}) = 0, \qquad \ker\bigl(p^{*} : H^{4}(B;\mathbb{Z}) \to H^{4}(P_E;\mathbb{Z})\bigr) = \mathbb{Z}q. \tag{25}$$

Indeed $q \neq 0$ and $H^{4}(B;\mathbb{Z})$ is free abelian, so d_4 is injective even when q is divisible. These statements concern integral cohomology and do not assume that q is primitive.

The projection p induces an isomorphism on π_2. Hurewicz, the universal coefficient theorem, and representability of cohomology therefore give

$$[P_E, B] = H^{2}(P_E;\Gamma) = \operatorname{Hom}(\Gamma,\Gamma).$$

Thus a map inducing F on π_2 projects to a map homotopic to $K(F,2)p$. After fixing that projected map, its lifts to the homotopy fibre are nullhomotopies of its composite with q. When lifts exist, their classes form a torsor under $[P_E, K(\mathbb{Z},3)] = H^{3}(P_E;\mathbb{Z})$, which is zero by (25). This lifting argument applies to arbitrary CW models, with no finiteness hypothesis on the domain. It follows that at most one homotopy class of self-maps induces any given F. In particular, there are no additional self-equivalences invisible on homotopy groups.

Naturality of the Postnikov class makes $F^{*}q = \epsilon q$ necessary for a self-equivalence inducing (F,ϵ); equivalently one can use naturality of the Whitehead form and the injectivity of integral polarization of quadratic polynomials, as in Proposition 7.30. Conversely this equality gives a homotopy-commutative square with $K(F,2)$ on B and multiplication by ϵ on $K(\mathbb{Z},4)$. The induced map of homotopy fibres acts by (F,ϵ) and is a homotopy equivalence. Existence uses representability by Eilenberg–Mac Lane spaces [35, Theorem 4.57]. Uniqueness was proved above, and composition respects both actions, giving the claimed group isomorphism. The target is simply connected, so forgetting basepoints does not identify any further classes. The principal-polarization assertion follows from Proposition 7.30. □

Corollary 7.33. *For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$, put $F_M(x,y) = (ax+by, cx+dy)$. There is a unique homotopy class of self-equivalences acting by F_M on π_2 and by $\det M$ on π_3. These classes give an injective homomorphism*

$$\mathrm{GL}_2(\mathbb{Z}) \longrightarrow \pi_0 \operatorname{haut}(P_E).$$

Its $\mathrm{SL}_2(\mathbb{Z})$ subgroup fixes π_3.

Proof. Alternation and commutativity in even cohomological degree give

$$E(au+bv,cu+dv)=(ad-bc)E(u,v).$$

Thus $F_M^{*}q=(\det M)q$. The proposition gives existence and uniqueness, which makes multiplication of matrices agree with composition of homotopy classes. Since $\Lambda\neq 0$, the action of matrices on $\Lambda\oplus\Lambda$ is faithful. $\square$

In particular, for the conventions

$$S=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\qquad T=\begin{pmatrix}1&0\\1&1\end{pmatrix},$$

the classes satisfy $[S]^4=1$ and $([S][T])^3=[S]^2$. This is a statement about components of the self-equivalence space; it does not by itself construct a homotopy coherent group action. The algebraic coherent action, when used, comes from the separate line-bundle constructions of Proposition 7.28.

Proposition 7.34. *Each line-corrected monoidal equivalence of Proposition 7.26 has a degreewise finite simplicial roof representing the class of Corollary 7.33. For any prescribed finite collection of such equivalences, the finite target model Y_X can be chosen in common for all the roofs.*

Proof. More generally, suppose a given compact semialgebraic monoidal equivalence has the form

$$(g,R)\longmapsto (F_Mg,R^{\otimes\epsilon}\otimes\mathcal{N}_g),\qquad \epsilon=\det M\in\{1,-1\}.$$

For negative exponent this is the covariant dual-inverse functor on line isomorphisms, so it sends a scalar phase z to z^{-1}. The tensorator has direction

$$\mathcal{N}_g\otimes\mathcal{N}_h\otimes\mathcal{C}(F_Mg,F_Mh)\longrightarrow\mathcal{C}(g,h)^{\otimes\epsilon}\otimes\mathcal{N}_{g+h}.$$

Adjoin phase frames $u_{ij}:\mathcal{N}_{g_{ij}}\to\mathbb{C}$ on every edge of the compact triangle nerve. Forgetting frames defines p_M, while applying the equivalence and conjugating by the frames defines q_M. In the notation of Proposition 7.25, the two principal actions of $U_n=C^1(\Delta^n;U(1))$ are now

$$v\cdot_p(g,\phi,u)=(g,\phi,vu),\qquad v\cdot_q(g,\phi,u)=(g,(\delta v)^{\epsilon}\phi,vu).$$

Indeed the target triangle depends on the source phase as ϕ^{ϵ}, while changing its edge frames contributes $(\delta v)^{-1}$. Their product is unchanged since $\epsilon^2=1$. The invertibility of F_M and of the phase map $z\mapsto z^{\epsilon}$ makes both actions

principal. The contractible realization of $U_\bullet$ proves that both maps realize to weak equivalences. The edge-frame comparison identifies the resulting roof with the given monoidal functor.

For a finite list of matrices, triangulate X_n compatibly with its faces, then each frame space Z_n^M compatibly with its own faces and the pullbacks of the chosen X_n cells under both p_M and q_M. All required families at each step are finite. The natural incidence comparisons and free-degeneracy diagonal construction give finite roofs

$$Y_X \xleftarrow{\widehat{p}_M} Y_{Z^M} \xrightarrow{\widehat{q}_M} Y_X$$

representing the desired classes. This is exactly the simultaneous carrier argument of Proposition 7.25. □

The abstract orthogonal-similitude description does not assert that every one of its elements comes from an algebraic monoidal symmetry. The finite roofs above apply to the supplied line-corrected functors. The next subsections sharpen both aspects of their scope: the shear cannot be a simplicial self-map of any degreewise finite model, but additional frame spaces for all matrix words supply coherent comparison diagrams in the localization.

7.19 An obstruction to strict simplicial operations

The use of finite roofs has an elementary necessity: an infinite-order homology automorphism cannot be induced by a simplicial self-map of any degreewise finite simplicial model. Write $H_n(Y)$ for $H_n(|Y|;\mathbb{Z})$.

Theorem 7.35. *Let Y be a nonempty simplicial set, with $m = |Y_n| < \infty$, and let $f : Y \to Y$ be a simplicial self-map. There are integers $a \geq 0$ and $b \geq 1$ such that*

$$(f_*)^{a+b} = (f_*)^a \quad \text{on } H_n(Y).$$

One may take $a \leq m - 1$ and take b to be the least common multiple of the cycle lengths of the finite function $f_n : Y_n \to Y_n$. In particular, b divides $\operatorname{lcm}(1, \dots, m)$. If f_* is an automorphism, then $(f_*)^b = \mathrm{id}$. If f_n itself is bijective, one may take $a = 0$.

More generally, for fixed simplicial sets Y, Z with Y_n, Z_n finite, the set of homomorphisms $H_n(Y) \to H_n(Z)$ induced by simplicial maps has cardinality at most $|Z_n|^{|Y_n|}$. Every strict group action on Y therefore has finite image on $H_n(Y)$; that image is a quotient of a subgroup of $\operatorname{Sym}(Y_n)$.

Proof. Every component of the directed graph of a function on a finite set consists of a cycle with trees feeding into it. Let a be the greatest distance of a point of Y_n from its eventual cycle, and let b be the least common multiple of the cycle lengths. Then $a \leq m-1$ and

$$f_n^{a+b} = f_n^a.$$

The normalized simplicial chain group is

$$C_n^N(Y;\mathbb{Z}) = \mathbb{Z}[Y_n]/D_n(Y),$$

where $D_n(Y)$ is generated by the degenerate simplices. A simplicial map preserves degeneracies, and its chain map in degree n is induced by the linearization of its function on Y_n. Thus the same power identity holds on $C_n^N(Y;\mathbb{Z})$, and hence on cycles modulo boundaries. This proves the assertion on $H_n(Y)$. When f_* is invertible, cancel $(f_*)^a$. The bounds and the bijective case follow from the same finite graph.

For maps $Y \to Z$, the induced degree-n chain map, and therefore the induced map on H_n, is determined by the function $Y_n \to Z_n$. There are at most $|Z_n|^{|Y_n|}$ such functions. Finally, a strict group action acts on Y_n by permutations. Elements acting trivially there act trivially on $H_n(Y)$, giving the asserted finite quotient. □

Remark 7.36 (Finite restrictions, rather than finite realizations). The argument requires finiteness only in the relevant simplicial degree. It imposes no bound on the number of nondegenerate simplices in higher degrees. The same powers also agree on the simplicial n-skeleton: for $j \leq n$, an iterated degeneracy $Y_j \to Y_n$ has a face retraction, and naturality transfers equality in degree n to equality in degree j. For a degreewise finite Y every fixed skeleton is therefore eventually periodic under iteration, although there need not be one period valid in all degrees simultaneously.

Corollary 7.37. *No simplicial self-map of any degreewise finite simplicial model of P_E, with $g \geq 1$, represents the shear*

$$T(x,y) = (x, x+y)$$

on $\pi_2(P_E) = \Lambda \oplus \Lambda$. In particular, no strict simplicial $\mathrm{SL}_2(\mathbb{Z})$-action on such a model can induce the standard slot action on π_2.

Proof. Since P_E is simply connected, Hurewicz identifies $H_2(P_E;\mathbb{Z})$ with $\pi_2(P_E) = \Lambda \oplus \Lambda$. The shear is an automorphism, and

$$T^k(x,y) = (x, kx+y).$$

As Λ is nonzero and torsion-free, this automorphism has infinite order. Its conjugate under any identification of a model's homology with $H_2(P_E)$ still has infinite order, contradicting Theorem 7.35. □

Corollary 7.38. *Let $r \geq 1$ and let Y be a degreewise finite simplicial model of $K(\mathbb{Z}, r)$. Multiplication by an integer k with $|k| > 1$ cannot be represented by a simplicial self-map of Y. There is consequently no simplicial map $\mu : Y \times Y \to Y$ whose realization represents addition on $K(\mathbb{Z}, r)$. No associativity or unit condition on μ is needed for this obstruction.*

Proof. Here $H_r(Y) \cong \mathbb{Z}$. Eventual periodicity of multiplication by k would give $k^{a+b} = k^a$, which is impossible for $|k| > 1$ and $b \geq 1$. If μ represented addition, its composite with the simplicial diagonal $Y \to Y \times Y$ would represent multiplication by 2, giving the same contradiction. □

These obstructions concern single simplicial maps on a fixed finite model. They do not apply to continuous maps of its realization or to roofs through finite refinements, and hence are consistent with Proposition 7.34 and the earlier refined addition operations. They also do not exclude the order-four Fourier class S: the obstruction detects infinite-order homology actions, rather than every nontrivial symmetry. Theorem 7.42 constructs strict actions of all finite matrix subgroups on suitable finite replacements, including the original Fourier C_4-action.

7.20 Coherent composition through finite frame spans

Individual roofs can be organized together with their composition comparisons. The useful finite object is now a diagram: its objects are degreewise finite simplicial sets, while a product of matrices is represented through a further common refinement. This construction does not turn a matrix into a simplicial self-map of the fixed base.

Use the normalized action of Proposition 7.28, and abbreviate its correction line $\widetilde{\mathcal{N}}_M$ to $\mathcal{K}_M$. Its coherent composition isomorphism has the form

$$\theta_{M,N}(g) : \mathcal{K}_N(g)^{\otimes \varepsilon_M} \otimes \mathcal{K}_M(Ng) \longrightarrow \mathcal{K}_{MN}(g), \qquad \varepsilon_M = \det M. \tag{26}$$

Let Z_M denote the corresponding frame span of Proposition 7.34, before replacement by finite simplicial sets:

$$X_\bullet \xleftarrow{p_M} Z_{M,\bullet} \xrightarrow{q_M} X_\bullet.$$

Proposition 7.39. *There is a simplicial compact semialgebraic space*

$$W_{M,N} = Z_N \times_{q_N, X, p_M} Z_M$$

and a simplicial semialgebraic map $c_{M,N} : W_{M,N} \to Z_{MN}$ such that

$$p_{MN} c_{M,N} = p_N \operatorname{pr}_1, \qquad q_{MN} c_{M,N} = q_M \operatorname{pr}_2 .$$

Both projections from $W_{M,N}$ and the contraction $c_{M,N}$ are weak equivalences after realization. Contractions of adjacent steps are associative, as equalities of maps on the corresponding three-step frame spaces.

Proof. The fibre product parametrizes two successive framed transports. On an edge with input label g, write their frames as

$$u_N : \mathcal{K}_N(g) \longrightarrow \mathbb{C}, \qquad u_M : \mathcal{K}_M(Ng) \longrightarrow \mathbb{C}.$$

Define the contracted frame by

$$u_{MN} = (u_N^{\otimes \varepsilon_M} \otimes u_M)\theta_{M,N}(g)^{-1}. \tag{27}$$

For negative exponent, the frame uses the covariant inverse dual. Monoidality of $\theta_{M,N}$ says that the transported output triangle is exactly the same for the two-step and contracted descriptions. This proves the two endpoint equalities. Restriction of edge data proves compatibility with faces; the unit normalizations prove compatibility with repeated vertices. Coherence of θ makes the two contractions of a three-step frame tuple equal, including their edge frames and triangles.

In simplicial degree n, write $U_n = C^1(\Delta^n; U(1))$. Both projections are pullbacks of the principal U_n-bundles p_M or q_N. The contraction is also a locally trivial principal U_n-bundle. Indeed, after fixing the input simplex and contracted frames, choose the frames u_N freely; formula (27) determines u_M uniquely. Its principal action is

$$(u_N, u_M) \longmapsto (v u_N, v^{-\varepsilon_M} u_M), \qquad v \in U_n,$$

with the intermediate simplex recomputed by the first transport. The final output is unchanged. This description is compatible with all simplicial operators. Local trivializations of the edge phase bundles give local sections for each principal map.

All spaces and maps are compact semialgebraic in each degree. Degeneracies have closed semialgebraic images and are cofibrations. The realization of $U_\bullet$ is contractible, by its extra degeneracy which prepends a vertex and assigns value 1 to its new edges. The principal-map argument in Proposition 7.25 applies unchanged and proves the three weak-equivalence assertions. □

All words and their faces. For a word $w = (M_1, \dots, M_r)$ in $\mathrm{GL}_2(\mathbb{Z})$, listed in the order of application, put

$$W_w = Z_{M_1} \times_X Z_{M_2} \times_X \cdots \times_X Z_{M_r}, \qquad W_\varnothing = X.$$

The product represented by w is $M_r \cdots M_1$. The outer word faces delete the first or last step. An inner word face replaces the adjacent steps M_i, M_{i+1} by $M_{i+1}M_i$, using (27). For a one-letter word, the two faces are q_M and p_M. These maps satisfy the semisimplicial identities strictly. For the inner identities this is the coherence just proved; the identities involving an outer face follow by forgetting the same initial or final transports. Word faces also commute with the original simplicial operators.

Every word face realizes to a weak equivalence. Deletion is a pullback of a principal map p_M or q_M, and contraction is the same principal frame calculation with the other steps retained. Each $W_{w,n}$ can also be described by an input X_n-simplex and independent edge frames at each stage. Thus it is compact semialgebraic, and its degeneracies are cofibrations. In particular, arbitrary finite compositions and all their associativity comparisons remain within this class.

Theorem 7.40. *There are simplicial sets Y_w, finite in every simplicial degree, for all finite matrix words w at once, with a single base $Y_\varnothing = Y_X$, and simplicial word-face maps satisfying the same strict semisimplicial identities. After realization this entire word diagram is naturally equivalent, through a span of objectwise weak equivalences, to $w \mapsto |W_w|$. Every word-face map of the finite diagram is a weak equivalence.*

Proof. Forget the original degeneracies temporarily. Index the spaces $W_{w,n}$ by (w, n) and give such an index rank $|w| + n$. Each generating simplicial face or word face lowers this rank, and each source has only finitely many outgoing generating faces. Inductively triangulate $W_{w,n}$ compatibly with the preimages of the open cells already chosen at the targets of all these faces.

There are countably many words, but the compatible family of semialgebraic subsets for each individual source is finite. Sources of the same rank can be treated independently. Consequently this induction chooses all the triangulations with the same previously chosen triangulations for X_n; it never asks for an infinite family of compatibility conditions on one source.

Let $P_{w,n}$ be the finite poset of nonempty open cells. Every generating map has its unique open-cell carrier, which is order-preserving by continuity and the closure relations of a triangulation. Uniqueness makes all actual

identities between composites hold for carriers. We therefore obtain finite-poset diagrams with the original simplicial faces and all word faces, and with their interchange identities.

Use the incidence categories from Theorem 7.15. Their comparison spans are natural for every displayed map f, which induces the functor

$$(F, z) \longmapsto (f^P F, f(z)).$$

This functor commutes with evaluation and with forgetting the point. Freely adjoin degeneracies in the original simplicial direction and take the diagonal. Explicitly,

$$(Y_w)_n = \coprod_{[n]\twoheadrightarrow[j]} N_n P_{w,j}.$$

This is finite for each w, n, and the word-face maps remain strictly compatible with one another. The natural incidence comparisons and the proper fat–thin comparisons identify their realizations with the full W-diagram. Since each word face in that diagram is a weak equivalence, the same is true of the corresponding map between the Y_w. □

In particular, the two-letter part has the strictly commuting comparison maps

$$Y_{(N)} \longleftarrow Y_{(N,M)} \longrightarrow Y_{(M)}, \qquad Y_{(N,M)} \longrightarrow Y_{(MN)}.$$

Together with the endpoint maps to Y_X, these compare the composite of the N and M roofs with the MN roof. The middle object need not be their ordinary simplicial fibre product. It is a model of their *homotopy* fibre product: the comparison square commutes and its maps are weak equivalences, so its map to the homotopy pullback is a weak equivalence. Three-letter and longer words supply all associativity diagrams, already commuting at the level of these finite simplicial sets.

Corollary 7.41. *Let $\mathcal{L}_{\mathrm{fin}}$ be the infinity-category obtained from degreewise finite simplicial sets by inverting weak equivalences, using simplicial localization [21], and let $\mathcal{L}_{\mathrm{fin}}^{\simeq}$ be its maximal infinity-groupoid. The finite word diagram defines a based map*

$$B\,\mathrm{GL}_2(\mathbb{Z}) \longrightarrow \mathcal{L}_{\mathrm{fin}}^{\simeq},$$

based at Y_X. This is a homotopy coherent matrix action on Y_X in that localization. Realization carries it to the action represented by the original coherent framed transports. For a matrix M its underlying equivalence is the roof $Y_X \xleftarrow{p_M} Y_{(M)} \xrightarrow{q_M} Y_X$.

Proof. Put $G = \mathrm{GL}_2(\mathbb{Z})$ and regard its nerve NG as a semisimplicial set by forgetting degeneracies. Use the inverse category of semisimplices

$$\mathcal{C} = (\Delta_{\mathrm{inj}}/NG)^{\mathrm{op}}.$$

Its objects are matrix words. A morphism to a face remembers the particular order-preserving injection selecting that face. It is essential to retain these operators: a one-letter word has two distinct morphisms to the empty word, even when its letter is the identity matrix. Thus $\mathcal{C}$ is not the poset of the words or of their possible face values.

Subdivision of the semisimplices identifies the classifying space of their category with the fat realization of NG, up to homotopy: a chain of injective face operators gives the flag of the corresponding faces in its largest simplex. Retaining the face operators makes these precisely the subdivision simplices, including when face values coincide. Passing to the opposite category reverses the order in each nerve simplex and does not change its realization. The discrete nerve is proper, so fat–thin comparison gives a based equivalence

$$B\mathcal{C} \simeq \|NG\| \simeq BG.$$

The empty word corresponds to the unique vertex of BG.

Let $\mathcal{W}_{\mathrm{fin}}$ be the category whose objects are degreewise finite simplicial sets and whose arrows are weak equivalences. Theorem 7.40 gives a strict functor $\mathcal{C} \to \mathcal{W}_{\mathrm{fin}}$. The localization functor sends every arrow of $\mathcal{W}_{\mathrm{fin}}$ to an equivalence, and hence induces a map of its nerve into $\mathcal{L}^{\simeq}_{\mathrm{fin}}$. Composing these maps with the displayed based equivalence gives the assertion. This argument uses no claim that localization of the finite objects is fully faithful in the infinity-category of all spaces.

For a one-letter word the loop starts at the empty word, traverses the p_M edge backwards, and returns along q_M. Its image is therefore $q_M p_M^{-1}$, with the required orientation. The two-letter faces supply the comparison between successive transports by N, M and transport by MN; all higher word faces supply their coherences. The natural incidence comparison is defined for this whole category. After realization it identifies the action with the original frame diagram, whose contractions are the composition isomorphisms of the specified normalized action. □

Units and the size of the whole diagram. Before carrier replacement, insertion of an identity matrix with the unit edge frames gives word degeneracies. The W-diagram is the nerve, in the word direction, of the groupoid of framed

transports. In particular it retains the unit data of the given normalized action. The finite replacement above preserves word faces; it does not require preservation of these word degeneracies. The category $\mathcal{C}$ retains all words involving identity matrices, and its classifying space is BG. Thus the localized action of Corollary 7.41 includes the unit constraints and their coherences. The finite word diagram itself is not asserted to be the strict unital nerve of a groupoid.

More explicitly, the degree-r word space is the disjoint union over $(M_1, \dots, M_r)$ of $|Y_{(M_1,\dots,M_r)}|$. The natural incidence spans give an objectwise weak equivalence with the corresponding unions of $|W_w|$, and hence a weak equivalence after fat realization. These comparisons respect the map to the word nerve of the discrete matrix group. This is a realization comparison of the full coherent transport diagrams, rather than merely a list of equalities in the homotopy category.

The distinction about finiteness is essential: every individual Y_w is degreewise finite, and every prescribed finite collection of composition diagrams uses finitely many such objects. The disjoint union over all matrix words is infinite; no degreewise finite model of that entire disjoint union is asserted. Moreover, for $g \geq 1$, Corollary 7.37 rules out a strict simplicial action on any degreewise finite model inducing this matrix action: even the single shear cannot be represented by a simplicial self-map. The coherent action constructed here therefore requires equivalences represented through roofs and their refinements.

7.21 Strict actions of finite matrix subgroups

The obstruction to a strict shear does not prevent strict finite group actions after changing the finite model. The coherent word diagram gives such models without equivariant triangulation. It gives both a free model and, after a pointed modification, a model with one vertex fixed by the whole finite group.

Theorem 7.42. *Let H be a finite subgroup of $\mathrm{GL}_2(\mathbb{Z})$. Choose the restricted normalized action, or any specified coherent family of line-corrected monoidal equivalences of $\mathcal{H}_L$ covering H, with the same normalized unit data. Its coherent action on P_E has a strict free simplicial H-model V_H, finite in every degree, with $|V_H| \simeq P_E$. It also has a strict pointed simplicial H-model V_H^*, finite in every degree and with exactly one vertex, again with $|V_H^*| \simeq P_E$. These strict actions recover the original action as an object of $\mathrm{Fun}(BH, \mathrm{Spaces})$, the infinity-category of homotopy coherent H-spaces. The model V_H^* also recovers its coherent*

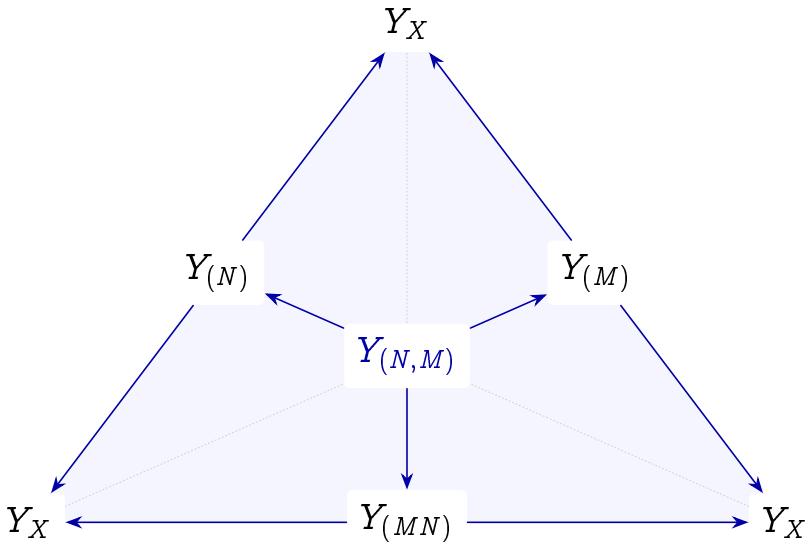


Figure 9: A finite word–face comparison for applying N and then M, whose product is MN. The labels lie at the barycentres of the triangle's faces, and every displayed arrow is a weak equivalence from a word model to a face model. The three occurrences of Y_X are retained separately: repeating the same object does not identify its distinct face maps. The triangle encodes the comparison between the composite roofs for N and M and the roof for MN. The faint dotted spokes only complete the geometric subdivision.

pointing.

Proof. Complete the finite word diagram. Restrict the diagram of Theorem 7.40 to words in H. For another coherent family, apply the identical frame-word construction using its composition isomorphisms. Use its canonical unit basepoints. At the triangulation stage, one may require compatibility also with the singleton consisting of zero edge labels, unit triangle phases, and unit frames. This is one further semialgebraic condition at each source. The carrier and incidence comparisons then preserve the unit points. The formal degeneracies of the degree-zero point give compatible simplicial basepoints in every Y_w. In particular $(Y_\varnothing)_0$ consists of one vertex. More explicitly, the basepoint of $(Y_w)_n$ lies in the $[n] \twoheadrightarrow [0]$ summand and is the constant nerve simplex of the unique degree-zero cell. The incidence comparisons and fat–thin maps send it to the degenerate unit simplex of W_w; thus the comparison of the entire word diagram is pointed.

Freely add degeneracies in the word direction. The resulting bisimplicial set is

$$\mathbb{Q}^H_{p,q} = \coprod_{[q]\twoheadrightarrow[r]} \coprod_{w\in H^r} (Y_w)_p. \tag{28}$$

Let $Q_H = \operatorname{diag} \mathbb{Q}^H$. The freely completed discrete word base is

$$(B^+_H)_q = \coprod_{[q]\twoheadrightarrow[r]} H^r.$$

Forgetting the Y_w coordinate defines $Q_H \to B^+_H$. There is a simplicial map $B^+_H \to NH$ sending a formal degeneracy of a word to the same word with identity letters inserted. Their composite will be denoted $b : Q_H \to NH$. Here NH is the usual nerve, with consecutive letters composed in the order of application. Since H is finite,

$$|(Q_H)_n| = \sum_{r=0}^{n} \binom{n}{r} \sum_{w\in H^r} |(Y_w)_n| < \infty, \qquad |(B^+_H)_n| = (1+|H|)^n. \tag{29}$$

Identify the homotopy fibre. After realizing the original simplicial direction, the map to the discrete word base is equifibered: over every map of word labels the map of fibres is a word-face weak equivalence, or an identity introduced by a formal degeneracy. Rezk's descent proposition therefore identifies its degree-zero fibre with the homotopy-colimit map [65, Proposition 2.3(2)]. The free word degeneracies are inclusions of whole components, so ordinary realization computes these homotopy colimits. The diagonal comparison and $|B^+_H| \simeq |NH| = BH$ give

$$|Y_\varnothing| \simeq \operatorname{hofib}(|b|). \tag{30}$$

Equivalently, subdivision by flags of faces identifies this construction with the homotopy colimit of the inverse word-face category used in Corollary 7.41. Thus the map to BH retains that coherent action, including its higher comparison maps.

Pull back the universal cover. Use the homogeneous model

$$(EH)_n = H^{n+1}, \qquad (h_0, \dots, h_n) \longmapsto (h_1 h_0^{-1}, \dots, h_n h_{n-1}^{-1}) \in (NH)_n,$$

with faces deleting and degeneracies repeating a coordinate. It is contractible, and $EH \to NH$ is a simplicial covering and a Kan fibration. Each fibre has exactly $|H|$ elements: the first vertex determines the remaining vertices uniquely. Form the ordinary simplicial pullback

$$V_H = Q_H \times_{NH} EH. \tag{31}$$

It has the strict free left action

$$a \cdot (z; h_0, \dots, h_n) = (z; h_0 a^{-1}, \dots, h_n a^{-1}).$$

The exact cardinality is

$$|(V_H)_n| = |H|\,|(Q_H)_n|. \tag{32}$$

This ordinary pullback computes the homotopy pullback, although Q_H need not be Kan. Indeed, factor $Q_H \to NH$ as a weak equivalence followed by a Kan fibration. Pulling the weak equivalence back along the fibration $EH \to NH$ preserves it by right properness of the Kan model structure; the resulting fibrant diagram computes the homotopy pullback. This factorization is used only to prove the claim and does not replace any finite set in (31). Since EH is contractible, (30) implies $|V_H| \simeq |Y_\varnothing| \simeq P_E$. The quotient is exactly $V_H/H = Q_H$, and the homotopy-orbit projection of this free action is equivalent over BH to $|Q_H| \to BH$.

Here the inverse in the displayed left action fixes the transport convention. To check it explicitly, let $i_h : Y_\varnothing \to V_H$ be the inclusion over the constant vertex h of EH; i_e is a weak equivalence by the homotopy-fibre comparison. The one-letter word gives a simplicial prism $Y_{(a)} \times \Delta[1] \to Q_H$ whose endpoints are p_a and q_a. Lift its edge along $(a^{-1}, e) \in (EH)_1$, which maps to $a \in (NH)_1$. The lifted prism gives

$$i_{a^{-1}} p_a \simeq i_e q_a.$$

Since the strict left action of a sends i_e to $i_{a^{-1}}$, its action under the marking i_e is $q_a p_a^{-1}$. Longer words give the same verification for all coherences.

Make the pointing strict. The compatible basepoints of the Y_w give a simplicial monomorphism $s : B_H^+ \to Q_H$, a section of $Q_H \to B_H^+$. It is important that the section is defined on B_H^+; no section $NH \to Q_H$ is required. Its pullback

$$E_H^+ = B_H^+ \times_{NH} EH \hookrightarrow V_H$$

is H-invariant. The map $E_H^+ \to EH$ is a weak equivalence, since $B_H^+ \to NH$ is a weak equivalence and $EH \to NH$ is a Kan fibration. Thus E_H^+ is contractible after realization. Its inclusion is a simplicial monomorphism, hence a cofibration. Consequently the quotient

$$V_H^* = V_H / E_H^+ \tag{33}$$

has a strict H-fixed vertex and the collapse $V_H \to V_H^*$ is an underlying weak equivalence. After realization this is the collapse of a contractible cofibrant subspace [35, Proposition 0.17]. Both $(V_H)_0$ and $(E_H^+)_0$ are the set H, so this is the only vertex. All degrees remain finite; more precisely,

$$|(V_H^*)_n| = |H|\,|(Q_H)_n| - |H|(1 + |H|)^n + 1.$$

To check the pointing, retain the equivariant square with upper arrow $E_H^+ \to V_H$ and lower arrow $* \to V_H^*$. The vertical maps are underlying weak equivalences. In the infinity-category of homotopy coherent H-spaces the map $E_H^+ \to *$ is therefore an equivalence, so the upper arrow is the original coherent basepoint, supplied by the unit sections of the word diagram. The square identifies it with the fixed vertex of V_H^*. The composite $Y_\varnothing \xrightarrow{i_e} V_H \to V_H^*$ gives the corresponding based weak equivalence of underlying spaces. □

Corollary 7.43. *The raw Fourier lift of the order-four matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the normalized lift of the order-six matrix $U = ST = \left(\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ each admit a strict simplicial realization on a degreewise finite model of P_E, inducing their usual slot actions on π_2 and the identity on π_3. The model can be chosen free for the cyclic group, or with one fixed vertex. The strict relations are $S^4 = 1$ and $U^6 = 1$ on their respective models.*

Proof. For the raw Fourier action, one can use the whole group of signed permutation matrices, isomorphic to D_8 of order 8. Every such matrix and every product has $\beta(M) = (ac, bd) = 0$. Hence the defect vanishes, as verified explicitly in Section 7.22. The raw lifts form a coherent action for arbitrary L, so the theorem applies to it and to its subgroup $\langle S \rangle \cong C_4$. For $\langle U \rangle \cong C_6$ use the normalized family; the matrix identities $U^3 = S^2 = -I$ verify the indicated orders. □

These finite subgroup models need not coincide. For $g \geq 1$ one degreewise finite model cannot even carry two simplicial self-maps f_S, f_U inducing the standard S, U on H_2 under a common marking. Indeed $(f_S^3 \circ f_U)_* = S^3 U = T$, which would realize the forbidden shear of Corollary 7.37. No strict finite-order relation on f_S or f_U is needed for this contradiction. The theorem preserves the coherent action, its homotopy orbits and homotopy fixed points, and its coherent basepoint. It does not assert equivalences on every subgroup's ordinary fixed-point space; the free and one-vertex models already have different fixed-point sets. Finally, strict order four on this topological model does not remove the derived shift: in the principally polarized case the Fourier functor still has fourth power $[-2g]$ [52, Theorem 2.2].

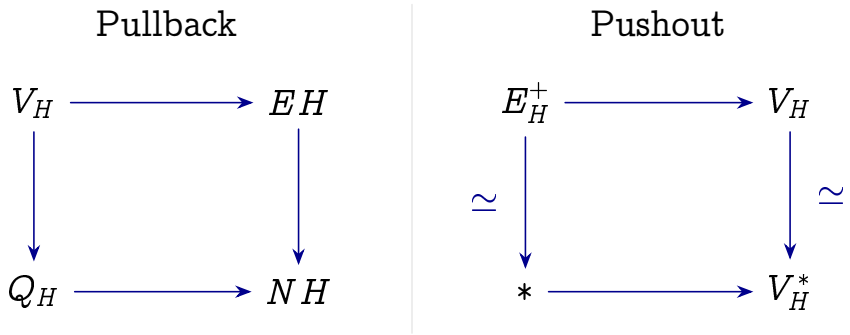


Figure 10: For finite H, pulling back the universal cover $EH \to NH$ gives V_H a free strict H-action. The invariant contractible simplicial subcomplex E_H^+ retains the coherent unit section. Collapsing it gives $V_H^* = V_H / E_H^+$ with a fixed vertex; the marked vertical maps are weak equivalences on underlying realizations. All displayed simplicial sets are finite in each degree. The collapse does not assert preservation of genuine fixed-point spaces.

7.22 Concrete finite matrix subgroups

The standard slot representation gives a precise boundary for strict actions on a degreewise finite simplicial model. The following explicit examples require no classification of the finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$.

Retain the rotation and lower shear, and put

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad U = ST = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Two useful reflections are

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 7.44. *These matrices give finite subgroups*

$$\langle S\rangle \cong C_4, \qquad \langle U\rangle \cong C_6, \qquad \langle S,R\rangle \cong D_8, \qquad \langle U,J\rangle \cong D_{12}.$$

Here D_{2m} denotes the dihedral group of order $2m$, with presentation

$$D_{2m} = \langle r, s \mid r^m = s^2 = 1,\ srs = r^{-1}\rangle.$$

In their Heisenberg actions, every rotation fixes the central band and every reflection acts on it by inversion.

Proof. Direct multiplication gives

$$S^2 = U^3 = -I, \qquad S^4 = U^6 = I, \qquad R^2 = J^2 = I,$$

and

$$RSR = S^{-1}, \qquad JUJ = U^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

The orders of S and U are exactly 4 and 6: $S^2 = -I \neq I$, while $U^3 = -I$ and $U^2 \neq I$. The reflection relations reduce every element of the two larger groups to S^i or S^iR, respectively U^i or U^iJ. All these elements are distinct: the rotation powers are distinct, and a reflected element has determinant -1, whereas a rotation has determinant 1. The orders are therefore 8 and 12. Finally the matrix lifts act on the central band by the determinant, as in Proposition 7.26. In particular the central element $-I$ fixes that band, although it negates both lattice summands. □

Corollary 7.45. *Let E be a nondegenerate integral alternating form on a free lattice Λ of rank $2g$, with $g \geq 1$, and let $H \leq \mathrm{GL}_2(\mathbb{Z})$. There exists a strict simplicial H-action on a degreewise finite model of P_E inducing the standard slot representation on $H_2(P_E;\mathbb{Z}) = \Lambda \oplus \Lambda$ if and only if H is finite.*

Proof. The standard representation $M \mapsto F_M$ on $\Lambda \oplus \Lambda$ is faithful because Λ is nonzero and torsion-free. Theorem 7.35 says that the homology image of any strict action is finite. Thus the isomorphic image of H must be finite. For the converse, realize E by a polarized complex abelian variety. In a symplectic normal basis write $E(e_i, f_j) = d_i\delta_{ij}$ with $d_i > 0$. The complex structure $Je_i = f_i$, $Jf_i = -e_i$ satisfies $E(Ju, Jv) = E(u, v)$ and $E(v, Jv) > 0$ for $v \neq 0$. The Riemann bilinear conditions give the required realization; one can equivalently take a product of elliptic curves with degree-d_i polarizations. This is the construction recalled in Proposition 7.48. The space P_E depends only on the marked form E. For finite H, restrict the normalized coherent action of Proposition 7.28 to H and apply Theorem 7.42. □

There is a useful refinement for the original Fourier lift. Every matrix in $D_8 = \langle S, R\rangle$ is a signed permutation matrix, so $\beta(M) = (ac, bd) = 0$ for every $M \in D_8$. The exact defect identity consequently gives

$$2k(M,N) = \beta(MN) - \varepsilon_M \beta(N) - \beta(M)N = 0 \qquad (M, N \in D_8).$$

Thus the original correction lines $\mathcal{N}_M$ already have trivial composition defect on this subgroup for arbitrary L. Uniqueness of origin-normalized isomorphisms gives their coherence. The finite replacement can therefore retain the earlier Fourier C_4-action with $\mathcal{N}_S = B_L(x,y)^{-1}$, without changing it to the normalized Fourier lift. For the displayed C_6 and D_{12} actions we use the normalized family; if L is symmetric, the original family is coherent on these groups as well.

The four displayed groups are concrete examples, rather than an asserted complete list. The obstruction concerns the faithful standard homology action and does not rule out actions of infinite groups whose image on that homology is finite.

7.23 The same hyperbolic lattice in Orlov's criterion

There is an exact lattice comparison behind the preceding connection with derived categories. Write $\Lambda^\vee = \operatorname{Hom}(\Lambda, \mathbb{Z})$, and equip $\Lambda \oplus \Lambda^\vee$ with the symmetric form

$$Q\big((x,\ell),(x',\ell')\big) = \ell(x') + \ell'(x).$$

This is the unimodular hyperbolic form of signature $(2g, 2g)$.

Proposition 7.46. *For a polarization form E on Λ, define*

$$\iota_E : \Lambda \oplus \Lambda \longrightarrow \Lambda \oplus \Lambda^\vee, \qquad (x,y) \longmapsto (x, \ell_y), \qquad \ell_y(z) = E(z,y).$$

Then the Whitehead form of P_E is

$$I = \iota_E^* Q. \tag{34}$$

If the polarization has type $(d_1, \dots, d_g)$, the image of ι_E has index $(d_1 \cdots d_g)^2$. In particular, a principal polarization gives an integral isometry $(\Lambda \oplus \Lambda, I) \cong (\Lambda \oplus \Lambda^\vee, Q)$.

Proof. The pullback calculation is

$$Q\big(\iota_E(x,y), \iota_E(x',y')\big) = E(x',y) + E(x,y') = I\big((x,y),(x',y')\big).$$

The second component of ι_E is the injective lattice map with matrix E. Its cokernel has order $|\det E| = (d_1 \cdots d_g)^2$. □

To fix the geometric sign, identify $H_1(A^\vee;\mathbb{Z})$ with $\Lambda^\vee$ by the convention

$$c_1(\mathcal{P})\big((z,0),(0,\ell)\big) = \ell(z).$$

The identity $B_L = (1\times\lambda_L)^*\mathcal{P}$ then gives $(\lambda_L)_*(y) = \ell_y$, because the mixed component of $c_1(B_L)$ is $E(z,y)$. Thus ι_E is the homology map of $1\times\lambda_L : A\times A \to A\times A^\vee$. The associated quadratic function also agrees:

$$\tfrac{1}{2}Q\big(\iota_E(x,y),\iota_E(x,y)\big) = E(x,y),$$

which is the quadratic function underlying the class $q = E(u\smile v)$.

Orlov uses this very form Q on $H_1(A\times A^\vee;\mathbb{Z})$. His Proposition 2.21 says that an *isomorphism of complex abelian varieties* $A\times A^\vee \to B\times B^\vee$ is isometric in his sense precisely when its map on first homology preserves Q [58, Remarks 2.2–2.3 and Proposition 2.21]. Derived equivalences produce such isomorphisms by his Theorem 2.19.

The algebraic-isomorphism hypothesis retains the missing period information. An integral lattice isomorphism must intertwine the complex structures on the two product tori before it defines a holomorphic, hence algebraic, group isomorphism. Equivalently it must respect their weight--1 Hodge structures. The identification of underlying lattices $H_1(A^\vee;\mathbb{Z}) \cong \Lambda^\vee$ should not erase the Tate twist: as Hodge structures the dual side is $H_1(A;\mathbb{Z})^\vee(1)$. Therefore (34) is an exact comparison of integral forms, not a principle that every integral Q-isometry is algebraic or arises from a derived equivalence. The Whitehead pairing records the form; Orlov's criterion also remembers how its lattice sits in a complex torus.

7.24 An obstruction to reconstruction from the Betti homotopy type

The loss of complex geometry can be demonstrated by two explicit principally polarized elliptic curves. The conclusion concerns the homotopy type of the model, even with the marked pairing retained.

Proposition 7.47. *There are principally polarized complex elliptic curves (A,L_A) and (B,L_B) with isomorphic marked integral pairings and equivalent Postnikov spaces P_E, but with A and B nonisogenous and*

$$D^b\operatorname{Coh}(A) \not\simeq D^b\operatorname{Coh}(B)$$

as $\mathbb{C}$-linear triangulated categories. Consequently the homotopy type of P_E, or of a degreewise finite simplicial model for it, does not determine either the complex abelian variety up to isomorphism or its derived category up to $\mathbb{C}$-linear exact equivalence.

Proof. Put $\omega = (-1+\sqrt{-3})/2$ and take

$$A = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z}), \qquad B = \mathbb{C}/(\mathbb{Z}+\omega\mathbb{Z}), \qquad L_A = \mathcal{O}_A([0]), \quad L_B = \mathcal{O}_B([0]).$$

These one-dimensional complex tori are elliptic curves, and the displayed degree-one lines give principal polarizations. Choose positive integral bases for their first homology. Both alternating forms then have the same matrix

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad [B_E] = u_1v_2 - u_2v_1.$$

Their Postnikov spaces are therefore homotopy fibres of the same degree-four class on $K(\mathbb{Z}^4, 2)$, and are equivalent over that common base. Corollary 7.19 supplies degreewise finite simplicial models of this homotopy type.

For a lattice $\Pi \subset \mathbb{C}$, the endomorphisms of $\mathbb{C}/\Pi$ are the scalars a satisfying $a\Pi \subseteq \Pi$; see [50, Chapter III, Proposition 3.17 and Example 3.18]. Since both displayed lattices contain 1 and are rings, this gives

$$\operatorname{End}(A) = \mathbb{Z}[i], \qquad \operatorname{End}(B) = \mathbb{Z}[\omega].$$

Thus their rational endomorphism algebras are respectively $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$, which are not isomorphic as $\mathbb{Q}$-algebras. An isogeny becomes invertible after tensoring homomorphisms with $\mathbb{Q}$; conjugating by it would identify the two rational endomorphism algebras. Hence A and B are not isogenous.

By Orlov's theorem [58, Theorem 2.19], an exact derived equivalence would induce an isomorphism of abelian varieties

$$A \times A^{\vee} \cong B \times B^{\vee}.$$

The principal polarizations identify the duals with the original curves, so this would give $A^2 \cong B^2$. Restrict such an isomorphism to the first factor A. At least one of its two projections to B is nonzero. Every nonzero homomorphism of elliptic curves is an isogeny, contradicting the preceding paragraph. □

This is an obstruction to *homotopy-invariant* reconstruction. It does not assert that the two chosen finite simplicial sets are isomorphic, or that arbitrary combinatorial data cannot encode additional information. A chosen presentation may carry markings, algebraic charts, or deliberately retained parameters beyond its homotopy type. The proposition says that recovering the variety or its derived category cannot factor through the Betti homotopy type alone. Retaining the algebraic group stack and its action is a different, richer construction.

7.25 The extra data that recover the abelian variety

The obstruction above identifies information that must be retained outside the bare homotopy type. There is a classical, explicit way to supply it. Keep the original lattice Λ, rather than only the isomorphism type of $\pi_2(P_E)$, and set $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 7.48 (Classical reconstruction data). *Suppose Λ has rank $2g$, $E : \Lambda \times \Lambda \to \mathbb{Z}$ is alternating, and $J : V \to V$ satisfies*

$$J^2 = -1, \qquad E(Ju, Jv) = E(u, v), \qquad E(v, Jv) > 0 \quad (v \neq 0).$$

Then $A = V_J/\Lambda$ is a complex abelian variety, and E determines a polarization of A. To specify an ample line bundle representing that polarization, retain in addition a semicharacter $\chi : \Lambda \to U(1)$ satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\exp\bigl(\pi i E(\lambda, \mu)\bigr). \tag{35}$$

The resulting data determine (A, L) up to isomorphism, with $c_1(L) = E$. A rigidification of L at the origin can be included.

Proof. The complex structure is multiplication by $i = J$ on V_J. The displayed conditions are the Riemann bilinear conditions, so the quotient is projective and its analytic group law is algebraic; see [49, Chapter I, Theorems 2.8–2.9]. The linear-first Hermitian convention there uses the alternating form $-E$. For the line, put

$$H_J(u, v) = E(u, Jv) + iE(u, v).$$

This form is conjugate-linear in its first variable, linear in its second, Hermitian, and positive definite. Define

$$j_\lambda(v) = \chi(\lambda)\exp\left(\pi H_J(\lambda, v) + \frac{\pi}{2} H_J(\lambda, \lambda)\right). \tag{36}$$

The functions j_λ are holomorphic and obey $j_{\lambda+\mu}(v) = j_\lambda(v+\mu)j_\mu(v)$. Indeed, the ratio of the right side to the left side is

$$\begin{aligned} &\frac{\chi(\lambda)\chi(\mu)}{\chi(\lambda+\mu)}\exp\left(\frac{\pi}{2}\bigl(H_J(\lambda,\mu) - H_J(\mu,\lambda)\bigr)\right) \\ &\qquad = \exp(-\pi i E(\lambda,\mu))\exp(\pi i E(\lambda,\mu)) = 1. \end{aligned}$$

Thus the action $\lambda \cdot (v, z) = (v + \lambda, j_\lambda(v) z)$ defines a holomorphic line bundle $L = (V_J \times \mathbb{C})/\Lambda$. Appell–Humbert gives $c_1(L) = E$ and classifies all the choices. The descended metric $e^{-\pi H_J(v,v)}|z|^2$ has positive Chern curvature,

so L is ample. By GAGA it is the analytification of an algebraic line bundle on A. The class of $(0,1)$ specifies a nonzero vector in L_0.

For the sign convention, Polishchuk uses a Hermitian form linear in its first variable and obtains $c_1 = -\operatorname{Im} H$ [61, Theorem 1.3(2) and Corollary 1.4]. Apply that statement to $H(u,v) = H_J(v,u)$, for which $\operatorname{Im} H = -E$. The factors are exactly (36); the semicharacter condition is unchanged because E is integral. □

Here the Hodge structure on $\Lambda = H_1(A;\mathbb{Z})$ has weight -1, with types $(-1,0)$ and $(0,-1)$. The familiar weight-one Hodge structure belongs to $\Lambda^\vee = H^1(A;\mathbb{Z})$. Remembering J is the corresponding period information. The integral form E, once J is fixed, determines $\lambda_L : A \to A^\vee$, but does not select L: two semicharacters satisfying (35) differ by a unitary character. These characters give precisely the $\mathrm{Pic}^0(A)$-torsor of line bundles with first Chern class E.

The scope of this enrichment is concrete. A lattice isomorphism intertwining J, preserving E, and carrying χ to χ' induces an isomorphism of the reconstructed pairs (A,L). Conversely, a group isomorphism of pairs induces these identifications. Thus the added data recover the algebraic input to the Heisenberg extension and its derived action. They are continuous geometric decorations: after choosing a basis, J is a real matrix and χ is specified by finitely many phases. They are not recovered from the finite model's homotopy type. Varying them leaves the formula $q = E(u \smile v)$ unchanged. No construction here equips the degreewise finite Eilenberg–Mac Lane models with an intrinsic Hodge structure, or proves an enriched spectrum reconstruction theorem.

7.26 Projective towers over Jacobians

The elliptic construction extends to every pointed smooth connected projective complex curve (C,p_0) of genus g. Put

$$L = \mathcal{O}_C(p_0), \qquad V_d = H^0(C,L^d), \qquad F_d = \mathbb{P}(V_d) = |dp_0| \quad (d \ge 0),$$

where projectivization parametrizes lines and F_0 is a point. Let s_0 be the canonical section of L with divisor p_0. Multiplication by s_0 gives a linear inclusion

$$i_d : F_d \hookrightarrow F_{d+1}, \qquad D \longmapsto D + p_0.$$

Riemann–Roch gives $F_d \cong \mathbb{CP}^{d-g}$ for $d \ge \max(0, 2g-1)$; in this range i_d is a hyperplane inclusion. Earlier inclusions can repeat a projective space. A compatible basis of the increasing section spaces identifies their weak colimit F_∞ homeomorphically with $\mathbb{CP}^\infty$.

Proposition 7.49. *Divisor addition defines morphisms*

$$\mu_{d,e} : F_d \times F_e \longrightarrow F_{d+e}, \qquad (D, D') \longmapsto D + D',$$

compatible with stabilization in either variable. They induce a continuous, strictly associative and commutative monoid operation on F_∞, with unit F_0. If $h_d = c_1(\mathcal{O}_{F_d}(1))$, then

$$\mu_{d,e}^* h_{d+e} = \mathrm{pr}_1^* h_d + \mathrm{pr}_2^* h_e.$$

Thus this operation represents addition on $K(\mathbb{Z}, 2)$.

Proof. Products of nonzero sections on the integral curve are nonzero. Section multiplication therefore defines the displayed morphism and the canonical line-bundle isomorphism

$$\mu_{d,e}^* \mathcal{O}_{F_{d+e}}(-1) \cong \mathcal{O}_{F_d}(-1) \boxtimes \mathcal{O}_{F_e}(-1).$$

Dualization proves the Chern-class formula. Multiplying either input by s_0 multiplies its output by s_0, proving compatibility. Associativity, commutativity and the unit hold on divisors exactly. Continuity uses the compactly generated weak colimit topology. The primitive class h characterizes addition on $K(\mathbb{Z}, 2)$. □

There is also an exact norm-line interpretation. On $\mathrm{Sym}^d C$, the permutation-equivariant line $\bigotimes_i \mathrm{pr}_i^* L$ on C^d descends to the incidence line $N_d = \mathcal{O}(X_{p_0})$, where X_{p_0} is the divisor of effective divisors containing p_0. For $d \geq 1$,

$$N_d|_{F_d} \cong \mathcal{O}_{F_d}(1) \otimes L^d|_{p_0}.$$

Indeed the universal divisor on $C \times F_d$ has line bundle $\mathrm{pr}_C^* L^d \otimes \mathrm{pr}_{F_d}^* \mathcal{O}_{F_d}(1)$; restrict it to $\{p_0\} \times F_d$. A choice of a nonzero vector in $L|_{p_0}$ removes the constant factors compatibly in all degrees. This line identity holds even when p_0 is a base point of $|dp_0|$; its incidence section then vanishes identically. The incidence locus is guaranteed to be a hyperplane in F_d for $d \geq \max(1, 2g)$.

The ambient geometry is the Abel–Jacobi map

$$a_d : \mathrm{Sym}^d C \longrightarrow J(C), \qquad D \longmapsto [\mathcal{O}_C(D - dp_0)], \qquad a_d^{-1}(0) = F_d.$$

Its fibres are complete linear systems [48, §5, especially Remark 5.6(c)]. For $d \geq \max(0, 2g - 1)$ it is a projective bundle: a normalized Poincaré line $\mathcal{P}$ on $C \times J(C)$ gives the vector bundle $\mathcal{E}_d = (\mathrm{pr}_J)_*(\mathcal{P} \otimes \mathrm{pr}_C^* L^d)$ of rank $d + 1 - g$, whose projectivization is $\mathrm{Sym}^d C$. The assertion follows from $H^1(C, M \otimes L^d) = 0$ for $M \in \mathrm{Pic}^0(C)$ and cohomology and base change. In

this range, stabilization gives exact sequences with constant one-dimensional quotients $\mathcal{E}_{d+1}/\mathcal{E}_d$. Continuous splittings and a trivializing cover for one initial bundle give compatible local trivializations of the whole tail. Thus a_∞ is a topological $\mathbb{CP}^\infty$-bundle, and its fibre F_∞ is also its homotopy fibre. Dold–Thom gives

$$\mathrm{Sym}^\infty(C, p_0) \simeq K(H_1(C;\mathbb{Z}),1) \times K(\mathbb{Z},2) \simeq J(C) \times \mathbb{CP}^\infty$$

[18, Theorems 6.10 and 7.1]. Here a_∞ induces the Abel–Jacobi isomorphism on π_1; the fibre contributes $\pi_2 = \mathbb{Z}$. No chosen product homeomorphism is asserted. For $g \geq 1$, the theta polarization belongs to the Jacobian [48, §6]: its pullback restricts trivially to each F_d, whereas h_d is the relative hyperplane class.

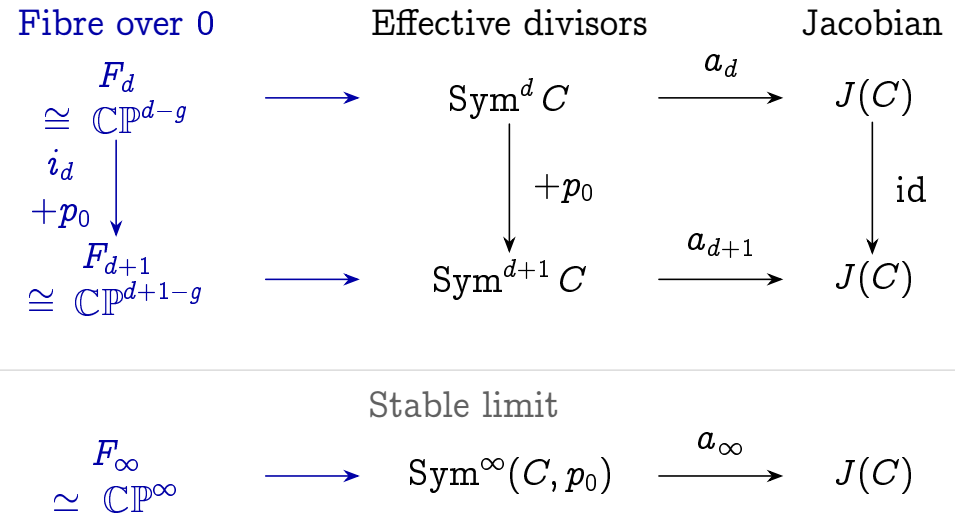


Figure 11: The curve tower for $d \geq \max(0, 2g-1)$. Each left-hand term is the fibre over $0 \in J(C)$. Adding p_0 gives a hyperplane inclusion on the projective fibres and leaves the Abel–Jacobi value unchanged. Both squares commute; the Jacobian remains the base in the stable limit.

Writing $\iota(p) = [\mathcal{O}_C(p - p_0)]$, the ordered fibre is

$$Z_d(C) = \{(p_1, \ldots, p_d) \in C^d : \textstyle\sum_i \iota(p_i) = 0\}, \qquad Z_d(C)/\mathfrak{S}_d = F_d.$$

For $g = 1$ and $p_0 = 0$, this is precisely the AM zero-sum elliptic torus, with $d = q + 1$ and $F_d = \mathbb{CP}^q$. For $g > 1$ the ordered fibre is generally not an abelian variety; the Jacobian remains its ambient parameter space for divisor classes. These results establish a geometric projective tower and its stable addition. They do not supply periodic polyhedra for $Z_d(C)$ or compatible finite flag presentations in higher genus. Such a comparison with the existing toric models requires a new construction.

8 Reordering, reorientation, and a finite symmetric-flag model

Reordering simplex vertices acts on the arrangement by signed changes of face coordinates. The existing walls already admit these changes. The ordinary nerve introduces a second ordering, namely the order of the faces in an incidence flag. Allowing arbitrary enumerations of a flag makes the two actions compatible on the diagonal, without changing its homotopy type or degreewise finiteness.

8.1 All vertex maps and their induced reorientations

Let $\mathcal{F}$ have the finite sets $[q] = \{0, \dots, q\}$ as objects and all set maps as morphisms. For a finite pointed input set S, put

$$V_q(S) = Z^k(\Delta[q]; \mathbb{R})^{S^\circ}, \qquad L_q(S) = Z^k(\Delta[q]; \mathbb{Z})^{S^\circ}.$$

Extend a normalized cochain alternatingly to all tuples of vertices, with value zero on tuples having repeated vertices. Every set map $f : [m] \to [n]$ then has a pullback

$$(f^* x)(i_0, \dots, i_k) = x(f(i_0), \dots, f(i_k)).$$

This is functorial and commutes with coboundary. Indeed the induced map on oriented simplex chains sends an oriented face to the oriented image face, or to zero if vertices coincide, and commutes with the alternating boundary.

If $J = \{j_0 < \cdots < j_k\} \subseteq [m]$ and $f|_J$ is injective, let $\epsilon(f, J)$ be the sign that sorts $(f(j_0), \dots, f(j_k))$. Then

$$\theta^s_J(f^* x) = \epsilon(f, J)\theta^s_{f(J)}(x). \tag{37}$$

The sign is the same for every input s. Hence every target wall pulls back to a source wall or to a constant equation:

$$\sum_{s \in U} \theta^s_J(f^* x) = \epsilon(f, J) \sum_{s \in U} \theta^s_{f(J)}(x).$$

If $f|_J$ is not injective, the left side is zero.

For clarity, write the covector coordinate as

$$X_x(U, J, a) = \operatorname{sgn}\left(\sum_{s \in U} \theta^s_J(x) - a \right), \qquad a \in \mathbb{Z}.$$

Equation (37) gives the exact reorientation rule

$$X_{f^*x}(U,J,a)=\begin{cases}\epsilon(f,J)X_x(U,f(J),\epsilon(f,J)a), & f|_J \text{ injective},\\ \operatorname{sgn}(-a), & f|_J \text{ not injective}.\end{cases} \tag{38}$$

Thus an open source face has one open carrier under f^*. The resulting order-preserving carrier maps

$$\rho_f:\widetilde{\mathcal{P}}_n^{(k)}(S)\longrightarrow\widetilde{\mathcal{P}}_m^{(k)}(S)$$

compose strictly. They are equivariant for $f^*:L_n(S)\to L_m(S)$ and therefore induce functors on the orbit incidence categories. The category-valued construction already extends from Δ^{op} to $\mathcal{F}^{\mathrm{op}}$. It still commutes with pointed-set fibre sums, since those sums commute with alternating vertex pullback.

This is the reorientation induced by reordering simplex vertices. It does not assert invariance under independently reversing selected inputs. For example, when $q=k$ and there are two inputs, reversing t_1 changes the wall $t_1+t_2\in\mathbb{Z}$ into $t_2-t_1\in\mathbb{Z}$, which is not one of the original wall families.

8.2 Why the ordinary diagonal needs a modification

The diagonal is $X_0^{(k)}(S)_n=N_nC_n^{(k)}(S)$. Its face map both restricts the vertex coordinates and deletes an entry of the incidence flag. Applying a vertex permutation only to the arrangement leaves the flag-entry positions fixed, so it need not satisfy the crossed simplicial identities.

Here is an explicit example with one input and $k=1$. In vertex degree two use $x=\theta_{01}$ and $y=\theta_{02}$, and the lifted flag

$$F_0=\{(0,0)\},\qquad F_1=\{0<x<1,\ y=0\},\qquad F_2=\{0<y<x<1\}.$$

Let $\tau=(1\ 2)$; its arrangement action interchanges x,y. Denote by T_τ its action on every entry of the flag without reindexing the entries. The injection identity $\tau\delta_2=\delta_1$ would require $d_2T_\tau=d_1$. But d_2T_τ gives the degenerate edge (v,v), whereas d_1 gives a nonidentity incidence (v,e) in the integer-wall circle category. These remain different after the lattice quotient. Thus the arrangement symmetry alone is not a symmetric structure on the ordinary diagonal.

8.3 Flags with arbitrary index order

Definition 8.1. For a poset P, let

$$\mathrm{Sym}_p(P)=\{(F_0,\dots,F_p):\text{the set of entries is a chain in } P\}.$$

The entries may occur in any order and may repeat. A set map $a : [r] \to [p]$ selects the tuple $(F_{a(0)}, \dots, F_{a(r)})$. This defines a functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Set}$, whose restriction to Δ^{op} is a simplicial set.

Lemma 8.2. *The inclusion $j : N(P) \to \mathrm{Sym}_\bullet(P)$ is a homotopy equivalence after realization, naturally for order-preserving maps of posets. Its geometric inverse and homotopy are also equivariant for every group acting on P by order automorphisms.*

Proof. Represent a point of $|\mathrm{Sym}_\bullet(P)|$ by a tuple $(F_0, \dots, F_p)$ and barycentric coordinates $(a_0, \dots, a_p)$. List its distinct entries in increasing order as $G_0 < \cdots < G_\ell$ and set

$$b_j = \sum_{i : F_i = G_j} a_i.$$

The ordered flag $(G_0, \dots, G_\ell)$ with weights $(b_0, \dots, b_\ell)$ defines a map r to $|N(P)|$, and $r|j|$ is the identity.

A homotopy from the identity to $|j|r$ is represented by the concatenated tuple

$$(F_0, \dots, F_p, G_0, \dots, G_\ell)$$

with weights

$$((1-t)a_0, \dots, (1-t)a_p, tb_0, \dots, tb_\ell), \qquad 0 \le t \le 1.$$

Its entries still form a chain. Removing an entry of weight zero removes a zero-weight entry in the first block and, if necessary, one in the sorted block. Combining adjacent equal entries, which is the degeneracy identification, combines their weights in the first block and leaves the aggregated weights unchanged. Thus the formula respects both face and degeneracy identifications. It is continuous on each realization simplex, hence continuous on the realization. No identification of nonadjacent repeated entries is being assumed.

An order-preserving map sends the sorted list to a weakly ordered list; any equal adjacent entries combine by a degeneracy. Therefore r, j, and the displayed homotopy are natural. Equivariance is the special case of order automorphisms. □

Define a bisimplicial set by

$$B^{\mathrm{sym}}_{p,q}(S) = \mathrm{Sym}_p(\widetilde{\mathcal{P}}^{(k)}_q(S))/L_q(S),$$

where the quotient uses one simultaneous translation of all entries. Both directions in fact admit all finite-set maps: one selects tuple entries, and the other applies the carrier maps ρ_f. These operations commute.

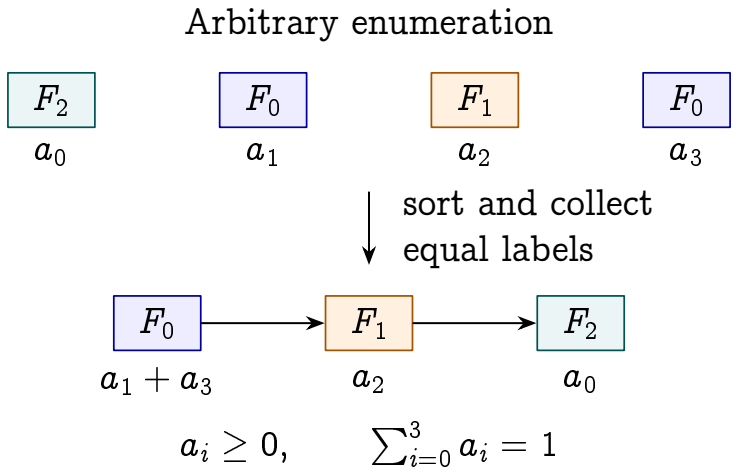


Figure 12: Geometric sorting for $F_0 < F_1 < F_2$. The two occurrences of F_0 contribute their combined barycentric weight $a_1 + a_3$.

Theorem 8.3. *The diagonal*

$$Y^{(k)}(S)_n = B^{\mathrm{sym}}_{n,n}(S)$$

is degreewise finite and extends to a functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Set}$. Explicitly, for $f : [m] \to [n]$,

$$f^*[F_0, \dots, F_n] = [\rho_f F_{f(0)}, \dots, \rho_f F_{f(m)}]. \tag{39}$$

There is a natural simplicial weak equivalence

$$X_0^{(k)}(S) \longrightarrow Y^{(k)}(S).$$

It is compatible with the pointed-set operators. In particular,

$$|Y^{(k)}(S)| \simeq K(\mathbb{Z}^{S^\circ}, k+1).$$

Proof. Every tuple in $\mathrm{Sym}_p(P)$ can be sorted into an ordinary weak flag by a permutation of its $p+1$ positions. Consequently there is a surjection of finite sets

$$\Sigma_{p+1} \times N_p \mathcal{C}^{(k)}_q(S) \longrightarrow B^{\mathrm{sym}}_{p,q}(S),$$

using the unique simultaneous-lift description of the ordinary nerve. This proves finiteness.

Formula (39) is independent of the common lift: translating the input by λ translates the output by $f^*\lambda$. Carrier functoriality and ordinary selection of tuple entries prove its composition identities.

A fixed vertex degree q

$$|N_\bullet \mathcal{C}_q^{(k)}(S)| \xrightarrow[\text{geometric}]{\cong} \mathbb{T}^{|S^\circ|\binom{q}{k}}$$

The diagonal models

$$\begin{array}{ccc} & |X_0^{(k)}(S)| & \\ \simeq \swarrow & & \searrow \chi_c \ \simeq \\ |Y^{(k)}(S)| & & |\mathcal{K}(\mathbb{Z}^{S^\circ}, k+1)| \end{array}$$

Figure 13: Two different comparisons. The solid arrow identifies a fixed polyhedral torus level geometrically; its chosen barycentric homeomorphism need not commute with vertex pullback. The dashed arrows are weak equivalences of realized diagonal models. Both flag models are degreewise finite, while the standard integral cocycle model on the right is not.

Apply Lemma 8.2 to $\widetilde{\mathcal{P}}_q^{(k)}(S)$. The retraction and homotopy are $L_q(S)$-equivariant. Since realization commutes with orbit quotients, they give a homotopy equivalence in every fixed vertex degree q:

$$|N_\bullet \mathcal{C}_q^{(k)}(S)| \longrightarrow |B_{\bullet,q}^{\mathrm{sym}}(S)|.$$

The inclusions form a map of bisimplicial sets, natural under carrier maps. The diagonal weak-equivalence lemma for bisimplicial sets gives the stated weak equivalence; equivalently one can first realize the nerve direction and then the vertex direction. Naturality for pointed-set carriers proves the remaining assertions. □

The symmetric crossed simplicial category $\Delta\mathfrak{S}$ has a forgetful functor to $\mathcal{F}$: it forgets the orders on the fibres of a morphism. Restricting the preceding finite-set functor along it gives an actual $\Delta\mathfrak{S}$-object. Its permutation action reorders the flag indices and simultaneously reorients the face evaluations by (38). This is a strict finite version of the reordering mechanism at the level of our models.

The result is a symmetric simplicial *set*, not a crossed simplicial *group*. It supplies neither a group law on its simplex sets nor a new crossed multiplication. The already constructed resolved addition is retained through the pointed-set operators. Identifying further group-like crossed data remains a separate construction.

8.4 Normalization and the exact correction law

Reordering interacts with the choice of a fundamental domain through an integer correction. Define $\nu_q(x) \in L_q(S)$ by specifying its anchor values:

$$\theta^s_{0,I}(\nu_q(x)) = \lfloor \theta^s_{0,I}(x) \rfloor.$$

The anchor coordinates freely specify an integral cocycle, so this defines ν_q even though the floors of *all* face values need not form a cocycle. Put $\mathsf{n}_q(x) = x - \nu_q(x)$. Its anchor coordinates lie in $[0, 1)$.

Let $T : V_q(S) \to V_r(S')$ be a composite of alternating vertex pullbacks and pointed-set fibre sums. It preserves the lattices and has the carrier maps proved above. Define

$$\kappa_T(x) = \nu_r(Tx) - T\nu_q(x) \in L_r(S'). \tag{40}$$

It is constant on every open source face and invariant under translation by $L_q(S)$: each relevant character has a constant integer part on that face, and an integral translation changes the two terms equally. It thus gives finite correction data on the finite set of orbit faces, for each fixed T.

Proposition 8.4. *For composable allowed maps T and U,*

$$\mathsf{n}_r(Tx) = T\mathsf{n}_q(x) - \kappa_T(x), \tag{41}$$
$$\kappa_{UT}(x) = U\kappa_T(x) + \kappa_U(Tx). \tag{42}$$

Proof. The first identity is the definition. For the second, expand the right side; the two terms $U\nu_r(Tx)$ cancel, leaving $\nu(UTx) - UT\nu_q(x)$. □

For addition in vertex degree $q = k$, one input has one coordinate. The correction is the familiar carry

$$b(x, y) = \lfloor x + y \rfloor - \lfloor x \rfloor - \lfloor y \rfloor.$$

For arbitrary real x, y, z, cancellation of integer parts gives

$$b(x, y) + b(x + y, z) = b(y, z) + b(x, y + z).$$

Together with integer-translation invariance, this is the same identity written using fractional representatives. In general degrees, evaluating (40) on the relevant anchor gives the cochains of Section 21.1.

Rebasing a circular configuration. For $k = 1$ in vertex degree two, set $x = z_{01}$ and $y = z_{02}$, so $z_{12} = y - x$. Interchanging vertices $0, 1$ acts by

$$(x, y) \longmapsto (-x, y - x).$$

On the normalized square the result is $(\{-x\}, \{y - x\})$, where $\{u\} = u - \lfloor u \rfloor$. For $0 < x, y < 1$, the carry vector is $(-1, 0)$ when $y > x$ and $(-1, -1)$ when $y < x$; on $y = x$ its second entry is zero. Thus the existing diagonal wall records the change of normalization. The reordering and correction are explicit, including their wall values.

Relation to the classical crossed mechanism. For a crossed simplicial group, unique factorization moves a group automorphism past an order-preserving vertex map. Its realization comparison also changes simplex coordinates; in a standard convention it has the form

$$|G \times_t K| \longrightarrow |G| \times |K|, \qquad [g, x; t] \longmapsto ([g; t], [x; g_* t]).$$

This is the mechanism in Krasauskas's group-realization argument [40]; see also [26, Section 5]. The sign rule (38), the symmetric flag operators, and (42) make corresponding data explicit for our arrangements. They do not by themselves produce unique group factorization or a product homeomorphism for the finite models. The following obstructions specify what remains beyond these formulas.

8.5 What reordering recovers geometrically

There are two different geometric issues: retaining the affine coordinates of a fixed torus level, and making the resulting construction compatible with the vertex direction before taking the diagonal. The following calculations separate them.

Proposition 8.5. *At vertex degree $q = k$, neither the ordinary carrier Segal map nor its symmetric-flag replacement realizes a homeomorphism. Sorting flag entries does not turn the carrier fold into affine addition under the usual barycentric evaluation.*

Proof. The single-input cover is the integer-wall line. With two inputs the cover has walls $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, and $x + y \in \mathbb{Z}$. Inside the unit square let

$$C_- = \{x > 0,\ y > 0,\ x + y < 1\}, \qquad C_+ = \{x < 1,\ y < 1,\ x + y > 1\}.$$

They are distinct open faces, even modulo the integer lattice. Both have projection carriers (e, e), where $e = (0, 1)$ is the open edge of the single-input arrangement. Thus their distinct nerve vertices have the same image under the carrier Segal map. The same is true in the symmetric nerve. On the diagonal, the constant length-$(k+1)$ flags at C_- and C_+ are distinct k-simplices with the same Segal image. Lemma F.1 therefore also rules out a homeomorphism for either diagonal Segal map.

For the fold, use the barycentres of the closed polytopes. The vertices labelled C_- and C_+ evaluate respectively to $(1/3, 1/3)$ and $(2/3, 2/3)$. Affine addition evaluates them to $2/3$ and $1/3$ modulo one. Their fold carrier is in both cases e, whose barycentre is $1/2$. Sorting a constant flag changes neither its cell label nor its weight, so cannot correct these discrepancies.

More generally, sorting is itself not an injective map on ordinary realizations. If $F < G$, the two nondegenerate symmetric-nerve edges (F, G) and (G, F) have disjoint interiors, but sorting identifies their interior points after reversing the weights. This is a statement about the specified carrier comparison and barycentric evaluation; it excludes no other geometric twist. □

Reorientation does nevertheless give an exact product and addition in this lowest torus level. The construction is particularly small. Let $C = \Delta[1]/\partial\Delta[1]$, with vertex v and nondegenerate edge e. Define a simplicial set P with one vertex, three nondegenerate edges a, b, d, and two nondegenerate 2-simplices satisfying

$$(d_0\alpha, d_1\alpha, d_2\alpha) = (d, b, a), \qquad (d_0\beta, d_1\beta, d_2\beta) = (a, b, d).$$

There are no further nondegenerate simplices. Geometrically these are the two ordered triangles

$$\alpha = [(0,0), (1,0), (0,1)], \qquad \beta = [(1,0), (0,1), (1,1)]$$

in the unit square, with opposite boundary edges identified. Consequently their affine characteristic maps give a homeomorphism

$$H : |P| \xrightarrow{\cong} |C| \times |C| = \mathbb{T}^2.$$

There is an ordinary simplicial map

$$\begin{gathered} m : P \longrightarrow C, \\ m(a) = m(b) = e, \qquad m(d) = s_0 v, \\ m(\alpha) = s_1 e, \qquad m(\beta) = s_0 e. \end{gathered}$$

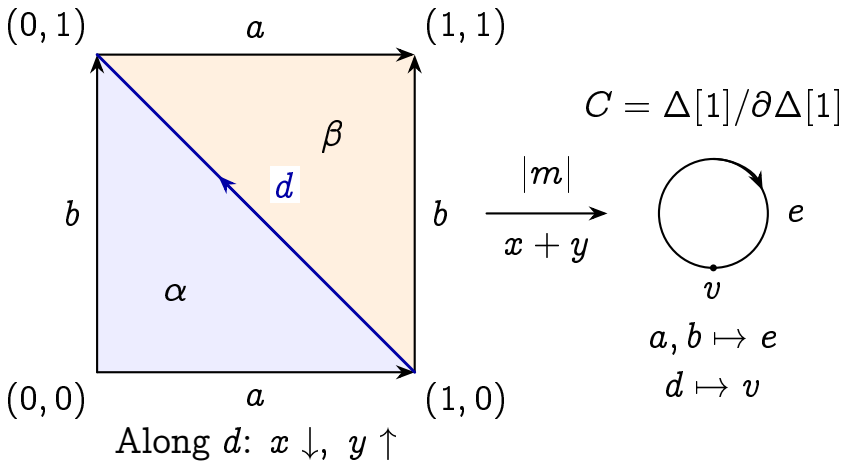


Figure 14: The reordered circle product. Addition is constant on the diagonal edge d; its first coordinate runs backwards. Edges bearing the same label are identified.

The displayed face formulas verify this assertion. In barycentric coordinates $(\lambda_0, \lambda_1, \lambda_2)$, the sum coordinate on α is $\lambda_1 + \lambda_2$ and on β is $1 + \lambda_2$. These are exactly the realizations of $s_1 e$ and $s_0 e$ modulo one. Hence

$$|m| = (+ \bmod \mathbb{Z}) \circ H.$$

Here H is not the realization of an ordinary simplicial map $P \to C \times C$: on the diagonal edge d, its first projection runs from 1 to 0, giving winding number -1, whereas a simplicial image of an edge in C is either e or degenerate, with winding number 1 or 0. The reversed orientation is exactly where this geometric product comparison escapes the ordinary-product obstruction.

This example is a finite presentation of the product and fold at the single torus level $q = k$. It does not provide vertex pullbacks compatible with these triangulations in every q, nor a crossed multiplication on $X_0^{(k)}(1_+)$ or $Y^{(k)}(1_+)$. In particular its circle C is a replacement presentation of the level torus, not the diagonal finite $K(\mathbb{Z}, k+1)$ model.

In every fixed vertex degree there is also an exact polyhedral description without such a compatibility claim for discrete simplex sets. Retain the closed characteristic polytopes and their actual affine maps. The subset-sum arrangement refines the product of the single-input arrangements, so forgetting its extra walls is the identity of the underlying product torus. The fold is the actual linear map on the covers, hence actual torus addition. These maps commute with vertex pullback at the level of the underlying simplicial topological tori. Passing only their carrier labels to nerves discards the point coordinates, as the preceding two-chamber calculation shows. A global crossed construction would have to retain or reconstruct the necessary

coordinate transformations compatibly in the vertex direction.

8.6 Testing the original coordinate-reordering formula

The classical comparison has more content than a permutation action. For a crossed simplicial group G and a simplicial set K, write π_g for the permutation of $[n]$ induced by $g \in G_n$ and put

$$(g^* u)_j = u_{\pi_g^{-1}(j)}, \qquad u \in \Delta^n.$$

Krasauskas defines the twisted product by

$$\begin{gathered} (G \times_t K)_n = G_n \times K_n, \\ d_i(g,x) = (d_i g, d_{\pi_g(i)} x), \qquad s_i(g,x) = (s_i g, s_{\pi_g(i)} x). \end{gathered}$$

Its realization has the natural homeomorphism

$$\begin{gathered} \Phi : |G \times_t K| \xrightarrow{\cong} |G| \times |K|, \\ [(g,x),u] \longmapsto ([g,u],[x,g^* u]). \end{gathered} \tag{43}$$

These are Definition 2.1, equation (14), and Theorem 2.3A of [40, pp. 51–52]. The underlying categorical condition is the unique expression of a morphism as an order-preserving map after a source automorphism [40, Definition 1.1, p. 47]. It gives the crossed relations used by the twisted faces and degeneracies.

To apply this argument to our flags, one would need a unique normal form with these compatibilities, or a replacement that proves the same geometric gluing statement. The symmetric-flag action alone supplies neither. In particular, two distinct chambers can have the same pair of projection carriers. The following test shows that merely assigning permutations to those chambers cannot repair the comparison.

Proposition 8.6. *Let $k \geq 1$. There is no injective map*

$$|X_0^{(k)}(2_+)| \longrightarrow |D_0^{(k)}|^2$$

whose value on each simplex uses its two projection-carrier flags and only permutes their barycentric coordinates. The same assertion holds for the symmetric-flag models. Allowing the induced vertex reorientations in addition to these coordinate permutations does not remove the obstruction below.

Proof. In vertex degree $q = k$, the one-input arrangement is the integer-wall line, and the two-input arrangement has walls $x \in \mathbb{Z}$, $y \in \mathbb{Z}$, and $x + y \in \mathbb{Z}$. Consider its two chambers

$$C_- = \{0 < x, y < 1,\ x + y < 1\}, \qquad C_+ = \{0 < x, y < 1,\ x + y > 1\}.$$

Let a_- and a_+ be the constant flags of length $k+1$ at these chambers. They give distinct nondegenerate diagonal k-simplices: below vertex degree k the cocycle space is zero, so every degeneracy from degree $k-1$ is supported at the zero cell. The two chambers are not lattice translates. Their interior points

$$p_\pm = [a_\pm, u_*], \qquad u_* = \left(\frac{1}{k+1}, \dots, \frac{1}{k+1}\right),$$

are therefore distinct in realization.

Both projections of either flag are the constant flag a_e at the open interval $e = (0, 1)$ modulo $\mathbb{Z}$. Every permutation fixes u_*. Consequently any map of the stated form sends both points to $([a_e, u_*], [a_e, u_*])$. In degree $q = k$ there is only one k-face character, and a vertex permutation changes it by a common sign. Reversing the sign replaces e by $(-1, 0)$, which has the same lattice orbit. Thus induced vertex reorientation also leaves this collision unchanged. The constant flags and the same argument apply in the symmetric nerve. □

This obstruction concerns a prescribed form of comparison, including the unchanged projection-carrier labels. It does not identify a general geometric change of coordinates with a coordinate permutation. To test whether such a broader change can recover the specified carrier fold, one needs a property preserved by arbitrary homeomorphisms. Openness supplies the stronger test below.

8.7 The crossed symmetric action on the original realization

There is a direct application of the classical reordering theorem to the original model, provided that the flag direction is realized first. Put

$$B_{p,q} = N_p \mathcal{C}_q(1_+), \qquad Q_q = |N\mathcal{C}_q(1_+)|, \qquad P = |D_0^{(k)}| \cong |Q_\bullet|.$$

Let Fin denote the category of the nonempty finite vertex sets $[q]$, with all set maps, and let $\Delta\mathfrak{S}$ be the crossed symmetric category, with $\mathfrak{S}_q = \Sigma_{q+1}$. Forgetting the orders on the fibres gives a functor $\Delta\mathfrak{S} \to \mathrm{Fin}$ whose restriction to Δ is the usual inclusion.

Proposition 8.7. *The row object $Q_\bullet$ extends to a functor* $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Top}$. *Consequently the original realization P carries a canonical continuous action of the topological group $|\mathfrak{S}_\bullet|$. This action is nontrivial for every $k \geq 1$, fixes the canonical basepoint, and acts trivially on the canonical cohomology classes.*

Proof. For a set map $\alpha : [m] \to [q]$, use the alternating pullback $\alpha^* : V_q \to V_m$. On an ordered k-face I, its evaluation is zero if α repeats a vertex of I; otherwise it is

$$\theta_I(\alpha^* x) = \epsilon(\alpha, I)\theta_{\alpha(I)}(x), \qquad \epsilon(\alpha, I) \in \{1, -1\},$$

where $\alpha(I)$ is written in increasing order. Thus every wall character pulls back to zero or to a signed wall character. This defines a carrier functor on lifted face posets. Its composition is strict: the sign or open unit interval of every target character is already determined by the source face, and the displayed substitutions compose exactly. Integral pullback also gives $\alpha^* L_q \subseteq L_m$. Translation equivariance therefore descends these functors to the incidence categories, including their separate lattice incidence arrows. Hence $\mathcal{C}_\bullet$, and then $Q_\bullet$, extend as asserted.

Restrict along $\Delta\mathfrak{S} \to \mathrm{Fin}$. The crossed relations give the usual evaluation map

$$\xi : \mathfrak{S} \times_t Q \longrightarrow Q, \qquad (g, x) \longmapsto \lambda_g x,$$

where λ_g is the row automorphism supplied by this contravariant functor, using the opposite-group convention for automorphisms of the crossed category. Krasauskas's realization theorem supplies a homeomorphism

$$\Phi : |\mathfrak{S} \times_t Q| \xrightarrow{\cong} |\mathfrak{S}| \times |Q|, \qquad [(g, x), u] \longmapsto ([g, u], [x, g^* u]).$$

Here, in the convention of Section 8.6, $(g^* u)_j = u_{\pi_g^{-1}(j)}$. The action is $a = |\xi|\Phi^{-1}$; equivalently,

$$a\big([g, u], [x, g^* u]\big) = [\lambda_g x, u]. \tag{44}$$

The homeomorphism and action statement are [40, Theorem 2.3A, pp. 51–52]. For completeness, one can apply that theorem to the crossed simplicial set $B_{p,\bullet}$ for each fixed p. Naturality in the flag variable p then gives the action after its realization; realization preserves products in compactly generated spaces. This proves the topological-row version without requiring a new crossed action on the diagonal simplicial set.

The zero cell in each degree is preserved by all these functors. Its realization is a single point, so the canonical basepoint is fixed by the entire group.

To see that the action is nevertheless nontrivial, take $q = k$, identify Q_k with $V_k/L_k = \mathbb{R}/\mathbb{Z}$ by the barycentric homeomorphism, and let g be a transposition. Vertex means are equivariant under this arrangement automorphism, so $\lambda_g(a) = -a$. For the uniform outer coordinate u_* one has $g^* u_* = u_*$, and (44) sends

$$[1/4, u_*] \longmapsto [3/4, u_*].$$

These points are distinct: below vertex degree k every row is a point, and its degeneracies give only the zero row point in degree k. Thus both displayed interior representatives are nondegenerate. Finally $\mathfrak{S}_0$ is the trivial group, so $|\mathfrak{S}|$ is path connected. A path from its identity to any given group element gives a based homotopy from the identity of P to that element's action. All cohomology classes are therefore preserved. □

The actual reordering step. Formula (44) applies to arbitrary pairs by a finite common refinement. Regard u as lengths of consecutive subintervals and g as a permutation of these blocks, with image lengths $g^* u$. Given a second point $[x, t]$, refine the image partition by the breakpoints of t. Transfer those cuts back to the source blocks. This gives a refined permutation g' and source lengths v representing the same group element, while the refined image lengths are $g'^* v$. If β sends each refined image interval to its original t-interval, then $x' = \beta^* x$ satisfies

$$[x, t] = [x', g'^* v], \qquad a([g, u], [x, t]) = [\lambda_{g'} x', v].$$

The map β is an ordinary monotone surjection. Further splitting gives the same realization point; the crossed relations also give the identity and composition laws. Zero-length blocks are handled by face identifications. This is the coordinate reordering mechanism in the classical theorem, now acting on the original carrier realization.

Changing normalized lattice representatives introduces precisely the correction κ_T of (40); the identities (41) and (42) make such changes compatible with the action. No additional choice of an anchor is part of the action on the quotient incidence categories.

The action uses vertex permutations after the independent flag coordinate has been realized. It neither makes an arbitrary permutation of the ordered flag into a diagonal simplicial operator nor supplies the missing local charts. In particular, the fixed basepoint cannot be moved into a new normal direction by this action. A proof of a local slice, or of compatible normal directions transverse to these orbits, would still be needed to deduce the unstabilized homeomorphism from it.

8.8 The fixed carrier fold is not open

A product comparison which is not simplicial avoids the earlier ordinary-product obstruction. There is nevertheless a topological obstruction to making the *existing* carrier fold a group multiplication: for every $k \geq 1$, this fold is not an open map.

Lemma 8.8. *Let $f : A \to B$ be a simplicial map. Suppose that $\sigma \in A_n$ and $\tau = f\sigma \in B_n$ are nondegenerate. Suppose that a nondegenerate $\gamma \in B_{n+1}$ satisfies*

$$d_0\gamma = \tau, \qquad d_j\gamma \neq \tau \quad (1 \leq j \leq n+1),$$

but there is no $\beta' \in A_{n+1}$ with $d_0\beta' = \sigma$ and $f\beta' = \gamma$. Then $|f|$ is not open.

Proof. Let U be the union of the interiors of all nondegenerate simplices of A having σ as a face, including σ itself. This is open: in each characteristic simplex its complement is the union of those closed faces whose nondegenerate images do not have σ as a face. This is a finite union of closed faces.

We claim that $|f|(U)$ misses the interior of γ. Otherwise some nondegenerate $\beta \in A_N$ having σ as a face has $f\beta = \varepsilon^*\gamma$, where $\varepsilon : [N] \twoheadrightarrow [n+1]$ is monotone. Choose an injection $a : [n] \hookrightarrow [N]$ with $a^*\beta = \sigma$. Since $(\varepsilon a)^*\gamma = \tau$ is nondegenerate, εa is injective. The unique-facet hypothesis therefore gives $\varepsilon a = \delta_0$. Choose an index j with $\varepsilon(j) = 0$; it lies before every index of $a([n])$. Restricting β to $\{j\} \cup a([n])$ produces a simplex β' with $d_0\beta' = \sigma$ and $f\beta' = \gamma$, a contradiction. This argument includes higher-dimensional source simplices whose images are degenerate.

The image $|f|(U)$ contains the interior of τ. Every neighborhood of a point in that interior meets the interior of γ, since $\tau = d_0\gamma$. Thus $|f|(U)$ is not open. □

Lemma 8.9. *Let W be a set of $k-1$ vertices, ordered before four further vertices $0, 1, 2, 3$. The formula*

$$J(a)_{W\cup\{i,j\}} = a_{ij}, \qquad J(a)_F = 0 \quad \text{if } W \not\subseteq F$$

embeds $Z^1(\Delta[3];\mathbb{R})$ into $Z^k(\Delta[k+2];\mathbb{R})$. Its image is the arrangement flat cut out by $\theta_F = 0$ for all F not containing W. The single-input and binary arrangements on this flat are exactly the corresponding degree-three arrangements for $k = 1$, and the same assertion holds for their integral lattices.

Proof. On a $(k+1)$-face consisting of W and three old vertices, the coboundary of $J(a)$ is $(-1)^{k-1}\delta a$. On every other $(k+1)$-face it is zero. Conversely, after setting all coordinates not containing W to zero, the remaining cocycle equations are exactly $\delta a = 0$. This proves the assertion over both $\mathbb{R}$ and $\mathbb{Z}$. Every remaining single-input or sum character restricts to the corresponding old edge character; all others vanish. Since the flat is itself an intersection of arrangement walls, its open cells are actual cells of the ambient arrangement, not just intersections with larger cells. □

Proposition 8.10. *For every $k \geq 1$, the binary carrier fold*

$$\mu : X_0^{(k)}(2_+) \longrightarrow D_0^{(k)}$$

has non-open realization.

Proof. First work with $k = 1$ in vertex degree three. A single-input cocycle is specified by anchors (x_1, x_2, x_3), with $x_0 = 0$ and $\theta_{ij} = x_j - x_i$. For two inputs take the point

$$x = (\tfrac{1}{2}, \tfrac{1}{2}, 0), \qquad y = (0, \tfrac{1}{2}, \tfrac{1}{2}).$$

It is a vertex F of the source arrangement: the six walls

$$x_3 = 0, \quad y_1 = 0, \quad x_1 = x_2, \quad y_2 = y_3,$$
$$x_1 + y_1 = x_3 + y_3, \qquad x_2 + y_2 = 1$$

have this unique intersection. Each is one of the permitted single-input or sum walls. Its fold lies in the open target edge

$$G = \{(t, 1, t) : 0 < t < 1\}.$$

Let $H = \{(0, 1, 0)\}$ be the endpoint at $t = 0$. For general k, apply Lemma 8.9 to both inputs and retain the names F, G, H for the resulting cells. Thus F is still a source vertex, G a target edge, and $H < G$ its endpoint, now in vertex degree $q = k + 2$. Let σ and τ be the constant length-$(q+1)$ flags at F and G, respectively.

The simplex τ is nondegenerate. Indeed, a degeneracy duplicating an adjacent vertex pair makes every k-face containing that pair zero modulo one. For every point of G, each adjacent pair inside W, between W and old vertex zero, or between old vertices zero and one is detected by the face $W \cup \{0, 1\}$, whose value is t. The other two adjacent pairs are detected by $W \cup \{1, 2\}$ and $W \cup \{2, 3\}$, with values $1 - t$ and $t - 1$. These are

nonzero modulo one for $0 < t < 1$. For $k = 1$ the same check just uses the three adjacent old edges. Since $\mu\sigma = \tau$, the source simplex σ is also nondegenerate.

Let $s_0H < s_0G$ denote the carriers after duplicating the first vertex, and form the degree-$(q+1)$ target simplex

$$\gamma = (s_0H, s_0G, \dots, s_0G).$$

Its zeroth face is τ. Its first face has entries $(H, G, \dots, G)$ and differs from τ. For $j \geq 2$, every entry of $d_j\gamma$ lies in the first-duplicate flat: all k-face characters containing the first two vertices are zero modulo one. The face $W \cup \{0, 1\}$ contains those vertices and has value t on G, so τ is not in that flat. Thus no other face equals τ. The simplex γ is nondegenerate: its nondegenerate zeroth face rules out all degeneracies except possibly $s_0\tau$, whose entries would all be s_0G.

Suppose a source simplex $\beta' = (E_0, \dots, E_{q+1})$ satisfied $d_0\beta' = \sigma$ and $\mu\beta' = \gamma$. Write ρ_0 for the carrier restriction deleting the first vertex, without deleting a flag entry. Then $\rho_0E_1 = F$ and $\rho_0E_0 \leq \rho_0E_1$. Since F is a vertex, $\rho_0E_0 = F$. Commutativity of restriction and fold would give

$$G = \mu(\rho_0E_0) = \rho_0(\mu E_0) = \rho_0(s_0H) = H,$$

which is impossible. The argument is unchanged in the orbit categories: incidences into a zero-dimensional face force the same face orbit. Lemma 8.8 now proves nonopenness. □

Corollary 8.11. *There is no homeomorphism*

$$H : |X_0^{(k)}(2_+)| \longrightarrow |D_0^{(k)}|^2, \qquad k \geq 1,$$

which turns this carrier fold into multiplication for a topological group structure on $|D_0^{(k)}|$.

Proof. Multiplication in a topological group is open: the map $(g, h) \mapsto (g, gh)$ is a homeomorphism, and multiplication is its composite with the second projection. If $|\mu| = m \circ H$, then $|\mu|$ would be open, contradicting Proposition 8.10. □

This obstructs an arbitrary geometric product comparison intertwining the fixed fold with a group law, including one that permutes barycentric coordinates. It does not rule out a homeomorphism of the underlying spaces without that compatibility, a replacement of the fold, or a multiplication required only to be homotopy-unital.

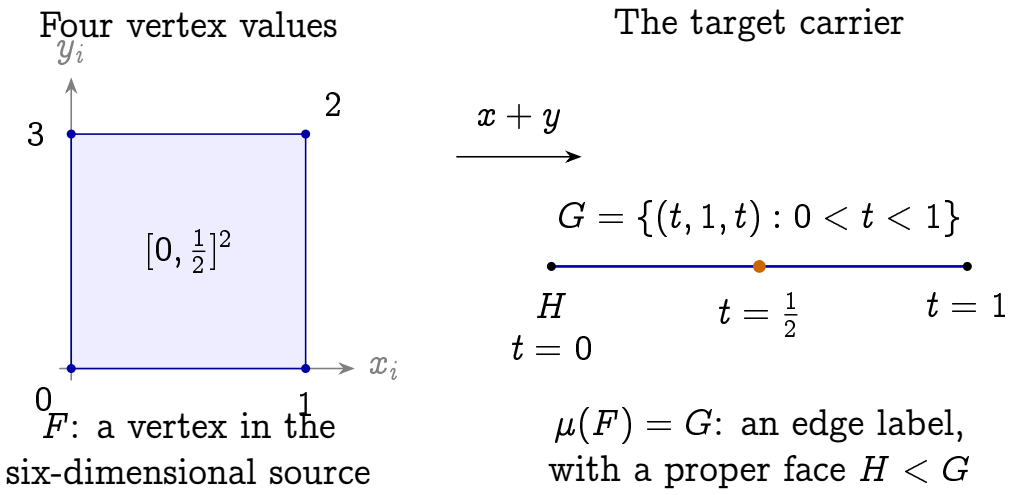


Figure 15: The obstruction already appears for $k = 1$ in vertex degree three. Reading the four corners in order gives $x = (1/2, 1/2, 0)$ and $y = (0, 1/2, 1/2)$. The affine sum is the marked point of G, whereas the carrier fold sends the vertex label F to the edge label G. The missing lift of the incidence $H < G$ makes the realized fold non-open. Adding the auxiliary vertices W carries the same configuration to every $k \geq 1$.

8.9 The limit of a literal crossed simplicial group

The classical crossed simplicial-group mechanism has a precise limitation in the present problem. Here a crossed simplicial group means an extension ΔG of Δ with the usual unique factorization of every arrow as a simplicial arrow after an automorphism. Its underlying simplicial set is $G_\bullet$, not the nerve of the whole category ΔG.

Theorem 8.12. *Let G be a crossed simplicial group whose groups G_q are finite for every q. Then*

$$\pi_i(|G|, 1) \text{ is finite for every } i \geq 2.$$

Consequently no such underlying simplicial set has the homotopy type $K(\mathbb{Z}, n)$ for $n \geq 2$.

Proof. The structure theorem of Krasauskas and Fiedorowicz–Loday gives an extension

$$1 \longrightarrow H \longrightarrow G \xrightarrow{p} Q \longrightarrow 1,$$

where H is an ordinary simplicial group and Q is one of the seven fundamental crossed simplicial groups [40, 26]; see also [22, Theorem I.7 and Table 2]. More precisely, H is the kernel and Q the image of the canonical map to

the Weyl crossed simplicial group. Elements of H have trivial crossing with every simplicial operator. Using the composition convention of [22], trivial crossing gives, for every simplicial operator α,

$$\alpha^*(hg) = \alpha^*(h)\alpha^*(g) \qquad (h \in H_q,\ g \in G_q).$$

Thus left multiplication makes $G \to Q$ a degreewise principal simplicial H-torsor.

For completeness, this torsor map is a Kan fibration even though G and Q need not be Kan. Given a horn above $y \in Q_q$, choose any lift $g \in G_q$ of y. Write its specified faces uniquely as

$$x_i = h_i(d_i g) \qquad (i \neq j),$$

where j is the missing face. The horn identities for the x_i, and freeness of the H-action, imply exactly the horn identities for the h_i. Since H is a simplicial group it is Kan. Choose $h \in H_q$ with $d_i h = h_i$ for $i \neq j$; then hg fills the original horn. No simplicial section of p is required.

It follows that H is the homotopy fibre over the identity vertex. One can justify this without assuming that Q is Kan by applying Ex^∞: this functor preserves Kan fibrations and finite limits, gives Kan replacements, and takes the strict fibre to $\mathrm{Ex}^\infty H$. The usual long exact sequence for Kan complexes therefore applies to the original homotopy types [42, 43].

The homotopy types of the seven possible bases are

Q	$\lvert Q\rvert$ up to homotopy
trivial	$*$
reflexive	$\mathbb{Z}/2$
cyclic	S^1
dihedral	$O(2)$
symmetric	$*$
reflexosymmetric	$\mathbb{Z}/2$
Weyl	$*$

In the cyclic case realization gives S^1; in the dihedral case it gives $O(2)$ [22, Theorem I.33]. The symmetric and Weyl simplicial sets are contractible: appending a new largest label at the end of the total-order word, with positive sign in the Weyl case, is an extra degeneracy contracting them. The reflexosymmetric realization has two components; its identity component is the symmetric realization, and translation in the realized topological group identifies the other component with it. These observations also directly verify the only property needed below:

$$\pi_j(\lvert Q\rvert, 1) = 0 \qquad (j \geq 2).$$

Hence the fibre sequence gives

$$\pi_i(|H|,1) \cong \pi_i(|G|,1) \qquad (i \geq 2).$$

Finally $H_i \subseteq G_i$ is finite. Every class in $\pi_i(H,1)$ has a representative which is an i-simplex of H with every face equal to the identity. There are only finitely many such representatives. Thus $\pi_i(H,1)$ is finite, proving the theorem. □

The cyclic model of $K(\mathbb{Z},1)$ is consistent with this theorem: the infinite fundamental group comes from the circle in the fundamental quotient Q. That exception does not extend to higher homotopy degrees.

Corollary 8.13. *Under the same finiteness hypothesis, neither $B|G|$ nor $|N(\Delta G)|$ has homotopy type $K(\mathbb{Z},n)$ for $n \geq 3$.*

Proof. For $n \geq 3$, the classifying-space identity gives $\pi_n(B|G|) = \pi_{n-1}(|G|)$, which is finite by the theorem. The equivalence $|N(\Delta G)| \simeq B|G|$ is [22, Theorem I.33(a)]. □

The cyclic example attains the preceding degree: $B|G| \simeq BS^1 \simeq K(\mathbb{Z},2)$. Thus the classical crossed-group route accounts for the circle and its first delooping, but cannot be repeated within the same finite-group framework to obtain the higher integral models.

The theorem concerns a literal crossed simplicial *group* on finite sets of simplices. It does not exclude a finite simplicial quotient of crossed groups, a finite model of an arbitrary classifying space, or a multiplication encoded by subdivisions, correspondences, or additional simplicial directions. In particular, it does not obstruct the finite models $D_0^{(k)}$ or their coherent addition. It rules out making their underlying finite simplex sets into crossed simplicial groups with the same higher Eilenberg–Mac Lane homotopy type. A successful extension of the reordering idea must therefore use a broader form of crossed multiplication.

8.10 A finite polyhedral prototype of the broader structure

The successful part of the crossed mechanism can be retained in every vertex degree if the charts keep their actual affine coordinates. This gives a precise prototype for a generalization, and locates the additional problem of presenting it by finite discrete simplex sets.

For each q, S, let $\mathfrak{P}_q(S)$ be the following finite polyhedral presentation. Choose one lifted closed polytope for each face orbit of $\widetilde{\mathcal{P}}_q^{(k)}(S)$ and glue

their faces using all the lattice-labelled incidence branches. Write $\|\mathfrak{P}_q(S)\|$ for the resulting polyhedral space. By construction,

$$\|\mathfrak{P}_q(S)\| = A_q(S) = Z^k(\Delta[q];\mathbb{T})^{S^\circ}.$$

There are finitely many charts and attaching maps in every fixed degree and arity, by Proposition 3.1.

Proposition 8.14. *Alternating vertex pullbacks and pointed-set fibre sums have strictly composable finite chart descriptions on $\mathfrak{P}$. On a canonical open source chart F, an allowed linear map T is represented in its canonical target chart by*

$$x \longmapsto Tx - \kappa_T(F). \tag{45}$$

The chart descriptions agree along the prescribed incidence identifications. Forgetting the extra subset-sum walls gives geometric product homeomorphisms

$$\|\mathfrak{P}_q(S)\| \cong \prod_{s\in S^\circ} \|\mathfrak{P}_q(1_+)\|,$$

natural in the vertex direction. Under them the fold is actual torus addition.

Proof. On the canonical source lift, $\nu_q(x) = 0$, so (41) is precisely (45). The correction is constant on the open face and the affine formula extends to its closure. When that closure meets another chosen chart, the two descriptions differ by the specified lattice translation. Since T preserves the lattices, their images agree in the target quotient.

The composition identity (42) proves strict composition in canonical charts; on common faces the same identity holds after the attaching translations. All charts and incidences involved are finite in number for fixed T. Finally, the multi-input arrangement is a subdivision of the product arrangement. Erasing its extra walls does not change any point of the product torus. The product comparison is therefore the identity on the underlying coordinates, and the fold is their sum. Actual linear pullbacks commute with these maps. □

In particular $A_\bullet(1_+)$, with its topological torus levels, is a simplicial topological abelian group. Realization preserves its products [19], so its realization has an exact topological abelian group law. The natural characteristic-cell comparison of Appendix A identifies its homotopy type with that of $D_0^{(k)}$, namely $K(\mathbb{Z}, k+1)$.

This prototype replaces the single permutation of simplex coordinates by a finite collection of affine chart maps and compatible subdivisions. It contains strict addition geometrically, and its lattice corrections compose by an explicit formula. Its levels, however, are tori presented by finitely many polytopes; they are not finite discrete sets. The finite simplex models are the carrier and symmetric flag constructions. Their comparison with the polyhedral group model is a weak equivalence, and the product comparison of the carrier models is not a homeomorphism.

Consequently the remaining target has a precise form: construct a degreewise finite combinatorial presentation whose geometric comparison retains the transformations (45), with refinements compatible with substitution of several inputs. The two-triangle example does this at one level. Reordering alone and canonical carrier labels do not yet accomplish it in all degrees. A structure with such refinements and coordinate transformations would generalize the role played by unique factorization in a crossed simplicial group. Section 9 gives an exact calculus of maps with refinement data and represents every integer operation by global degreewise finite simplicial spans. Corollary 8.11 shows that an exact construction cannot keep the current carrier fold: that map is not open. Retaining the actual affine fold, as the polyhedral prototype does, therefore requires changing the discrete presentation or its multiplication. The next theorem rules out one possible shortcut, namely strictly natural finite ordinary triangulations of all the torus levels.

8.11 An obstruction to compatible finite row triangulations

There is a further obstruction to recovering an exact geometric group law by choosing better triangulations of the torus levels. For $k \geq 2$, one cannot triangulate all those levels by ordinary simplicial sets with a finite vertex set in the circle level, while making the geometric identifications strictly natural. This assertion concerns that particular route to a product homeomorphism; it does not exclude a different crossed construction or a homeomorphic group model obtained in another way.

Lemma 8.15. *Let $f : P \to Q$ be a map of ordinary simplicial sets, and let v be a vertex of Q. The inverse image $|f|^{-1}(v)$ is the realization of the simplicial subset $f^{-1}(\langle v \rangle)$, where $\langle v \rangle$ consists of v and all its degeneracies. Finite intersections of such inverse images are likewise realizations of simplicial subsets. Every isolated point of such an intersection is a vertex of P.*

Proof. A point of $|P|$ has a unique expression in the interior of a nondegener-

ate simplex. Its barycentric coordinates are strictly positive. After applying f and deleting degeneracies, those coordinates are combined into positive sums. The result can therefore be the vertex v only if the whole simplex maps to a degeneracy of v. This proves the first assertion and, by the same description of the supporting simplex, the assertion about intersections. A point in the interior of a positive-dimensional simplex of the resulting subset is not isolated. $\square$

Lemma 8.16. *For every prime p there is a finite two-dimensional simplicial complex K_p with complete 1-skeleton and*

$$H_1(K_p;\mathbb{Z}) \cong \mathbb{Z}/p.$$

Proof. We give the finite triangulation input explicitly. Take a circle with three vertices a_0, a_1, a_2 and a circle with $3p$ vertices $u_0, \dots, u_{3p-1}$. The simplicial covering $u_i \mapsto a_{i \bmod 3}$ has degree p. A triangulation of its mapping cylinder, followed by coning off the source circle with a new vertex z, has triangles

$$\{u_i, u_{i+1}, a_{i+1}\}, \qquad \{u_i, a_i, a_{i+1}\}, \qquad \{z, u_i, u_{i+1}\}, \qquad 0 \le i < 3p.$$

Here the subscripts on u are read modulo $3p$, and those on a modulo 3. The first two triangles subdivide each cylinder strip, and the third cones its upper boundary edge. Distinct lifts of a target edge meet along that edge, exactly as in the mapping cylinder. This is a finite abstract simplicial complex M_p realizing the mapping cone. The usual one-cell structures on the circles give its cellular chain model with boundary map $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ in dimensions two and one, so $H_1(M_p;\mathbb{Z}) = \mathbb{Z}/p$.

Choose a spanning tree T in the 1-skeleton of M_p. Adjoin a new vertex v and the cone $v * T$. Every old vertex is now joined to v. For each missing edge $\{i, j\}$ between old vertices, adjoin that edge and the triangle $\{v, i, j\}$. The two other edges of this triangle already exist. The resulting complex K_p has complete 1-skeleton.

Each newly added old–old edge is a free face of its unique new triangle, so these pairs collapse away. The cone on a tree collapses onto its base tree relative to that base, by removing leaves successively. Consequently K_p collapses to M_p, and its first homology is as asserted. $\square$

Theorem 8.17. *Fix $k \ge 2$ and put*

$$A_q^{(k)} = Z^k(\Delta[q];\mathbb{T}).$$

There is no bisimplicial set $B_{\bullet,\bullet}$ with $B_{0,k}$ finite and homeomorphisms

$$h_q : |B_{\bullet,q}| \xrightarrow{\cong} A_q^{(k)}$$

natural for every monotone vertex map. In particular, the simplicial topological tori cannot be replaced by compatible finite ordinary simplicial row triangulations with these natural geometric identifications. Finiteness of the higher-dimensional simplex sets is not needed for the obstruction.

Proof. Since $A_0^{(k)}$ is a point, $B_{\bullet,0}$ has exactly one vertex. The degeneracy from degree zero to degree k sends it to a vertex of $B_{\bullet,k}$ which h_k identifies with $0 \in A_k^{(k)} = \mathbb{T}$. Every k-face $F \subseteq [q]$ defines a monotone injection $[k] \to [q]$. Its pullback on the torus levels is the evaluation map

$$\theta_F : A_q^{(k)} \longrightarrow \mathbb{T}.$$

Naturality of h makes this the realization of an ordinary simplicial map between the corresponding rows. Lemma 8.15 therefore applies to every finite intersection of its zero fibres.

First take $k = 2$. Label the vertices of the complex K_p from Lemma 8.16 by $[q]$ and set

$$Z_p = \{x \in A_q^{(2)} : \theta_F(x) = 0 \text{ for every triangle } F \in K_p\}.$$

Exactness of the cochain complex of the simplex identifies

$$C^1(\Delta[q];\mathbb{T})/B^1(\Delta[q];\mathbb{T}) \xrightarrow{\delta} Z^2(\Delta[q];\mathbb{T}).$$

Because K_p contains every edge, the equations defining Z_p say precisely that the representing one-cochain is a cocycle on K_p. The coboundaries are unchanged. Hence

$$Z_p \cong H^1(K_p;\mathbb{T}) \cong \mathrm{Hom}(H_1(K_p;\mathbb{Z}),\mathbb{T}) \cong \mathbb{Z}/p.$$

Thus Z_p consists of p isolated points, and the preceding lemma forces all of them to be vertices of the row model. Face evaluations jointly distinguish the points of $A_q^{(2)}$. At least one triangle evaluation is therefore nonzero on Z_p; since p is prime, its restriction is injective and its image has p elements. All these elements must be vertices of the fixed circle row $B_{\bullet,2}$. Taking a prime $p > |B_{0,2}|$ is a contradiction.

For general $k \geq 2$, adjoin to the old vertex set $[q]$ a set W of $k-2$ new vertices, ordered before the old vertices. In the torus of k-cocycles on this enlarged simplex, impose the equations

$$\theta_G = 0 \qquad \text{whenever } W \not\subseteq G.$$

Every remaining k-face has the form $W \cup \{a, b, c\}$. Writing its value as x_{abc} identifies the resulting subgroup exactly with $Z^2(\Delta[q]; \mathbb{T})$. Indeed, a cocycle equation indexed by $k+2$ vertices is identically zero unless those vertices contain W. If they do contain W, the four old vertices contribute $(-1)^{k-2}\delta x$, while deleting a vertex of W contributes zero. These are precisely the two-cocycle equations and no others.

Now also impose

$$\theta_{W \cup F} = 0 \qquad \text{for every triangle } F \in K_p.$$

The full intersection is the same p-point group Z_p as before. It is defined entirely by zero fibres of actual k-face evaluations, so all its points must again be vertices. For some triangle F of the full old simplex, necessarily outside K_p, the evaluation $\theta_{W \cup F}$ maps it injectively to the circle. This forces at least p vertices in $B_{\bullet,k}$ for every prime p, contradicting the assumed finiteness of $B_{0,k}$. □

The theorem does not merely obstruct a natural choice of barycentres: it allows arbitrary row homeomorphisms and still rules out their strict naturality. Its hypothesis is stronger than a homeomorphism between total realizations, or a product comparison involving a crossed construction. Neither of those possibilities is excluded here. The degreewise finite carrier and symmetric-flag models also remain unaffected: their characteristic-cell comparisons are weak equivalences, not the natural row homeomorphisms required in the theorem. The case $k = 1$ is not covered by this argument.

9 A crossed law with finite refinements

The polyhedral prototype admits an exact composition calculus. The additional datum is a subdivision: moving a map past a subdivision pulls its walls back. This section makes that rule precise. Section 9.3 then represents every integer matrix by an actual span of degreewise finite simplicial sets in all vertex degrees. The common-subdivision principle is classical PL topology [10, Section 2]; the wall description below makes its composition rule explicit for these models. This extends the geometric mechanism of crossed factorization without asserting the unique factorization axiom of a crossed simplicial group [40, 26, 22].

9.1 Maps with their required walls

Let L be a finite-rank free abelian group and put $V = L \otimes \mathbb{R}$. A *wall presentation* $(L, \mathcal{H})$ is a locally finite, L-periodic set of rational affine hyperplanes, with finitely many translation orbits. We require that it include the integer walls of some lattice coordinate basis. Its cells therefore have bounded closures, and their orbit-incidence category is finite. Rank zero is allowed, with the empty arrangement presenting a point. The presentation retains the affine coordinates and all lattice incidence identifications, so its underlying space is V/L.

For two arrangements on V, write $\mathcal{H} \vee \mathcal{R} = \mathcal{H} \cup \mathcal{R}$ for their common refinement. If $T : L \to L'$ is a homomorphism, let $T^*\mathcal{J}$ be the collection of proper hyperplanes among the inverse images of the walls of $\mathcal{J}$. Empty and whole-space inverse images impose no cut and are discarded. The map T is still retained: an image contained in a target wall is not thereby forgotten.

Lemma 9.1. *The union of finitely many wall presentations is again finite modulo the lattice. Pullback along a lattice homomorphism has finitely many wall orbits and preserves local finiteness. Moreover,*

$$T^*(\mathcal{J} \vee \mathcal{S}) = T^*\mathcal{J} \vee T^*\mathcal{S}, \qquad T^*U^*\mathcal{Q} = (UT)^*\mathcal{Q}.$$

Proof. One target orbit can be written as $\ell(y) = a + j$, $j \in \mathbb{Z}$, with ℓ primitive integral and $a \in \mathbb{Q}$. If $\ell T = 0$ there are no proper inverse-image walls. Otherwise $(\ell T)(L) = d\mathbb{Z}$ for some $d \geq 1$. The pulled-back walls have exactly d translation orbits and only finitely many meet any compact set. This proves the finiteness assertions. The identities follow from inverse images of sets, including the stated convention for constant equations. □

Definition 9.2. A map with refinement data from $(L, \mathcal{H})$ to $(L', \mathcal{J})$ is a pair $(T, \mathcal{R})$, where $T : L \to L'$ and $\mathcal{R}$ is a wall presentation satisfying

$$\mathcal{R} \supseteq \mathcal{H} \vee T^*\mathcal{J}.$$

It represents the geometric span

$$(L, \mathcal{H}) \xleftarrow{\text{ref}} (L, \mathcal{R}) \xrightarrow{T} (L', \mathcal{J}).$$

The first map is the identity on the underlying torus. The second is the actual linear map on each lifted cell, with the specified lattice identifications on its boundary.

Proposition 9.3. *These maps form a category under the rule*

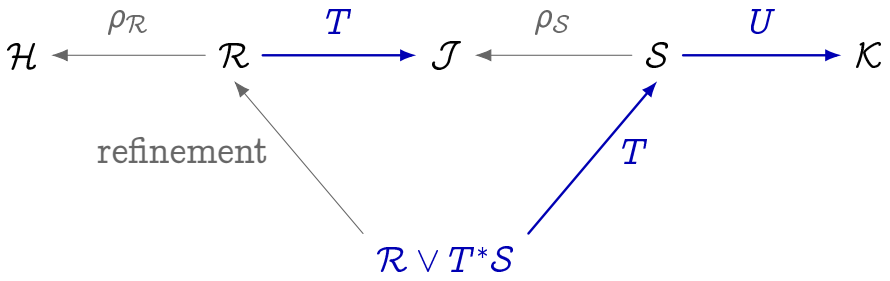
$$(U, \mathcal{S}) \circ (T, \mathcal{R}) = (UT, \mathcal{R} \vee T^{*}\mathcal{S}). \tag{46}$$

The identity of $(L, \mathcal{H})$ *is* $(\mathrm{id}, \mathcal{H})$. *Increasing* $\mathcal{R}$ *gives compatible refinement comparisons between maps with the same underlying* T.

Proof. If $(U, \mathcal{S})$ has target $(L'', \mathcal{Q})$, then $\mathcal{S} \supseteq \mathcal{J} \vee U^{*}\mathcal{Q}$. Thus the displayed refinement contains both $\mathcal{H}$ and $(UT)^{*}\mathcal{Q}$, so the composite is defined. Three maps with linear parts T, U, W and refinements $\mathcal{R}, \mathcal{S}, \mathcal{P}$ give, in either order,

$$(WUT, \mathcal{R} \vee T^{*}\mathcal{S} \vee (UT)^{*}\mathcal{P}).$$

This proves literal associativity. The unit equations use the containments already required in the definition. Unions and pullbacks preserve inclusion, so refinement comparisons also compose. There is at most one comparison for a specified inclusion; consequently all diagrams made of these comparisons commute. Figure 16 displays the construction geometrically. □



Common subdivision of $\mathcal{R}$
and the pullback of $\mathcal{S}$

Figure 16: Composition by common refinement. The maps ρ forget a subdivision and are the identity on the underlying torus. Cutting the source additionally by $T^{*}\mathcal{S}$ makes T compatible with the second roof. The composite is represented by $\mathcal{H} \leftarrow \mathcal{R} \vee T^{*}\mathcal{S} \xrightarrow{UT} \mathcal{K}$, using the actual affine maps on the characteristic polytopes.

We direct a refinement comparison from the finer decoration to the coarser one: if $\mathcal{R} \subseteq \mathcal{R}'$, then $(T, \mathcal{R}') \Rightarrow (T, \mathcal{R})$. It is represented by the actual refinement map between their source presentations. With these comparisons, each hom-category is ordered by reverse inclusion, and the composition just proved is a strict 2-categorical composition.

9.2 The abelian law after identifying subdivisions

Identify $(T, \mathcal{R})$ and $(T, \mathcal{R}')$ through the common refinement $\mathcal{R} \vee \mathcal{R}'$. The quotient category has precisely the homomorphisms $T : L \to L'$ as its morphisms: every T has the minimal refinement $\mathcal{H} \vee T^* \mathcal{J}$, any two refinements have a common refinement, and no refinement changes T. Equivalently, these morphisms are the continuous torus homomorphisms. The category of choices over each fixed T has the minimal arrangement as a terminal object, hence a contractible nerve.

Products are presented by

$$\mathcal{H} \boxtimes \mathcal{J} = \mathrm{pr}_1^* \mathcal{H} \vee \mathrm{pr}_2^* \mathcal{J}$$

on $L \oplus L'$. Addition $L \oplus L \to L$, zero $0 \to L$, and inversion $-1 : L \to L$ therefore give an abelian group object in this quotient category. The group identities are equalities of the underlying integer matrices. In the category with refinement data, the two sides may use different walls; their union supplies the comparison. This is an explicit finite chart presentation of the usual abelian torus law.

On normalized coordinates the charts retain the integer correction

$$\kappa_T(x) = \nu(Tx) - T\nu(x), \qquad \mathsf{n}(Tx) = T\mathsf{n}(x) - \kappa_T(x).$$

Here ν takes the integer parts in lattice coordinates. The source and pulled-back target coordinate walls make the correction constant on every open refined cell. It is invariant under source lattice translations, and

$$\kappa_{UT}(x) = U\kappa_T(x) + \kappa_U(Tx). \tag{47}$$

For normalized x, set $F_T(x) = Tx - \kappa_T(x)$. Lattice invariance then gives $F_U(F_T(x)) = F_{UT}(x)$. This equality concerns normalized representatives. The half-open fundamental cube is not a global continuous section; continuity holds on closed affine charts glued by their lattice incidences.

For a sum of m inputs, all binary trees have the same total carry

$$b_m(x_1, \dots, x_m) = \nu\left(\sum_i x_i\right) - \sum_i \nu(x_i). \tag{48}$$

The carries at internal vertices telescope to this expression. For disjoint-input substitutions, the original all-subset-sum arrangement already contains every partial-sum wall. Thus it simultaneously compares all parenthesizations in a fixed arity, including the five vertices and every edge of the four-input pentagon. Each route is the same sum map after this refinement. Inversion and input repetition are covered by the more general integer coefficient families below.

What these coherence statements mean. Associativity of the decorated composition is strict, and diagrams of its refinement comparisons commute strictly. The abelian group law is strict after identifying subdivisions. These assertions do not choose inverse weak equivalences between simplicial models, or assert a particular higher action on their original simplex sets. The next construction supplies actual finite simplicial spans and states exactly what their comparisons prove.

9.3 Integer operations by global finite refinements

The refinement construction can be made simultaneously in every vertex degree using a finite family of integer coefficient vectors. Fix $k \geq 1$. For $m \geq 0$, let $C \subset \mathbb{Z}^m \setminus \{0\}$ be finite and contain the standard basis vectors. For $m = 0$ take $C = \varnothing$. In

$$V_q(m) = Z^k(\Delta[q]; \mathbb{R}^m), \qquad L_q(m) = Z^k(\Delta[q]; \mathbb{Z}^m),$$

use the periodic walls

$$a \cdot \theta_F \in \mathbb{Z}, \qquad a \in C, \tag{49}$$

where F ranges over the nondegenerate k-faces of $\Delta[q]$. Let $\widetilde{\mathcal{P}}_q(C)$ be their face poset, let $\mathcal{C}_q(C) = \widetilde{\mathcal{P}}_q(C)/L_q(m)$ retain all orbit incidences, and put

$$X_C^{(k)}{}_n = N_n \mathcal{C}_n(C).$$

Repeated vectors can be removed; zero vectors give no walls. One must not divide a coefficient vector by its common divisor: for example, $2\theta_F \in \mathbb{Z}$ includes the half-integer cuts which $\theta_F \in \mathbb{Z}$ does not.

Theorem 9.4. *Every $X_C^{(k)}$ is degreewise finite and has homotopy type $K(\mathbb{Z}^m, k+1)$. An inclusion $C \subseteq R$ induces an ordinary simplicial weak equivalence*

$$r_{R,C} : X_R^{(k)} \longrightarrow X_C^{(k)}.$$

For an integer matrix $M : \mathbb{Z}^m \to \mathbb{Z}^r$ and a target family $D \subset \mathbb{Z}^r$, set $R = C \cup M^{\mathrm{t}}D$, discarding zero vectors. There is an ordinary simplicial roof

$$X_C^{(k)} \xleftarrow{\ r_{R,C}\ } X_R^{(k)} \xrightarrow{\ f_M\ } X_D^{(k)}. \tag{50}$$

It represents the map $M : K(\mathbb{Z}^m, k+1) \to K(\mathbb{Z}^r, k+1)$. All these constructions use the same finite coefficient families in every simplicial degree.

Proof. The basis vectors supply all anchor integer walls. Thus each closed lifted face is a bounded rational polytope, the lattice acts freely on faces, and there are finitely many face and incidence orbits in every fixed q. Proposition 3.1 applies without change and proves degreewise finiteness.

Vertex pullback sends a face character to a face character or zero (and to its negative if arbitrary vertex reorderings are also considered). The integer walls are preserved. Matrix pullback sends a target character $d\cdot\theta_F$ to $(M^{\mathrm{t}}d)\cdot\theta_F$. Therefore refinement and matrix maps have strictly composable carriers. Taking nerves and the diagonal gives the asserted ordinary simplicial maps.

To prove the refinement assertion, fix q and choose lattice-equivariant barycentres for both arrangements. These give geometric homeomorphisms from the two cover nerves to $V_q(m)$. For a fine flag $F_0 \le \cdots \le F_p$, write G_i for its coarse carriers. The fine barycentric evaluation and the evaluation of its carrier image both lie in $\overline{G}_p$. Straight-line interpolation in this convex polytope defines a homotopy between the two evaluations. The formulas agree on flag faces and degeneracies and commute with lattice translation. On quotient nerves the refinement map is consequently homotopic, through these homeomorphisms, to the identity of the torus. It is a homotopy equivalence in every fixed vertex degree. The diagonal weak-equivalence lemma for bisimplicial sets proves the claim; no naturality of the chosen barycentric homeomorphisms is required.

For the basis family $E_m = \{e_1, \dots, e_m\}$ the arrangement is a product of m single-input arrangements. Its incidence category and diagonal nerve are products, so $X^{(k)}_{E_m} = (D^{(k)}_0)^m$. Refinement to E_m and Theorem 4.2 identify the homotopy type of $X^{(k)}_C$.

For completeness, the matrix action can be checked directly on the canonical classes. On a lifted degree-k flag with first cell F_0, write $\theta = \theta_{0\dots k}|_{F_0}$ and take componentwise floors. The integer cochain

$$b_M = \lfloor M\theta \rfloor - M\lfloor \theta \rfloor$$

is constant on F_0, since R contains both the basis vectors and the rows of M. It is lattice-invariant and normalized, so descends to $X^{(k)}_R$. The canonical upstairs primitive is $a = \lfloor \theta \rfloor$; hence

$$\delta b_M = f_M^* c_D - M c_R, \qquad c_R = r^*_{R,C} c_C.$$

Here c_C denotes the vector of canonical degree-$(k{+}1)$ cocycles. Thus the roof acts on the fundamental classes by exactly the specified integer matrix. □

There is an explicit composition rule. Decorate a matrix $M : C \to D$ by any finite $R \supseteq C \cup M^{\mathrm{t}}D$. For a second decorated matrix $(N, S) : D \to E$,

put

$$(N, S) \circ (M, R) = (NM,\ \ R \cup M^{\mathrm{t}} S). \tag{51}$$

The composite source refines R and maps by M to the presentation with coefficient family S. The resulting carrier square commutes strictly, so this formula represents composition of the roofs after inverting refinement maps. Pullback of coefficient families preserves unions and composition, making (51) strictly associative, with identity (I, C).

For a fixed matrix, $R \subseteq R'$ gives a refinement comparison $(M, R') \Rightarrow (M, R)$, represented by $X^{(k)}_{R'} \to X^{(k)}_{R}$. Thus the hom-categories use reverse inclusion. Any two decorations have the common refinement $R \cup R'$. These form a thin category: there is at most one refinement arrow between given decorations. Consequently the comparison diagrams for any finite collection of specified composite expressions commute after a common finite refinement. Equivalently, matrices with decorations and these refinement arrows form a strict 2-category. After identifying decorations through common refinements, the morphisms are exactly integer matrices; sending a decoration to (50) defines their realization in the homotopy category of simplicial sets.

Products use the union of the two coefficient families embedded in the separate coordinate blocks. Addition, zero, inversion, diagonals, and multiplication by every integer are therefore supported by global degreewise finite simplicial roofs, with their abelian identities compared by the preceding refinements.

Corollary 9.5. *Let* $\mathsf{Mat}_{\mathbb{Z}}$ *have objects* $m \geq 0$ *and integer* $r \times m$ *matrices as morphisms* $m \to r$*, with products given by addition of ranks. There is a product-preserving functor*

$$\Phi_k : \mathsf{Mat}_{\mathbb{Z}} \longrightarrow \mathsf{Ho}(\mathsf{sSet}), \qquad \Phi_k(m) = (D^{(k)}_0)^m,$$

represented on every matrix by a span of degreewise finite simplicial sets. This is a realization of the Lawvere theory of abelian groups in the homotopy category with explicit finite representatives for every operation.

Proof. Choose $C = E_m$, $D = E_r$, and $R = E_m \cup M^{\mathrm{t}} E_r$. The roof defines $\Phi_k(M) = [f_M][r_{R,E_m}]^{-1}$. For a second matrix use the common source family $R \cup M^{\mathrm{t}} S$. The carrier square just constructed commutes, and every refinement arrow is a weak equivalence. It follows that $\Phi_k(NM) = \Phi_k(N)\Phi_k(M)$. The identity matrix gives the identity roof. Coordinate projections require no extra walls and are the literal projections of $(D^{(k)}_0)^m$, proving product preservation. This proof does not compose arbitrary spans by an ordinary pullback; it uses the specified common-refinement diagrams. □

Carries for all integer operations. On the source family $R \cup M^{\mathrm{t}}S$, cancellation of integer parts gives the exact cochain identity

$$b_{NM} = M^{*}b_{N} + Nb_{M}. \tag{52}$$

Here M^{*} is pullback along the carrier map to $X_{S}^{(k)}$; all cochains are pulled to the common refinement. The identity is the equality $\lfloor NM\theta \rfloor - NM\lfloor \theta \rfloor = \lfloor NM\theta \rfloor - N\lfloor M\theta \rfloor + N(\lfloor M\theta \rfloor - M\lfloor \theta \rfloor)$. The resulting correction in any anchor coordinate obeys

$$\sum_{i} \min(M_{ji}, 0) \;\le\; b_{M,j} \;\le\; \sum_{i} \max(M_{ji}, 0).$$

These bounds depend only on the matrix, not on the vertex degree.

For inversion one has $b_{-1}(x) = -1$ on $0 < x < 1$ and $b_{-1}(0) = 0$. Doubling uses the global family $\{1, 2\}$; its carry is zero on $0 \le x < 1/2$ and one on $1/2 \le x < 1$. Duplication itself is the ordinary diagonal $D_{0}^{(k)} \to (D_{0}^{(k)})^{2}$ and has zero carry. Composing it with the refined addition produces the doubling roof, including its additional walls, in every simplicial degree.

Proposition 9.6. *Write $D = D_{0}^{(k)}$, $Z = X_{0}^{(k)}(2_{+})$, and let $p : Z \to D^{2}$ and $\mu : Z \to D$ be the carrier projections and fold. Negation gives an ordinary simplicial involution $\iota : D \to D$. There are ordinary simplicial maps $h, u_{L}, u_{R} : D \to Z$ satisfying the literal identities*

$$\iota^{2} = \mathrm{id}, \qquad ph = (\mathrm{id}, \iota), \qquad \mu h = 0,$$
$$pu_{L} = (\mathrm{id}, 0), \qquad pu_{R} = (0, \mathrm{id}), \qquad \mu u_{L} = \mu u_{R} = \mathrm{id}.$$

The interchange of inputs gives an involution $\tau : Z \to Z$ with $\mu\tau = \mu$.

Proof. Negation preserves the single-input wall set, so its carriers compose to the identity. The coefficient family of Z is $\{(1, 0), (0, 1), (1, 1)\}$. Pulling it back along $x \mapsto (x, -x)$ gives coefficients $1, -1, 0$, hence no new walls. This gives h. The maps $x \mapsto (x, 0)$ and $x \mapsto (0, x)$ similarly give u_{L}, u_{R} without refinement. Input interchange preserves the family of Z. Strict carrier composition now proves every displayed identity. In particular, the zero linear composite has the zero cell as carrier in every degree and gives the constant basepoint map. □

Thus inversion, cancellation, the unit comparisons and symmetry already have strict finite simplicial representatives within the addition span. The map p remains a weak equivalence, and these identities do not supply an inverse simplicial map $D^{2} \to Z$. In this affine carrier construction the repeated-input map $x \mapsto (x, x)$ requires the additional coefficient 2, explaining the different behaviour of doubling.

The original pointed-set model is the case where C consists of all nonzero zero–one vectors. The construction here allows signed coefficients and repeated variables. It also admits the symmetric-flag replacement of Theorem 8.3, because arbitrary vertex pullbacks still preserve every coefficient family up to sign and zero. Thus it is compatible with the previously constructed reordering action.

The result supplies explicit comparisons between finite representatives and a product-preserving functor to the homotopy category. The next subsection specifies the localization and upgrades this to coherent integer operations in the localization of degreewise finite models. This still does not assert a strict ordinary multiplication on the original finite simplex sets.

9.4 Coherent integer operations through localization

The preceding roofs also admit a coherent interpretation. The useful intermediate object is an ordinary category of presentations, so no composition rule for arbitrary spans is required. Fix $k \geq 1$ throughout this subsection.

Definition 9.7. Let $\mathcal{P}$ have objects (m, C), where $m \geq 0$ and C is a finite coefficient family containing the coordinate vectors, as in Theorem 9.4. A morphism

$$M : (m, C) \longrightarrow (r, D)$$

is an integer matrix such that $M^{\mathrm{t}} D \setminus \{0\} \subseteq C$. Composition is matrix multiplication. Let $W \subset \mathcal{P}$ be the subcategory of identity-matrix arrows; these are precisely the refinements $(m, R) \to (m, C)$ with $C \subseteq R$.

Every morphism of $\mathcal{P}$ has an ordinary simplicial carrier map. Strict carrier composition therefore gives a functor

$$X^{(k)} : \mathcal{P} \longrightarrow \mathrm{sSet}_*, \qquad (m, C) \longmapsto X_C^{(k)},$$

where the basepoint is the zero-cell flag. Its values are degreewise finite, and it sends W to weak equivalences. The category $\mathcal{P}$ is countable; its families need not be uniformly bounded. Write $\mathcal{L}_{\mathrm{fin},*}$ for the localization of pointed degreewise finite simplicial sets at weak equivalences, the pointed version of $\mathcal{L}_{\mathrm{fin}}$ in Corollary 7.41.

Theorem 9.8. *Projection to the underlying matrix induces an equivalence*

$$N\mathcal{P}[W^{-1}] \simeq N\mathsf{Mat}_{\mathbb{Z}}$$

of infinity-categories. Consequently the carrier construction induces an infinity-functor

$$\Phi_k^{\mathrm{fin}} : N\mathsf{Mat}_{\mathbb{Z}} \longrightarrow \mathcal{L}_{\mathrm{fin},*}, \qquad m \longmapsto X_{E_m}^{(k)}. \tag{53}$$

Its composite with the functor to pointed spaces $\mathcal{S}_*$ preserves products and is naturally equivalent to the usual Eilenberg–Mac Lane functor $m \mapsto K(\mathbb{Z}^m, k+1)$, with fundamental classes identified by the canonical integral cocycles. The functor is presented by the hammock localization of $\mathcal{P}$: each simplex in a hammock mapping space is a finite array of the degreewise finite models $X_C^{(k)}$.

Proof. We verify the right calculus of fractions, with fractions written as roofs $A \leftarrow A' \rightarrow B$ whose left arrow belongs to W. Given

$$(m, C) \xrightarrow{M} (r, D) \xleftarrow{w} (r, D'), \qquad D \subseteq D',$$

put $C' = C \cup M^{\mathrm{t}} D'$, omitting zero vectors. There is a commuting square with top arrow $(m, C') \xrightarrow{M} (r, D')$, bottom arrow M, and both vertical arrows refinements directed towards the bottom. This is the required right Ore square. If postcomposition of two parallel arrows by a member of W makes them equal, their matrices were already equal, so the arrows themselves were equal. The subcategory W contains identities and is closed under composition. The same axioms hold for (W, W), using unions of coefficient families for the Ore squares.

Dwyer–Kan's comparison theorem for classical fractions [20, §7, Propositions 7.2–7.3], in its right-fraction form, now identifies the hammock localization of $(\mathcal{P}, W)$ with its ordinary localization: its mapping spaces have contractible components. In the ordinary localization a matrix $M : (m, C) \to (r, D)$ is represented by

$$(m, C) \longleftarrow (m, C \cup M^{\mathrm{t}} D) \xrightarrow{M} (r, D).$$

Two roofs with the same matrix agree after the union refinement; distinct matrices remain distinct under the projection. Thus all localized hom-sets are exactly the appropriate sets of integer matrices, and projection is essentially surjective. This proves the asserted equivalence of infinity-categories.

Since $X^{(k)}$ takes values in pointed degreewise finite simplicial sets and sends refinements to weak equivalences, the universal property of localization gives a functor to $\mathcal{L}_{\mathrm{fin},*}$. Transport across the preceding equivalence gives (53). The comparison of its composite to $\mathcal{S}_*$ with the usual functor is independent of any choice of inverses to the refinement maps. Put $n = k + 1$. The full infinity-category of pointed spaces $K(\mathbb{Z}^m, n)$ has discrete mapping-space homotopy types: the components are $\mathrm{Hom}(\mathbb{Z}^m, \mathbb{Z}^r)$, and for $i > 0$ their homotopy groups are

$$\widetilde{H}^{\,n-i}(K(\mathbb{Z}^m, n); \mathbb{Z}^r) = 0.$$

Here negative cohomology is zero and reduced degree-zero cohomology vanishes. The primitive cocycles identify the degree-n homotopy group with $\mathbb{Z}^m$, and the carry formula

$$\delta b_M = f_M^* c_D - M c_C$$

for an admissible carrier arrow shows that its action is precisely M. Hence these identifications give a natural equivalence with the usual Eilenberg–Mac Lane functor, not merely equality in the homotopy category. That functor preserves products, proving the assertion. One can also see the products directly: coordinate-block unions are products in $\mathcal{P}$, and $X^{(k)}$ takes them to literal products of simplicial sets.

Finally a simplex in a hammock mapping space is a finite array of arrows of $\mathcal{P}$, with the specified backward and vertical arrows in W. Applying the strict carrier functor gives a finite array of ordinary simplicial maps between degreewise finite objects. This proves the asserted finiteness of hammock simplices. □

Hammock mapping spaces need not themselves be Kan. To represent higher fillers, apply Ex^∞ to these mapping spaces. This functor preserves finite products, hence the enriched composition. Each simplex in this Kan replacement appears at some finite stage Ex^s, and therefore corresponds to a map from the finite simplicial set $\mathrm{Sd}^s \, \Delta[d]$ to the original hammock mapping space. A finite collection of such simplices uses a common finite stage. Thus its certificates consist of finitely many hammocks after finite subdivision; no assertion that an arbitrary filler is a single hammock is needed.

What is finite, and what is coherent. No full-faithfulness assertion about $\mathcal{L}_{\mathrm{fin},*} \to \mathcal{S}_*$ is needed, and product preservation is asserted here for the composite to pointed spaces. The complete localization is infinite. The assertion is that each hammock, hence each specified localization certificate, uses finitely many coefficient families, and every entry is finite in each simplicial degree. There is no assertion that one fixed finite family supports every matrix by a direct simplicial map. For example, closure under repeated pullback by doubling would require the coefficients $1, 2, 4, 8, \ldots$.

The compositional correction remains explicit. On the common refinement for M and N, equation (52) reads

$$b_{N\,M} = M^* b_N + N b_M.$$

Thus the chain-homotopies identifying the primitive integral classes obey the composition law exactly. The infinity-categorical assertion above is obtained

from localization; it does not infer all higher coherence merely from this cochain identity.

This construction concerns linear integer operations at one fixed Eilenberg–Mac Lane degree. It does not itself construct coherence between different deloopings, cup products, or nonlinear cohomology operations, nor a strict multiplication on the original finite simplex sets. Its contribution is an explicit finite presentation of the usual coherent linear structure, using the same wall families in every simplicial degree.

9.5 Exactly which linear diagrams admit finite strict models?

The integer-wall construction gives a converse to the obstruction in Theorem 7.35. Here a strict diagram means ordinary simplicial maps whose specified compositions agree literally, on one chosen model for each object.

Fix $n = k + 1 \geq 2$. Let I be a small category with finitely many objects, and let $M : I \to \mathrm{Mat}_{\mathbb{Z}}$ be a functor. Write $A_i = \mathbb{Z}^{m_i}$, with its given basis, and set

$$\mathcal{M}_{ij} = \{M(\alpha) : \alpha \in \mathrm{Hom}_I(i, j)\}.$$

The category may have infinitely many arrows; the condition below concerns their distinct matrix values.

Theorem 9.9 (Finite matrix image criterion). *The following conditions are equivalent:*

1. *Every set $\mathcal{M}_{ij}$ is finite.*
2. *There is a strict based diagram $Y : I \to \mathrm{sSet}$ of degreewise finite models of $K(A_i, n)$, inducing M on π_n under specified identifications.*

When these conditions hold, one may take $Y_i = X^{(k)}_{C_i}$, where

$$C_i = \bigcup_j \bigcup_{A \in \mathcal{M}_{ij}} A^{\mathrm{t}} E_{m_j} \setminus \{0\}, \qquad |C_i| \leq \sum_j m_j |\mathcal{M}_{ij}|. \tag{54}$$

Here E_m is the standard basis family. The coefficient families are finite once and for all, and work in every simplicial degree.

Proof. For necessity, the Hurewicz isomorphism identifies $H_n(Y_i; \mathbb{Z})$ with A_i. A simplicial map $Y_i \to Y_j$ induces its degree-n chain map from the function $(Y_i)_n \to (Y_j)_n$. Only finitely many such functions exist. Consequently there are at most $|(Y_j)_n|^{|(Y_i)_n|}$ induced homomorphisms on H_n, so $\mathcal{M}_{ij}$ is finite.

This argument applies to arbitrary degreewise finite simplicial models, not only to integer walls.

For sufficiency, identity arrows ensure $E_{m_i} \subseteq C_i$. If $\beta : i \to j$, then

$$M(\beta)^{\mathrm{t}} C_j \subseteq C_i \cup \{0\}.$$

Indeed, the pullback of a row of $M(\alpha)$, for $\alpha : j \to \ell$, is the corresponding row of $M(\alpha\beta)$. Thus the matrix map sends every open source cell into a single open target cell. Its carriers define an ordinary simplicial map $f_\beta : X^{(k)}_{C_i} \to X^{(k)}_{C_j}$. The carrier of a composite is the composite of its carriers, giving $f_{\alpha\beta} = f_\alpha f_\beta$ and strict identities. The zero cell is preserved, so these maps are based. Theorem 9.4 identifies their homotopy types and matrix actions. Finiteness and the displayed bound follow directly from the union defining C_i. □

As before, zero coefficients are discarded but a nonzero coefficient must not be divided by its common divisor. For example, a row $2e_j$ requires half-integer walls, even if e_j is already present.

Primitive classes and exact carry bookkeeping. Let $c_i \in Z^n(Y_i; A_i)$ be the vector of canonical primitive classes. For an arrow $\alpha : i \to j$, the degree-$(n-1)$ carry on a lifted flag, with full-face coordinate θ, is

$$b_\alpha = \lfloor M(\alpha)\theta \rfloor - M(\alpha)\lfloor \theta \rfloor.$$

The chosen source walls make it constant on the first open cell; it descends to a normalized integral cochain on Y_i. The equations from Section 9.3 now hold on the chosen objects themselves, without a further common refinement:

$$\delta b_\alpha = f_\alpha^* c_j - M(\alpha) c_i, \tag{55}$$

$$b_{\beta\alpha} = f_\alpha^* b_\beta + M(\beta) b_\alpha, \qquad b_{\mathrm{id}} = 0. \tag{56}$$

The parameter-simplex formula of Theorem 22.5 therefore applies unchanged to every composable chain in I. It gives coherent comparisons with the standard integral cocycle diagram using these primitive cocycles and carries. The comparison target is the usual infinite integral cocycle model; the flag models and each finite parameter domain are degreewise finite.

Corollary 9.10 (Self-maps and monodromy). *An integer matrix $A : \mathbb{Z}^m \to \mathbb{Z}^m$ is induced by a simplicial self-map of some degreewise finite model of $K(\mathbb{Z}^m, n)$ if and only if its powers form a finite set. Equivalently, $A^{a+b} = A^a$ for some $a \geq 0$, $b \geq 1$. The finite model can be chosen*

so that the same identity holds literally on its simplicial self-map. A representation $G \to \mathrm{GL}_m(\mathbb{Z})$ admits a strict action on such a model if and only if its image is finite.

Proof. Apply the theorem to the one-object category generated by A, or to the one-object group category G. The constructed carriers depend only on the matrix, so every equality between matrix powers holds literally. Conversely the theorem forces finite matrix image. □

For $m > 0$, the power condition is equivalent to the minimal polynomial of A over $\mathbb{Q}$ dividing $t^a(t^b - 1)$. In spectral terms its nonzero eigenvalues are roots of unity with no nontrivial Jordan blocks; nilpotent blocks at zero are allowed. In particular an invertible matrix is allowed exactly when it has finite order. Large coefficients alone cause no obstruction: a nilpotent matrix with an entry 2 still satisfies the criterion, whereas multiplication by 2 does not.

For a finite group H acting on $A = \mathbb{Z}^m$, the usual diagonal Borel construction has q-simplices $Y_q \times H^q$ and is degreewise finite. Its realization has $\pi_1 = H$, $\pi_n = A$ with the specified monodromy, and no other higher homotopy groups. The fixed basepoint gives a section over BH, so its Postnikov k-invariant is zero. Thus the classical split Postnikov construction acquires an explicit finite flag presentation with a prescribed finite monodromy action.

Finite computations need not be finite dynamical systems. Every finite acyclic diagram of integer matrices satisfies the criterion: it has only finitely many directed paths. In particular, two successive doublings admit the strict diagram

$$X^{(k)}_{\{1,2,4\}} \xrightarrow{f_2} X^{(k)}_{\{1,2\}} \xrightarrow{f_2} X^{(k)}_{\{1\}}, \qquad f_2 f_2 = f_4.$$

All three objects model $K(\mathbb{Z}, n)$, but they are different presentations. Turning the chain into an indefinitely repeatable self-map would require all powers of 2, and is impossible on any single degreewise finite simplicial model. The earlier roofs provide the comparison between the different presentations.

This criterion treats linear operations in one fixed Eilenberg–Mac Lane degree. It does not assert literal product preservation by the chosen models, or realize general nonlinear k-invariants and Sullivan differentials. It supplies strict finite geometry for any prescribed finite linear computation and identifies exactly when its feedback or monodromy can remain on fixed finite models.

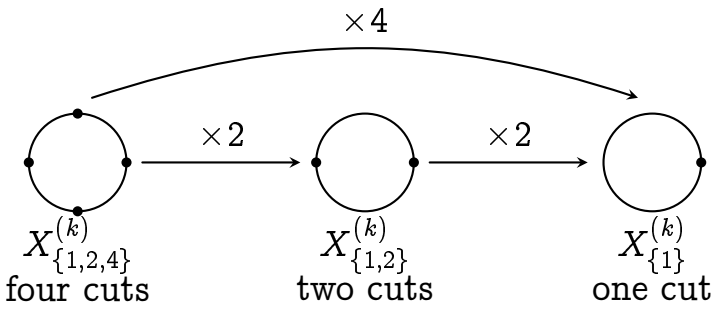


Figure 17: The lowest nontrivial torus row, in simplicial degree k, for two successive doublings. The wall coefficients give quarter, half, and integer cuts on $\mathbb{R}/\mathbb{Z}$. Both arrows and their composite have strict carrier maps on flags. A finite chain requires only finitely many cuts; a repeatable doubling loop on a fixed finite model cannot exist.

9.6 Finite arithmetic for every coherence dimension

The cocycle and carry formulas have bounds independent of the length of an operation diagram. First, the three-value result of Proposition 15.1 extends to every degree, including refinements by arbitrary finite coefficient families.

Proposition 9.11. *For $k \geq 1$, each coordinate of the canonical primitive cocycle c_{k+1} takes exactly the values*

$$\left\{ -\left\lceil \frac{k}{2} \right\rceil, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

Thus its alphabet has $k+1$ integers and its absolute value is at most $\lceil k/2 \rceil$.

Proof. Make the first cell F_0 of a lifted $(k+1)$-flag canonical. The character $\theta_{1\dots k+1}$ is an alternating sum of $k+1$ anchor coordinates in $[0,1)$, with $\lceil (k+1)/2 \rceil$ positive and $\lfloor (k+1)/2 \rfloor$ negative terms. Its value on F_0 therefore lies in

$$\left(-\left\lceil \frac{k}{2} \right\rceil, \left\lfloor \frac{k}{2} \right\rfloor + 1 \right).$$

Every integer level is a wall. Since $F_0 \leq F_1$, the strip containing F_1 has closure meeting this open interval. Its floor cannot fall outside the asserted alphabet, even when F_0 is on a wall and F_1 leaves through the lower adjacent strip. Formula (2) proves the bound. Varying the anchor coordinates in $(0,1)$ reaches every intervening integer strip; the constant flag at its containing open cell realizes the corresponding value. Further walls do not alter this argument. □

For a row $a = (a_i)$ of an integer matrix put $P_a = \sum_i \max(a_i, 0)$ and $N_a = -\sum_i \min(a_i, 0)$. The carry $b_a = \lfloor a\theta \rfloor - a\lfloor \theta \rfloor$ has the sharp range

$$b_a \in \begin{cases} \{-N_a, \dots, P_a - 1\}, & P_a > 0, \\ \{-N_a, \dots, 0\}, & P_a = 0. \end{cases} \tag{57}$$

Indeed it is $\lfloor a\{\theta\} \rfloor$ on the half-open anchor cube. Every listed integer occurs on a constant flag already in vertex degree k. In the purely negative case the extra endpoint 0 is attained at the origin; the zero row has only this value. These bounds sharpen the bounds following (52).

Now consider any integer-matrix diagram with finitely many objects and finitely many composite matrices between each pair, equipped with simultaneous finite wall models. Along a chain of length r write M_{ij} for the composite matrix from object i to object j, and μ_i for its carrier map from object 0. On the initial model set

$$q_i = M_{ir}\mu_i^* c^i, \qquad H_{ij} = M_{jr}\mu_i^* b_{M_{ij}}.$$

The integer carry identity gives $\delta H_{ij} = q_j - q_i$ and $H_{i\ell} = H_{ij} + H_{j\ell}$. Consequently the proof of Theorem 22.5 applies verbatim:

$$\mathcal{C} = q_0 + \delta \sum_{i=1}^{r} H_{0i} \smile \alpha_i \quad \text{on } X_0 \times \Delta[r].$$

There is a particularly short evaluation of this cocycle.

Proposition 9.12. *Put $n = k + 1$. For an n-simplex (σ, τ) of this product, set $\ell = \tau(n-1)$ and $j = \tau(n)$. Then*

$$\mathcal{C}(\sigma, \tau) = q_j(\sigma) + (-1)^k H_{\ell j}(\sigma|_{[0,\dots,n-1]}). \tag{58}$$

Let $L \geq 1$ bound the absolute row sums of all composite matrices, and let B bound all their sharp absolute carry values in (57). Every coordinate of every coherence cocycle then satisfies

$$|\mathcal{C}| \leq L\big(\lceil k/2 \rceil + B\big),$$

independently of the chain length r. Rank-zero objects contribute no coordinates; $L = 1$, $B = 0$ is allowed if all ranks are zero.

Proof. In the Alexander–Whitney cup convention, $\delta(H \smile \alpha) = \delta H \smile \alpha + (-1)^k H \smile \delta\alpha$. The first term evaluates α at the last parameter vertex j, changing q_0 to q_j. The second evaluates $\delta\alpha$ on the last parameter edge (ℓ, j) and gives $(-1)^k(H_{0j} - H_{0\ell}) = (-1)^k H_{\ell j}$ on the front face. This proves the formula, including a constant last edge. The two terms are bounded respectively by $L\lceil k/2 \rceil$ and LB. □

Thus arbitrary coherence dimension introduces no new integer alphabet: an evaluation reads one endpoint cocycle and one last-edge carry. The source flag sets still grow with degree. This is a uniform arithmetic bound for the specified diagrams, not a claim that a single finite model supports all integer matrices or that arbitrary coherence boundaries are fillable.

The certificate `verify_last_edge.py` compares (58) with the full coboundary formula as universal integer linear polynomials. It checks 552090 cases in cocycle degrees two through five and parameter dimensions zero through four, including repeated source vertices and rebased and repeated parameter labels. Omitting the edge term or reversing its sign is detected in 4790 cases each. These bounded checks support the evaluation formula; the proofs above give the uniform and all-degree assertions.

9.7 Why one fixed partition cannot suffice

There is a simple distinction between finiteness for an operation and a single finite presentation preserved by every operation. The duplication map $x \mapsto (x, x)$ followed by addition is $x \mapsto 2x$. At the lowest torus level, the sum wall $x_1 + x_2 \in \mathbb{Z}$ pulls back to $2x \in \mathbb{Z}$. Besides the original cut at zero this requires a cut at $1/2$; see Figure 18.

Proposition 9.13. *No finite circle partition with a nonempty vertex set makes doubling a map carrying each open cell into one open cell of that same partition.*

Proof. If $B \subset \mathbb{R}/\mathbb{Z}$ is the vertex set, every inverse image of a vertex must be a vertex. Otherwise doubling, which is a local homeomorphism, maps an interior point of an edge onto a vertex and nearby points onto edges, contradicting the carrier condition. Thus $2^{-1}B \subseteq B$. But $|2^{-1}B| = 2|B| > |B|$. □

Every fixed iterate nevertheless has a finite presentation: $x \mapsto 2^j x$ uses the cuts $a/2^j$ modulo one. Every finite list of expressions and its finite comparisons can likewise use finite common refinements. This does not give a single finite partition invariant under an expanding loop, even when that loop is the only arrow in a proposed diagram. General simultaneous triangulation questions also require care [10, Section 2].

The global coefficient-family theorem has a different, positive scope: each expression is represented in every simplicial degree by a span whose three objects are degreewise finite. Its comparisons are explicit and its action on

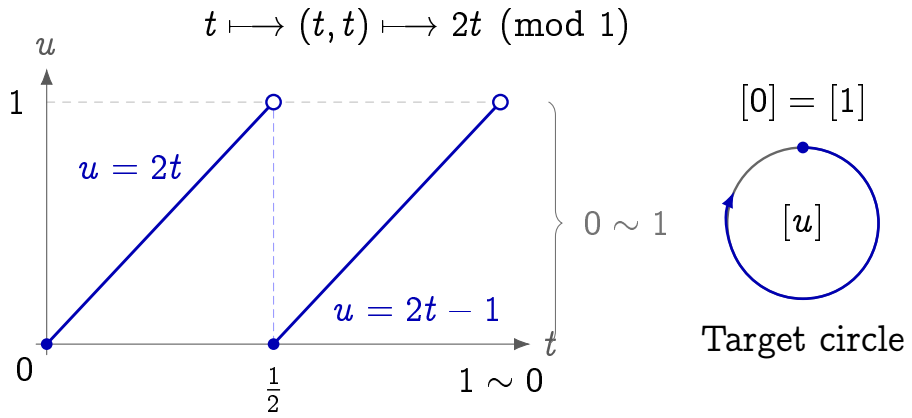


Figure 18: Duplication followed by addition gives the degree-two circle map. Pulling the sum wall $x + y = 1$ back along $t \mapsto (t, t)$ introduces the cut $t = 1/2$. The graph uses half-open coordinates $t, u \in [0, 1)$: open circles mark excluded endpoints and their one-sided limits. After identifying $u = 0$ with $u = 1$, both affine branches fit together continuously, each going once around the target circle.

fundamental classes is the prescribed integer matrix. Carrier refinement arrows realize weak equivalences; their geometric counterparts are subdivision homeomorphisms, but these are not the same maps.

The remaining target is an exact crossed construction with a replacement fold or presentation, retaining the affine coordinate changes. The present finite spans provide composition, inversion, and all integral combinations with coherent refinement comparisons. For the existing carrier fold, an intertwining homeomorphism is impossible by Corollary 8.11. The strictly natural ordinary row-triangulation route is also excluded for $k \geq 2$ by Theorem 8.17. These results do not rule out a broader crossed construction, a bare homeomorphism of total spaces, or a suitable replacement of the multiplication. In particular, Proposition 9.13 does not exclude subdivisions and coordinate changes depending on the operation. Section 10 constructs an explicit coherent deformation of the row operators to the affine diagram and gives a finite operad organizing these refinement choices. The resulting exact group structure belongs to the transported polyhedral realization. Section 11 identifies the original one-input realization homeomorphically with its affine group model. That identification does not turn the fixed carrier fold into a group multiplication.

10 Coherent geometric deformation and operadic refinement

10.1 Transported affine maps and an explicit coherent comparison

Retaining the geometric coordinates gives a positive construction on the existing row polyhedra. The affine torus maps can be transported through their barycentric homeomorphisms, and the comparison with the original carrier maps has explicit homotopies in every dimension. The transported endpoint is a strict simplicial topological group. Its vertex operators are generally piecewise linear maps, rather than ordinary simplicial maps on the fixed nerves.

The two diagrams on the covers. Use the category whose objects are wall presentations $(L_a, \mathcal{H}_a)$ and whose arrows $T : a \to b$ are lattice homomorphisms for which every open source face maps into one open target face. The condition $\mathcal{H}_a \supseteq T^{*}\mathcal{H}_b$ is sufficient. This category contains the vertex pullbacks and pointed-set sums of the original arrangements, and the refinement and matrix arrows used in the integer-operation roofs. Write $V_a = L_a \otimes \mathbb{R}$ and $\widetilde{\mathcal{P}}_a$ for the cover face poset.

For each closed face take the arithmetic mean of its vertices. These points define a lattice-equivariant barycentric homeomorphism

$$e_a : |N\widetilde{\mathcal{P}}_a| \xrightarrow{\cong} V_a.$$

The coordinates are rational, and translation commutes with their choice. If ρ_T is the carrier map, define maps in vector-space coordinates by

$$f_T = e_b |N\rho_T| e_a^{-1}, \qquad g_T = T.$$

Both assignments are strictly functorial:

$$f_U f_T = f_{UT}, \qquad g_U g_T = g_{UT}.$$

Moreover, for $\lambda \in L_a$,

$$f_T(x + \lambda) = f_T(x) + T\lambda, \qquad g_T(x + \lambda) = g_T(x) + T\lambda.$$

The distinction is that f_T records carriers, whereas g_T retains the actual affine image.

Proposition 10.1. *For every composable string*

$$a_0 \xrightarrow{T_1} a_1 \xrightarrow{T_2} \cdots \xrightarrow{T_n} a_n$$

there is an explicit map $H_{T_1,\dots,T_n} : \Delta^n \times V_{a_0} \to V_{a_n}$. These maps give a homotopy-coherent comparison between the two diagrams, with identity object components. They are equivariant for the total lattice homomorphism and descend to the quotient tori.

Proof. For $0 \le j \le n$ put

$$P_j(x) = g_{T_n} \cdots g_{T_{j+1}} f_{T_j} \cdots f_{T_1}(x),$$

where an empty composite is the identity. Define

$$H_{T_1,\dots,T_n}(t_0, \dots, t_n; x) = \sum_{j=0}^{n} t_j P_j(x). \tag{59}$$

For a string of length zero this is the identity of its object. For a single arrow it is the straight segment from $g_T(x)$ to $f_T(x)$.

Here are all the coherence equations. On the face $t_i = 0$ the formula becomes, with the remaining barycentric coordinates understood,

$$\begin{cases} H_{T_2,\dots,T_n}(-; f_{T_1}(x)), & i = 0, \\ H_{T_1,\dots,T_{i+1}T_i,\dots,T_n}(-; x), & 0 < i < n, \\ g_{T_n}\big(H_{T_1,\dots,T_{n-1}}(-; x)\big), & i = n. \end{cases}$$

The first and the interior identities use strict composition of the f's and g's. The last also uses linearity of g_{T_n}, which commutes with the convex combination. Inserting an identity arrow repeats two adjacent P_j's; merging their barycentric coordinates gives exactly the degeneracy equation. Thus the formula specifies all faces and degeneracies, not just separate homotopies for individual arrows.

Every P_j has the same lattice equivariance, with linear part $T_n \cdots T_1$. Since $\sum_j t_j = 1$, their convex combination has that equivariance as well. Hence it descends to the tori and, by the e_a, to the original row polyhedra.

There is also a common open geometric carrier. Let F be the largest face having positive coefficient in the barycentric support of x. Then x lies in the relative interior of F. The carrier prefix sends it into the relative interior of the corresponding intermediate carrier: the coefficient of its largest image face remains positive. The remaining linear maps send that open cell into the open total carrier. Hence every $P_j(x)$, and their convex combination, lie in the same open face $\rho_{T_n \cdots T_1}(F)$. Relative interiors of convex polytopes are convex. The formulas agree on flag faces and are continuous. □

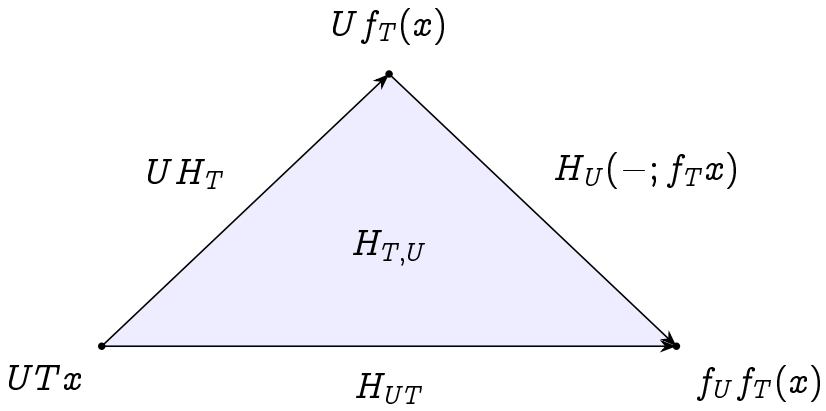


Figure 19: The parameter triangle for the coherent geometric deformation. Its long edge deforms the composite; the other two deform the successive maps. The convex formula fills the triangle, and the same formula supplies all higher parameter simplices. Its image in the target polytope may have lower dimension.

In particular, every further vertex restriction R has constant integer-wall status throughout this comparison: for each target face character θ_I and every j,

$$\theta_I\big(RH_{T_1,\dots,T_n}(t;x)\big) \in \mathbb{Z} \quad \Longleftrightarrow \quad \theta_I\big(RP_j(x)\big) \in \mathbb{Z}.$$

Both points lie in the same open carrier after applying R. Thus the comparison detects exactly the same zero restrictions on the quotient tori, even though their nonintegral values vary. This is useful control of the row homotopies. It does not yet produce maps of the two total realizations whose inverse image of the canonical point is exactly that point: such maps must also respect the identifications where outer simplex weights vanish.

For two arrows the comparison is particularly visible: its triangle has vertices

$$UTx, \qquad U f_T(x), \qquad f_U f_T(x).$$

One boundary edge is the homotopy for the composite UT. The other two are the successive homotopies for T and U, with the stated precomposition and postcomposition. The triangle compares them. Formula (59) supplies every higher comparison in the same manner. It does not assert that simultaneous linear interpolation of all arrow maps is strictly functorial at each intermediate parameter.

Finiteness and the geometry of the formulas. For a fixed chain of maps, vertex degree, and arity, the formulas use finitely many rational charts modulo the source lattice. Each P_j is piecewise affine. On a common source subdivision, (59) is polynomial in the barycentric parameters and the source

coordinates, with rational coefficients. It is piecewise linear in the source variable for each fixed parameter, but generally is not jointly piecewise linear in the parameters and source variable: products $t_j x_i$ occur. Finite exact formulas are therefore being asserted, not a finite triangulation on which every parameterized homotopy is affine.

Some comparisons are already constant. If T maps each closed source face affinely and isomorphically onto its target carrier, then it carries the vertex mean to the vertex mean, so $f_T = g_T$. This applies to arrangement isomorphisms, including the permitted input and vertex permutations and simultaneous inversion. For the subset-sum arrangements it also applies to insertion of zero inputs: its image is a face subcomplex with the original induced arrangement. In particular the fold comparison is constant on its unit faces. Independent input reversals still require the appropriate signed coefficient refinements when they do not preserve the arrangement.

The strict geometric endpoint. Let $\mathcal{C}_a$ be the orbit-incidence category of $\widetilde{\mathcal{P}}_a$ under L_a, retaining every lattice-labelled incidence branch, and put

$$P_a = |N\mathcal{C}_a|, \qquad \bar{e}_a : P_a \xrightarrow{\cong} V_a/L_a.$$

On these same row polyhedra replace the carrier operator by

$$G_T = \bar{e}_b^{-1}\,\overline{T}\,\bar{e}_a. \tag{60}$$

These operators compose strictly. They have finite rational piecewise linear descriptions: intersect each source barycentric simplex with the inverse images of the target barycentric simplices, and use the actual linear map on each resulting chart. Only finitely many target charts meet the image of a compact source chart.

In particular, the one-input spaces P_q with the operators G_α form a simplicial topological abelian group. Its levelwise group law is

$$x \oplus_q y = \bar{e}_q^{-1}\big(\bar{e}_q(x) + \bar{e}_q(y)\big).$$

The maps $\bar{e}_q$ now form an actual isomorphism of simplicial topological groups with $A_\bullet^{(k)}$. Its realization is therefore homeomorphic to $|A_\bullet^{(k)}|$ and carries its exact topological abelian group law. For the multi-input presentations the transported product comparison is the actual homeomorphism

$$P_q(S) \xrightarrow{\bar{e}_q(S)} A_q^{(k)}(S) \xrightarrow{\prod_s \bar{e}_q(1_+)^{-1}} \prod_{s \in S^\circ} P_q(1_+).$$

Under this comparison the transported fold is addition. These homeomorphisms commute with the new vertex operators.

This strengthens the polyhedral prototype by comparing its operators coherently with the old carrier operators on the same row spaces. It changes the vertex operators. The identity homeomorphisms of the individual row spaces, together with the homotopies above, are not a strict natural homeomorphism between the two simplicial diagrams. Consequently no homeomorphism from the original finite diagonal $|D_0^{(k)}|$ to the new realization is supplied here. The construction also does not contradict Theorem 8.17: the new row maps are piecewise linear maps, not ordinary simplicial maps on a compatible collection of finite row triangulations.

Why the arrowwise interpolation is not a strict diagram. The higher comparisons are necessary even for refinement arrows whose underlying linear maps are identities. Write

$$r_T(x) = f_T(x) - Tx, \qquad h_s(T)(x) = (1-s)f_T(x) + sTx.$$

Strict composition of the carrier maps gives

$$r_{UT}(x) = Ur_T(x) + r_U(f_T(x)).$$

Consequently the defect of simultaneous interpolation is exactly

$$\begin{aligned} &h_s(U)\big(h_s(T)(x)\big) - h_s(UT)(x) \\ &\qquad = (1-s)\big[r_U(h_s(T)(x)) - r_U(f_T(x))\big]. \end{aligned} \tag{61}$$

This follows by substituting $h_s(T) = T + (1-s)r_T$ on both sides; no linearity of the piecewise affine function r_U is assumed.

Here is an exact example on $\mathbb{R}/\mathbb{Z}$. On the cover use, successively, the quarter, half, and integer grids

$$\mathcal{Q} = \tfrac{1}{4}\mathbb{Z}, \qquad \mathcal{R} = \tfrac{1}{2}\mathbb{Z}, \qquad \mathcal{H} = \mathbb{Z}.$$

The refinement arrows $T : \mathcal{Q} \to \mathcal{R}$ and $U : \mathcal{R} \to \mathcal{H}$ both have underlying linear map $\mathrm{id}_{\mathbb{R}}$. Transport the carrier nerve maps through the barycentric evaluations that send an interval to its midpoint. For $0 \le x \le 1$ the second map is

$$f_U(x) = \begin{cases} 2x, & 0 \le x \le \tfrac{1}{4}, \\ \tfrac{1}{2}, & \tfrac{1}{4} \le x \le \tfrac{3}{4}, \\ 2x - 1, & \tfrac{3}{4} \le x \le 1. \end{cases}$$

The first map is its rescaling to each half-period: $f_T(x) = \frac{1}{2} f_U(2x)$ for $0 \le x \le \frac{1}{2}$, extended by $f_T(x + \frac{1}{2}) = f_T(x) + \frac{1}{2}$. In particular $f_T(\frac{1}{8}) = \frac{1}{4}$ and $f_{UT}(\frac{1}{8}) = f_U f_T(\frac{1}{8}) = \frac{1}{2}$. Hence

$$\begin{aligned} h_{1/2}(T)(\tfrac{1}{8}) &= \tfrac{3}{16}, \\ h_{1/2}(U)\big(h_{1/2}(T)(\tfrac{1}{8})\big) &= \tfrac{9}{32}, \\ h_{1/2}(UT)(\tfrac{1}{8}) &= \tfrac{5}{16}. \end{aligned}$$

The defect is $-\frac{1}{32}$, which remains nonzero on the circle. These are presentations from the coefficient families $\{1, 2, 4\}$, $\{1, 2\}$, and $\{1\}$ in vertex degree $q = k$.

The comparison triangle in Proposition 10.1 has vertices UTx, $U f_T(x)$, and $f_U f_T(x)$. Its boundary compares the direct interpolation for UT with the successive interpolations, first from $f_U f_T$ to $U f_T$, then from $U f_T$ to UT. The simultaneous path $h_s(U)h_s(T)$ is not a boundary edge of that triangle. It is homotopic relative endpoints to the successive path through the square $(s, t) \mapsto h_t(U)h_s(T)$. Thus the triangle and square account for the composition defect; they do not make that defect identically zero.

10.2 Semidetnov's operad and finite refinement choices

Semidetnov's construction suggests organizing the choices of presentations by an operad. His hypotheses include left and right insertions and a compatibility between block insertion and transport [72, Definitions 2.1–2.4]. Under these hypotheses, Theorem 3.6 constructs an operad in action groupoids. In the symmetric case, its arity-m groupoid is the chaotic groupoid on Σ_m: its objects are permutations, and there is exactly one arrow between any two objects. Its nerve is the standard Barratt–Eccles E_∞ operad [72, Remark 3.7]. We use this established symmetric case as an external source of ordering parameters. We do not apply the theorem directly to our incidence categories, which are not the required action groupoids.

Here arity m, nerve degree p, and vertex degree q are three different indices. Semidetnov's shifted indexing uses the group G_{m-1} in arity m; it does not identify arity with our q. The finite wall choices below supply a separate operad, with an elementary proof of its substitution law.

Definition 10.2. For $m \ge 1$, let $\mathsf{P}(m)$ be the poset of families C of nonempty subsets of $\{1, \dots, m\}$ which contain every singleton and the total set. Order these families by reverse inclusion: there is an arrow $C' \to C$ when $C' \supseteq C$. Set $\mathsf{P}(0) = \{\varnothing\}$.

For $C \in \mathsf{P}(m)$ and $D_i \in \mathsf{P}(n_i)$, put $N = \sum_i n_i$ and partition $\{1, \dots, N\}$ into consecutive blocks $B_1, \dots, B_m$ of sizes $n_1, \dots, n_m$. Some blocks may be

empty. With D_i embedded in its block, define

$$C \circ (D_1, \dots, D_m) = \left\{ \bigcup_{i \in U} B_i : U \in C \right\}_{\neq \varnothing} \cup \bigcup_i D_i. \tag{62}$$

The subscript means that the empty set is discarded. Permutations act by relabelling the underlying set.

Proposition 10.3. *These posets and substitutions form a symmetric operad in finite categories, including nullary operations. Each* $\mathsf{P}(m)$ *has a terminal object: the family of singletons together with the total set, or the empty family when* $m = 0$.

Proof. Every singleton of the final set occurs in one of the D_i. The total set occurs as the image of the total set in C, unless $N = 0$, when the result is the empty family. Thus substitution is well defined. Increasing any family increases the resulting family, so substitution is a functor for the specified order.

For associativity, write the first term of (62) as pullback along the block-collapse map $b : \{1, \dots, N\} \to \{1, \dots, m\}$. In a three-level substitution, either order of evaluation produces the union of the pulled-back outer family, the pulled-back middle families, and the innermost families. Pullback commutes with union and with composition. Discarding empty sets commutes with these operations, so the argument includes empty blocks.

The unique family in $\mathsf{P}(1)$ is the unit. Inserting it in each input adds only singletons already present. Substituting D into it adds only the total set already present in D, with the empty result when $D \in \mathsf{P}(0)$. Substituting nullaries in every input likewise gives the empty family. Relabelling commutes with unions and block pullback, proving symmetric equivariance. Finally, every family contains the stated minimal family, which is therefore terminal in the reverse-inclusion order. □

The terminal choices are not themselves closed under substitution. For example, the two binary parenthesizations on three inputs give

$$C_L = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 2, 3\}\},$$
$$C_R = \{\{1\}, \{2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Both have refinement arrows to the minimal three-input family. Their union refines both. Keeping these arrows records the comparison rather than imposing equality of the two presentations.

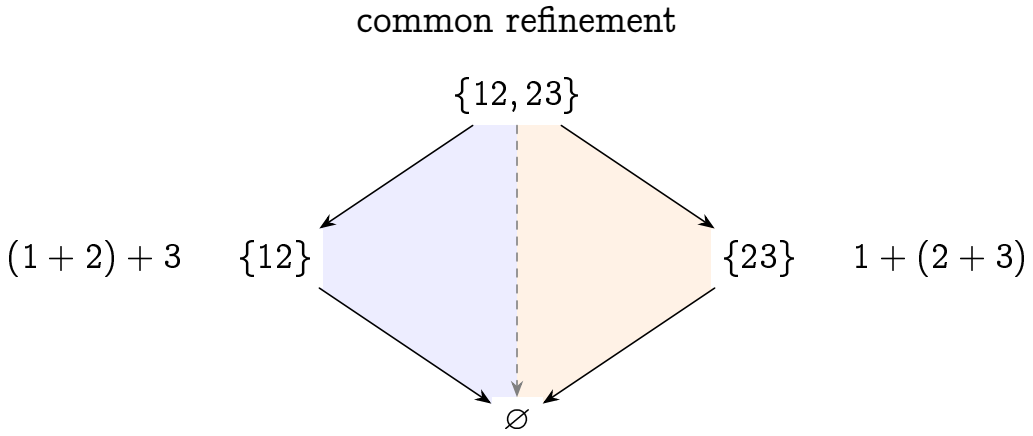


Figure 20: A square in the eight-object refinement poset $\mathsf{P}(3)$. Labels list only optional pair walls; the singleton and total-sum walls are always present. Arrows forget extra walls. The two shaded nerve triangles compare the binary presentations through their common refinement and the minimal presentation.

Let $\mathcal{E}\Sigma_m$ be the chaotic groupoid on Σ_m, with the usual substitution of permutation blocks. Define

$$\mathcal{O}(m) = N\big(\mathcal{E}\Sigma_m \times \mathsf{P}(m)\big). \tag{63}$$

For arity zero both factors are the terminal category.

Theorem 10.4. *The simplicial operad $\mathcal{O}$ is an E_∞ operad, finite in each arity and simplicial degree. Its exact counts are*

$$|\mathcal{O}(m)_p| = (m!)^{p+1}(p+2)^{2^m-m-2} \qquad (m \geq 2). \tag{64}$$

The simplicial sets $\mathcal{O}(0)$ and $\mathcal{O}(1)$ are points.

Proof. Products of categorical operads are categorical operads, and the nerve preserves their products and substitution maps. The groupoid $\mathcal{E}\Sigma_m$ is contractible, and $\mathsf{P}(m)$ is contractible because it has a terminal object. The diagonal Σ_m-action on their product nerve is free, since it is free on the permutation entries in the first factor. Thus each operation space is contractible with free symmetric group action, as required for an E_∞ operad.

For $m \geq 2$ there are $2^m - m - 2$ optional subsets: all the nonempty subsets other than the singletons and the total set. In a weak chain $C_0 \supseteq \cdots \supseteq C_p$, any optional subset can survive for a prefix of length $0, 1, \ldots, p+1$. These

$p+2$ choices are independent. A p-simplex of the chaotic groupoid nerve is an arbitrary list of $p+1$ permutations. Multiplying the two counts gives (64). □

What the operation labels present. Fix $k \geq 1$ and abbreviate $D = D_0^{(k)}$. Identify a subset $U \subseteq \{1, \dots, m\}$ with its characteristic vector in $\mathbb{Z}^m$. A family $C \in \mathsf{P}(m)$ then gives the finite model $X_C^{(k)}$ of Section 9.3 and the operation roof

$$D^m \xleftarrow{s_C} X_C^{(k)} \xrightarrow{\mu_C} D. \tag{65}$$

The left map uses the singleton coefficients and is a weak equivalence; the right map uses the total-set coefficient and is the carrier fold. A refinement arrow in $\mathsf{P}(m)$ gives a strictly commuting comparison of these roofs. All its middle objects remain degreewise finite.

For $E = C \circ (D_1, \dots, D_m)$, block summation and block projection give a strictly commuting square

$$\begin{array}{ccc} X_E^{(k)} & \xrightarrow{b_*} & X_C^{(k)} \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s_C} \\ \displaystyle\prod_i X_{D_i}^{(k)} & \xrightarrow{\prod_i \mu_{D_i}} & D^m. \end{array} \tag{66}$$

Indeed, the pullback of an outer subset-sum wall is the wall for the union of its input blocks, and every inner wall is also included in E. These are exactly the two terms of (62). The composite fold satisfies $\mu_C b_* = \mu_E$ strictly. The same calculation for three levels makes all substitution comparisons commute, including those with empty input blocks.

Both vertical maps in (66) are weak equivalences. For the left map, the union of the embedded inner families presents the product of their arrangements, and E refines that union. Thus the square is homotopy cartesian: projection from its homotopy pullback to the lower left corner is a weak equivalence, and the map from the upper left corner has the same property. This does not identify $X_E^{(k)}$ with the ordinary categorical pullback; the carrier maps have not been asserted to be fibrations.

Accordingly, $\mathcal{O}$ organizes finite operation presentations and their refinement comparisons. Its permutation factor is the standard Barratt–Eccles factor, and its second factor records the explicit wall choices. No priority assertion about an E_∞ operad is intended. The construction does not give an ordinary simplicial action $\mathcal{O}(m) \times D^m \to D$. Indeed, a unital action would supply an ordinary simplicial binary multiplication with a unit, excluded by Theorem F.2. Transporting the resolved products to geometric

operations on $|D|$ would require compatible homotopy inverses and their coherence data. Semidetnov's perspective helps organize those parameters; it does not by itself prove a product homeomorphism or retain the fixed carrier fold as an exact group multiplication.

11 Homeomorphisms with the affine group realization

11.1 A degreewise finite contractible stabilizer

There is a simple way to add an infinite-dimensional Euclidean factor without losing finiteness in any simplicial degree. This construction will allow a homeomorphism result after an explicit enlargement of the finite models.

For $n \geq 2$, let $\Lambda^n_0 \subset \Delta[n]$ be the zeroth horn and form the pointed simplicial quotient

$$C^{(n)} = \Delta[n]/\Lambda^n_0, \qquad C = \prod_{n\geq 2} C^{(n)}.$$

The product is taken in simplicial sets. Its apparent infinite size is harmless in a fixed degree: $C^{(n)}_q$ is a singleton if $n > q+1$. In particular, C has one vertex.

Proposition 11.1. *The simplicial set C is degreewise finite, with*

$$\#C_q = \prod_{n=2}^{q+1} \left(1 + \binom{q+1}{n}\right). \tag{67}$$

There is a homeomorphism

$$|C| \cong \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty = \varinjlim_N \mathbb{R}^N$ has the direct-limit topology. Consequently C is contractible after realization. For every degreewise finite simplicial set X, the product $X \times C$ remains degreewise finite and the projection $X \times C \to X$ is a weak equivalence.

Proof. A simplex $[q] \to [n]$ lies outside the zeroth horn precisely when its image contains all the vertices $1, \dots, n$. Its image is therefore either $\{1, \dots, n\}$ or $\{0, \dots, n\}$. The numbers of monotone surjections onto these ordered sets are respectively $\binom{q}{n-1}$ and $\binom{q}{n}$. Adding the collapsed basepoint gives

$$\#C^{(n)}_q = 1 + \binom{q}{n-1} + \binom{q}{n} = 1 + \binom{q+1}{n}.$$

Here a binomial coefficient is zero outside its usual range. This proves the degreewise assertion and (67).

We next identify the realization of each factor, including its basepoint. Write the geometric simplex as the cone on its facet F opposite vertex 0. The geometric horn is the cone on ∂F. Thus

$$|C^{(n)}| = \Delta^n / |\Lambda^n_0| = \operatorname{cone}(F)/\operatorname{cone}(\partial F).$$

The complement of the collapsed point is

$$\operatorname{int}(F) \times (0,1] \cong \mathbb{R}^{n-1} \times [0,\infty).$$

The compact quotient is its one-point compactification. Inverse stereographic projection identifies the one-point compactification of this closed half-space with a closed hemisphere of S^n; the added point lies on the boundary. We therefore have a pointed homeomorphism

$$(|C^{(n)}|, *) \cong (I^n, 0),$$

where $I = [0,1]$ and 0 is a corner of the cube.

Put $C_{\le N} = \prod_{n=2}^{N} C^{(n)}$ and include it into the next product by adjoining the basepoint. In each degree these products eventually stabilize, so that

$$C = \varinjlim_N C_{\le N}.$$

Realization preserves this colimit and finite products. Each finite product here is a finite simplicial set, so its realization is compact and the product homeomorphism is the ordinary one. Choosing the pointed factor homeomorphisms above gives compatible homeomorphisms

$$|C| \cong \varinjlim_N \prod_{n=2}^{N} I^n = \varinjlim_N I^{d_N},$$

$$d_N = \sum_{n=2}^{N} n = \frac{N(N+1)}{2} - 1.$$

This is a cofinal subsequence of the standard cube tower, with all new coordinates zero. Hence $|C| \cong I^\infty$. Sakai's Corollary 6.4.3 identifies I^∞ with $\mathbb{R}^\infty$ [68].

Finally, a degreewise finite X has countable CW realization. Such a space is a direct limit of a tower of compact finite subcomplexes. Finite products of these direct limits have the ordinary product topology [68, Proposition 6.1.3(2)]; equivalently, the usual compactly generated

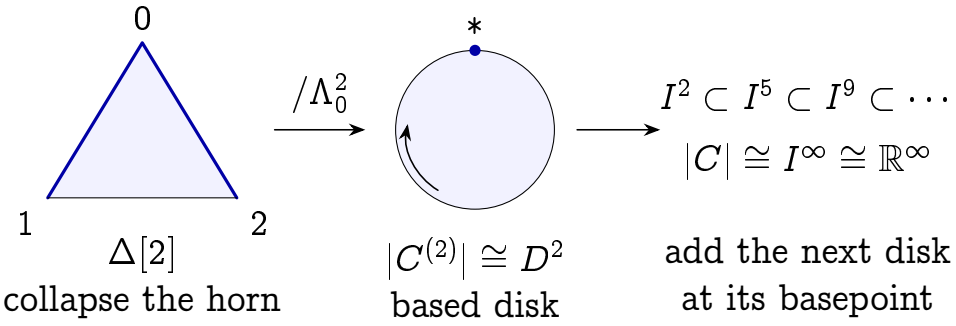


Figure 21: The contractible factor supplies extra coordinates without losing degreewise finiteness. The first horn quotient is a disk; the successive finite products are cubes of dimensions $2, 5, 9, \ldots$. A factor $C^{(n)}$ first contributes a nonconstant simplex in degree $n-1$, so only finitely many factors are visible in any fixed degree.

realization-product homeomorphism is an ordinary homeomorphism in this case. Thus

$$|X \times C| \cong |X| \times \mathbb{R}^\infty.$$

The factor $\mathbb{R}^\infty$ is contractible, so projection is a homotopy equivalence on realizations. □

The topology in this proposition is essential: I^∞ is the direct limit of finite cubes with appended zero coordinates, not the countable product cube with its product topology. Likewise, $\mathbb{R}^\infty$ denotes the direct limit of finite-dimensional Euclidean spaces. The construction adds simplices in arbitrarily high degrees, while keeping every individual degree finite.

Preserving a prescribed finite skeleton. For any integer $b \geq 0$ the same argument applies to the tail

$$C_{>b} = \prod_{n \geq b+2} \Delta[n]/\Lambda^n_0.$$

It still has realization $\mathbb{R}^\infty$, but $(C_{>b})_q = *$ for every $q \leq b$. Thus $X \times C_{>b}$ agrees canonically with X in all degrees through b. Its projection has a section given by the basepoint, so X is also a simplicial retract of the enlarged model. Every homeomorphism conclusion below holds with $C_{>b}$ in place of C. In particular, the enlargement can be postponed beyond any prescribed finite skeleton; the transported group law is not asserted to preserve the old carrier formulas on that skeleton.

11.2 An actual homeomorphism after a finite stabilization

The contractible finite factor just constructed permits a positive homeomorphism theorem. The extra factor has a geometric purpose: it supplies fresh coordinates in which points can be separated, without changing any prescribed compact part of an embedding. The direct-limit space $\mathbb{R}^\infty$ introduced above is a topological vector space.

We use two classical results of infinite-dimensional topology. Among direct limits of finite-dimensional compact metrizable spaces, Sakai's recognition theorem characterizes $\mathbb{R}^\infty$-manifolds by extension of embeddings from closed compact pairs to neighbourhoods. His classification theorem says that such manifolds of the same homotopy type are homeomorphic [68, Theorems 6.4.8 and 6.5.1]. We verify the recognition hypothesis explicitly, so that no regularity or triangulability of the original CW structure is being assumed.

Lemma 11.2. *If M is a nonempty countable CW complex, then $M \times \mathbb{R}^\infty$ is an $\mathbb{R}^\infty$-manifold. Consequently, weakly homotopy equivalent countable CW complexes M and N satisfy*

$$M \times \mathbb{R}^\infty \;\cong\; N \times \mathbb{R}^\infty.$$

Proof. Exhaust M by finite subcomplexes $M_1 \subseteq M_2 \subseteq \cdots$. Each M_i is a finite-dimensional compact metrizable ANR. This standard fact about finite CW complexes does not require their cells to be regular. Every compact subset of M lies in some finite subcomplex. The product topology satisfies

$$M \times \mathbb{R}^\infty = \operatorname*{colim}_i \big(M_i \times [-i,i]^i\big),$$

by the product theorem for towers of locally compact spaces [68, Proposition 6.1.3(2)]. Thus the required finite-dimensional compact exhaustion is available.

Let B be a closed subset of a finite-dimensional compact metrizable space A, and let $f : B \hookrightarrow M \times \mathbb{R}^\infty$ be an embedding. The empty case is immediate. Compactness puts its image in $M_i \times \mathbb{R}^m$ for some i, m. Write $f = (u, v)$. The ANR property extends u to a map $\widetilde{u} : U \to M_i$ on an open neighbourhood U of B in A. Coordinatewise Tietze extension extends v to $\widetilde{v} : A \to \mathbb{R}^m$. Choose an embedding $e : A \hookrightarrow \mathbb{R}^a$, a metric on A, and set $\delta(x) = d(x, B)$. The fresh-coordinate map

$$h(x) = \big(\delta(x), \delta(x) e(x)\big) \in \mathbb{R}^{a+1}$$

vanishes exactly on B and is injective on $A \setminus B$. Choose a closed neighbourhood V of B with $V \subseteq U$. Then

$$x \longmapsto \big(\widetilde{u}(x), \widetilde{v}(x), h(x)\big), \qquad x \in V,$$

extends f, using coordinates beyond the first m. It separates points of B because f does, separates all remaining points by h, and separates B from its complement by δ. It is an embedding because V is compact and the target is Hausdorff. Restriction to the interior of V proves the embedding-neighbourhood extension property. Sakai's recognition theorem therefore applies.

A weak equivalence between CW complexes is a homotopy equivalence. The factor $\mathbb{R}^\infty$ is contractible, so the two stabilized manifolds have the same homotopy type. Their homeomorphism follows from classification. $\square$

Theorem 11.3. *Fix $k \geq 1$, write $D = D_0^{(k)}$ and $Z = X_0^{(k)}(2_+)$, and let C be the degreewise finite contractible simplicial set of Proposition 11.1, with $|C| \cong \mathbb{R}^\infty$. Put $D^{\mathrm{st}} = D \times C$. There are homeomorphisms*

$$|D^{\mathrm{st}}| \cong |K(\mathbb{Z}, k+1)| \times \mathbb{R}^\infty,$$
$$|Z \times C| \cong |D^{\mathrm{st}}| \times |D^{\mathrm{st}}|.$$

The model D^{st} is degreewise finite and has the homotopy type $K(\mathbb{Z}, k+1)$. Its realization admits the structure of a topological abelian group.

Proof. The realizations of D, Z, and $D \times D$ are countable CW complexes. The standard simplicial abelian group $K(\mathbb{Z}, k+1)$ is countable in each degree, so its realization is also a countable CW complex. Apply Lemma 11.2 first to the classifying weak equivalence of Theorem 4.1, and then to the carrier weak equivalence $Z \to D \times D$. Realization preserves finite products [19, Section 1]. Here the ordinary product topology also agrees with the compactly generated one: each factor is a countable direct limit of compact finite subcomplexes, and the same product theorem applies. Finally, interlacing coordinates gives $\mathbb{R}^\infty \times \mathbb{R}^\infty \cong \mathbb{R}^\infty$. These facts give both displayed homeomorphisms.

The first target is a topological abelian group: its first factor is the realization of a simplicial abelian group, and its second factor has vector addition. Transport this law along the first homeomorphism. Translating that homeomorphism in its target, if necessary, makes the chosen basepoint the identity. Degreewise finiteness follows directly from that of D and C, while contractibility of $|C|$ preserves the homotopy type. $\square$

The theorem proves existence of homeomorphisms without selecting a preferred formula. The transported group law is continuous. It is not claimed to arise from an ordinary simplicial multiplication on D^{st}. Such a multiplication with a homotopy unit is excluded by Theorem F.2. Nor do these homeomorphisms identify the original carrier fold with the new group law. Removing $\mathbb{R}^{\infty}$ requires a separate argument. For the original one-input spaces this is provided by Theorem 11.36 and Proposition 11.38 below.

11.3 A homeomorphism to the affine group realization

The affine realization itself has the local topology needed for the classification theorem. Thus the contractible finite factor can be placed entirely on the finite simplicial model.

Theorem 11.4. *For $k \geq 1$ and $r \geq 1$, put*

$$A_n = Z^k(\Delta[n]; (\mathbb{R}/\mathbb{Z})^r), \qquad G_r = |A_\bullet|.$$

Then G_r is an $\mathbb{R}^{\infty}$-manifold.

Proof. Write

$$V_n = Z^k(\Delta[n]; \mathbb{R}^r), \qquad L_n = Z^k(\Delta[n]; \mathbb{Z}^r), \qquad d_n = \dim V_n = r\binom{n}{k}.$$

The anchor coordinates give $A_n = V_n/L_n$. Let $K_n \subseteq G_r$ be the image of the realization through simplicial degree n. These are compact metrizable finite-dimensional ANRs, with closed inclusions and direct-limit topology. To check the ANR assertion without assuming compatible ordinary row triangulations, observe that the latching subset

$$L_nA = \bigcup_{i=0}^{n-1} s_i A_{n-1} \subseteq A_n$$

is a finite union of rational subtori. Set

$$P_n = A_n \times \Delta^n, \qquad B_n = (A_n \times \partial\Delta^n) \cup (L_nA \times \Delta^n).$$

The pair (P_n, B_n) is a finite polyhedral pair. The skeletal pushout expresses K_n as the adjunction of P_n to K_{n-1} along a continuous map $B_n \to K_{n-1}$. This is a compact metrizable adjunction space: its defining equivalence relation on the compact metric space $P_n \amalg K_{n-1}$ is closed. Starting from

$K_0 = \{0\}$, Whitehead's compact-ANR adjunction theorem therefore proves the ANR assertion by induction [79, Theorem 1].

Finite dimensionality follows separately, without assuming that a quotient preserves dimension. Let $q_n : P_n \to K_n$ be the characteristic quotient map. Its restriction to $P_n \setminus B_n$ is a homeomorphism onto $K_n \setminus K_{n-1}$. Choose compact sets $C_j \subseteq P_n \setminus B_n$ exhausting this open subset, for example the sets at distance at least $1/j$ from B_n. Then K_n is the countable closed union

$$K_{n-1} \ \cup \ \bigcup_{j \geq 1} q_n(C_j).$$

Each $q_n(C_j)$ is homeomorphic to C_j, so has dimension at most $\dim P_n = d_n + n$. The countable closed-sum theorem for metric dimension [68, Theorem 1.12.8] gives

$$\dim K_n \leq \max\{\dim K_{n-1}, d_n + n\} < \infty.$$

No CW structure on the adjunction space and no PL structure on its quotient map are needed.

Every compact subset of G_r lies in some K_n. For completeness, if a compact subset were not contained in any stage, one could choose distinct points $x_j \notin K_j$. Every subset of $\{x_j : j \geq 0\}$ has finite intersection with each K_n, hence is closed in the direct-limit topology. This would give an infinite closed discrete subspace of a compact space, a contradiction.

Directions avoiding a prescribed finite stage. For any M and positive integer ℓ, there are an integer N, an ℓ-dimensional subspace $W \subseteq V_N$, a neighborhood B_W of zero in W, and a continuous additive map

$$g : W \longrightarrow G_r$$

such that

$$g(v) \notin K_M \qquad (0 \neq v \in B_W). \tag{68}$$

Indeed choose $N > M$ with $d_N \geq d_M + \ell$. For every monotone surjection $\alpha : [N] \twoheadrightarrow [m]$, $m \leq M$, let

$$S_\alpha = \alpha^* V_m \subseteq V_N.$$

There are finitely many such subspaces and each has dimension at most d_M. A generic ℓ-plane W meets each S_α only at zero.

The same avoidance holds locally modulo the lattice. Choose a monotone section $\delta_\alpha : [m] \hookrightarrow [N]$ and set

$$Q_\alpha = 1 - \alpha^* \delta_\alpha^*.$$

This integer linear map has kernel S_α. If $v + L_N \in \alpha^* A_m$, then $Q_\alpha v \in L_N$. Choose a sufficiently small neighborhood B of zero in V_N on which every anchor coordinate of every $Q_\alpha v$ has absolute value less than one. The preceding condition then forces $Q_\alpha v = 0$. Consequently

$$(B \cap W) \cap \{v : v + L_N \in \alpha^* A_m \text{ for some } m \le M\} = \{0\}.$$

Fix $w \in \operatorname{int} \Delta^N$ and define

$$g(v) = [v + L_N, w].$$

This is continuous and additive: two inputs with the same barycentric parameter w add in simplicial degree N. The uniqueness of the nondegenerate representative in a realization shows that $[v + L_N, w] \in K_M$ precisely when $v + L_N$ comes from one of these degeneracy images. Taking $B_W = B \cap W$ proves (68).

The embedding extension property. Let P be a finite-dimensional compact metric space, $B \subseteq P$ closed, and $f : B \to G_r$ an embedding. Assume first $B \neq \varnothing$. Its image lies in a stage K_d. Since K_d is an ANR, f extends to a continuous map $f' : U \to K_d$, where U is a closed neighborhood of B in P. Thus U is compact. Choose M containing the compact difference set $f'(U) - f'(U)$.

Choose an embedding $e : P \hookrightarrow \mathbb{R}^a$, and apply the preceding construction with $\ell = a + 1$. After identifying $\mathbb{R}^{a+1}$ linearly with W, define

$$h(x) = \varepsilon\big(\operatorname{dist}(x, B),\, \operatorname{dist}(x, B) e(x)\big) \in W, \qquad x \in U.$$

It vanishes exactly on B and is injective on $U \setminus B$. Choose $\varepsilon > 0$ so small that $h(U) - h(U) \subseteq B_W$. Set

$$F(x) = f'(x) + g(h(x)).$$

Then $F|_B = f$. If $F(x) = F(y)$ and $h(x) \neq h(y)$, additivity would put $g(h(x) - h(y))$ in $f'(U) - f'(U) \subseteq K_M$, contrary to (68). If $h(x) = h(y)$, either $x = y$ or both points lie in B; the latter case also gives $x = y$ because f is an embedding. Thus F is injective. As U is compact and G_r Hausdorff, it is an embedding of a neighborhood of B, extending f. When $B = \varnothing$, take $f' = 0$ on P and use the tag $h(x) = \varepsilon(1, e(x))$; the same argument embeds P.

We have verified the finite-dimensional compact embedding extension property, usually denoted (E_f). Together with the exhaustion by finite-dimensional compacta, Sakai's characterization theorem identifies G_r as an $\mathbb{R}^\infty$-manifold [68, Theorem 6.4.8]. □

Corollary 11.5. *Let C be the degreewise finite contractible simplicial set of Proposition 11.1. For every $k \geq 1$ there is a homeomorphism*

$$|D_0^{(k)} \times C| \;\cong\; |A_\bullet^{(k)}|.$$

More generally, for every $r \geq 1$ there is a homeomorphism

$$|X_0^{(k)}(r_+) \times C| \;\cong\; |Z^k(\Delta[\bullet]; (\mathbb{R}/\mathbb{Z})^r)|.$$

Proof. Proposition 11.1 and Lemma 11.2 make the spaces on the left $\mathbb{R}^\infty$-manifolds. Theorem 11.4 does the same for the spaces on the right. The finite-model comparison identifies their common homotopy type as $K(\mathbb{Z}^r, k+1)$. Sakai's classification theorem therefore gives the asserted homeomorphisms [68, Theorem 6.5.1]. □

Transporting the affine addition through such a homeomorphism gives an exact topological abelian group structure on $|D_0^{(k)} \times C|$. The simplicial set $D_0^{(k)} \times C$ remains finite in every degree. The classification theorem does not supply a natural homeomorphism for the original carrier maps, nor an ordinary simplicial multiplication. The factor C has been retained on the finite model; no factor is added to the affine group realization.

11.4 Homeomorphisms preserving the integral markings

The homeomorphisms can be chosen to preserve the canonical integral classes. With this choice the geometric replacement carries all integer operations simultaneously, and its multiplication agrees up to homotopy with the original finite carrier fold.

Fix $k \geq 1$ and write $n = k + 1$. Put

$$P_0 = *, \qquad P_r = |X_0^{(k)}(r_+) \times C| \quad (r \geq 1), \qquad G = |A_\bullet^{(k)}|.$$

Thus $P_1 = |D_0^{(k)} \times C|$. Let $\mathbf{c}_r \in H^n(P_r; \mathbb{Z}^r)$ be the ordered vector of canonical classes, pulled back along the projection that forgets C. Choose the generator $u \in H^n(G; \mathbb{Z})$ corresponding to the canonical integral marking, and write $\mathbf{u}_r$ for its coordinate copies on G^r. The basepoints of P_r are the zero flags and the basepoint of C; the basepoint of G^r is its group identity.

Theorem 11.6. *There are based homeomorphisms*

$$h_r : P_r \longrightarrow G^r \quad (r \geq 0), \qquad h_r^* \mathbf{u}_r = \mathbf{c}_r,$$

where h_0 is the identity of a point. For an integer $s \times r$ matrix M, let $M_G : G^r \to G^s$ be its ordinary action by sums and integer multiples, and define

$$T_M = h_s^{-1} M_G h_r. \tag{69}$$

These continuous maps satisfy

$$T_{I_r} = \mathrm{id}_{P_r}, \qquad T_N T_M = T_{NM}, \qquad T_M^{*} \mathbf{c}_s = M \mathbf{c}_r.$$

They give a product-preserving functor from $\mathsf{Mat}_{\mathbb{Z}}$ to topological spaces. Its product identifications are the homeomorphisms

$$H_r = (h_1^{-1})^r h_r : P_r \longrightarrow P_1^r. \tag{70}$$

Under the canonical identifications in the homotopy category, T_M represents the finite integer-operation roof of Theorem 9.4.

Proof. For $r \geq 1$, the preceding results make P_r and G^r $\mathbb{R}^\infty$-manifolds of homotopy type $K(\mathbb{Z}^r, n)$. They are connected and paracompact: P_r is a countable CW complex, while G^r is a Hausdorff topological group with a countable compact exhaustion, hence regular and Lindelöf. Choose a based homotopy equivalence $e_r : P_r \to G^r$ carrying $\mathbf{u}_r$ to $\mathbf{c}_r$. Such a map exists because maps into an Eilenberg–Mac Lane space represent integral cohomology, and this prescribed marking is an isomorphism on the only nonzero homotopy group. The spaces have CW homotopy type, so the resulting weak equivalence is a homotopy equivalence.

Heisey's classification theorem supplies a homeomorphism $\widetilde{h}_r$ homotopic to e_r [37, Theorem C]. This is stronger than the assertion that the spaces are merely homeomorphic. To make the choice based, take a homotopy J_r from e_r to $\widetilde{h}_r$, and replace it by

$$J_r'(x,t) = J_r(x,t) - J_r(*,t).$$

Its endpoint $h_r(x) = \widetilde{h}_r(x) - \widetilde{h}_r(*)$ is a based homeomorphism, and J_r' is a based homotopy from e_r to h_r. Consequently $h_r^{*}\mathbf{u}_r = \mathbf{c}_r$.

Formula (69) gives the composition and unit identities by cancelling the intervening $h_s h_s^{-1}$. It also gives the displayed cohomology identity. More explicitly,

$$H_s T_M = \big((h_1^{-1})^s M_G h_1^r \big) H_r.$$

The expression in parentheses is the ordinary matrix action on the powers of the topological abelian group P_1 transported from G by h_1. Coordinate projections therefore exhibit the topological product, while the zero-rank object is terminal.

Finally, maps from $K(\mathbb{Z}^r, n)$ to $K(\mathbb{Z}^s, n)$ are determined up to based homotopy by their action on π_n, equivalently by the pullback of the vector of fundamental classes. Both T_M and the finite roof have pullback $M\mathbf{c}_r$, by Theorem 9.4. The contractible factor C does not change this identification. Thus their classes in the homotopy category coincide. $\square$

Comparison with the original binary fold. Write $Z = X_0^{(k)}(2_+)$ and $D = D_0^{(k)}$. Its carrier maps are $p = (p_1, p_2) : Z \to D^2$ and $\mu : Z \to D$. The ordinary simplicial map

$$Z \times C \longrightarrow (D \times C)^2, \qquad (x, z) \longmapsto \big((p_1 x, z), (p_2 x, z)\big)$$

realizes to a based weak equivalence $F : P_2 \to P_1^2$. It uses the diagonal of the contractible factor. Let $\widehat{\mu} = |\mu \times \mathrm{id}_C| : P_2 \to P_1$. The markings give based homotopies

$$H_2 \simeq F, \qquad T_{(1\,1)} \simeq \widehat{\mu}. \tag{71}$$

Indeed $(h_1 \times h_1)F$ and h_2 pull back the same ordered vector of classes; similarly, the canonical carry identity makes $\widehat{\mu}^*\mathbf{c}_1$ the sum of the two classes on P_2. The same Eilenberg–Mac Lane classification of maps proves both homotopies.

The multiplication

$$m = T_{(1\,1)} H_2^{-1} : P_1 \times P_1 \longrightarrow P_1$$

is therefore strictly associative and commutative, with strict zero and inverse. It is exactly $h_1^{-1}(h_1(-) + h_1(-))$, and satisfies

$$m H_2 \simeq \widehat{\mu}.$$

This is a comparison with the original fold through a homotopy. The chosen homeomorphism does not turn the fixed carrier fold into the group multiplication by equality.

The theorem realizes the whole abelian theory by actual continuous maps on the enlarged finite realizations. The choices of h_r are not canonical, and the construction gives no ordinary simplicial action on their finite simplex sets. Likewise, the individual homotopies to the earlier roofs have not been furnished with a specified system of higher coherences. Strict identities belong to the transported continuous maps themselves.

11.5 Degeneracies agree; face attachments remain

There is an exact agreement between the carrier and affine models which narrows the homeomorphism problem to the face attachments. Put

$$V_q = Z^k(\Delta[q];\mathbb{R}), \quad L_q = Z^k(\Delta[q];\mathbb{Z}), \quad A_q = V_q/L_q, \quad Q_q = |NC_q(1_+)|.$$

Choose the barycentre of every lifted closed cell to be the arithmetic mean of its vertices. These choices give the row homeomorphisms $\bar{e}_q : Q_q \to A_q$ used above.

Lemma 11.7. *For every $k \geq 1$, every $q \geq 0$, and every elementary degeneracy $s_i : Q_q \to Q_{q+1}$, the row homeomorphisms satisfy*

$$\bar{e}_{q+1} s_i = s_i^{\mathrm{aff}} \bar{e}_q.$$

The assertion also holds for any number of inputs, and for the arrangements defined by finite integer coefficient families containing the coordinate vectors.

Proof. Let $\sigma : [q+1] \twoheadrightarrow [q]$ identify the adjacent vertices $i, i+1$, so the underlying linear map is the injective pullback $\sigma^* : V_q \to V_{q+1}$. Its image is precisely the flat

$$W_i = \{\theta \in V_{q+1} : \theta_F = 0 \text{ whenever the } k\text{-face } F \text{ contains both } i, i+1\}.$$

One inclusion follows because those faces become degenerate under σ. For the converse, the cocycle relation on a vertex set $J \cup \{i, i+1\}$ shows that the coordinates of the two k-faces obtained by replacing i with $i+1$ are equal: all the other terms contain both vertices and vanish. Here J has k vertices and contains neither of the repeated vertices. Since $i, i+1$ are adjacent, their positions in the two ordered faces agree. The remaining coordinates are unchanged. Thus the cocycle descends uniquely along σ; equivalently, pullback along any monotone section of σ gives its inverse on W_i. This argument includes $k = 1$, when the defining equation is $\theta_{i,i+1} = 0$. In degrees where there are no relevant faces, the corresponding cocycle spaces are zero and the assertion has the same interpretation.

The flat W_i is an intersection of arrangement walls. Each ambient face character restricts to either an original face character or zero, and every original character occurs by choosing a section of σ. Hence the arrangement restricted to W_i is exactly the image of the original arrangement. In particular, σ^* maps each open cell onto an open cell of the full arrangement,

contained in this flat. It maps their closed polytopes and vertex sets bijectively. Taking the arithmetic mean of vertices therefore commutes with σ^*. The barycentric evaluation of every flag commutes with it as well. Passing to lattice quotients proves the displayed identity. The same argument applies componentwise; the coordinate vectors ensure that the equations defining W_i remain arrangement walls for a general coefficient family. □

By the usual diagonal–iterated-realization homeomorphism, the original finite realization can also be written as

$$|D_0^{(k)}| \cong |Q_\bullet|.$$

After using $\bar{e}_q$ to identify the rows, this simplicial space and $A_\bullet$ have exactly the same degeneracies. Their nondegenerate row elements are consequently the same, and the uniqueness of nondegenerate representatives gives a set bijection

$$\beta : |Q_\bullet| \longrightarrow |A_\bullet|, \qquad [a,t] \longmapsto [a,t] \quad \big(a \text{ nondegenerate},\ t \in \operatorname{int}\Delta^q\big).$$

This statement concerns the underlying sets. It does not identify the realization topologies.

To see the precise issue, denote the carrier face operator in these coordinates by f_i and the affine one by d_i^{aff}. As t approaches an interior point t' of the ith facet, the source attaches the simplex to $[f_i(a), t']$, whereas the affine target attaches it to $[d_i^{\mathrm{aff}}(a), t']$. If these two boundary points differ, β is not continuous along that approach. Agreement of degeneracies gives no agreement of these face attachments, and thus supplies no continuity proof. This observation makes no assertion that such a discrepancy occurs in every k. It identifies the remaining geometric task: a homeomorphism of the original realizations must accommodate the face attachments, rather than merely identify their row tori or their nondegenerate representatives.

11.6 Arbitrarily large collars over a single open simplex

The original realization already has arbitrarily many transverse directions over each individual open simplex. This gives a partial form of the embedding property, without using the stabilizing factor. The compatibility of these directions across different simplex strata is a separate question.

Lemma 11.8. *Every nondegenerate n-simplex σ of $D_0^{(k)}$ is a face of nondegenerate simplices of arbitrarily large dimension.*

Proof. Choose a lifted flag $F_0 \leq \cdots \leq F_n$ representing σ. For $N \geq n$, let $\alpha : [N] \twoheadrightarrow [n]$ fix $[n]$ and send all additional vertices to n. Let

$$s = \alpha^* : V_n \longrightarrow V_N, \qquad r : V_N \longrightarrow V_n$$

be the corresponding injection and restriction, so $rs = 1$. The image of s is an arrangement flat, and s identifies the arrangement on V_n with its restriction to that flat. In particular every sF_i is an actual open face.

The relatively open convex polyhedron $\Omega = r^{-1}(F_n)$ is a union of open faces of the arrangement on V_N. Its dimension is

$$\dim F_n + \binom{N}{k} - \binom{n}{k}.$$

There is a face E of this dimension contained in Ω whose closure contains sF_n. Indeed one can choose, arbitrarily near a point of sF_n, a point of Ω avoiding all the finitely many additional local walls. The interval of faces between sF_n and E has saturated chains of length $\binom{N}{k} - \binom{n}{k}$.

For arbitrarily large N we have $\binom{N}{k} - \binom{n}{k} \geq N - n$. Put $b = N - n$ and choose the first b terms of such a saturated chain:

$$sF_n < E_1 < \cdots < E_b.$$

Then

$$\beta = (sF_0 \leq \cdots \leq sF_n < E_1 < \cdots < E_b)$$

defines an N-simplex of $D_0^{(k)}$. Restricting to the first $n+1$ vertex and flag positions gives σ. No diagonal degeneracy in positions $i \geq n$ is possible, since the corresponding consecutive flag entries are distinct. A degeneracy in a position $i < n$ would restrict to the same degeneracy of σ. Thus β is nondegenerate. □

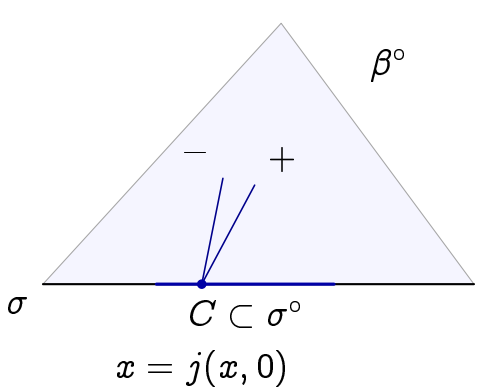
Proposition 11.9. *Let $P = |D_0^{(k)}|$, let C be a compact subset of one open nondegenerate simplex of P, and let a and M be nonnegative integers. There is an embedding*

$$j : C \times B^a \longrightarrow P$$

such that $j(x, 0) = x$ and

$$j\big(C \times (B^a \setminus \{0\})\big) \cap P^{(M)} = \varnothing.$$

Here B^a is the closed Euclidean unit ball and $P^{(M)}$ is the ordinary M-skeleton.



$j(x,z) \in \beta^\circ$ for $z \neq 0$

$\dim \beta > M$

$j(x,z) \notin P^{(M)}$

Each compact piece of one open simplex has arbitrarily many transverse directions.

Figure 22: A collar over one open simplex, shown schematically. The two rays depict the two signs of a one-dimensional normal coordinate; in the proof their directions lie strictly inside the complementary barycentric simplex. Higher-dimensional normal spheres are placed there in the same way. All nonzero normal coordinates enter one sufficiently high open simplex. The picture gives no gluing across the boundary of σ.

Proof. The case $a = 0$ is immediate. Write σ for the n-simplex containing C. Choose a nondegenerate coface β of dimension $N > M$ with $b = N - n \geq a + 1$, as in the lemma. In barycentric coordinates its distinguished σ-face occupies the first $n + 1$ positions. Choose an embedding

$$v : S^{a-1} \hookrightarrow \operatorname{int} \Delta^{b-1}.$$

For $u \in C \subset \operatorname{int} \Delta^n$ and $z \in B^a$ define, with $0 < \varepsilon < 1$,

$$j(u,z) = \begin{cases} [\beta, ((1 - \varepsilon\|z\|)u, \varepsilon\|z\| v(z/\|z\|))], & z \neq 0, \\ [\sigma, u], & z = 0. \end{cases}$$

The formula is continuous also at $z = 0$. If $z \neq 0$, every barycentric coordinate is positive, so the image lies in the open N-simplex of β. Uniqueness of nondegenerate interior representatives recovers u, $\|z\|$, and $v(z/\|z\|)$, hence z. At $z = 0$ injectivity follows from the characteristic map on the open simplex of σ. The compact-to-Hausdorff criterion now makes j an embedding. Its off-zero image lies outside $P^{(M)}$ because $N > M$. $\square$

Corollary 11.10. *Let A be a finite-dimensional compact metric space, $B \subseteq A$ closed, and $f : B \hookrightarrow P$ an embedding whose image is contained in one open simplex. Then f extends to an embedding of a neighborhood of B in A.*

Proof. If $B = \varnothing$, embed A into a finite-dimensional ball and then into a collar over any chosen point of P. Assume henceforth that $B \neq \varnothing$. The open simplex is an ANR, so f extends to a map $f' : U \to P$ from a compact neighborhood of B, with compact image C still inside that open simplex. Choose an embedding $e : A \hookrightarrow \mathbb{R}^b$ and a positive constant η small enough that

$$h(x) = \eta\big(d(x, B), d(x, B)e(x)\big) \in B^{b+1}, \qquad x \in U.$$

The map h vanishes exactly on B and is injective on $U \setminus B$. Compose (f', h) with the collar from the proposition. The result extends f and is injective: equal tags off B force equality, and on B the map f is injective. Compactness makes the result an embedding. □

The collar is not claimed to be an open neighborhood, and its construction is not natural as C approaches the boundary of its simplex. For a compact subset meeting several strata, the choices of high cofaces must still be made compatible along their common faces. Neither the lemma nor the proposition supplies that compatibility.

11.7 Why unrestricted coface amalgamation cannot supply the collars

One possible way to glue the preceding local collars would be to amalgamate every finite collection of cofaces over their common face into a single larger simplex. For $k \geq 2$ this property fails. This is an obstruction to that proposed proof strategy, not an obstruction to the desired homeomorphism.

Proposition 11.11. *For $k \geq 2$, there are simplices β_2, β_3 of $D_0^{(k)}$ with a common specified face σ, such that no simplex γ has both as faces by injections which agree on their specified σ-faces.*

Proof. The torsion construction in the proof of Theorem 8.17, applied with $p = 2$ and $p = 3$, gives an arrangement vertex v_p in a torus level $A_{q_p}^{(k)}$ and a k-face injection $a_p : [k] \hookrightarrow [q_p]$ such that

$$a_p^* v_p = 1/p \quad \text{in } \mathbb{R}/\mathbb{Z}.$$

Here the vertex assertion concerns the original periodic arrangement: the point is isolated in an intersection of its integer walls, so its open arrangement cell is zero-dimensional. The generator of the p-point subgroup can be chosen so that the nonzero face evaluation equals $1/p$.

Let β_p be the constant flag at v_p, repeated $q_p + 1$ times. Its specified a_p-face is the constant flag of length $k+1$ at the unique open edge $e = (0, 1)$

of the integer-wall circle. Thus both specified faces are the same simplex $\sigma = (e, \dots, e)$.

Suppose a simplex $\gamma = (H_0 \to \cdots \to H_m)$ and injections $b_p : [q_p] \hookrightarrow [m]$ satisfied

$$b_p^* \gamma = \beta_p, \qquad b_2 a_2 = b_3 a_3 = c.$$

The same flag position $c(0)$ occurs in both restrictions. The carrier of $H_{c(0)}$ under b_p^* is the vertex v_p. Therefore the affine image of every point of that open cell under c^* is the single circle point $a_p^* v_p = 1/p$. For $p = 2$ and $p = 3$ this requires the same nonempty open cell to have two distinct constant values under the same face evaluation, a contradiction. □

The simplices in this argument need not be declared nondegenerate: the statement already disproves the proposed universal amalgamation property for the simplex category. The geometric cause is that a carrier forgets where inside an open target cell the affine image lies. Equal carriers do not ensure compatible affine extension data. A global collar construction must therefore retain additional geometry or use more than one larger simplex.

11.8 Point fibres of the carrier face maps

The loss of affine positions in a carrier does not create an additional homotopy type in an individual lifted face-map fibre. The following statement concerns the row realizations; it does not identify links in the diagonal realization.

Lemma 11.12. *Let $\rho : P \to Q$ be an order-preserving map of locally finite, finite-dimensional face posets. Suppose that the nerves of*

$$\rho^{-1}(F), \qquad \{H \geq E : \rho(H) = F\} \quad \big(F \geq \rho(E)\big)$$

are nonempty and contractible. Then every point inverse of $|N\rho| : |NP| \to |NQ|$ is contractible.

Proof. Let a target point have nondegenerate support $F_0 < \cdots < F_m$ and positive barycentric weights $\lambda_0, \dots, \lambda_m$. A point in its inverse image is a source chain, partitioned into nonempty consecutive blocks whose carriers are $F_0, \dots, F_m$. The weights in block j sum to λ_j. Dividing by these sums identifies the fibre with the iterated realization of the multisimplicial set whose $(a_0, \dots, a_m)$-simplices are arrays

$$E_{j,0} \leq \cdots \leq E_{j,a_j}, \qquad \rho(E_{j,l}) = F_j,$$
$$E_{j,a_j} \leq E_{j+1,0} \quad (0 \leq j < m).$$

This identification respects all zero weights and repeated entries. It is a homeomorphism on the product-of-simplices cell structures.

Proceed by induction on m. For $m = 0$ the fibre is $|N\rho^{-1}(F_0)|$. For a fixed first block ending at E, the remaining blocks give the shorter-flag fibre for the restricted map on $P_{\geq E}$. This restriction satisfies the same hypotheses: its exact fibres over the remaining F_j are the displayed upper fibres, and its upper fibres above any $H \geq E$ are unchanged. Consequently the realization of the remaining blocks is contractible by induction. Projecting to the first block is therefore a degreewise weak equivalence after realizing the other directions. The usual diagonal weak-equivalence lemma for multisimplicial sets makes the full realization weakly equivalent to $|N\rho^{-1}(F_0)|$. Finally the point fibre is a polyhedral CW space, so weak contractibility implies contractibility. □

Proposition 11.13. *Let $r : V_N \to V_n$ be restriction to a specified ordered vertex subset, and let $\rho : \widetilde{\mathcal{P}}_N \to \widetilde{\mathcal{P}}_n$ be its carrier map for the original single-input arrangement. Every point fibre of*

$$|N\rho| : |N\widetilde{\mathcal{P}}_N| \longrightarrow |N\widetilde{\mathcal{P}}_n|$$

is contractible. The same assertion holds for any fixed number of inputs, or for one fixed integer coefficient family containing the coordinate vectors in both degrees.

Proof. Restriction is surjective: a monotone retraction onto the specified vertex subset gives a linear section by pullback. Every target wall pulls back to a source wall. We verify both hypotheses of Lemma 11.12.

First let F be an open target cell and put $\Omega = r^{-1}(F)$. This is a nonempty convex, relatively open polyhedron. It is the union of precisely those open source cells E with $\rho(E) = F$. Use the barycentric triangulation of the source arrangement. At a point $x \in \Omega$, discard the barycentric weights at cells whose carrier is not F, and renormalize the remaining weights. The remaining sum is positive: the maximal cell in the barycentric support is the open source cell containing x, and its carrier is F. Its coefficient remains positive throughout straight-line interpolation to the renormalized point. The interpolation therefore stays in this same open source cell, hence in Ω. This defines a strong deformation retraction

$$\Omega \longrightarrow |N\rho^{-1}(F)|.$$

The local finiteness of the arrangement ensures continuity of the barycentric formulas. In particular this nerve is contractible.

Next fix a source cell E, put $G = \rho(E)$, and let $F \geq G$. If $F = G$, the cell E is initial in the required upper fibre. Suppose $F > G$ and choose $x \in E$. Locally at x, cofaces of E are the cones of the central arrangement consisting of the walls through E, modulo its tangent space. Similarly the target coface F corresponds at $r(x) \in G$ to a nonzero relatively open convex cone $C_{F,G}$ modulo the tangent space of G. The induced linear map

$$\bar{r} : V_N / \operatorname{lin}(E - E) \longrightarrow V_n / \operatorname{lin}(G - G)$$

is well-defined and surjective. Thus $\bar{r}^{-1}(C_{F,G})$ is a nonempty convex cone excluding zero. It is the union of the local source cones indexed by

$$H \geq E, \qquad \rho(H) = F.$$

This also proves that every such target coface has a source coface: a direction into F can be lifted at the chosen point of E, even when $r(E)$ is a proper subset of G.

Intersect this convex cone with a unit sphere. Its intersection is contractible, by normalized straight-line contraction to any one of its points. Apply the same barycentric weight-deletion retraction in the spherical face decomposition of the local central arrangement. It retracts this intersection to the nerve of the indicated upper fibre. Hence that nerve is contractible. The preceding lemma now proves the claim for all target points, including points supported on a nontrivial barycentric flag. □

Corollary 11.14. *On the lattice quotients, every point fibre of the corresponding row carrier face map $Q_N \to Q_n$ has homotopy type*

$$K(\ker(L_N \to L_n), 1) \simeq (\mathbb{R}/\mathbb{Z})^{\binom{N}{k} - \binom{n}{k}}.$$

For a inputs the exponent is multiplied by a. These row maps are quasifibrations.

Proof. The integral restriction $L_N \to L_n$ is surjective, with an integral section. Choose a lift of the target point to the covering row realization. The original fibre is the quotient of its lifted point fibre by $\ker(L_N \to L_n)$. The latter lattice acts freely and properly, and the lifted fibre is contractible by the proposition. Thus the quotient fibre is the displayed aspherical polyhedral space.

The row spaces Q_N and Q_n are tori under the barycentric homeomorphisms, and their carrier map is homotopic to the linear restriction. The homotopy fibre is therefore connected and aspherical with fundamental group

$\ker(L_N \to L_n)$. The inclusion of the actual fibre induces this same kernel inclusion on fundamental groups, as is seen from its restricted covering space. Its map to the homotopy fibre is consequently a weak equivalence. This is the quasifibration condition. □

The lifted point fibres need not be compact; the term "cell-like map" is therefore not being used here. On the quotient, the kernel torus is essential and cannot be replaced by a contractible fibre. A quasifibration need not be a locally trivial bundle. These statements provide information about the row face maps, but they do not establish compatible neighborhoods or contractible links in the diagonal realization.

11.9 Punctured carrier fibres and their stable connectivity

The following calculation allows many source cells with the same carrier to vary together. In particular, it is stronger than placing one chosen simplex in one high coface. It is a calculation on the lifted arrangements, and does not identify these spaces with links in the diagonal realization.

Proposition 11.15. *Fix an open lifted face F in $V_n = Z^k(\Delta[n];\mathbb{R})$. For $N \geq n$, let*

$$s_N : V_n \longrightarrow V_N, \qquad r_N : V_N \longrightarrow V_n$$

be repeated-last-vertex pullback and restriction to the first $n+1$ vertices, respectively. Thus $r_N s_N = 1$. Let $\mathcal{P}_N(F)$ be the subposet of lifted arrangement faces defined by

$$\mathcal{P}_N(F) = \{E : \rho_{r_N}(E) = F,\ E \neq s_N F\}.$$

Put $b_N = \binom{N}{k} - \binom{n}{k}$. If $b_N > 0$, then

$$|N\mathcal{P}_N(F)| \simeq S^{b_N - 1}.$$

More precisely, barycentric evaluation embeds this nerve as a strong deformation retract of

$$\Omega_N \setminus s_N F, \qquad \Omega_N = r_N^{-1}(F).$$

These embeddings and deformation retractions commute with further repeated-last-vertex injections $V_N \hookrightarrow V_{N+1}$. Consequently the union of the nerves under these injections is contractible, with the weak topology of its simplicial complex.

Proof. The set Ω_N is a union of open arrangement faces. The degeneracy image $s_N(V_n)$ is an arrangement flat, and

$$\Omega_N \cap s_N(V_n) = s_N F.$$

Thus the faces belonging to $\Omega_N \setminus s_N F$ are exactly the elements of $\mathcal{P}_N(F)$.

Use vertex-mean barycentres to identify the nerve of the full lifted face poset with V_N. Write a point x of $\Omega_N \setminus s_N F$ in its unique open barycentric simplex:

$$x = \sum_{i=0}^{m} t_i b(E_i), \qquad E_0 < \cdots < E_m, \quad t_i > 0, \quad \sum_i t_i = 1.$$

The point x lies in the open face E_m. Therefore $E_m \in \mathcal{P}_N(F)$. In particular

$$a(x) = \sum_{E_i \in \mathcal{P}_N(F)} t_i > 0.$$

Define

$$R_N(x) = a(x)^{-1} \sum_{E_i \in \mathcal{P}_N(F)} t_i b(E_i), \qquad H_N(x,u) = (1-u)x + u R_N(x).$$

These formulas agree on common barycentric faces. The arrangement triangulation is locally finite, so they are continuous. Throughout the homotopy the coefficient of $b(E_m)$ is strictly positive. Consequently the point remains in E_m, and therefore remains in $\Omega_N \setminus s_N F$. The map R_N fixes its image pointwise. This proves the asserted strong deformation retraction.

The affine splitting gives a homeomorphism

$$\Omega_N \setminus s_N F \longrightarrow F \times (\ker r_N \setminus \{0\}), \qquad x \longmapsto (r_N x, x - s_N r_N x).$$

Its inverse sends (y,z) to $s_N y + z$. The open convex polyhedron F is contractible and $\dim \ker r_N = b_N$, proving the sphere homotopy type.

Let $j_N : V_N \hookrightarrow V_{N+1}$ duplicate the last vertex. Then

$$r_{N+1} j_N = r_N, \qquad j_N s_N = s_{N+1}.$$

The map j_N identifies its arrangement with the induced arrangement on an actual flat. It sends each face to a face, preserves incidence, and sends its vertex-mean barycentre to the vertex-mean barycentre of its image. Moreover

$$E \in \mathcal{P}_N(F) \quad \Longleftrightarrow \quad j_N E \in \mathcal{P}_{N+1}(F).$$

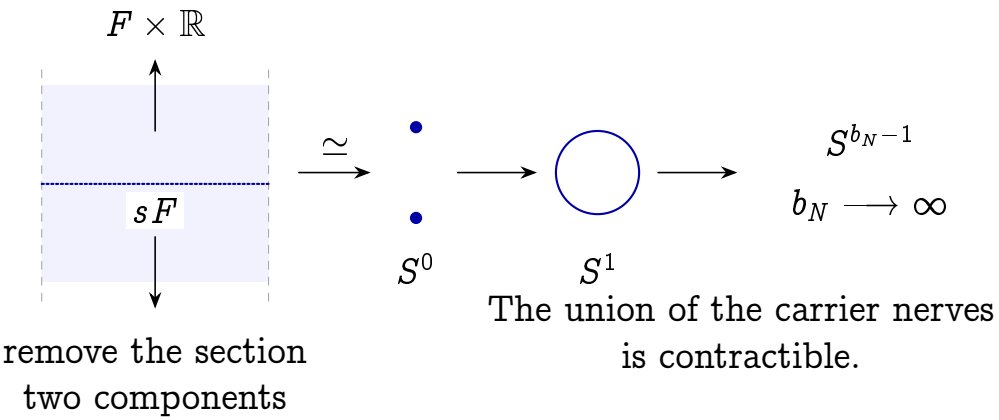


Figure 23: Extra vertices resolve the two sides of a punctured carrier fibre. On the real cover, the region over an open face F splits as $F \times \ker r_N$. Removing its canonical section has homotopy type $S^{b_N - 1}$, where $b_N = \dim \ker r_N$. The arrows between spheres indicate successive vertex-degree inclusions up to these homotopy types. This calculation concerns auxiliary carrier fibres; their identification with diagonal links is not used in the later local quasifibration proof.

It follows directly from the weight-deletion formulas that $R_{N+1} j_N = j_N R_N$ and $H_{N+1}(j_N x, u) = j_N H_N(x, u)$.

The nerves form an increasing union of simplicial subcomplexes. Every map from a sphere into their weak union has image in a finite subcomplex, hence in one stage. Since $b_N \to \infty$, its image is nullhomotopic in a sufficiently large later stage. The union is nonempty and path connected by the same argument. It is therefore a weakly contractible CW complex, hence contractible. □

The calculation includes cells whose affine images are distinct proper subsets of the same open carrier: it does not require their simultaneous occurrence in one larger simplex. The unresolved step is to compare a full diagonal link, with all of its flag-entry, vertex-face, and degeneracy identifications, to an appropriate stable punctured-fibre complex. Contractibility of the latter does not by itself establish that comparison.

11.10 The stable carrier fibre is an infinite-dimensional manifold

The same stabilization gives more than contractibility. All the simplex links of the stable carrier-fibre nerve are contractible, so Sakai's theorem applies to this auxiliary complex itself.

Theorem 11.16. *Fix an open lifted face $F \subset V_n$. Under repeated-last-vertex inclusions, form the direct-limit poset*

$$\mathcal{Q}_\infty(F) = \operatorname*{colim}_{N \geq n} \{E \subset V_N : \rho_{r_N}(E) = F\}.$$

Let $K(F) = N\mathcal{Q}_\infty(F)$, regarded as an abstract simplicial complex of finite strict chains and given its weak realization topology. Then

$$|K(F)| \cong \mathbb{R}^\infty.$$

The full subcomplex obtained by deleting the canonical vertex $s_N F$ (the same vertex in the direct limit) is likewise homeomorphic to $\mathbb{R}^\infty$.

Proof. All the face posets are countable, and the repeated-last maps are order embeddings onto arrangement flats. Thus the direct-limit nerve is a countable simplicial complex.

Upper intervals. Fix $E \in \mathcal{Q}_\infty(F)$, represented in degree N. In each later degree M, retain the notation E for its image. Set

$$\Omega_M = r_M^{-1}(F), \qquad d_M = \dim F + \binom{M}{k} - \binom{n}{k} - \dim E.$$

The set Ω_M is relatively open in the affine space $r_M^{-1}(\operatorname{aff} F)$. At a point in the relative interior of E, the arrangement induces a finite complete central fan in

$$\operatorname{lin}(\Omega_M - \Omega_M)/\operatorname{lin}(E - E),$$

of dimension d_M. Its cones correspond exactly to the faces $E' \geq E$ whose carrier under r_M equals F. Indeed a sufficiently small neighborhood of the chosen point in $\operatorname{aff} \Omega_M$ remains in Ω_M, and the walls containing E determine all incident face germs. The fan is pointed because E is an arrangement face.

Consequently, whenever $d_M > 0$, the nerve of the strict upper interval

$$\{E' > E : \rho_{r_M}(E') = F\}$$

is the barycentric subdivision of the fan's spherical cell decomposition, hence is homeomorphic to $S^{d_M - 1}$. The dimensions d_M tend to infinity. Any sphere map into the stable strict upper interval has compact image in a finite subcomplex and hence in one degree. Its inclusion in a sufficiently large later degree is nullhomotopic. The stable strict upper interval is nonempty and path connected for the same reason. Its nerve is therefore contractible by the CW Whitehead theorem.

Links. The link of a strict chain $E_0 < \cdots < E_l$ in an order complex is the join of the nerves of

$$\mathcal{Q}_{<E_0}, \quad (E_0, E_1), \quad \ldots, \quad (E_{l-1}, E_l), \quad \mathcal{Q}_{>E_l}.$$

An empty interval contributes the usual empty-complex join unit. The final factor is nonempty and contractible by the preceding paragraph. Thus every simplex link is nonempty and contractible. Sakai's link characterization and its identification with $\mathbb{R}^\infty$-manifolds show that $|K(F)|$ is an $\mathbb{R}^\infty$-manifold [68, Theorems 6.11.1 and 6.12.2].

The carrier-fibre nerve in each fixed degree is a strong deformation retract of the convex set Ω_M, by the barycentric weight-deletion argument. Hence the stable nerve is contractible: every sphere map lies in one degree, where it is already nullhomotopic. Sakai's classification theorem now identifies $|K(F)|$ with $\mathbb{R}^\infty$ [68, Theorem 6.5.1].

Finally the canonical cell $s_M F$ is a minimal element of each carrier-F poset. Every proper face of $s_M F$ maps to a proper face of F, so has a different carrier. Removing this vertex therefore does not change the strict upper interval of any remaining element. The same link argument shows that the deleted-vertex subcomplex is an $\mathbb{R}^\infty$-manifold. Its contractibility is Proposition 11.15. Applying the classification theorem again proves the second assertion. □

This theorem concerns the nerve of a stabilized, fixed-carrier poset. Its stabilization raises vertex degree while retaining a fixed target face and allowing arbitrary flag length. The original diagonal ties flag length to vertex degree and imposes additional face and degeneracy identifications. An identification, or a homotopy-cofinal comparison, of its local links with these stable carrier-fibre complexes has not been established.

11.11 Avoiding every row degeneracy below a fixed degree

The preceding puncture can be enlarged to remove all row data coming from a bounded simplicial degree, including their lattice translates. Write $V_q = Z^k(\Delta[q]; \mathbb{R})$ and $L_q = Z^k(\Delta[q]; \mathbb{Z})$, and retain a fixed lifted open face $F \subset V_n$. For $N \geq n$ set

$$\Omega_N = r_N^{-1}(F), \qquad d_N = \dim F + \binom{N}{k} - \binom{n}{k}.$$

For a nonnegative integer M, define

$$\mathcal{A}_N^{\leq M} = \bigcup_{m=0}^{\min(M,N)} \bigcup_{\alpha:[N]\twoheadrightarrow[m]} \bigcup_{\lambda \in L_N} (\lambda + \alpha^* V_m),$$

where the middle union is over monotone surjections. Repeated affine flats are counted only once. Let

$$\mathcal{G}_N(F; M) = \{E : \rho_{r_N}(E) = F,\ E \not\subseteq \mathcal{A}_N^{\leq M}\}.$$

The binomial coefficient $\binom{M}{k}$ is understood to be zero when $M < k$.

Proposition 11.17. *Put $c_N = d_N - \binom{M}{k} - 2$. Whenever $c_N \geq -1$, both*

$$\Omega_N \setminus \mathcal{A}_N^{\leq M} \quad \text{and} \quad |N\mathcal{G}_N(F; M)|$$

are c_N-connected; here (-1)-connected means nonempty. There is a barycentric strong deformation retraction from the first space onto the second. These deformation retractions commute with repeated-last-vertex injections when M is fixed. Consequently

$$\left| N\operatorname*{colim}_{N \geq n} \mathcal{G}_N(F; M) \right| \ \cong\ \mathbb{R}^\infty.$$

The realizations and direct limits use the weak topology of the corresponding countable simplicial complexes. No nonemptiness is asserted for an early stage with $c_N < -1$.

Proof. Each $\alpha^* V_m$ is a rational arrangement flat: a degeneracy identifies the smaller arrangement with the arrangement induced on that flat. Translation by $\lambda \in L_N$ preserves the full periodic arrangement. Thus $\mathcal{A}_N^{\leq M}$ is an arrangement subcomplex. It is closed and locally finite as a union of distinct affine flats. Indeed, for each fixed rational linear subspace $\alpha^* V_m$, its lattice translates form a locally finite family: their images in the quotient vector space form a discrete lattice. There are only finitely many choices of m and α. Every such flat has dimension at most $\binom{M}{k}$.

The set Ω_N is an open convex subset of its d_N-dimensional affine hull. Its intersections with the removed flats have dimensions at most $\binom{M}{k}$. The stated connectivity is relative general position for this locally finite family of affine subspaces. Here is a direct argument. Nonemptiness follows because finitely many proper affine subspaces cannot fill an open ball. Given a sphere map of dimension $j \geq 0$ with

$$j + 1 + \binom{M}{k} < d_N,$$

approximate it by a sufficiently close piecewise linear map and use the straight-line small homotopy in the complement. Its image can be coned

to a point of Ω_N because that set is convex. A compact neighborhood of this cone meets only finitely many removed flats. Choose the cone point generically in a sufficiently small open ball. For each boundary simplex and each relevant flat, the forbidden cone points lie in an affine subspace of dimension at most $j + 1 + \binom{M}{k} < d_N$. Hence the perturbed cone misses every removed flat. This gives the required nullhomotopy.

An open arrangement face is either contained in $\mathcal{A}_N^{\leq M}$ or disjoint from it. For a point in $\Omega_N \setminus \mathcal{A}_N^{\leq M}$, the largest face in its open barycentric support belongs to $\mathcal{G}_N(F; M)$. Delete all other face weights and normalize the retained weights, as in Proposition 11.15. The largest face retains a strictly positive coefficient throughout the straight-line homotopy. Thus the homotopy stays in that same open face and in the complement. This is the asserted strong deformation retraction.

Let $j_N : V_N \hookrightarrow V_{N+1}$ duplicate the last vertex. We have the exact equivalence

$$x \in \mathcal{A}_N^{\leq M} \quad \Longleftrightarrow \quad j_N x \in \mathcal{A}_{N+1}^{\leq M}.$$

The forward implication follows by composing the corresponding monotone surjections and applying j_N to the lattice translation. For the converse, pull back along a monotone section of the last duplication. The resulting monotone map $[N] \to [m]$ need not be surjective, but factors through its image $[m']$ with $m' \leq m \leq M$. The pulled-back translation remains integral. This proves the equivalence. It also shows that the good-cell predicate is preserved and reflected, so the barycentric deformation retractions commute with the inclusions.

Since $c_N \to \infty$ for a fixed M, every sphere map into the stable nerve becomes nullhomotopic in a later degree. Compactness places its image in a finite subcomplex and hence in one degree. The stable nerve is therefore a contractible CW complex.

In fact it is an $\mathbb{R}^\infty$-manifold. The good cells form an upward-closed subposet of the carrier-F poset: if $E \leq E'$ and E' lay in the removed closed arrangement subcomplex, then E would lie there as well. Thus the strict upper interval of every good cell is unchanged. The complete-normal-fan and chain-link argument of Theorem 11.16 applies without alteration. Sakai's characterization and classification now identify the contractible stable nerve with $\mathbb{R}^\infty$. □

The lattice translates in this statement are essential. In the torus $A_N = V_N / L_N$, a point comes from degree at most M precisely when any real lift belongs to $\mathcal{A}_N^{\leq M}$. Omitting the translates would only test a chosen lift against the intrinsic linear degeneracy flats. This is a statement about ordinary row

degeneracies; it does not identify the removed set with the finite skeleton of the diagonal realization, whose degeneracies also repeat flag positions.

Corollary 11.18. *At a fixed degree $N \geq \max\{n, 1\}$, remove every proper row degeneracy by taking $M = N - 1$. Whenever the displayed number is at least -1, the remaining part of Ω_N, and its good-cell nerve, are*

$$\left(\binom{N-1}{k-1} - \binom{n}{k} + \dim F - 2 \right) \text{-connected.}$$

For $k \geq 2$ this lower bound tends to infinity with N.

Proof. Apply the proposition at the single degree N with $M = N - 1$ and use $\binom{N}{k} - \binom{N-1}{k} = \binom{N-1}{k-1}$. □

This last corollary is a sequence of finite-degree assertions. Repeated-last duplication sends every row point to a proper degeneracy in the next degree, so it does not give maps between these genuinely nondegenerate complements. No direct-limit identification for the varying cutoff $M = N-1$ is asserted. For $k = 1$ the stated bound is $\dim F - n - 1$ and does not grow.

For example, take $k = 2$, $n = 2$, and the open edge $F = (0, 1) \subset V_2 = \mathbb{R}$. The complement of every proper row degeneracy over this edge is $(N - 3)$-connected. Thus in degree $N = 5$ it is already simply connected with vanishing π_2, and every fixed higher connectivity requirement is met in a later degree. This supplies fillings once the prescribed family has been represented inside that carrier strip. It does not construct the transition maps needed to represent a whole diagonal link there.

11.12 An exact join and a manifold theorem with a retained anchor

Keeping one distinguished vertex gives an exact geometric operation which is absent from the unmarked carrier construction. In this subsection $k \geq 2$ and $Q_q = |NC_q(1_+)|$ denotes the original single-input row realization. Define the lower decalage by

$$E_q = (\mathrm{Dec}_0\, Q)_q = Q_{q+1}, \qquad d_i^E = d_{i+1}^Q, \qquad s_i^E = s_{i+1}^Q, \qquad E = |\,\mathrm{Dec}_0\, Q|.$$

The vertex 0 in each row is retained as an anchor; the other $q + 1$ vertices carry the coordinates of Δ^q in this realization.

Lemma 11.19. *Let $S = \{a\} \amalg I$ and $T = \{a\} \amalg J$ be ordered vertex sets with their common anchor first and otherwise disjoint. On $U = \{a\} \amalg I \amalg J$, let*

$p : U \to S$ collapse J to a and let $q : U \to T$ collapse I to a. Alternating pullback defines an injective map

$$\iota : V_S \oplus V_T \longrightarrow V_U, \qquad (x, y) \longmapsto p^* x + q^* y.$$

Its image is an arrangement flat, its lattice is $\iota(L_S \oplus L_T)$, and its induced periodic arrangement is exactly the product arrangement. Consequently it induces an embedding

$$\omega : Q_S \times Q_T \hookrightarrow Q_U$$

natural for vertex maps preserving the anchor and the two blocks. Restriction to either anchored block recovers the corresponding factor. The row barycentric homeomorphisms commute with ω and the affine map ι.

Proof. Use the integral anchor coordinates θ_{aF}, with F a k-element subset of the unmarked vertices. The image of ι is defined by

$$\theta_{aF} = 0 \qquad (F \cap I \neq \varnothing,\ F \cap J \neq \varnothing).$$

All remaining anchor coordinates are the independent coordinates of x or y. This proves the flat and lattice assertions.

On a $(k+1)$-vertex face omitting a, both pullbacks could be nonzero only if at most one of its vertices belongs to I and at most one belongs to J. Because $k + 1 \geq 3$, this is impossible. On a face containing a, a nonzero first pullback requires that the face miss J, and a nonzero second pullback requires that it miss I, so again they cannot both be nonzero. Each ambient face character therefore restricts to one input face character, possibly with its orientation sign, or to zero. Every input face character occurs on its own anchored block. The wall arrangement on the flat is thus the product arrangement. Its cells are the images of products of input cells. Their vertex barycentres are the images of the corresponding pairs of vertex barycentres, proving the final assertion. Pullback is natural under the indicated block maps, which also proves the stated naturality after passing to nerves and lattice quotients. □

For $x = [u, t] \in E$, with $u \in Q_{q+1}$ and $t \in \Delta^q$, and $y = [v, s] \in E$, with $v \in Q_{r+1}$ and $s \in \Delta^r$, set

$$J_\lambda(x, y) = [\omega(u, v), ((1 - \lambda)t, \lambda s)] \in E, \qquad 0 \leq \lambda \leq 1. \tag{72}$$

The row on the right has one anchor and $q + r + 2$ unmarked vertices. Block naturality makes the formula respect all face and degeneracy relations in

both inputs. Zero weights at either endpoint remove the corresponding block, so

$$J_0(x,y) = x, \qquad J_1(x,y) = y.$$

It defines a continuous map $E \times E \times I \to E$. One may check continuity on finite stages and then use their compact direct-limit topology and its finite-product property. The join is ordered: with three inputs, both ways of concatenating the blocks give the same row and the same point whenever their three total barycentric weights agree. No commutativity is asserted.

Lemma 11.20. *Fix $r+1 \geq k$, a weight vector $s \in \operatorname{int}\Delta^r$, and an open set B in the tag row Q_{r+1} on which every anchored k-face character is nonzero modulo $\mathbb{Z}$. The map*

$$\begin{aligned} E \times B \times (0,1) &\longrightarrow E, \\ (x, v, \lambda) &\longmapsto J_\lambda(x, [v, s]) \end{aligned}$$

is injective. If the minimal outer simplicial degree of x is q, the minimal outer degree of its image is $q + r + 1$.

Proof. Choose the unique nondegenerate outer representative $[u, t]$ of x, with all entries of t positive. Degeneracies of the rows agree with affine duplication under the row homeomorphisms (Lemma 11.7). A row coming from duplication of two adjacent vertices has zero value on every k-face containing both of those vertices.

No two adjacent tag vertices can be duplicated: place that pair in an anchored k-face, using $k-2$ further tag vertices. Its value is nonzero by hypothesis. Nor can the last old vertex and the first tag vertex be duplicated. Take the face consisting of that old vertex and any k tag vertices including the first one. The old pullback vanishes on this face, while the tag pullback is, up to sign, the nonzero anchored coordinate on those k tags. Finally, a duplication wholly inside the old block would restrict to a duplication of u on that block, which is impossible. Conversely an old duplication before taking the minimal representative is carried to the same duplication by block naturality. Thus there is no additional identification hidden by a different presentation of x.

The displayed output row is therefore nondegenerate, and all its $q+r+2$ barycentric weights are positive. Its minimal representative has degree $q + r + 1$. From that representative, its last $r+1$ weights have sum λ. Normalize the preceding weights to recover t, and restrict the row to the anchor plus the preceding vertices to recover u. Restriction to the anchor plus the last $r+1$ vertices recovers v. These operations prove injectivity. □

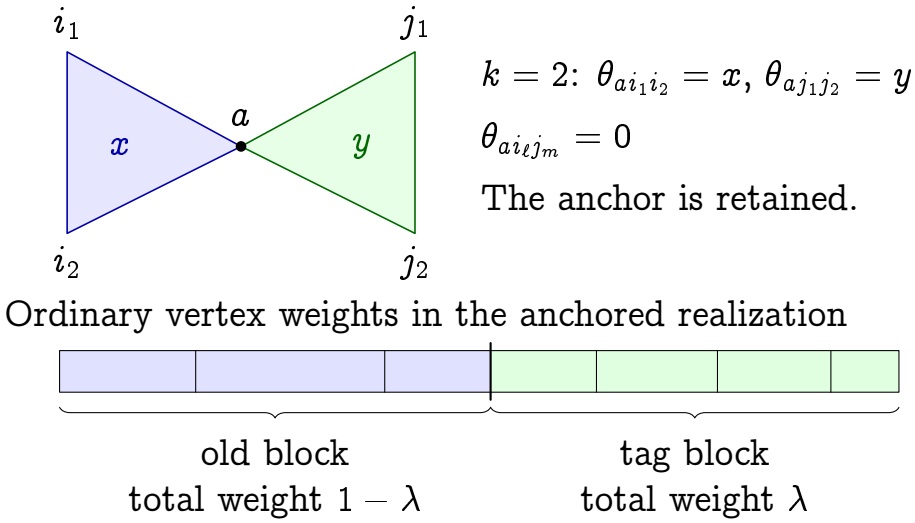


Figure 24: The anchored product and its ordered join. The triangles depict two cocycle blocks sharing a retained vertex a. Mixed anchored coordinates vanish. In the realization the anchor has no outer barycentric weight. Concatenating the two lists of ordinary weights gives the join. A tag chosen with nonzero anchored coordinates cannot merge with the old block under a degeneracy, so its final vertices recover the parameter λ.

Theorem 11.21. *For $k \geq 2$, the original anchored realization satisfies*

$$|\operatorname{Dec}_0 Q| \cong \mathbb{R}^\infty.$$

Proof. The bisimplicial set with (p, q)-simplices $N_p C_{q+1}(1_+)$ is finite in each bidegree. The diagonal–iterated-realization homeomorphism therefore presents E as the realization of a degreewise finite simplicial set. In particular it has a countable CW structure with compact finite skeleta. Compact subsets lie in finite subcomplexes, and these are finite-dimensional compact metrizable ANRs.

Let A be a finite-dimensional compact metric space, let $B \subseteq A$ be closed and nonempty, and let $f : B \hookrightarrow E$ be an embedding. Its image lies in a finite subcomplex. The ANR property extends it to a continuous map $f' : U \to E$, where U is a compact neighborhood of B and the image remains in a finite subcomplex. In particular its image has bounded outer degree, say at most M: this follows directly from the diagonal realization on every simplex of that finite subcomplex.

Choose an embedding $e : A \hookrightarrow \mathbb{R}^a$. Take r so large that $r > M$, $r + 1 \geq k$, and

$$\binom{r+1}{k} \geq a.$$

The tag row is a torus of the displayed dimension. Its independent anchor coordinates contain a nonempty open box avoiding zero modulo $\mathbb{Z}$ in every coordinate. Choose a small ball in that box and, after rescaling e, an embedding $v : A \hookrightarrow B_{\mathrm{tag}} \subset Q_{r+1}$ into it. Fix $s \in \operatorname{int} \Delta^r$ and choose $\varepsilon > 0$ so that $\lambda(x) = \varepsilon \operatorname{dist}(x, B) < 1$ on A. Define

$$F(x) = J_{\lambda(x)}(f'(x), [v(x), s]), \qquad x \in U.$$

This is continuous and extends f. At points outside B, the tag recovery lemma proves injectivity, since the recovered tag $v(x)$ determines x. Their images have outer degree at least $r + 1 > M$, so cannot meet $F(B) = f(B)$. On B the map is the given embedding. Thus F is injective, and compactness of U makes it an embedding. If B is empty, the map $x \mapsto [v(x), s]$ embeds A directly into a sufficiently large nondegenerate tag row.

This proves the compact embedding extension property (E_f). Sakai's recognition theorem makes E an $\mathbb{R}^\infty$-manifold [68, Theorem 6.4.8]. The augmentation of the decalage to $Q_0 = \{0\}$ has extra degeneracy s_0^Q, which duplicates the anchor, and therefore contracts E. Sakai's classification now gives $E \cong \mathbb{R}^\infty$ [68, Theorem 6.5.1]. □

This result concerns the original row maps with an anchor retained throughout the realization. It does not identify the unmarked realization $|Q_\bullet|$ with a manifold. Deleting the anchor need not carry the product flat to an arrangement flat; its image can be specified by equalities between distinct face characters which the integer-wall arrangement does not retain. The carrier then forgets these equalities. The local quasifibration theorem in Section 11.18 supplies the additional control needed to transfer fibre avoidance to the basepoint link of $|Q_\bullet|$.

11.13 The exact zero fibre of anchor deletion

Write $Q_q^{(k)}$ for the original single-input row in degree q, and let

$$\pi_k : |\operatorname{Dec}_0 Q^{(k)}| \longrightarrow |Q^{(k)}|$$

be induced by deleting the retained vertex 0. For the degree-zero endpoint define $Q_q^{(0)}$ using constant real cocycles and their integer lattice; its periodic arrangement is the integer subdivision of $\mathbb{R}$, and $Q_\bullet^{(0)}$ is the constant circle.

Proposition 11.22. *For every $k \geq 1$, there is a natural homeomorphism*

$$\pi_k^{-1}(*) \cong |Q_\bullet^{(k-1)}|.$$

In particular the actual fibre is $|D_0^{(k-1)}|$ for $k \geq 2$, and is the circle for $k = 1$.

Proof. In row degree q, let

$$r : V_{q+1}^{(k)} \longrightarrow V_q^{(k)}$$

delete the retained anchor. Regard the remaining vertices as $1, \dots, q+1$. The linear map

$$\begin{aligned} \phi_q : V_q^{(k-1)} &\longrightarrow \ker r, \\ (\phi_q c)(0, i_1, \dots, i_k) &= c(i_1, \dots, i_k), \\ (\phi_q c)(i_0, \dots, i_k) &= 0 \end{aligned}$$

is an isomorphism. The cocycle equations containing 0 say exactly that c is a $(k-1)$-cocycle; the equations omitting 0 are identically zero. This includes $k = 1$, where those equations say that the zero-cocycle is constant. The same formulas identify the integral lattices and commute with every map of the unmarked vertices.

The kernel is an actual arrangement flat, cut out by the zero values of all old k-face characters. The other characters are exactly the $(k-1)$-face characters of c. Thus its induced periodic arrangement is precisely the lower-degree arrangement, including its face incidences and its lattice labels.

Under the row vertex-mean homeomorphisms, the inverse image of the zero row point under the carrier restriction is exactly

$$\{x \bmod L_{q+1}^{(k)} : r(x) \in L_q^{(k)}\}.$$

To check this, a point in an open source cell maps into the open target carrier. That carrier is an integral zero-cell if and only if all old face characters take integral values on the source cell. Integral restriction is surjective, so subtracting an integral extension identifies the displayed torus kernel with

$$\ker r / (\ker r \cap L_{q+1}^{(k)}).$$

The flat description therefore identifies the actual row fibre with $Q_q^{(k-1)}$, not merely with a space of the same homotopy type. At the incidence level the zero-cell has only its identity incidence, so the fibre retains precisely the incidences of the kernel arrangement modulo its kernel lattice. These identifications are natural for the unmarked faces and degeneracies.

It remains to check that realization introduces no further points into the fibre. A point of the anchored realization has a unique minimal outer representative with all outer weights positive. Its image represents $*$ precisely when its deleted row point is zero: a positive-weight nonzero row point cannot become the zero point by degeneracies. Thus the set-theoretic fibre is

exactly the realization of the row fibres just identified. This identification is also topological. The row fibres are closed subpolyhedra, and on every finite outer stage their realizations are compact and map bijectively onto the corresponding closed fibre. Compact-to-Hausdorff gives a homeomorphism there. The finite-stage weak topology, equivalently the countable compact direct-limit topology of these realizations, gives the claimed homeomorphism on the full fibre. □

This is an actual fibre calculation for the original carrier maps. It does not assert that π_k is a fibre bundle, a Serre fibration, or a map admitting continuous local sections. In particular the fibre calculation alone does not descend the anchored manifold theorem to the unmarked realization. Theorem 11.33 will prove the required quasifibration property over every open subset.

11.14 Ambient shuffles avoid compacta away from the fixed point

The crossed symmetric action supplies a global avoidance statement which is compatible with all realization identifications. Unlike the carrier-strip calculations, it does not require a compact family to lie in one row or one flag. Its limitation is exactly the canonical fixed point.

Lemma 11.23. *Let $x \in Z^k(\Delta[S];\mathbb{R}/\mathbb{Z})$, where $k \geq 1$. Declare $a \sim_x b$ if x factors through the set quotient identifying a with b. This is an equivalence relation. If $\alpha : T \twoheadrightarrow S$ is any surjection, then*

$$a \sim_{\alpha^* x} b \quad \Longleftrightarrow \quad \alpha(a) \sim_x \alpha(b).$$

On an ordered vertex set, the minimal ordinary row degree of the row point represented by x is the number of adjacent changes of its $\sim_x$-classes.

Proof. Let I_S be the augmentation kernel of the free abelian group on S. A normalized k-cocycle is equivalently an alternating k-form on I_S with values in $\mathbb{R}/\mathbb{Z}$, with

$$x(i_0,\dots,i_k) = b_x\big((i_1 - i_0) \wedge \cdots \wedge (i_k - i_0)\big).$$

The factorization condition is that $a - b$ belong to the subgroup

$$\operatorname{rad}(b_x) = \{v \in I_S : b_x(v \wedge w) = 0 \text{ for all } w \in \Lambda^{k-1} I_S\}.$$

This proves transitivity. A surjection induces a surjection of augmentation kernels and of their exterior powers, proving the displayed equivalence. An elementary ordinary degeneracy identifies two adjacent equivalent vertices. Collapsing them introduces no new equivalences by the same reflection statement. Thus successive collapses remove exactly the consecutive repetitions of classes. The row identifications used here agree with affine pullback for permutations and degeneracies, hence for all surjections. □

Proposition 11.24. *Let $P = |D_0^{(k)}|$, $k \geq 1$, and let $*$ be its canonical basepoint. For every compact $K \subset P \setminus \{*\}$ and integer $M \geq 0$, there is a path*

$$g : [0,1] \longrightarrow |\mathfrak{S}_\bullet|, \qquad g(0) = 1,$$

such that $g(t)K$ is disjoint from the part of the realization of row degree at most M for every $t > 0$. In particular it is disjoint from the diagonal M-skeleton.

Proof. The empty case is immediate, so assume $K \neq \varnothing$. Compactness in the CW realization puts K in a finite subcomplex, and hence in bounded row degree d. The characteristic map

$$\pi_d : Q_d \times \Delta^d \longrightarrow P$$

covers all row degrees at most d, by using degeneracies to raise lower degrees. Its inverse image $\widehat{K}$ is compact. Write x also for the torus cocycle corresponding to a point of Q_d under the vertex-mean homeomorphism. On this compact set consider the continuous function

$$m(x,u) = \max_{\substack{I \subseteq [d] \\ |I| = k+1}} \left(\prod_{i \in I} u_i \right) \operatorname{dist}\big(\theta_I(x), \mathbb{Z}\big).$$

It is positive on $\widehat{K}$. Indeed restriction to the vertices with positive weight represents the basepoint precisely when all of its k-face evaluations are integral. To check this assertion for carrier restriction, the image of a point in an open source cell lies in the open target carrier: the maximum cell in a barycentric support has positive weight. That carrier is an integral zero-cell precisely under the displayed integrality condition. A nonzero row point with all outer coordinates positive cannot degenerate to the zero row point.

Consequently there is $\delta > 0$ such that every representative in $\widehat{K}$ has a set I of $k+1$ vertices with $u_i \geq \delta$ for $i \in I$ and $\theta_I(x) \neq 0$ in $\mathbb{R}/\mathbb{Z}$. Restrict by the carrier map to all positive-weight vertices, and call the resulting torus

cocycle y. Its evaluation on the corresponding face I is still nonzero. Its numerical value need not be the former value: what is preserved is membership in the union of integer walls, by the open-carrier observation above. The vertices in this I are consequently pairwise inequivalent for $\sim_y$: identifying any two would make their alternating face evaluation zero.

View u as a partition of $[0,1]$ into successive labelled intervals. Choose a fixed uniform grid with spacing $h < \delta/4$, with sample centres at its midpoints, and put $R > M$. Around each grid centre choose a sample interval of length $th/2$, for $0 < t \le 1$. Split each sample into R equal pieces. There are two types of blocks: the complementary gaps and these sample pieces. Perform the following interval permutation: first list all gaps in their original order, then all first sample pieces in grid order, then all second pieces in grid order, and so on through all R pieces. Let $g(t)$ be the crossed symmetric group element which performs this reordering on realization points, with the permutation convention adjusted as in (44).

This is a fixed finite permutation with continuously varying block lengths. At $t = 0$ the sample pieces all have zero length, and the remaining gap blocks retain their order. Face identifications in $|\mathfrak{S}_\bullet|$ therefore give the identity element. This defines a continuous path with $g(0) = 1$.

Every original interval of length at least δ contains an entire sample interval, for every $0 < t \le 1$, by the grid choice. Choose two of the inequivalent labels provided above. In each of the R terminal groups of sample pieces, both labels occur with positive length. Common refinement with the original partition, as in the crossed action, may introduce more cuts, but does not remove either occurrence. Each such group therefore contains an adjacent change of equivalence class: otherwise all its labels would be equivalent by transitivity. The R groups give at least R distinct essential changes.

The resulting row cocycle is the pullback of y along a surjection from refined, reordered vertices onto the positive-weight vertices. Lemma 11.23 shows that these essential changes cannot be removed by ordinary degeneracies. Its minimal row degree is at least $R > M$. This holds uniformly for every point of K and every $t > 0$. □

Corollary 11.25. *Let $C \subset P \setminus \{*\}$ be compact. If A is compact metrizable, $B \subset A$ is closed, and $f : A \to P$ satisfies $f(B) \cap C = \varnothing$, then f has arbitrarily small homotopies fixed on B whose positive-time images miss C. If $f(A)$ misses $*$, so does the entire homotopy.*

Proof. Choose M so that $f(A)$ lies in the diagonal M-skeleton, and apply the proposition to $K = C$. Then

$$g(s)C \cap f(A) = \varnothing \qquad (s > 0).$$

Choose a continuous $\tau : A \to [0,1]$ which vanishes exactly on B; when $B = \varnothing$ take $\tau = 1$. Define

$$f_t(a) = g\big(t\tau(a)\big)^{-1} \cdot f(a).$$

For $a \notin B$ and $t > 0$, membership $f_t(a) \in C$ would imply $f(a) \in g(t\tau(a))C$, a contradiction. On B the homotopy is constant and already misses C. Continuity of the group action and its inverse makes f_t uniformly close to f in the open-cover sense when t is uniformly small. Finally every group element fixes $*$, so its inverse preserves the punctured space. □

The corollary crosses all carrier flags and all face and degeneracy identifications. It still excludes C containing the fixed point. It asserts relative avoidance for compact source spaces; a global self-map approximation definition of a Z-set is not being asserted without a separate recognition argument. An ambient homeomorphism also cannot by itself separate two domain points already identified by a map; passage from these avoidance properties to a manifold recognition theorem needs its stated hypotheses, such as an actual triangulation.

11.15 The original realization away from its fixed point

The ambient shuffle theorem has a local consequence stronger than avoidance of finite skeleta. It identifies the topology at every point except the canonical fixed point.

Lemma 11.26. *Let X be the weak realization of a countable simplicial complex. Suppose that, for every two compact subsets $C, A \subseteq X$, there is a continuous path of ambient homeomorphisms h_t, $0 \le t \le 1$, with $h_0 = \mathrm{id}$ and*

$$h_t(A) \cap C = \varnothing \quad (t > 0).$$

Then every simplex of X has nonempty contractible link. Consequently X is an $\mathbb{R}^\infty$-manifold.

Proof. Fix a simplex σ and let U be its open star, namely the points whose barycentric coordinates at every vertex of σ are positive. If $L = \mathrm{Lk}(\sigma)$, then

$$U \setminus \sigma \cong \operatorname{int}\sigma \times |L| \times (0,1).$$

Here the last coordinate is the total weight outside σ; normalizing the two sets of coordinates gives the displayed homeomorphism. In particular projection to $|L|$ is a homotopy equivalence whenever the link is nonempty.

Apply the hypothesis with $C = \sigma$ and A its barycenter. For sufficiently small positive t, the moved barycenter still lies in U, but is outside σ. Hence L is nonempty. Now let $f : S^j \to |L|$, $j \geq 0$. Include this sphere in $U \setminus \sigma$ by assigning total weight $1/2$ to f and weight $1/2$ to the barycenter of σ. Cone it to that barycenter by allowing the outside weight to run from 0 to $1/2$. This gives a map $F : D^{j+1} \to U$ with compact image.

Choose the path in the hypothesis with $A = F(D^{j+1})$ and $C = \sigma$. By continuity on compacta, there is $t_0 > 0$ so small that $h_t F(D^{j+1}) \subseteq U$ for $0 \leq t \leq t_0$, and

$$h_t F(S^j) \subseteq U \setminus \sigma \quad (0 \leq t \leq t_0).$$

The disk $h_{t_0} F$ lies entirely in $U \setminus \sigma$. Adjoining the indicated boundary homotopy gives a nullhomotopy of the original sphere there. Projection to $|L|$ proves that every sphere in $|L|$ is nullhomotopic, including the assertion of path connectedness when $j = 0$. The link is a CW complex, so it is contractible. Sakai's link characterization now applies [68, Theorem 6.12.2]. □

Theorem 11.27. *For every $k \geq 1$, the original carrier realization satisfies*

$$|D_0^{(k)}| \setminus \{*\} \quad \textit{is an } \mathbb{R}^\infty\textit{-manifold.}$$

Proof. Write $P = |D_0^{(k)}|$. A simplicial-set realization is triangulable: normal subdivision is regular, and its realization admits a triangulation [28, Corollary 4.6.12]. The construction is countable since $D_0^{(k)}$ is degreewise finite. A stellar subdivision at the point corresponding to $*$, if necessary, makes it a vertex.

We record explicitly why deleting this vertex remains triangulable with the subspace topology. Write its closed star as the cone $* * L$, with radial coordinate $s \in [0, 1]$ measured from the vertex. Replace the punctured cone by

$$|L| \times (0, 1] = \bigcup_{j \geq 1} |L| \times [1/(j+1), 1/j].$$

Triangulate these prisms compatibly, and attach the end at $s = 1$ to the subcomplex of simplices not containing $*$. This gives a countable simplicial complex. On each old simplex the radial bands are locally finite after the vertex is removed, so their weak topology is the ordinary topology of the punctured simplex. The weak topology over the old simplices is exactly the subspace topology on $P \setminus \{*\}$. Thus this construction is a triangulation of the required puncture, rather than merely a homotopy model for it.

Let C, A be compact subsets of $P \setminus \{*\}$. Choose M large enough that A lies in row degree at most M. Proposition 11.24, applied to C, gives $g_t C \cap A = \varnothing$ for $t > 0$. Set $h_t = g_t^{-1}$ acting on P. Since the canonical point is fixed, these are homeomorphisms of the puncture, and

$$h_t(A) \cap C = \varnothing \quad (t > 0).$$

The preceding lemma proves the claim. □

Thus this argument reduces the local-manifold question to the canonical point. The proof does not remove that point from compact images: the ambient crossed symmetric action fixes it. No assertion about its link, or about the homotopy type of the puncture, is needed for this theorem.

Corollary 11.28. *Let $G = |A_\bullet^{(k)}|$ be the affine group realization. The following assertions are equivalent:*

1. *$P \cong G$;*
2. *P is an $\mathbb{R}^\infty$-manifold;*
3. *the canonical point $\{*\}$ is a Z-set in P.*

Here the last assertion uses the global open-cover definition: for every open cover $\mathcal{U}$ of P there is a continuous map $z : P \to P \setminus \{*\}$ which is $\mathcal{U}$-close to the identity. It is not being replaced by an assertion only about maps with compact domain.

Proof. The affine realization is an $\mathbb{R}^\infty$-manifold. If P is one as well, the known homotopy equivalence $P \simeq G$ and manifold classification give $P \cong G$ [68, Theorem 6.5.1]. Every compact subset of an $\mathbb{R}^\infty$-manifold is a Z-set [68, Proposition 6.5.6]. It remains to prove that the third assertion makes P a manifold.

Use the countable triangulation of P above and fix a simplex σ, now possibly containing $*$. Work in its open star U. Given a sphere in its link, form the compact radial cone $F : D^{j+1} \to U$ as in Lemma 11.26. The global Z-set assumption gives a map z sufficiently close to the identity that

$$zF(D^{j+1}) \subseteq U, \qquad zF|S^j \simeq F|S^j \text{ within } U \setminus \sigma.$$

Here is an explicit justification of the boundary homotopy. Cover its compact image by finitely many sets $W_v = U \cap \{x : x_v > 0\}$, where v is a vertex of the link. Refine this cover, together with the complement of that image, by $\{U, P \setminus F(D^{j+1})\}$. Closeness for this global cover keeps the disk in

U and makes each boundary pair $F(a), zF(a)$ share some W_v. A partition of unity on S^j subordinate to these common-vertex sets gives a point supported on vertices common to the two link-coordinate supports. Straight interpolation through that common face gives the boundary homotopy after normalizing the link coordinates; the remaining open-star coordinates can be interpolated in $\operatorname{int}\sigma \times (0,1)$.

The image $zF(D^{j+1})$ is compact and misses $*$. Apply the forward shuffle of Proposition 11.24 to this image, choosing a finite row stage containing σ. At sufficiently small positive time it gives a disk missing σ, still in U, and its boundary homotopy stays in $U \setminus \sigma$. Together with the preceding boundary homotopy this contracts the original sphere in the link. The same argument starting from the barycenter supplies a point of $U \setminus \sigma$, so the link is nonempty. All links are therefore contractible CW complexes, and [68, Theorem 6.12.2] proves the manifold assertion. □

11.16 The basepoint link is itself an infinite-dimensional manifold

Proposition 11.29. *Let $P = |D_0^{(k)}|$, $k \geq 1$, and choose a countable triangulation $P \cong |K|$ in which the canonical point is a vertex v. Then the link $L = \operatorname{Lk}(v, K)$ is nonempty and $|L|$ is an $\mathbb{R}^\infty$-manifold. Its contractibility will follow from the further arguments below.*

Proof. The space P is connected and contains more than one point, so v is not an isolated vertex. Hence L is nonempty.

Fix a nonempty simplex $\tau \in L$ and write $\sigma = v * \tau \in K$. There is an equality of simplicial complexes

$$\operatorname{Lk}(\tau, L) = \operatorname{Lk}(\sigma, K).$$

Let U be the open star of σ in $|K|$, defined by strict positivity of all coordinates at vertices of σ. Since τ is nonempty, $v \notin U$.

First consider the barycenter b_σ. Apply the forward shuffle avoidance theorem to the compact set $\{b_\sigma\}$, using a finite row stage containing σ. At sufficiently small positive time its image is still in U and lies outside σ. Therefore $\operatorname{Lk}(\sigma, K)$ is nonempty.

For any map $f : S^j \to |\operatorname{Lk}(\sigma, K)|$, include its image in U by assigning outside weight $1/2$ and the remaining weight to b_σ. Cone this sphere to b_σ, using outside weights in $[0, 1/2]$. This gives a compact disk image $A \subset U \subset P \setminus \{v\}$. Forward shuffle avoidance moves A off a finite row stage containing

σ. For sufficiently small time the moved disk remains in U, and the entire boundary homotopy remains in $U \setminus \sigma$. Adjoining that boundary homotopy therefore fills the original sphere in $U \setminus \sigma$.

The space $U \setminus \sigma$ is homeomorphic to

$$\operatorname{int}\sigma \times |\operatorname{Lk}(\sigma, K)| \times (0,1).$$

Projection to the link shows that this nonempty link is weakly contractible, including path connectedness from the case $j = 0$. It is a CW complex, hence contractible. Thus every simplex of the countable complex L has nonempty contractible link. Sakai's characterization proves that $|L|$ is an $\mathbb{R}^\infty$-manifold [68, Theorem 6.12.2]. □

The open star of v is the open cone on $|L|$. The local problem is therefore reduced to the homotopy type of this manifold link. The same forward-shuffle argument applies to every simplex $\sigma \neq \{v\}$ of K, because its open star misses v.

Corollary 11.30. *For the link L above, the following are equivalent:*

$$P \cong |A^{(k)}_\bullet| \quad \Longleftrightarrow \quad |L| \text{ is contractible} \quad \Longleftrightarrow \quad |L| \cong \mathbb{R}^\infty.$$

Proof. All links in K other than the one at v are already nonempty and contractible by the preceding argument. Sakai's link characterization therefore makes contractibility of $|L|$ equivalent to the manifold property of P. Apply Corollary 11.28. Since $|L|$ is itself an $\mathbb{R}^\infty$-manifold, its contractibility is equivalent to its being homeomorphic to $\mathbb{R}^\infty$ by manifold classification. □

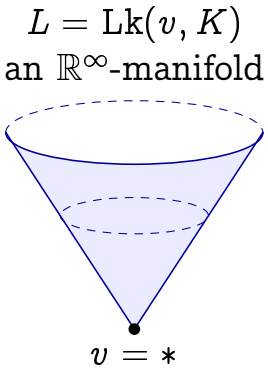
This reduces the comparison to a homotopy calculation for an actual link. Theorem 11.36 performs that calculation for $k \geq 2$ using anchor deletion; Proposition 11.38 gives a direct homeomorphism for $k = 1$. The link is not being identified with an anchored fibre or with the contractible decalage realization.

11.17 The lifted original realization is an infinite-dimensional manifold

Let $\widetilde{D}^{(k)}$ denote the original diagonal formed from the lifted arrangement face posets, and put $\widetilde{P} = |\widetilde{D}^{(k)}|$. Thus the original simplicial lattice torsor is

$$L^{(k)}_\bullet \longrightarrow \widetilde{D}^{(k)} \longrightarrow D^{(k)}_0.$$

Write $0 \in \widetilde{P}$ for the point defined by the zero cocycle in every degree. The distinction between this single point and the entire fibre $|L^{(k)}_\bullet|$ above the basepoint of P is essential.



$\mathrm{st}^\circ(v) = \mathrm{Cone}^\circ(L)$

Every other simplex has
a nonempty contractible link.

Anchor deletion and fibre avoidance:
$|L|$ is contractible ($k \geq 2$).

Figure 25: The local geometry of the original realization. The ellipse schematically represents its infinite-dimensional basepoint link. For $k \geq 2$, the local quasifibration argument proves its contractibility. For $k = 1$, the weighted-centre homeomorphism gives the manifold property directly, and the link criterion again implies contractibility.

Theorem 11.31. *For every $k \geq 1$,*

$$\widetilde{P} \cong \mathbb{R}^\infty.$$

Proof. The lifted row spaces are

$$\widetilde{Q}_q = |N\widetilde{\mathcal{P}}_q| \cong V_q.$$

They extend to all finite vertex maps, just as the quotient rows do: a face character pulls back to a signed face character or zero. Hence the crossed symmetric group acts continuously on $\widetilde{P}$, fixing 0.

The shuffle avoidance proof has a real-valued version. We give the compactness modification needed because these rows are no longer compact. Let $K \subset \widetilde{P} \setminus \{0\}$ be compact. It lies in a finite subcomplex of the countable lifted diagonal realization. Choose finitely many closed characteristic simplices covering that subcomplex. The diagonal-to-iterated realization map represents each of these compact simplices in $\widetilde{Q}_d \times \Delta^d$ for its fixed diagonal degree d. Restrict its parameter space to the closed inverse image of K. We obtain a finite family of compact representative spaces covering K. We do not use the full inverse image of K in a lifted row, which can be noncompact because zero-weight vertices can carry unbounded cocycle coordinates.

On every member of this finite family use the continuous witness

$$m(x, u) = \max_{\substack{I \subseteq [d] \\ |I| = k+1}} \left(\prod_{i \in I} u_i \right) |\theta_I(x)|.$$

It is positive. Indeed, after restriction to the positive-weight vertices, the resulting carrier row is zero if and only if all the corresponding real evaluations are zero. The zero equalities are arrangement walls, and the maximum-face argument proves this equivalence exactly as in the quotient proof. A positive-weight representative of a nonzero row cannot degenerate to the zero row. The finitely many compact representative spaces therefore give a uniform positive lower bound for m, and an upper bound for all the finitely many evaluations appearing in it. Consequently every representative has a nonzero face evaluation on $k+1$ intervals whose lengths are bounded below by one common positive number.

Vertex equivalence is now defined over $\mathbb{R}$ instead of $\mathbb{R}/\mathbb{Z}$. The same augmentation-kernel exterior-form proof shows that it is an equivalence relation, reflected by surjective vertex pullbacks. A nonzero face evaluation makes all its vertices pairwise inequivalent. The finite-grid sampling and terminal interleaving from Proposition 11.24 therefore give a path g_t, $g_0 = 1$, which moves K beyond any prescribed finite row degree for every $t > 0$.

It follows, by applying the inverse action to a second compact set, that $\widetilde{P} \setminus \{0\}$ has the compact ambient pushing property of Lemma 11.26. The lifted diagonal has countably many simplices. Its realization and its puncture admit countable triangulations by the same relative triangulation and radial-band construction used for P. Thus that lemma proves

$$\widetilde{P} \setminus \{0\} \text{ is an } \mathbb{R}^\infty\text{-manifold.}$$

The realized simplicial lattice acts by homeomorphisms on $\widetilde{P}$. It takes the point 0 to the corresponding point of its orbit $|L^{(k)}_\bullet|$. This orbit contains a nonzero point: choose a nonzero integral k-cocycle on $\Delta[k]$ and positive outer coordinates. Its constant lifted face flag is a nondegenerate outer representative, since all lower degrees of $L^{(k)}_\bullet$ are zero. Translation by this group element takes 0 to a point already known to have a manifold neighborhood. Translating that neighborhood back proves the manifold property also at 0.

Finally $\widetilde{P}$ is the known contractible total space of the original simplicial lattice torsor. The classification of $\mathbb{R}^\infty$-manifolds therefore identifies it with $\mathbb{R}^\infty$ [68, Theorem 6.5.1]. □

Corollary 11.32. *For every $k \geq 1$, the topological group $|L^{(k)}_\bullet|$ is an $\mathbb{R}^\infty$-manifold, of homotopy type $K(\mathbb{Z}, k)$.*

Proof. The integer cocycle groups extend to all finite vertex maps, so their realization has the same crossed symmetric action. Every compact subset is covered by finitely many compact characteristic simplices. Away from

zero, on each such simplex the witness used above is positive: some integral face value on the positive-weight vertices is nonzero. Real cocycle vertex equivalence is still reflected by every surjective pullback. Thus the identical finite-grid shuffle proof supplies compact ambient pushing on $|L^{(k)}_\bullet| \setminus \{0\}$.

The lattice simplicial set is countable in every degree, and its realization and puncture are countably triangulable. The link argument proves the manifold property away from zero; translation by any nonzero group element proves it at zero as well. Its homotopy type is the standard Dold–Kan identification. No homeomorphism with the contractible space $\mathbb{R}^\infty$ is being asserted for this group. □

This theorem alone does not prove the manifold property of the base. The closed fibre above $*$ is all of $|L^{(k)}_\bullet|$, not a single point. Avoiding one point upstairs does not avoid this entire fibre. A countable union of compact Z-sets need not be a Z-set in an $\mathbb{R}^\infty$-manifold: the whole manifold is itself such a union. Nor is $|L^{(1)}_\bullet|$ being identified homeomorphically with a circle merely from its homotopy type. The proof below instead uses the anchored join and the local quasifibration property of anchor deletion.

11.18 Anchor deletion is a quasifibration over every open set

Put $B_\bullet = Q^{(k)}_\bullet$, $T_\bullet = \mathrm{Dec}_0 Q^{(k)}_\bullet$, and let $p_\bullet : T_\bullet \to B_\bullet$ delete the retained anchor. Thus $T_n = Q^{(k)}_{n+1}$ and $p_n = d_0$. All realizations below are ordinary realizations in compactly generated Hausdorff spaces. Write $p : |T_\bullet| \to |B_\bullet|$.

Theorem 11.33. *For every $k \geq 1$ and every open subset $U \subseteq |B_\bullet|$, the restriction*

$$p^{-1}(U) \longrightarrow U$$

is a quasifibration. In particular this assertion concerns the actual inverse image with its subspace topology, not a replacement up to homotopy.

We give the localization details because a quasifibration need not remain a quasifibration over an arbitrary open subset, and a realization fibration of simplicial spaces does not by itself settle this issue.

Lemma 11.34. *A quasifibration between finite polyhedra which is a PL map is a quasifibration over every open subset of its target.*

Proof. Choose compatible triangulations making the map simplicial, with any specified target point b a vertex. For its open star W, group the source

barycentric coordinates according to their image vertices. On the inverse image of W the sum of the coordinates mapping to b is positive. Delete the other coordinates and renormalize. Straight interpolation defines a strong deformation retraction of the inverse image onto the actual fibre F_b, covering the straight contraction of W to b.

Use this contraction to map the inverse image of W to the homotopy fibre of the original map at b. Its restriction to F_b is the quasifibration equivalence, so the entire map is a weak equivalence. For any $y \in W$, its restriction to F_y is likewise a weak equivalence: it is the quasifibration equivalence at y, followed by transport along the chosen path from y to b. Consequently $F_y \to p^{-1}(W)$ is a weak equivalence, proving the quasifibration assertion over the contractible set W.

Compatible subdivisions give a basis of distinguished open stars. For any open subset U of the target, take all these stars contained in U. They cover U and satisfy the local intersection-refinement hypothesis of Dold–Thom's theorem: if $y \in W_1 \cap W_2$, choose a smaller distinguished star W_3 with $y \in W_3 \subseteq W_1 \cap W_2$. Satz 2.2 of [18, p. 245] therefore proves that $p^{-1}(U) \to U$ is a quasifibration. The theorem requires this refinement property, not closure of the family under intersections. The familiar two-open-set criterion is its subsequent corollary. For the distinguished-basis terminology see also [54, Definition 2.7.12 and Proposition 2.7.13]. □

Proof of Theorem 11.33. Step 1: the row maps. The finite row incidence nerves are non-singular: the dimensions of the cells strictly increase along a nonidentity flag. Their realizations and carrier maps consequently have their usual finite polyhedral structures. Corollary 11.14 proves that each p_n is a quasifibration. The preceding lemma proves the same statement after restriction to every open subset of B_n. Equivalently, each strict pullback square along an open inclusion into B_n is a homotopy pullback.

Step 2: homotopy pullbacks and subdivision. A map of simplicial spaces is called a realization fibration if homotopy realization takes every degreewise homotopy pullback square along it to a homotopy pullback square. Rezk's criterion applies whenever the target's component simplicial set is constant [65, Proposition 5.4]. Here all target rows are connected tori, so $p_\bullet$ is a realization fibration. The stability under homotopy base change is recorded there immediately after Definition 1.1.

Let sd_r denote r-fold edgewise subdivision, defined by precomposition with the ordinal-sum functor

$$[n] \longmapsto [r(n+1)-1].$$

There are natural compatible realization homeomorphisms $|\operatorname{sd}_r X| \cong |X|$ [67, Theorem 1.5]. The same coend construction applies to simplicial spaces. On a representing row, the new weights are repeated r times and divided by r. The map $\operatorname{sd}_r p$ again has connected rows, and its level maps are among the same carrier restrictions. Thus all assertions of Steps 1 and 2 apply after every edgewise subdivision.

Step 3: the simplicial open pieces. For an open $U \subseteq |B|$, define

$$O^{(r)}_n = \{x \in (\operatorname{sd}_r B)_n : \chi_x(\Delta^n) \subseteq U\}, \qquad T^{(r)}_U = (\operatorname{sd}_r T) \times_{\operatorname{sd}_r B} O^{(r)},$$

where χ_x uses the subdivision homeomorphism. Compactness of Δ^n shows that $O^{(r)}_n$ is open. Faces and degeneracies preserve the displayed condition, so these are simplicial spaces. By Step 1 their defining strict pullback is a degreewise homotopy pullback. Thus $T^{(r)}_U \to O^{(r)}$ is a realization fibration.

The ordinary realizations here compute homotopy realizations. Indeed, the original degeneracies are inclusions of finite subpolyhedra. The new spaces are open subsets of the rows, and their degeneracy images are the intersections with these same subpolyhedra; the corresponding closed cofibrations remain cofibrations on open restriction. The same observation applies to finite unions of degeneracy images. The point-fibre simplicial spaces used below also have polyhedral rows and polyhedral degeneracy inclusions.

If r divides s, subdividing $O^{(r)}$ gives a simplicial subspace of $O^{(s)}$, since a subdivided characteristic simplex lies in its old characteristic simplex. The realization homeomorphisms consequently give compatible maps

$$|O^{(r)}| \longrightarrow |O^{(s)}| \longrightarrow U, \qquad |T^{(r)}_U| \longrightarrow |T^{(s)}_U| \longrightarrow p^{-1}(U).$$

These maps are injective on underlying sets. We do *not* assume that $|O^{(r)}|$ has the topology inherited from U; realization of a degreewise open subspace need not have that property.

Step 4: continuous factorization of compact families. For $r_j = r_0 2^j$, every map from a compact Hausdorff space into U factors continuously through some $|O^{(r_j)}|$. The same is true for maps into $p^{-1}(U)$ and the corresponding $|T^{(r_j)}_U|$. Here is the topology argument.

Every compact subset of $|B|$ lies in a finite outer stage. Such a stage is the image of the compact space $B_n \times \Delta^n$: lower-degree representatives may be increased to degree n by degeneracies. Let $K \subset U$ be compact in that stage, and take its full compact inverse image $C \subset B_n \times \Delta^n$. The inverse image of U is an open neighborhood of C. On the standard simplex the meshes of the edgewise subdivisions tend uniformly to zero: within each subdivided simplex the cumulative barycentric coordinates vary by at most

$1/r_j$, so each ordinary coordinate varies by at most $2/r_j$. Therefore, for sufficiently large j, whenever $(x,t) \in C$, every subdivided closed simplex containing t has its entire characteristic image for x inside U.

There are finitely many such subdivided simplices. On the closed subset of C over each of them, the corresponding refined characteristic map lands continuously in $|O^{(r_j)}|$. These maps agree on overlaps, by the realization identities, so finite pasting gives a continuous map $C \to |O^{(r_j)}|$. It is constant on the fibres of $C \to K$, because the inclusion into $|B|$ is injective on underlying sets. The latter surjection is compact-to-Hausdorff and therefore a quotient map. Thus it descends to a continuous map $K \to |O^{(r_j)}|$. Composing gives the stated factorization for any compact-source map with image K.

For the total space, use its compact finite-stage representative space $T_n \times \Delta^n$ and apply the same argument to the characteristic maps followed by p. This proves the second assertion. The argument applies equally to compact homotopies. It proves continuous factorization, which is stronger than merely putting a compact set inside the image of a subdivided open piece.

Step 5: make a specified point a vertex. Fix $y \in U$ and use its positive-weight outer representative $[x;(u_0,\dots,u_{r_0-1})]$. An increasing homeomorphism $\varphi : [0,1] \to [0,1]$ acts naturally on every realization by

$$(u_0,\dots,u_m) \longmapsto (\varphi(s_0), \varphi(s_1)-\varphi(s_0), \dots, 1-\varphi(s_{m-1})), \quad s_i = \sum_{a=0}^{i} u_a.$$

This commutes with faces, degeneracies, and every realized simplicial map; compare the interval action in [19]. Choose φ taking the cumulative sums of this representative to the uniform subdivision points. Replacing U and its inverse image by their images under these commuting homeomorphisms, we may suppose $y = [x;(1/r_0,\dots,1/r_0)]$. Now y is the degree-zero point x in $\mathrm{sd}_{r_0} B$, and remains a degree-zero point after every further subdivision in the sequence $r_j = r_0 2^j$. Since $y \in U$, these vertices lie in $O_0^{(r_j)}$.

The constant degenerate vertex therefore defines an honest simplicial map $* \to O^{(r_j)}$. Its strict pullback along $T_U^{(r_j)} \to O^{(r_j)}$ is a degreewise homotopy pullback, by Step 1. Step 3 and the realization-fibration property show that the realized square is a homotopy pullback. Ordinary realization preserves strict finite limits in the compactly generated category; see [44, Corollary 11.6]. Hence its realized strict fibre is the actual topological fibre $F_y = p^{-1}(y)$, and

$$F_y \longrightarrow \mathrm{hofib}_y\big(|T_U^{(r_j)}| \longrightarrow |O^{(r_j)}|\big)$$

is a weak equivalence for every j.

Step 6: pass to the actual open set. A compact-source map to $\mathrm{hofib}_y(p^{-1}(U) \to U)$ consists of a map into $p^{-1}(U)$ and a compact family of paths in U ending at y. Step 4 factors both maps continuously through one common stage of the sequence. Their endpoint identities still hold at that stage, because the maps from the subdivided open pieces are injective on underlying sets. The same statement applies to homotopies and to relative compact parameter spaces. Consequently weak homotopy groups of this homotopy fibre are the filtered colimits of those in Step 5. The actual fibre F_y is unchanged throughout this system, and every comparison in Step 5 is a weak equivalence. Its comparison with the homotopy fibre over U is therefore a weak equivalence. This holds at every $y \in U$, proving the theorem. □

11.19 Removing the last local obstruction

In this subsection $k \geq 2$. Put

$$P = |Q_\bullet^{(k)}|, \qquad E = |\operatorname{Dec}_0 Q_\bullet^{(k)}|, \qquad \pi : E \longrightarrow P, \qquad F = \pi^{-1}(*).$$

The exact zero-fibre calculation identifies $F \cong |Q_\bullet^{(k-1)}|$, so F is nonempty and path connected.

Lemma 11.35. *For every open neighborhood U of $*$ in P, the inclusion*

$$\pi^{-1}(U) \setminus F \longrightarrow \pi^{-1}(U)$$

is a weak homotopy equivalence.

Proof. Choose once and for all an anchored tag $\eta = [v, s] \in E$ whose outer weights are positive and whose restriction after deleting the anchor is nonzero. Equivalently, choose a tag row with a nonintegral *unanchored* k-face character; at least $k+1$ unmarked tag vertices suffice. Thus $\pi(\eta) \neq *$. Nonzero anchored characters alone would not be the required condition.

The anchored product-flat construction, valid for $k \geq 2$, gives the continuous join $J_t(x, \eta)$, with $J_0(x, \eta) = x$. For every $x \in E$ and every $0 < t < 1$,

$$J_t(x, \eta) \notin F.$$

Indeed, restriction of the joined row to its unmarked tag vertices recovers the nonzero deleted tag row. These vertices have positive outer weights. A point over $*$ would have zero deleted row, whose restriction to this block

would also be zero, a contradiction. This uses the actual carrier restrictions and their block naturality.

Write $V = \pi^{-1}(U)$. For a continuous map $f : C \to V$ from a compact Hausdorff space, continuity of J and openness of V give $\varepsilon > 0$ such that

$$J_t(f(C), \eta) \subset V \qquad (0 \le t \le \varepsilon).$$

For $t > 0$ this entire image lies in $V \setminus F$. If a specified closed subset of C already maps into $V \setminus F$, its whole homotopy remains there, including at time zero. Applying this to points, paths, spheres, disks, and their boundary pairs proves the weak-equivalence assertion. Basepoint tracks lie in $V \setminus F$ and may be adjoined to make the homotopy-group comparisons based. □

Theorem 11.36. *For the original one-input construction and every $k \ge 2$, the link of the canonical point is contractible. Consequently P is an $\mathbb{R}^\infty$-manifold and*

$$|D_0^{(k)}| \cong |A_\bullet^{(k)}|.$$

Proof. Choose the countable triangulation of P in which $*$ is a vertex, and let U be its open star. Thus U is contractible. Let

$$B = U \setminus \{*\}, \qquad V = \pi^{-1}(U), \qquad V' = V \setminus F.$$

By Theorem 11.33, both $V \to U$ and $V' \to B$ are quasifibrations with their actual subspace topologies. The first has connected fibre F over $*$ and contractible base; hence V and all its fibres are path connected. The preceding lemma makes $V' \to V$ a weak equivalence, so V' is nonempty and path connected. Since $V' \to B$ is surjective, B is path connected as well.

Fix $b \in B$ and a basepoint in its fibre F_b. The inclusions give a commuting diagram of quasifibration homotopy exact sequences, with the identity map on F_b and the weak equivalence $V' \to V$ on total spaces. For degrees at least two, the five-lemma argument gives isomorphisms $\pi_j(B, b) \to \pi_j(U, b)$. For degree one, connectedness of F_b identifies the two base fundamental groups with

$$\pi_1(B, b) = \pi_1(V')/\operatorname{im}\pi_1(F_b), \qquad \pi_1(U, b) = \pi_1(V)/\operatorname{im}\pi_1(F_b).$$

The total-space isomorphism identifies these images, and therefore these quotients. The path-connectedness already established handles degree zero. Thus $B \to U$ is a weak homotopy equivalence, and B is weakly contractible.

If L is the link of the canonical vertex, then $B \cong |L| \times (0, 1)$. Consequently $|L|$ is weakly contractible; as a CW complex it is contractible. The preceding

shuffle argument already gives nonempty contractible links for every simplex other than the canonical vertex. All links in this triangulation now satisfy Sakai's characterization, so P is an $\mathbb{R}^\infty$-manifold [68, Theorem 6.12.2].

The affine realization is an $\mathbb{R}^\infty$-manifold of the same homotopy type. Manifold classification therefore gives the displayed homeomorphism [68, Theorem 6.5.1]. □

This proves a homeomorphism of the one-input underlying spaces. It does not identify the fixed carrier fold with the affine group law, and it makes no assertion here about multi-input realizations.

11.20 Weighted centres for the circle arrangement

For $k = 1$, a row point is a configuration of labelled phases $x_0, \dots, x_q$ modulo simultaneous rotation, with $x_0 = 0$. The integer-wall arrangement records their circular weak order. An open face is specified by a cyclically ordered partition

$$C_0, C_1, \dots, C_r, \qquad 0 \in C_0,$$

together with its lattice lift. Its closed polytope has gap coordinates $g_0, \dots, g_r \geq 0$, with $\sum_j g_j = 1$; here g_j is the gap immediately before cluster C_j. For a zero-dimensional face there is one cluster and no free coordinate. In particular every closed lifted face is a simplex.

Given outer weights $t \in \Delta^q$, define the weighted centre $b_t(F)$ of this closed face by

$$g_j(b_t(F)) = \sum_{i \in C_j} t_i.$$

Use the specified lattice lift to interpret the point on the cover. For t in the interior of Δ^q every displayed gap is positive, so $b_t(F)$ lies in the open face F. The construction is affine in t on each fixed lifted face and commutes with lattice translations.

For a row point represented by a barycentric flag $F_0 < \cdots < F_l$ with weights λ_j, put

$$e_{q,t}(x) = \sum_{j=0}^{l} \lambda_j b_t(F_j) \pmod{L_q}.$$

Zero flag weights and repeated flag entries cause no ambiguity, so this is a continuous map $Q_q \times \Delta^q \to A_q$.

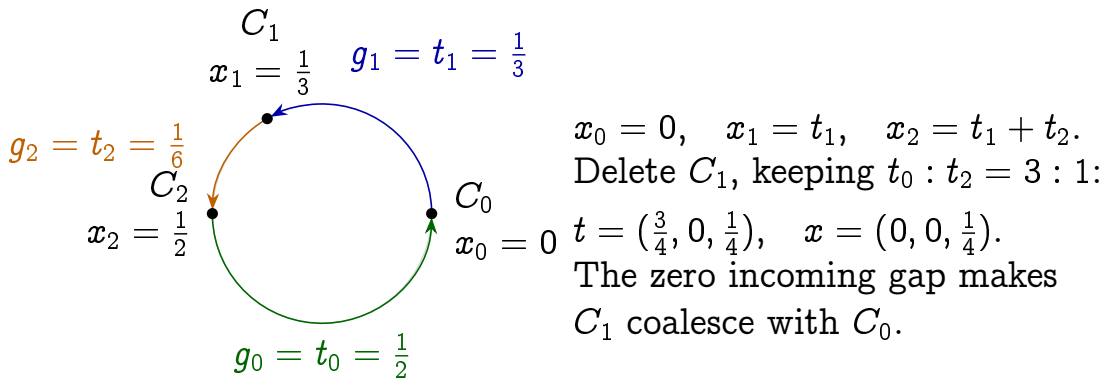


Figure 26: The weighted centre for three singleton clusters when $k = 1$. Each gap immediately *before* a cluster equals that cluster's outer weight. The weights $(1/2, 1/3, 1/6)$ therefore give the phases $(0, 1/3, 1/2)$. When the weight of a deleted cluster tends to zero, its incoming gap vanishes. Removing that cluster leaves the weighted centre of the restricted configuration, exactly as required by the face identity.

Lemma 11.37 (Weighted restriction identity). *For every monotone map $\alpha : [m] \to [q]$, every $s \in \Delta^m$, and every row point $x \in Q_q$, one has*

$$e_{m,s}(f_\alpha x) = T_\alpha\big(e_{q,\alpha_* s}(x)\big), \qquad (\alpha_* s)_i = \sum_{\alpha(j)=i} s_j.$$

Proof. It suffices to check a face centre. The clusters in the restricted face are the nonempty sets $\alpha^{-1}(C_j)$, in their inherited circular order. Their masses are exactly

$$\sum_{a \in \alpha^{-1}(C_j)} s_a = \sum_{i \in C_j} (\alpha_* s)_i.$$

An omitted cluster has zero mass. Removing it therefore leaves unchanged every phase difference between retained clusters. Changing the anchor subtracts the phase of the new anchor, which is precisely the affine cocycle pullback T_α. Repeated vertices only divide a cluster's mass among its labels. This proves the identity for centres modulo the lattice. Linearity of T_α proves it for flag combinations, including flags whose restricted entries coincide. □

Proposition 11.38. *The formula*

$$\Phi : |Q^{(1)}_\bullet| \longrightarrow |A^{(1)}_\bullet|, \qquad [x, t] \longmapsto [e_{q,t}(x), t]$$

defines a homeomorphism preserving the canonical point.

Proof. The weighted restriction identity is exactly the relation needed for both the face and degeneracy identifications in thin realization. Thus Φ is well-defined and continuous.

Fix interior outer weights t. Choosing one relative-interior point in every face of a polyhedral complex and joining these along flags gives its barycentric subdivision. Consequently $e_{q,t} : Q_q \to A_q$ is a piecewise linear homeomorphism, taking each open arrangement face onto itself. In particular it preserves and reflects every row degeneracy flat. The unique minimal outer representatives, whose outer weights are positive, therefore prove that Φ is bijective.

It preserves the finite outer-degree filtration and restricts to a bijection on every stage: on a minimal representative the inverse uses $e_{q,t}^{-1}$ in the same degree. Each such stage is compact, being a quotient of finitely many compact row tori times standard simplices; the affine realization is Hausdorff. Hence each finite-stage restriction is a homeomorphism. The weak finite-stage topology of both realizations makes the full map a homeomorphism. Integral zero-faces are unchanged, so the canonical point is preserved. □

The dependence on the outer weights is essential. This is not a strict natural homeomorphism between the row diagrams, and it does not turn the fixed carrier fold into group multiplication.

11.21 The realized abelian law and its precise scope

Corollary 11.39. *For every $k \geq 1$, the original one-input realization $P = |D_0^{(k)}|$ is an $\mathbb{R}^\infty$-manifold homeomorphic to $G = |A_\bullet^{(k)}|$. There is a based homeomorphism $h : P \to G$ preserving the canonical integral generator. Consequently*

$$m_P(x, y) = h^{-1}(h(x) + h(y)), \qquad \iota_P(x) = h^{-1}(-h(x))$$

make P a topological abelian group with identity the canonical point. These operations represent the previously constructed coherent addition and inversion in the homotopy category. Moreover $P \cong P \times \mathbb{R}^\infty$.

Proof. For $k \geq 2$ apply Theorem 11.36; for $k = 1$ apply Proposition 11.38. The affine realization is an $\mathbb{R}^\infty$-manifold, so P is one in both cases. Choose a homotopy equivalence $P \to G$ carrying the generator to the prescribed canonical class. Heisey's theorem replaces it by a homeomorphism in that homotopy class [37, Theorem C]. Subtracting its value at the basepoint makes it based and does not change its homotopy class, as in the proof of

Theorem 11.6. Transport of the group structure proves the strict abelian identities and continuity.

For the binary carrier projection p and fold μ, we have

$$m_P \circ |p| \simeq |\mu|$$

because both sides pull the fundamental integral class back to the sum of the two input classes. Since the target has homotopy type $K(\mathbb{Z}, k+1)$, equality of these classes gives the displayed homotopy. The inversion statement is identical. Finally P and $P \times \mathbb{R}^\infty$ are $\mathbb{R}^\infty$-manifolds of the same homotopy type, so manifold classification gives their homeomorphism [68, Theorem 6.5.1]. □

This conclusion concerns the original one-input space, without adjoining the contractible factor C. For $k = 1$ the preceding weighted-centre construction gives an explicit homeomorphism. For $k \geq 2$, the proof uses local quasifibrations, fibre avoidance, and manifold classification; an explicit finite formula for h or for m_P is not supplied.

The fixed carrier fold remains nonopen, by Proposition 8.10; hence it cannot itself become group multiplication under any product homeomorphism. Likewise, the absence of an ordinary unital simplicial multiplication is unchanged. Simplicially, addition still uses the refinement diagrams and their coherent comparisons. Geometric realization and the choice of h give the ordinary continuous abelian law. No simultaneous homeomorphism theorem for the original multi-input realizations is asserted here; the marked multi-input result of Theorem 11.6 still uses the enlargement.

The distinction between the two comparison proofs is useful. The weighted centres exploit the cyclic partition geometry specific to $k = 1$. The higher-degree proof instead controls the actual inverse image of every open set under anchor deletion. It needs neither local sections of that map nor a continuous section of a comparison span. In particular, no cancellation principle for arbitrary spaces is being assumed.

12 Finite fibres of Bockstein operations

Let

$$\beta_m : H^t(-; \mathbb{Z}/m) \longrightarrow H^{t+1}(-; \mathbb{Z})$$

be the integral Bockstein for $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0$. We now give a finite simplicial construction for the homotopy fibre of an operation represented by $\beta_m q$.

Theorem 12.1. *Fix integers $t \geq 1$ and $m \geq 2$. Let X be degreewise finite and choose a cocycle $q \in Z^t(X;\mathbb{Z}/m)$. Suppose Y is a degreewise finite model of $K(\mathbb{Z},t)$ equipped with a primitive normalized integral cocycle $e \in Z^t(Y;\mathbb{Z})$. Define*

$$P_n = \left\{ (x,y,b) : \begin{array}{l} x \in X_n, \quad y \in Y_n, \\ b \in C^{t-1}(\Delta[n];\mathbb{Z}/m), \\ \delta b = q_x - \bar{e}_y \end{array} \right\}, \tag{73}$$

where $\bar{e}$ is the reduction of e modulo m. Simultaneous pullback of the three entries makes P a degreewise finite simplicial set, with strict projections to X and Y. It satisfies

$$|P| \simeq \mathrm{hofib}\big(\beta_m[q] : |X| \longrightarrow K(\mathbb{Z},t+1)\big),$$

and its exact degreewise cardinality is

$$|P_n| = |X_n|\,|Y_n|\,m^{\binom{n}{t-1}}. \tag{74}$$

Proof. The cocycles q and $\bar{e}$ define simplicial maps $X \to \mathcal{K}(\mathbb{Z}/m,t)$ and $Y \to \mathcal{K}(\mathbb{Z}/m,t)$. Equation (73) describes the ordinary pullback of the fibration in Lemma 2.1 along their difference. It therefore models the homotopy fibre of that difference. In an abelian target this is equivalently the homotopy pullback

$$X \xrightarrow{q} \mathcal{K}(\mathbb{Z}/m,t) \xleftarrow{\bar{e}} Y.$$

This argument also proves that the formula is valid for the non-Kan toric models.

Because e is primitive, the right-hand map represents the usual coefficient reduction $K(\mathbb{Z},t) \to K(\mathbb{Z}/m,t)$, after choosing the orientation of the generator of Y. The coefficient sequence gives a homotopy fibre sequence

$$\mathcal{K}(\mathbb{Z},t) \xrightarrow{\rho_m} \mathcal{K}(\mathbb{Z}/m,t) \xrightarrow{\beta_m} \mathcal{K}(\mathbb{Z},t+1).$$

Pulling this sequence back along q identifies the homotopy pullback with the asserted fibre of $\beta_m[q]$. In particular, the construction identifies the actual classifying invariant; it does not merely produce a space with the expected homotopy groups.

The right side of the equation for b is a cocycle on $\Delta[n]$, so a solution always exists. The set of solutions is a coset of $Z^{t-1}(\Delta[n];\mathbb{Z}/m)$, which has $m^{\binom{n}{t-1}}$ elements. This proves (74). Pullback commutes with δ, q, and $\bar{e}$, so it preserves the defining equation and gives the stated simplicial operators. □

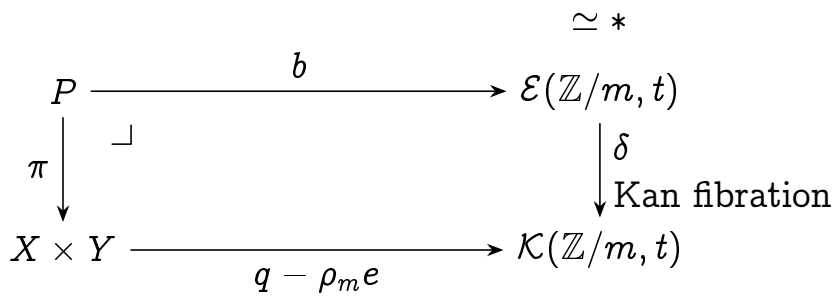


Figure 27: The finite construction is a strict pullback against a Kan fibration. Its simplices retain a solution of $\delta b = q_x - \rho_m e_y$. Thus the square computes the homotopy pullback even for non-Kan X and Y; the resulting projection to X models the fibre of $\beta_m[q]$. Every corner is degreewise finite when X and Y are.

There is also a cochain record of the nullhomotopy. The coordinate b defines a tautological $(t-1)$-cochain on P. Choose its integral lift B with values in $\{0, \dots, m-1\}$, and use the same convention for an integral lift Q of q. On P, where pullback notation is suppressed, the cochain

$$\frac{Q - e - \delta B}{m}$$

is integral and its coboundary is $\delta Q/m$, the standard cocycle representing $\beta_m[q]$.

Corollary 12.2. *Let X be degreewise finite and let $\kappa \in H^{t+1}(|X|; \mathbb{Z})$ be a finite-order class, with $t \geq 1$. Then the homotopy fibre of a map representing κ has an explicit degreewise finite simplicial model. Consequently finite iterations of untwisted integral Postnikov extensions with torsion k-invariants preserve degreewise finiteness.*

Proof. Choose $m \geq 2$ with $m\kappa = 0$. Exactness of the coefficient sequence gives a class $[q] \in H^t(X; \mathbb{Z}/m)$ with $\beta_m[q] = \kappa$. At the cochain level, choose an integral cocycle c for κ and solve

$$\delta Q = mc.$$

The torsion assumption guarantees a solution. Then $q = Q \bmod m$ is the required cocycle. This is a finite integral linear problem: only the simplices in degrees t and $t+1$ enter the equation. For $t \geq 2$, take the finite toric model $Y = D_0^{(t-1)}$; for $t = 1$, take $Y = \Delta[1]/\partial\Delta[1]$, with e equal to one on its nondegenerate edge. Applying Theorem 12.1 proves the assertion. The construction can be repeated at each stage. □

Here “untwisted” means that the added group is the constant coefficient group $\mathbb{Z}$; no nontrivial fundamental-group action is being included. The corollary makes no assertion about arbitrary infinite-order invariants. It is also an existence and construction statement, not a complexity estimate: the finite linear systems and the finite toric models can be large. This distinction is relevant when comparing the construction with algorithmic Postnikov methods such as [11].

13 The fibre of the universal degree-three square

The first example combines a familiar cohomology operation with the finite models constructed above. Set

$$X = D_0^{(2)}, \qquad Y = D_0^{(4)},$$

so that $|X| \simeq K(\mathbb{Z},3)$ and $|Y| \simeq K(\mathbb{Z},5)$. Let $c \in Z^3(X;\mathbb{Z})$ and $e \in Z^5(Y;\mathbb{Z})$ be their primitive normalized integral cocycles. They are given by Theorem 4.1. A finite Smith reduction of the normalized cochain matrices is an alternative way to choose them; a full enumeration of these matrices is not needed for the construction below.

For a three-cocycle c, use the degree-five cup-one cochain

$$\begin{aligned} h(012345) = {} & c(0345)c(0123) \\ & + c(0145)c(1234) \\ & + c(0125)c(2345). \end{aligned} \tag{75}$$

Here a string such as 0345 denotes restriction to that face of the given simplex. With these signs, the cup-one identity is

$$\delta h = -2(c \smile c). \tag{76}$$

It follows either from the usual cochain homotopy for graded commutativity, or by expanding the seven faces of a six-simplex and using $\delta c = 0$. Cup-one operations and their role in Steenrod squares are classical [74, 9].

Reducing (75) modulo two gives the cocycle

$$q = \overline{h} \in Z^5(X;\mathbb{F}_2), \qquad [q] = Sq^2\rho_2[c].$$

Equation (76) gives

$$\beta_2 Sq^2\rho_2[c] = -[c]^2 = [c]^2.$$

The final equality uses $2[c]^2 = 0$, also witnessed by (76). The following short calculation proves that the universal square is nonzero, independently of any finite-model computation.

Proposition 13.1. *For a primitive class $u \in H^3(K(\mathbb{Z},3);\mathbb{Z})$,*

$$H^6(K(\mathbb{Z},3);\mathbb{Z}) = \mathbb{Z}/2\langle u^2\rangle.$$

Proof. Put $B = K(\mathbb{Z},3)$ and use the multiplicative integral cohomological Serre spectral sequence of the path fibration

$$K(\mathbb{Z},2) \simeq \mathbb{CP}^\infty \longrightarrow PB \longrightarrow B.$$

The spectral sequence of a fibration is classical [73, Chapter II]. Here its second page is $H^p(B;\mathbb{Z}) \otimes \mathbb{Z}[t]$, with t in bidegree $(0,2)$. All odd fibre rows vanish, so $d_2 = 0$. Since B is 2-connected, $H^3(B;\mathbb{Z}) = \mathbb{Z}$, and PB is contractible, $d_3(t)$ is a generator; choose its sign to be u. The derivation rule gives

$$d_3(t^2) = 2ut, \qquad d_3(ut) = -u^2.$$

Thus the relevant part of the third page is

$$\mathbb{Z}\langle t^2\rangle \xrightarrow{2} \mathbb{Z}\langle ut\rangle \xrightarrow{1\mapsto -u^2} H^6(B;\mathbb{Z}).$$

No later differential enters or leaves bidegree $(3,2)$: the potential source has negative first degree, or the target has negative second degree. Contractibility therefore says that the kernel of the second arrow is exactly $2\mathbb{Z}\langle ut\rangle$. For bidegree $(6,0)$, the only additional possible incoming differentials have odd fibre-degree sources, or the source $(1,4)$, which vanishes because $H^1(B;\mathbb{Z}) = 0$. There is no outgoing differential. Contractibility therefore makes the second arrow surjective. Its kernel and image give the asserted group and generator. □

Theorem 13.2. *Define*

$$P_n = \left\{ (x,y,b) : \begin{array}{l} x \in X_n, \quad y \in Y_n, \\ b \in C^4(\Delta[n];\mathbb{F}_2), \\ \delta b = \overline{h}_x + \overline{e}_y \end{array} \right\}. \tag{77}$$

Then P is a degreewise finite simplicial model of

$$\operatorname{hofib}\bigl(u^2 : K(\mathbb{Z},3) \longrightarrow K(\mathbb{Z},6)\bigr).$$

Its realization is a two-stage Postnikov system, with

$$\pi_j|P| = \begin{cases} \mathbb{Z}, & j = 3,5, \\ 0, & j \neq 3,5, \end{cases} \qquad (j \geq 1),$$

and is connected. Its projection to X has homotopy fibre $K(\mathbb{Z},5)$ and k-invariant u^2. Moreover

$$|P_n| = |X_n|\,|Y_n|\,2^{\binom{n}{4}}.$$

Proof. Apply Theorem 12.1 with $t = 5$, $m = 2$, and $q = \overline{h}$. Subtraction and addition agree over $\mathbb{F}_2$, so its formula is exactly (77). The preceding cochain identity identifies its classifying invariant with u^2. The homotopy exact sequence of the resulting fibre sequence gives the stated homotopy groups. The cardinality is (74). □

The model carries a concrete integral nullhomotopy of the pulled-back square. Let B be the integral lift, with values zero or one, of its tautological four-cochain b. Suppress the projections when writing c, e, h on P. Then

$$T = \frac{\delta B - h - e}{2} \tag{78}$$

is an integral normalized five-cochain: its numerator is even by (77). Its coboundary is

$$\delta T = -\frac{\delta h}{2} = c \smile c.$$

Thus the finite model supplies both the space and the cochain data expressing the vanishing of its pulled-back k-invariant. As a further check, the projection $P \to Y$ induces multiplication by two, up to generator orientation, on π_5: its image is the kernel of reduction $\mathbb{Z} \to \mathbb{F}_2$.

Proposition 13.3. *The cochains c and T give an explicit weak equivalence from P to the standard integral cochain fibre*

$$\mathcal{F}_n = \{(z, t) : z \in Z^3(\Delta[n]; \mathbb{Z}),\ t \in C^5(\Delta[n]; \mathbb{Z}),\ \delta t = z \smile z\}.$$

The map is $(x, y, b) \mapsto (c(x), T(x, y, b))$, with each cochain evaluated on all faces of the simplex.

Proof. Equation (78) and $\delta T = c \smile c$ show that the map is defined and simplicial. The projection $\mathcal{F} \to \mathcal{K}(\mathbb{Z}, 3)$ is the pullback of the contractible cochain fibration $\mathcal{E}(\mathbb{Z}, 6) \to \mathcal{K}(\mathbb{Z}, 6)$ along the square map. It therefore models the stated homotopy fibre.

On the base, the comparison is the primitive class map $X \to \mathcal{K}(\mathbb{Z}, 3)$, a weak equivalence. On its homotopy fibre the comparison has degree -1, after compatible choices of generators. Indeed, the fibre over the basepoint of X is modelled by $\operatorname{hofib}(Y \to \mathcal{K}(\mathbb{F}_2, 5))$. Its projection to Y has image $2\mathbb{Z}$ on π_5. On this fibre $c = h = 0$, and $2T = \delta B - e$. Thus $[T] = -[e]/2$ is primitive there.

For completeness, the fibre calculation is legitimate even though Y is not Kan. Replace it by its class map $Y \to \mathcal{K}(\mathbb{Z}, 5)$. Pulling this weak equivalence back along the finite-coefficient Kan fibration gives a weak equivalence

of the two total models, and also of their fibres over the basepoint of X. The replacement model projects to X by a Kan fibration. Hence these strict fibres compute the homotopy fibres used above. The comparison is an isomorphism on π_3 and π_5, and both spaces have no other nonzero homotopy groups. It is a weak equivalence. $\square$

14 A local formula for the first prime-three operation

Let $c \in Z^3(X;\mathbb{Z})$ be a normalized cocycle. For a simplex with ordered vertices, write c_{ijkl} for evaluation on the indicated three-dimensional face. Products of these values in the formulas below are ordinary products of integers. Define an integer seven-cochain by

$$Q(c) = -\sum_{(\epsilon,I,J,K)\in\mathcal{T}} \epsilon\, c_I c_J c_K, \qquad q_3(c) = \rho_3 Q(c),$$

where the following table specifies $\mathcal{T}$ on $[0,\dots,7]$. Thus $Q(c)$ is an integer polynomial lift of $q_3(c)$; its values need not lie in $\{0,1,2\}$.

ϵ	I	J	K
$-$	0345	0567	0123
$-$	0456	0167	1234
$-$	0567	0127	2345
$-$	0145	1567	1234
$+$	0156	1267	2345
$-$	0167	1237	3456
$-$	0125	2567	2345
$-$	0126	2367	3456
$-$	0127	2347	4567
$+$	0123	3456	0167
$+$	0234	4567	0127
$+$	0123	3456	1267
$-$	0134	4567	1237
$+$	0123	3456	2367
$+$	0124	4567	2347
$+$	0123	3456	3467
$-$	0123	3567	3457
$+$	0123	3456	4567
$+$	0123	3467	4567

The same example in [39] also gives the three face products used for the Bockstein operation. In our ordinary integral cochain convention they give the much shorter eight-cochain

$$\begin{aligned} R(c)_{012345678} = {} & - c_{0678}c_{0123}c_{3456} + c_{0178}c_{1234}c_{4567} \\ & - c_{0128}c_{2345}c_{5678}. \end{aligned} \tag{79}$$

Proposition 14.1. *These are natural normalized cochains, and they satisfy*

$$\delta Q(c) = 3R(c), \qquad \delta R(c) = 0.$$

With the usual integral Bockstein $\beta_3[\bar{a}] = [\delta a/3]$, their classes are

$$[q_3(c)] = \mathcal{P}^1\rho_3[c], \qquad [R(c)] = \beta_3\mathcal{P}^1\rho_3[c].$$

Proof. The nineteen signed triples are the degree-three formula in [39, Theorem 24 and Example 27]. That theorem identifies the natural simplicial cochain operation with the classical reduced power. The displayed tensors there use graded tensor evaluation. Three degree-three cochains evaluated on three degree-three chains contribute the Koszul sign $(-1)^{27} = -1$, explaining the minus sign in Q. There is also a direct calibration: for a suspension cochain $c_{0ijk} = a_{ijk}$, with $c_{ijkl} = 0$ when the apex 0 is absent, only the first row contributes, and

$$Q(c)_{01234567} = a_{123}a_{345}a_{567}.$$

This agrees with stability and the normalization $\mathcal{P}^1[a] = [a]^3$ for a degree-two class.

The integral identity admits a finite polynomial certificate. On Δ^8 put $x_{ijk} = c_{0ijk}$ for $1 \le i < j < k \le 8$. Every integral three-cocycle has the unique expression

$$c_{ijkl} = x_{jkl} - x_{ikl} + x_{ijl} - x_{ijk} \quad (0 < i < j < k < l).$$

Substitution in the nineteen-term formula, followed by the alternating sum over the nine faces, gives exactly $3R(c)$. In these 56 independent variables the expanded polynomial has 36 monomials, all with coefficient $+3$ or -3; division by three agrees term by term with (79). The accompanying standard-library Python verifier performs this integer calculation exactly. It also verifies normalization on all 127 surjective monotone maps $[7] \to [m]$ with $m < 7$, and the suspension calibration. Face naturality is immediate from the face formulas; the normalization checks supply the degeneracy

condition for Q. Normalization of R then follows from $\delta Q = 3R$. Finally $3\delta R = \delta^2 Q = 0$ implies $\delta R = 0$ since integer cochain groups are torsion-free. Reducing Q modulo three and using the definition of the Bockstein proves the second class identity. □

If $\widetilde{q}_3$ denotes instead the pointwise residue lift of q_3 with values in $\{0, 1, 2\}$, then $\widetilde{q}_3 = Q + 3L$ for an integer seven-cochain L. Consequently $\delta\widetilde{q}_3/3 = R + \delta L$. Either lift gives the same integral operation, while the polynomial lift gives the particularly short exact identity above.

15 Incidence labels and face-insertion formulas

The canonical cocycle on $X = D_0^{(2)}$ has a small alphabet. Its values can be read from integer labels on the incidence flags, so the cochain operations below have a direct evaluation in the finite toric model.

Proposition 15.1. *The canonical cocycle $c \in Z^3(X; \mathbb{Z})$ takes precisely the three values $-1, 0, 1$.*

Proof. Lift a three-simplex to a flag $F_0 \le F_1 \le F_2 \le F_3$ whose first cell is canonical. On F_0 the three anchor coordinates lie in $[0, 1)$, and the real cocycle identity gives

$$\theta_{123} = \theta_{023} - \theta_{013} + \theta_{012} \in (-1, 2).$$

The canonical formula reads $c = \lfloor \theta_{123}|_{F_1} \rfloor$. Because every integer level of θ_{123} is an arrangement wall and F_0 is a face of $\overline{F_1}$, this floor cannot be at most -2 or at least 2: the corresponding closed strips do not meet $(-1, 2)$. Thus $c \in \{-1, 0, 1\}$.

All three values occur on constant flags. For example, take the cells containing the points with anchor coordinates $(\theta_{012}, \theta_{013}, \theta_{023})$ equal respectively to

$$(1/10, 7/10, 1/10), \qquad (1/10, 1/10, 1/10), \qquad (7/10, 1/10, 7/10).$$

The corresponding values of θ_{123} are $-1/2$, $1/10$, and $13/10$. □

There is also a formula for every face of a higher simplex without separately choosing canonical lifts of its faces. Choose any lifted flag $F_0 \le \cdots \le F_n$ and write

$$L^r_{ijk} = \lfloor \theta_{ijk}|_{F_r} \rfloor.$$

For $i<j<k<l$, the descended cocycle has face value

$$c_{ijkl} = L^{j}_{jkl} - L^{i}_{ikl} + L^{i}_{ijl} - L^{i}_{ijk}. \tag{80}$$

Indeed, the lifted two-cochain on the face ijk is $A_{ijk} = L^{i}_{ijk}$, since the diagonal face operation retains the nerve objects with indices i, j, k. Equation (80) is exactly $c = \delta A$ on the lifted flag. Translation by an integral two-cocycle changes A by that cocycle, so this expression is independent of the chosen lift. The incidences in the flag remain essential data.

For each fixed triangle, the sequence of labels L^{r}_{ijk} is nonincreasing and can decrease at most once, by one. A character that is already strictly between two consecutive integers stays in that same strip on every coface. A character on an integer wall can leave the wall through either adjacent strip; only departure through the lower strip changes its floor. If F_0 is canonical, all the labels L^{r}_{ijk} lie in $\{-1, 0, 1\}$, and the anchor labels L^{r}_{0ij} lie in $\{-1, 0\}$.

Substituting (80) into the cup-one face-insertion formulas expresses the operations entirely through these finite incidence labels. This uses the ordered simplex faces and their incidences in the model. It does not identify the additive Γ-structure by itself with the higher diagonals defining Steenrod operations, nor does it assert that arbitrary assignments of the displayed labels form valid simplices.

15.1 Deriving the operation by face insertions

The three-term formula for $R(c)$ has a direct explanation in terms of the ordinary cup-one product. This also gives a shorter integral primitive, using only the same face evaluations that define the canonical toric cocycle.

We use the following sign convention. For normalized cochains a and b of positive degrees p and q, respectively,

$$(a \smile_1 b)_{0\cdots p+q-1} = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} a_{0\cdots j,\, j+q\cdots p+q-1}\, b_{j\cdots j+q}. \tag{81}$$

This is the convention of [62, Eq. (4.5)]. Its coboundary identity is

$$\begin{aligned} \delta(a \smile_1 b) = {} & \delta a \smile_1 b + (-1)^{p} a \smile_1 \delta b \\ & + (-1)^{p+q+1}\bigl(a \smile b - (-1)^{pq} b \smile a\bigr). \end{aligned} \tag{82}$$

See [62, Eq. (3.1)]. One can also check it directly in (81): the internal face terms cancel with the coboundaries of the two factors, and the two remaining endpoint terms give the displayed commutator.

For our integral 3-cocycle c, the earlier cochain h is precisely $c \smile_1 c$, so $\delta h = -2c^2$. The same insertion formula gives

$$\begin{aligned} c \smile_1 (c^2) = \; & - c_{0678}c_{0123}c_{3456} + c_{0178}c_{1234}c_{4567} \\ & - c_{0128}c_{2345}c_{5678} = R(c). \end{aligned} \tag{83}$$

In particular, the cocycle equation for R follows immediately from (82) and associativity of the cup product: $\delta R = c \smile c^2 - c^2 \smile c = 0$. The three terms correspond to the three possible positions of one six-dimensional block in the insertion formula.

Proposition 15.2 (A twelve-term integral primitive). *Put*

$$A = c \smile_1 h, \qquad B = h \smile_1 c, \qquad S = c \smile h + h \smile c.$$

Define the degree-seven cochain J by

$$\begin{aligned} J(c)_{01234567} = \; & c_{0123}c_{0367}c_{3456} + c_{0123}c_{0347}c_{4567} \\ & + c_{0147}c_{1234}c_{4567}. \end{aligned} \tag{84}$$

Then

$$B - A = 2J, \qquad Q_{12}(c) := A - J = \frac{3A - B}{2}$$

is an integral normalized cochain, with exactly twelve distinct cubic face monomials, and

$$\delta Q_{12}(c) = 3R(c).$$

The integrality identity $B - A = 2J$ holds even before imposing the cocycle equation on c.

Proof. The left Hirsch identity, or a direct division of the insertion sum into its first and second halves, gives

$$(c^2) \smile_1 c = c \smile h + h \smile c = S;$$

the relevant Hirsch sign is $(-1)^{3(3-1)} = 1$ [62, Eq. (3.2)]. Applying (82) to A and B now yields

$$\delta A = 2R - S, \qquad \delta B = -3S.$$

It remains to explain why division by two is integral. Expand $B = (c \smile_1 c) \smile_1 c$ by two successive face insertions. When the second insertion lies inside the first, the resulting terms are exactly $A = c \smile_1 (c \smile_1 c)$. When the insertions have disjoint interiors, either insertion can be performed first.

These two orders give the same scalar face product, with the same sign: all three inserted cochains have odd degree, so the relevant insertion signs are positive. There are exactly three such products, those in (84). Thus $B = A + 2J$ as an identity of cochains, without requiring $\delta c = 0$.

Consequently $Q_{12} = A - J$ is integral. For clarity, its nine positive terms can be kept in the factored form

$$A_{01234567} = c_{0567}h_{012345} + c_{0167}h_{123456} + c_{0127}h_{234567}.$$

Each occurrence of h has three terms. Subtracting the three distinct terms of J gives twelve terms with coefficients $+1$ or -1. Finally,

$$2\delta Q_{12} = 3(2R - S) - (-3S) = 6R,$$

and integral cochain groups are torsion-free, so $\delta Q_{12} = 3R$. □

Every formula above is natural under simplicial pullback. Normalization is also visible directly: for each summand in (81), every adjacent edge of the original simplex occurs in at least one of the two evaluated faces. If that edge is collapsed by a degeneracy, the corresponding normalized factor is zero. Thus cup-one preserves normalized cochains, and $Q_{12} = A - J = (3A - B)/2$ is normalized as well.

This derivation is universal for integral 3-cocycles. In the finite toric model it is applied to the canonical cocycle obtained from integer parts of lifted face characters; hence all twelve terms are explicit integer face evaluations. The reduction from nineteen terms to twelve comes from separating nested insertions from disjoint ones, rather than from an additional relation special to a toric chamber.

The standalone verifier `verify_geometric_cup1.py` checks the displayed integer identities in independent anchor variables, checks the general coboundary signs on arbitrary cochains in all degree pairs $1 \le p, q \le 6$, and checks normalization on all 127 proper surjective monotone maps out of $[7]$. These are exact polynomial checks, with no random sampling or enumeration of toric simplices.

15.2 Comparison with the reduced-power representative

Write $Q_{19} = Q$ for the nineteen-term integer formula in Section 14.

Proposition 15.3. *For every normalized integral three-cocycle c, the short formula $Q_{12}(c)$ and the nineteen-term formula $Q_{19}(c)$ represent the same class modulo three. In particular,*

$$[\rho_3 Q_{12}(c)] = \mathcal{P}^1 \rho_3 [c].$$

This assertion applies to reductions of integral cocycles. Equivalently, with $\bar{c} = \rho_3 c$,

$$\mathcal{P}^1[\bar{c}] = \big[(\bar{c} \smile_1 \bar{c}) \smile_1 \bar{c}\big]. \tag{85}$$

Proof. Both integer formulas satisfy $\delta Q_i(c) = 3R(c)$. Evaluate their difference on the tautological integral cocycle of $\mathcal{K}(\mathbb{Z}, 3)$. It is an integral seven-cocycle, and

$$H^7(K(\mathbb{Z}, 3); \mathbb{Z}) = 0.$$

Here is a short independent verification of this vanishing. In the integral cohomology spectral sequence of the path fibration, the fibre $K(\mathbb{Z}, 2)$ has cohomology $\mathbb{Z}[t]$, $|t| = 2$, and $d_3 t = u$, where u generates $H^3(K(\mathbb{Z}, 3); \mathbb{Z})$. The base is two-connected. No differential can enter $(4, 0)$, so contractibility of the total space gives $H^4 = 0$. The only potential nonzero sources entering $(7, 0)$ are $(4, 2)$, $(2, 4)$, and $(0, 6)$. The first two vanish, and the last is killed already by the injective differential $d_3(t^3) = 3ut^2$. Sources with odd fibre degree vanish as well. Thus $(7, 0)$ has no incoming or outgoing differential and must itself be zero.

Consequently $Q_{12} - Q_{19} = \delta L$ for an integral normalized six-cochain on $\mathcal{K}(\mathbb{Z}, 3)$. Pulling L back along the classifying map of c, then reducing modulo three, proves the claim. The argument identifies the operation without requiring an explicit polynomial formula for L. Finally, the identity $B - A = 2J$ gives $B - Q_{12} = 3J$. Thus $\rho_3 Q_{12} = \rho_3 B$ as actual cochains, proving (85). □

15.3 A strict finite target for the integral Bockstein

Fix $t \geq 1$. Write $\mathbb{T}_\delta$ for the circle group with its discrete topology. There is a simplicial carrier map

$$j : \mathcal{K}(\mathbb{T}_\delta, t) \longrightarrow D_0^{(t)}.$$

Indeed, the coefficient exact sequence and $H^{t+1}(\Delta[n]; \mathbb{Z}) = 0$ identify $Z^t(\Delta[n]; \mathbb{T})$ with $Z^t(\Delta[n]; \mathbb{R})/Z^t(\Delta[n]; \mathbb{Z})$. A cocycle z is therefore a point of this torus. Let $C(z)$ be its open-cell carrier, interpreted in the quotient incidence category. Define

$$j_n(z) = (C(z) \xrightarrow{\mathrm{id}} \cdots \xrightarrow{\mathrm{id}} C(z)),$$

a constant flag with n arrows. Every target wall character pulls back under a monotone vertex map to a source character or zero. Therefore carrier

functors carry $C(z)$ to $C(\alpha^* z)$, including for maps moving the first vertex. Constant nerve flags commute with all nerve operators, so j is strictly simplicial. The torus point and its carrier are independent of a lattice lift; this is why the construction uses the quotient incidence category.

Let A be the normalized real t-cochain on $\mathcal{K}(\mathbb{T}_\delta, t)$ given by the residue lift in $[0,1)$ of the tautological cocycle. Then δA is integral. The canonical generator satisfies the exact identity

$$j^* c_{t+1} = -\delta A.$$

To see this, use canonical anchor coordinates $A_{0I} \in [0,1)$. On a $(t+1)$-simplex their real cocycle extension has full non-anchor coordinate

$$\theta_{1\dots t+1} = \sum_{i=1}^{t+1} (-1)^{i+1} A_{0\dots\widehat{i}\dots t+1} = A_{1\dots t+1} - (\delta A)_{0\dots t+1}.$$

Since $A_{1\dots t+1} \in [0,1)$, its integer part is $-(\delta A)_{0\dots t+1}$. This also holds when the point lies on walls. Thus j represents the negative of the connecting map for the discrete coefficient sequence $0 \to \mathbb{Z} \to \mathbb{R}_\delta \to \mathbb{T}_\delta \to 0$.

For $m \geq 2$ compose j with the coefficient embedding $\mathbb{Z}/m \to \mathbb{T}$, $z \mapsto -z/m$, to obtain

$$b_m : \mathcal{K}(\mathbb{Z}/m, t) \longrightarrow D_0^{(t)}.$$

Both simplicial sets are finite in each degree, and b_m represents the positive integral Bockstein. More precisely, let $q = \rho_m Q \in Z^t(X; \mathbb{Z}/m)$ and suppose $\delta Q = mR$. Set $S = (-Q) \bmod m \in \{0, \dots, m-1\}$ pointwise. Then $S = -Q + m\lceil Q/m \rceil$, and for $g = b_m \circ \widehat{q}$ one has

$$g^* c_{t+1} = -\frac{\delta S}{m} = R - \delta\lceil Q/m \rceil. \tag{86}$$

All cochains in this identity are normalized, since Q is normalized and $\lceil 0 \rceil = 0$.

In particular, the prime-three formulas give a strict map

$$D_0^{(2)} \xrightarrow{\widehat{q_3}} \mathcal{K}(\mathbb{F}_3, 7) \xrightarrow{b_3} D_0^{(7)}$$

representing $\beta_3 \mathcal{P}^1 \rho_3 u$ and satisfying (86) with $R = R(c)$. This assertion concerns maps and their cohomology classes; b_m is not asserted to be a Kan fibration. Ordinary pullbacks against this finite non-Kan target still require a separate homotopy-pullback argument.

More generally, every finite-order integral class in $H^{t+1}(X; \mathbb{Z})$ on a degreewise finite X admits such a strict finite-target representative after choosing a Bockstein factorization. The choices need not be canonical.

16 A second nontrivial stage at the prime three

Write $X = D_0^{(2)}$, and let $c \in Z^3(X;\mathbb{Z})$ be its canonical integral generator. The first finite stage is

$$P_{1,n} = \{(x, y_5, b_4) : \ \delta b_4 = \rho_2 h_x + \rho_2 (e_5)_{y_5}\},$$

where $y_5 \in (D_0^{(4)})_n$ and $b_4 \in C^4(\Delta[n];\mathbb{Z}/2)$. Thus $|P_1|$ models the homotopy fibre of the square of the universal degree-three class. Recall that it carries the integral cochain

$$T = \frac{\delta B_4 - h - e_5}{2}, \qquad \delta T = c \smile c.$$

All classes and cochains on X in the following formulas are understood to be pulled back when necessary.

Use the explicit operations of Section 14 to define

$$q_3 = \rho_3 Q(c) \in Z^7(X;\mathbb{Z}/3), \qquad [q_3] = \mathcal{P}^1(\rho_3[c]).$$

The integral three-term cocycle

$$K_3 = R(c) = \frac{\delta Q(c)}{3} \in Z^8(X;\mathbb{Z})$$

represents the integral operation $\beta_3 \mathcal{P}^1 \rho_3 [c]$.

Proposition 16.1. *The pullback of $[K_3]$ to P_1 is nonzero and has order three.*

Proof. Let $G = C_3 \times C_3$. Its mod-three cohomology is

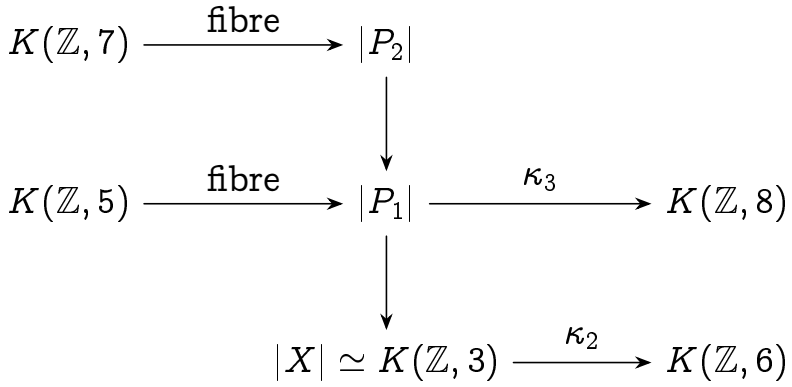
$$H^*(BG;\mathbb{Z}/3) = \mathbb{F}_3[x,y] \otimes \Lambda(a,b), \qquad |a| = |b| = 1, \quad |x| = |y| = 2,$$

with mod-three Bockstein $\overline{\beta}(a) = x$ and $\overline{\beta}(b) = y$; see [35, Example 3E.2]. Set $v = \beta_3(ab) \in H^3(BG;\mathbb{Z})$, so $\rho_3 v = xb - ay$. Graded commutativity gives $2v^2 = 0$, while $3v = 0$ gives $3v^2 = 0$. Hence $v^2 = 0$. The classifying map of v therefore lifts, up to homotopy, to $|P_1|$.

Naturality, the Cartan formula, and the instability identities [35, Section 4.L] give

$$\mathcal{P}^1(xb - ay) = x^3 b - a y^3, \qquad \overline{\beta}\mathcal{P}^1(xb - ay) = x^3 y - x y^3 \neq 0.$$

Since $\rho_3 \beta_3 = \overline{\beta}$, the pullback of $[K_3]$ along this lift is nonzero. Thus $[K_3]$ remains nonzero on P_1. Its order divides three by the coefficient exact sequence, so its order is exactly three. $\square$



$$\kappa_2 = [c]^2 = \beta_2 Sq^2 \rho_2[c] \quad \text{(order 2)}$$
$$\kappa_3 = \beta_3 \mathcal{P}^1 \rho_3[c] \quad \text{(order 3)}$$

Figure 28: The first two nontrivial stages, with $X = D_0^{(2)}$. Each descending arrow has the homotopy fibre indicated on its left. The second invariant is pulled back from X to P_1 and remains nonzero there. The only positive homotopy groups of $|P_2|$ are copies of $\mathbb{Z}$ in degrees 3, 5, and 7. The finite models also retain the witnesses $\delta T = c^2$ and $\delta V = K_3$.

Let $e_7 \in Z^7(D_0^{(6)};\mathbb{Z})$ be the canonical integral generator. Define the second stage by

$$P_{2,n} = \{(x, y_5, b_4, y_7, b_6) : (x, y_5, b_4) \in P_{1,n}, \quad y_7 \in (D_0^{(6)})_n,$$
$$b_6 \in C^6(\Delta[n];\mathbb{Z}/3), \qquad \delta b_6 = (q_3)_x - \rho_3(e_7)_{y_7}\}.$$

Every simplicial operator pulls back all five entries. The finite Bockstein-fibre theorem gives a homotopy fibre sequence

$$K(\mathbb{Z},7) \longrightarrow |P_2| \longrightarrow |P_1| \xrightarrow{[K_3]} K(\mathbb{Z},8).$$

In particular, this is a second nontrivial integral Postnikov extension. Its only nonzero homotopy groups are

$$\pi_3(|P_2|) = \pi_5(|P_2|) = \pi_7(|P_2|) = \mathbb{Z}.$$

The exact degreewise count is

$$|P_{2,n}| = |(D_0^{(2)})_n|\,|(D_0^{(4)})_n|\,|(D_0^{(6)})_n|\,2^{\binom{n}{4}}3^{\binom{n}{6}}.$$

Indeed, after the three toric entries have been chosen, the two independent cochain equations have respectively $2^{\binom{n}{4}}$ and $3^{\binom{n}{6}}$ solutions.

The construction also supplies an integral nullcochain for the second invariant. Let B_6 be the lift of the tautological b_6 with values in $\{0,1,2\}$. Then

$$V = \frac{Q(c) - e_7 - \delta B_6}{3} \in C^7(P_2;\mathbb{Z}), \qquad \delta V = K_3.$$

Thus both Postnikov equations have concrete integral witnesses on the final space: $\delta T = c^2$ and $\delta V = K_3$.

Proposition 16.2 (The secondary cocycle is the next invariant). *On the first finite stage P_1, let c be the pulled-back normalized integral three-cocycle and let T be its specified square primitive, $\delta T = c \smile c$. Put*

$$M = c \smile T + T \smile c.$$

Then the exact integral cochain identity

$$\delta(c \smile_1 T) = -M - R(c) \tag{87}$$

holds. Hence $[M] = -[K_3]$ has exact order three on P_1. On P_2, where $\delta V = K_3$, it has the explicit primitive

$$M = \delta\big(-V - c \smile_1 T\big).$$

Proof. For the cup-one convention used above, its coboundary formula in degrees three and five, with $\delta c = 0$, is

$$\delta(c \smile_1 T) = -c \smile_1 \delta T - c \smile T - T \smile c.$$

Substitute $\delta T = c \smile c$ and the three-term identity $R(c) = c \smile_1 (c \smile c)$. This proves (87) over the integers. Proposition 16.1 gives the asserted order; the displayed primitive on P_2 follows by taking its coboundary. □

For fixed c, changing the shared nullcochain to $T+z$, with $\delta z = 0$, changes M by the explicit coboundary $-\delta(c \smile_1 z)$. Thus this symmetric secondary class is independent of that choice. The prime-three stage already kills this secondary class; it is not an additional independent invariant.

17 Lifting equations and further finite extensions

The preceding constructions turn a lifting question into a finite cochain equation on a finite test simplicial set B. It is useful to distinguish a homotopy lift from a strict map into the chosen finite model.

Proposition 17.1 (Integral obstruction equation). *Let $q \in Z^t(X;\mathbb{Z}/m)$, $t \geq 1$, and let Q be an integral lift of q. Put $K = \delta Q/m$. For a simplicial map $f : B \to X$, the following conditions are equivalent:*

1. *$|f|$ lifts up to homotopy to the finite Bockstein fibre $|P| \to |X|$;*
2. *$[f^*K] = 0$ in $H^{t+1}(B;\mathbb{Z})$;*
3. *there is $v \in C^t(B;\mathbb{Z})$ with $\delta v = f^*K$.*

If these conditions hold, the homotopy classes of lifts with the chosen identification over $|f|$ form a torsor for $H^t(B;\mathbb{Z})$.

Proof. The first equivalence is the defining property of the homotopy fibre of the map represented by K, and the second is the definition of a zero cohomology class. The difference of two nullhomotopies is a map to $K(\mathbb{Z},t)$, which gives the stated torsor. One can also see existence directly in the standard cochain model. Replace the finite auxiliary model Y by $K(\mathbb{Z},t)$ and write

$$\widehat{P}_n = \{(x,z,b) : z \in Z^t(\Delta[n];\mathbb{Z}), \quad b \in C^{t-1}(\Delta[n];\mathbb{Z}/m), \quad \delta b = q_x - \rho_m z\}.$$

The map $\widehat{P} \to X$ is a Kan fibration, and the class map of Y gives a weak equivalence $P \to \widehat{P}$ over X. Given v as above, the integral cochain $z = f^*Q - mv$ is closed and satisfies $\rho_m z = f^*q$. Thus $(f,z,0)$ is a strict lift to $\widehat{P}$. Conversely, any lift (f,z,b), with an integral lift B_b of b, gives

$$v = \frac{f^*Q - z - \delta B_b}{m}, \qquad \delta v = f^*K.$$

□

For the chosen finite model P, a strict lift is exactly the additional data of a simplicial map $g : B \to Y$ and a cochain $b \in C^{t-1}(B;\mathbb{Z}/m)$ satisfying $\delta b = f^*q - \rho_m g^*e$. Vanishing of the obstruction guarantees a homotopy lift, but does not by itself guarantee such a strict g on the original triangulation, because Y need not be Kan. For finite B, a homotopy lift can be represented by an actual map from a sufficiently iterated simplicial subdivision of B to P; its projection represents the original base map up to homotopy. This follows from the adjunction $\mathrm{Sd} \dashv \mathrm{Ex}$ and the fact that a map from a finite simplicial set into $\mathrm{Ex}^\infty P$ factors through some finite $\mathrm{Ex}^r P$. This is an existence statement, not a bound on the required number of subdivisions.

For example, a map $f : B \to X = D_0^{(2)}$ lifts up to homotopy through both displayed stages precisely when one can solve the two integral equations

$$\delta t = (f^*c) \smile (f^*c), \qquad \delta v = R(f^*c), \qquad |t| = 5, \quad |v| = 7.$$

The second obstruction is independent of the chosen first lift, because its defining operation was pulled back from X. For finite B, each equation is a finite integer linear system once the cocycle on its right has been evaluated. The integral solution sets modulo coboundaries are torsors for $H^5(B;\mathbb{Z})$ and $H^7(B;\mathbb{Z})$, respectively. Thus the finite presentation makes the obstruction calculation explicit even though the integral cohomology groups need not be finite.

Proposition 17.2 (Finite coefficient stages). *Let A be a finite abelian group, let $t \geq 0$, and let $k \in Z^{t+1}(W;A)$ be a normalized cocycle on a degreewise finite simplicial set W. Then*

$$R_n = \{(w,b) : w \in W_n, \quad b \in C^t(\Delta[n];A), \quad \delta b = k_w\}$$

is a degreewise finite model of the homotopy fibre of the classifying map $[k] : |W| \to K(A,t+1)$, with

$$|R_n| = |W_n|\,|A|^{\binom{n}{t}}.$$

For every simplicial map $f : B \to W$, a strict lift to R exists if and only if $f^*[k] = 0$ in $H^{t+1}(B;A)$.

Proof. This is the pullback of the contractible Kan cochain resolution $\delta : \mathcal{E}(A,t+1) \to \mathcal{K}(A,t+1)$. The equation is solvable on each simplex, and its solution set is a torsor for $Z^t(\Delta[n];A) \cong A^{\binom{n}{t}}$. A map $B \to R$ over f is precisely a normalized cochain $b \in C^t(B;A)$ with $\delta b = f^*k$, proving the last claim. □

Thus finite coefficient Postnikov stages and integral stages with finite-order, untwisted invariants can be interleaved while preserving degreewise finiteness. These statements use constant coefficient groups; they make no assertion about general local coefficient systems or integral invariants of infinite order.

18 A degreewise finite tower over all primes

The preceding construction can be continued through every prime. Write $p_1 = 2 < p_2 = 3 < p_3 = 5 < \cdots$, put $P_0 = X = D_0^{(2)}$, and retain the canonical cocycle $c \in Z^3(X;\mathbb{Z})$. Choose normalized cocycles

$$q_p \in Z^{2p+1}(X;\mathbb{Z}/p), \qquad [q_p] = \begin{cases} Sq^2\rho_2[c], & p = 2, \\ \mathcal{P}^1\rho_p[c], & p \text{ odd}. \end{cases}$$

For $p = 2, 3$ use the formulas already fixed above; for other primes one can use the natural cochain reduced-power construction of [39]. These choices are made on X and pulled back to every later stage. Put

$$\kappa_p = \beta_p[q_p] \in H^{2p+2}(X;\mathbb{Z}), \qquad Y_p = D_0^{(2p)},$$

and let $e_p \in Z^{2p+1}(Y_p;\mathbb{Z})$ be its canonical primitive cocycle. Thus $\kappa_2 = [c]^2$.

Define P_j recursively by adjoining to P_{j-1} the entries

$$y_{p_j} \in (Y_{p_j})_n, \qquad b_{p_j} \in C^{2p_j}(\Delta[n];\mathbb{Z}/p_j),$$
$$\delta b_{p_j} = (q_{p_j})_x - \rho_{p_j}(e_{p_j})_{y_{p_j}},$$

where $x \in X_n$ is the common base coordinate. The bonding map $P_j \to P_{j-1}$ forgets these two entries. This notation agrees with the stages P_1 and P_2 above. Finally take the inverse limit in simplicial sets,

$$P_\infty = \varprojlim_j P_j.$$

Theorem 18.1. *The simplicial set P_∞ is degreewise finite, with*

$$|(P_\infty)_n| = |X_n| \prod_{\substack{p \text{ prime} \\ 2p \le n}} |D_{0,n}^{(2p)}|\, p^{\binom{n}{2p}}. \tag{88}$$

Its realization is simply connected and has homotopy groups

$$\pi_i|P_\infty| = \begin{cases} \mathbb{Z}, & i = 3 \text{ or } i = 2p+1 \text{ for a prime } p, \\ 0, & \text{otherwise}, \end{cases} \qquad i \ge 1.$$

For $j \ge 1$, the map $|P_\infty| \to |P_j|$ is a Postnikov truncation in degree $2p_j + 1$; the map to $|P_0|$ is a truncation in degree three. At each finite stage,

$$K(\mathbb{Z}, 2p_j+1) \longrightarrow |P_j| \longrightarrow |P_{j-1}|$$

has a nonzero k-invariant of exact order p_j, namely the pullback of κ_{p_j}.

Proof. Theorem 12.1 identifies $|P_j|$ with the homotopy fibre of the pulled-back class κ_{p_j}. Induction using the homotopy exact sequence shows that the only positive homotopy groups of $|P_j|$ are one copy of $\mathbb{Z}$ in degree three and in degrees $2p_i + 1$, $1 \le i \le j$. Indeed, these new degrees increase strictly, and at each step the new fibre is $K(\mathbb{Z}, 2p_j+1)$.

We check that none of these invariants vanishes. The assertion for $p = 2$ is Proposition 13.1. For an odd prime p, use the detector from the prime-three argument with $A = B(C_p \times C_p)$:

$$H^*(A; \mathbb{Z}/p) = \mathbb{F}_p[x, y] \otimes \Lambda(a, b), \qquad v = \beta_p(ab), \qquad \rho_p v = xb - ay.$$

Here $|a| = |b| = 1$, $|x| = |y| = 2$, and the mod-p Bockstein $\overline{\beta}$ sends a, b to x, y [35, Example 3E.2]. The class v is killed by p, so v^2 is killed by both p and 2 and is zero. Its classifying map therefore lifts through the first, square-fibre stage. For each previously used odd prime $\ell \neq p$, $\rho_\ell v = 0$: multiplication by p is invertible over $\mathbb{Z}/\ell$. Hence $\beta_\ell \mathcal{P}^1 \rho_\ell v = 0$. Inductively the detector lifts through all stages preceding the prime p. Along any such lift, the mod-p reduction of the new invariant is, by the reduced-power identities [35, Section 4.L],

$$\overline{\beta}\mathcal{P}^1(xb - ay) = \overline{\beta}(x^p b - a y^p) = x^p y - x y^p \neq 0.$$

Thus the invariant is nonzero; its order is exactly p because it is an integral Bockstein image.

It remains to justify the infinite limit. For $k > n$, the vector space $Z^k(\Delta[n]; \mathbb{R})$ is zero, so the toric face category has one object and only its identity. Thus $D^{(k)}_{0,n}$ is a singleton for $n < k$. Also $C^k(\Delta[n]; \mathbb{Z}/p) = 0$ for $n < k$. Consequently, when $2p_j > n$, forgetting the newly adjoined entries gives a bijection

$$(P_j)_n \xrightarrow{\cong} (P_{j-1})_n.$$

For each fixed n, the tower therefore stabilizes exactly as sets in degree n. The finite-stage cardinality formula gives (88), a finite product.

For any integer m, choose j to include every prime with $2p \leq m$. The projection $P_\infty \to P_j$ is then an isomorphism in every simplicial degree at most m. Its realization identifies the m-skeleta. Cellular approximation therefore implies an isomorphism on π_i for $i < m$: maps of S^i and homotopies between them can be computed in the $(i+1)$-skeleton. Taking $m = i + 1$ and using the finite-stage computation proves the displayed homotopy groups of $|P_\infty|$. Connectedness also follows directly from its single vertex.

Fix a finite stage j and put $N = 2p_j + 1$ (or $N = 3$ if $j = 0$). To check its projection is an isomorphism on π_i for $i \leq N$, choose a later stage $l \geq j$ that already agrees with P_∞ through degree $i + 1$. The map $|P_\infty| \to |P_l|$ induces an isomorphism on π_i by the preceding paragraph. The finite bonding maps from P_l to P_j also do so, by their homotopy fibre sequences: all their fibres occur in degrees greater than N. Thus $|P_\infty| \to |P_j|$ induces the required isomorphisms, while $|P_j|$ has no homotopy above N. This is precisely the

claimed Postnikov truncation property. In particular, no interchange of realization and inverse limits has been used. □

19 Toric incidence data and Postnikov twisting

The phrase "tame tori with intricate gluing" has a precise interpretation in these constructions. The integral building blocks come from finite incidence categories on compact tori. Coefficient reduction comes from finite covers of those same tori. A Postnikov invariant then changes the way the diagonal simplicial faces are attached. We first identify the covers, then locate the twisting in one face operator. This leads to families with the same finite incidence data and cell counts but different homotopy types.

The underlying twisting mechanism is classical: a twisted Cartesian product has a product of sets in each degree and a modified first face [43, Section 18]. Cochain-resolution pullbacks are used explicitly for Postnikov systems in [11, Section 3.9]. The purpose here is to identify that mechanism in the finite integral toric models and to extract its geometric and counting consequences.

19.1 Coefficient reduction is a finite torus cover

Fix $r \geq 2$ and $m \geq 2$. Put $Y = D_0^{(r-1)}$, with canonical cocycle $e = c_r$, and abbreviate

$$V_d = Z^{r-1}(\Delta[d]; \mathbb{R}), \qquad L_d = Z^{r-1}(\Delta[d]; \mathbb{Z}).$$

Let $\widetilde{\mathcal{P}}_d$ be the lifted integer-wall face poset used to construct Y. In a quotient below, retain orbit incidences as arrows, including distinct branches with the same endpoints. Define

$$\mathcal{C}_d^{[m]} = \widetilde{\mathcal{P}}_d/(mL_d), \qquad Y_d^{[m]} = N_d\mathcal{C}_d^{[m]}.$$

Carrier maps preserve the sublattices mL_d, so these categories give a bisimplicial nerve and the displayed diagonal.

Proposition 19.1 (The geometric reduction torsor). *There is a simplicial isomorphism over Y*

$$Y^{[m]} \;\cong\; R_m^+(Y), \qquad R_m^+(Y)_d = \{(y, b) : b \in C^{r-1}(\Delta[d]; \mathbb{Z}/m), \quad \delta b = \overline{e}_y\}. \tag{89}$$

It is a principal torsor for the degreewise finite simplicial abelian group $K(\mathbb{Z}/m, r-1)$. For each fixed vertex degree d, the map of row realizations

is the finite torus covering

$$V_d/(mL_d) \longrightarrow V_d/L_d, \qquad \text{degree } m^{\binom{d}{r-1}}.$$

The diagonal map realizes, up to weak equivalence and compatible choices of generators, to multiplication by m on $K(\mathbb{Z}, r)$.

Proof. The proof of Theorem 4.1 supplies a normalized integral cochain a on the lifted diagonal $\widetilde{Y}$, with $\delta a = p^* e$. Evaluating a on all faces gives $\widehat{a}(\widetilde{y}) \in C^{r-1}(\Delta[d]; \mathbb{Z})$, and

$$\widehat{a}(\lambda + \widetilde{y}) = \widehat{a}(\widetilde{y}) + \lambda, \qquad \lambda \in L_d.$$

Reduction modulo m therefore descends to $\widetilde{Y}/(mL)$. Both its fibre over y and the solution set in (89) are torsors for

$$L_d/mL_d \cong Z^{r-1}(\Delta[d]; \mathbb{Z}/m).$$

Here reduction of integral simplex cocycles is surjective, as is immediate in anchor coordinates. The equivariant map $[\widetilde{y}] \mapsto (y, \overline{\widehat{a}(\widetilde{y})})$ is consequently a bijection in every degree. It is simplicial because a is a cochain on the lifted simplicial set. Lifting incidence strings identifies $\widetilde{Y}/(mL)$ with the diagonal defining $Y^{[m]}$.

The barycentric subdivision of the lifted arrangement is V_d. Quotienting by mL_d or L_d proves the assertion about row realizations and the covering degree. Finally, (89) is the homotopy fibre of $\rho_m e : Y \to \mathcal{K}(\mathbb{Z}/m, r)$. The coefficient fibre sequence identifies its total homotopy type with $K(\mathbb{Z}, r)$ and its map to Y with multiplication by m. □

Write $R^-_m(Y)$ for the equation $\delta b = -\overline{e}_y$. The map $(y, b) \mapsto (y, -b)$ identifies it with $R^+_m(Y)$; its toric-cover coordinate is $b = -\overline{\widehat{a}}$. For $r = 1$ the analogous geometric object is the ordinary m-fold covering of a simplicial circle.

Thus "toric" refers to actual compact tori in each row of the presentation. The lifted face closures are compact convex polytopes, with induced affine face identifications on the quotient; quotient closures can have identified boundary faces. No assertion is made that the diagonal realization is a finite-dimensional torus, or that a general Postnikov operation is a map of row tori.

19.2 An anchored normal form for the attaching maps

Let X, Y, q, e satisfy Theorem 12.1, with $t = r \geq 1$, and write $P(q)$ for its model. Set $A = \mathbb{Z}/m$ and $B = X \times Y$. The projection $P(q) \to B$ is the pullback of

$$\delta : \mathcal{E}(A, r) \longrightarrow \mathcal{K}(A, r)$$

along the cocycle $q - \bar{e}$. It is therefore a principal simplicial $\mathcal{K}(A, r-1)$-torsor: the action adds a closed cochain to b. This finite coefficient torsor over B and the integral extension $P(q) \to X$ are two different projections. The latter is not asserted to be a strict torsor for the non-Kan simplicial set Y.

Let $\Delta_0 \subset \Delta$ be the subcategory with all objects $[d]$ and those monotone maps $\alpha : [j] \to [d]$ satisfying $\alpha(0) = 0$. It contains every degeneracy operator and every face d_i with $i > 0$ in the contravariant convention. For $z \in Z^r(\Delta[d]; A)$ define its vertex-zero primitive by

$$(h_d z)(i_0, \dots, i_{r-1}) = \begin{cases} z(0, i_0, \dots, i_{r-1}), & i_0 > 0, \\ 0, & i_0 = 0. \end{cases} \tag{90}$$

All evaluations use normalized cochains; repeated vertices give zero. The cone identity and its anchored naturality are

$$\delta h_d z = z, \qquad \alpha^* h_d z = h_j \alpha^* z \quad \text{if } \alpha(0) = 0.$$

The first follows by evaluating $\delta z = 0$ on $(0, i_0, \dots, i_r)$, with the case $i_0 = 0$ evaluated directly. The second follows by substitution, including the cases where a pullback repeats a vertex.

Theorem 19.2 (All twisting can be placed in the first face). *The degreewise bijections*

$$\Phi_d : P(q)_d \longrightarrow P(0)_d, \qquad (x, y, b) \longmapsto (x, y, b - h_d q_x) \tag{91}$$

give an isomorphism after restriction to Δ_0. For an arbitrary monotone $\alpha : [j] \to [d]$, the transported operator on $P(0)$ is

$$(x, y, b_0) \longmapsto \big(\alpha^* x, \alpha^* y, \alpha^* b_0 + \tau_\alpha(q_x)\big), \quad \tau_\alpha(z) = \alpha^* h_d z - h_j \alpha^* z. \tag{92}$$

The correction is a closed $(r-1)$-cochain and satisfies

$$\tau_{\alpha\beta}(z) = \beta^* \tau_\alpha(z) + \tau_\beta(\alpha^* z). \tag{93}$$

In particular only d_0 changes among the elementary simplicial operators. In dimension $d = r$ its correction recovers the chosen representative:

$$\tau_{d^0}(q_x)(0, \dots, r-1) = q_x(0, \dots, r). \tag{94}$$

Proof. Subtracting $h_d q_x$ changes $\delta b = q_x - \overline{e}_y$ to $\delta b_0 = -\overline{e}_y$, and adding it gives the inverse. Pulling back and applying these coordinate changes gives (92). Taking a coboundary yields $\alpha^* z - \alpha^* z = 0$; substituting the definition twice proves (93). If $\alpha(0) = 0$, anchored naturality makes the correction zero. All degeneracies and all positive faces have this property. For $d^0 : [r-1] \hookrightarrow [r]$, the first term in the correction is $q_x(0, \dots, r)$, while the second is zero because $\Delta[r-1]$ has no nonzero normalized r-cochains. This proves the last formula. □

This is an exact coordinate form of the classical twisted Cartesian product. Using $b - h_d(q_x - \overline{e}_y)$ instead of (91) identifies the graded set with $B_d \times \mathcal{K}(A, r-1)_d$ itself. The relative form used here has an additional geometric advantage: when Y is toric, $P(0) = X \times R_m^-(Y)$ already carries the torus-cover incidence presentation of Proposition 19.1.

The first-face correction recovers q, not just its Bockstein class. Different representatives can describe isomorphic or homotopy-equivalent extensions. Precisely, if $q' - q = \delta a$ for $a \in C^{r-1}(X; A)$, then

$$P(q) \xrightarrow{\cong} P(q'), \qquad (x, y, b) \longmapsto (x, y, b + a_x)$$

is a simplicial isomorphism over $X \times Y$. These maps compose by addition of a. Because X is degreewise finite, the cocycles q and such cochains a form a finite groupoid. Its components are $H^r(X; A)$; the integral extension class is the further image under β_m. Thus equality of the integral extension classes is a coarser condition than the displayed change of cocycle.

19.3 The same cells can carry different Postnikov invariants

Corollary 19.3 (Counts and degeneracies are independent of the twist). *For fixed X, Y, r, m, every $P(q)$ has*

$$|P(q)_d| = |X_d|\,|Y_d|\,m^{\binom{d}{r-1}}.$$

The number of nondegenerate simplices is also independent of q in every dimension. Indeed the bijections (91) preserve all degeneracies. Writing f_d for this number gives the explicit formula

$$f_d = \sum_{j=0}^{d} (-1)^{d-j} \binom{d}{j} |X_j|\,|Y_j|\,m^{\binom{j}{r-1}}.$$

Consequently the realizations have identical numbers of cells in every dimension in their standard CW structures.

Proof. The solution set for b is a torsor for $Z^{r-1}(\Delta[d];A) \cong A^{\binom{d}{r-1}}$. The anchored bijection and its inverse commute with degeneracies, so they identify their images and hence the complementary nondegenerate simplices. Alternatively, the unique degeneracy decomposition gives $|P_d| = \sum_{j=0}^{d} \binom{d}{j} f_j$; binomial inversion gives the formula. Realization has one open cell for each nondegenerate simplex. □

Here is a concrete pair. Take $X = D_0^{(2)}$, $Y = D_0^{(4)}$, $r = 5$, $m = 2$, and let c be the canonical integral three-cocycle on X. Compare $q = 0$ with a cocycle representing $Sq^2\rho_2[c]$. Their realizations are respectively

$$K(\mathbb{Z},3) \times K(\mathbb{Z},5), \qquad \operatorname{hofib}\big([c]^2 : K(\mathbb{Z},3) \to K(\mathbb{Z},6)\big).$$

They have identical ordinary and nondegenerate simplex counts, and the same homotopy groups, but

$$H^6(|P(0)|;\mathbb{Z}) = \mathbb{Z}/2, \qquad H^6(|P(q)|;\mathbb{Z}) = 0. \tag{95}$$

For the first equality, use the product and the low-degree cohomology of the factors. For the second, the Serre spectral sequence of $K(\mathbb{Z},5) \to |P(q)| \to K(\mathbb{Z},3)$ has $d_6 : \mathbb{Z} \to \mathbb{Z}/2$ sending the fibre generator to $[c]^2$, hence surjectively. In total degree six the only other possible fibre term is $H^6(K(\mathbb{Z},5);\mathbb{Z}) = 0$, and the mixed term has $H^1(K(\mathbb{Z},3);\mathbb{Z}) = 0$. The stated low-degree vanishing follows, for example, from a CW construction of $K(\mathbb{Z},5)$ starting with S^5 and attaching cells only in dimensions at least seven. Thus (95) distinguishes the unmarked homotopy types.

The torsor also retains a witness for the next stage. With the integral lifts Q, B used above,

$$v = \frac{Q - e - \delta B}{m}, \qquad \delta v = \frac{\delta Q}{m}.$$

This is the integral nullcochain of the pulled-back k-invariant. It explains why subsequent operations can use actual cochain witnesses. In the square example, choosing $Q = h$ with $\delta h = -2c^2$ gives the earlier primitive $T = -v - e$, so $\delta T = c^2$. This sign and closed-cochain adjustment keeps the conventions consistent. No new independent obstruction is inferred merely from having chosen such a primitive.

19.4 One toric presentation and a family of Postnikov towers

The all-prime construction admits a stronger variation. Retain $X = D_0^{(2)}$, the chosen cocycles q_p, and $Y_p = D_0^{(2p)}$ from Section 18. For a set S of primes

put $\epsilon_p = 1$ if $p \in S$ and $\epsilon_p = 0$ otherwise. At *every* prime p, adjoin the same coordinates y_p, b_p as before, with

$$\delta b_p = \epsilon_p (q_p)_x - \rho_p (e_p)_{y_p}. \tag{96}$$

Even when $\epsilon_p = 0$, the stage and its integral homotopy group are retained. Denote the finite stages by $P_{S,j}$ and their degreewise inverse limit by P_S.

Theorem 19.4 (Identical toric flags and different homotopy types). *The following statements hold.*

1. *Every P_S is degreewise finite. All the P_S have a common restriction to Δ_0, up to the explicit isomorphisms of Theorem 19.2. This common object is*

$$U = X \times \prod_{p \text{ prime}} R_p^-(Y_p). \tag{97}$$

   The product has only finitely many nontrivial factors in each simplicial degree.

2. *Every $|P_S|$ is simply connected, with*

$$\pi_i |P_S| = \begin{cases} \mathbb{Z}, & i = 3 \text{ or } i = 2p+1 \text{ for a prime } p, \\ 0, & \text{otherwise}, \end{cases} \qquad i \geq 1.$$

   All ordinary simplex counts, all nondegenerate simplex counts, and hence all CW cell counts agree for all S.

3. *The canonical Postnikov invariant in degree $2p+2$ vanishes exactly when $p \notin S$. If $p \in S$, it has exact order p. Distinct sets S therefore give non-homotopy-equivalent realizations, even without a specified identification of their homotopy groups.*

In particular this gives $2^{\aleph_0}$ homotopy types with the same graded homotopy groups and the same cell counts. For the first N primes alone it gives 2^N different finite Postnikov-stage homotopy types with these same agreements at that truncation.

Proof. Apply (91) simultaneously to the coordinates in (96):

$$(x, (y_p, b_p)_p) \longmapsto (x, (y_p, b_p - \epsilon_p h_d(q_p)_x)_p).$$

For $2p > d$, both the toric factor and the cochain factor are singletons. Thus this is a finite product of bijections in degree d, compatible with all operators

in Δ_0. It proves the common presentation and the equality of ordinary and nondegenerate counts. In particular their common ordinary count is

$$|X_d| \prod_{2p \le d} |(Y_p)_d|\, p^{\binom{d}{2p}}.$$

Every bonding map realizes to an extension with fibre $K(\mathbb{Z}, 2p+1)$, whether its invariant is zero or nonzero. The degrees increase strictly. The homotopy exact sequence therefore computes the finite-stage groups as in Theorem 18.1. The tower stabilizes exactly in each simplicial degree, so the same skeleton argument proves the displayed limit groups and identifies $|P_{S,j}|$ with the corresponding Postnikov truncation. No commutation of realization with an arbitrary inverse limit is being used.

An inactive invariant is zero by construction. The active invariant at $p = 2$ is nonzero by Proposition 13.1. For odd p, use $B(C_p \times C_p)$ and $v = \beta_p(ab)$ as in Theorem 18.1. The square of v vanishes: it is killed by both 2 and p. For every earlier odd prime $\ell \neq p$, one has $\rho_\ell v = 0$, so the ℓ-invariant pulls back to zero. The detector therefore lifts through every earlier stage, independently of which earlier invariants were activated. On such a lift the mod-p reduction of the active invariant is

$$\overline{\beta}\mathcal{P}^1(xb - ay) = x^p y - x y^p \neq 0.$$

It consequently has order exactly p.

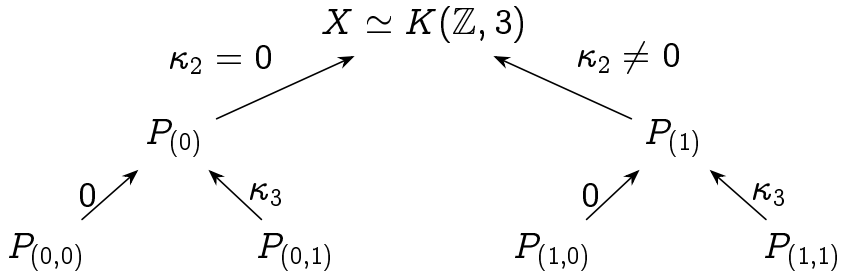
The vanishing or nonvanishing of a canonical Postnikov invariant is intrinsic to the homotopy type. Changing the equivalence of the lower truncation or the generator of its coefficient group preserves zero. If $S \neq S'$, choose their first differing prime. One space has zero invariant at that stage and the other has a nonzero invariant. Their canonical truncations cannot be equivalent. This proves the unmarked distinction for both the finite and infinite families. □

The common object (97) is itself toric in a precise rowwise sense. By Proposition 19.1, its degree d simplices are the d-flags in the product category

$$\mathcal{C}_d^{(2)} \times \prod_{2p \le d} \left(\widetilde{\mathcal{P}}_d^{(2p)} / p L_d^{(2p)}\right).$$

The nerve of this category realizes to a torus of dimension

$$\binom{d}{2} + \sum_{2p \le d} \binom{d}{2p}.$$



All four bottom spaces have $\pi_3 = \pi_5 = \pi_7 = \mathbb{Z}$
and the same cell counts, but different Postnikov data.

Figure 29: Independent choices of the first two prime invariants; the binary subscripts record activation in prime order. Each downward step in the construction retains the same new fibre, even when the invariant is zero. The arrows drawn are the Postnikov projections. Continuing this construction over all primes gives Theorem 19.4.

All families use these same flags, positive faces, and degeneracies. Only the transported first face varies. In the covering-flag coordinate its correction is the deck translation $-\epsilon_p \tau_{d^0}((q_p)_x)$; the minus sign comes from $b = -\widehat{a}$. This correction can depend on the full diagonal flag. It is not claimed to arise from a functor on the row categories. That is the precise limit of the torus description for general operations.

19.5 Classical context and the remaining scope

Neither the first-face form of a twisted product nor uncountably many homotopy types sharing homotopy groups is a new general phenomenon. Finite computability of Postnikov complexes was already studied by Brown [8]. The references above supply the twisting formalism. Rector's 1971 construction [64, pp. 99–100] is an especially close warning against a broad novelty claim: independent prime choices give uncountably many deloopings of the homotopy type of S^3, with the same homotopy groups and CW models having one cell in every dimension $4j$. Thus agreement of CW cell counts is classical as well. The particular statements proved here keep additional data fixed: explicit finite toric flags in every degree, all degeneracies, all positive faces, and all cell counts. The zero/nonzero prime invariants then distinguish the homotopy types. These are consequences of the displayed construction; historical priority for the combination requires a separate comparison with earlier examples. For the Δ_0 assertion itself, classical finite-coefficient twisted Cartesian products are an even closer precedent; see Section 25.7.

The accompanying standard-library verifier `verify_toric_postnikov_`

`twists.py` checks the anchor and translation identities over integral cocycle bases in degrees one through three. It tests 4574 closed defects, 2374 anchored pullbacks, and 27701 composition identities in its stated dimension ranges, as well as 24 exhaustive small finite-cochain enumerations. Three negative controls detect the omission of the first-face correction. These bounded checks support the local formulas; the infinite-family assertion uses the proof above.

There is also a distinction between finiteness and effectivity. An arbitrary set of primes need not be decidable. The continuum family is a set-theoretic family of degreewise finite presentations, not a continuum of finite algorithms. Every prescribed finite pattern gives an explicit finite-stage construction.

Finally, this section treats constant finite coefficient torsors and integral stages whose invariants are Bocksteins. It does not extend the torsion theorem to all integral invariants or to all local coefficient systems. The Heisenberg examples provide a separate infinite-order family. Their geometric input must be retained if one seeks a corresponding extension of the toric presentation theorem.

20 Uniform organization of operations and homotopies

The preceding formulas are instances of a general construction. There are three parts: cocycle-valued operations and their parameter spaces; explicit homotopies for known relations; and the full cochain action inherited from classical operads. The finiteness statements below apply to any degreewise finite source, including the toric models and their finite Postnikov extensions.

20.1 Finite parameter spaces for operations and their coherences

All cochains below are normalized, and mapping spaces are unpointed unless a base point is specified. Let $K(A,t)_n = Z^t(\Delta[n]; A)$, including $t = 0$.

Proposition 20.1. *Let X be a degreewise finite simplicial set, let A be a finite abelian group, and let $t \geq 0$. The simplicial mapping space*

$$\mathcal{M}^t_A(X) = \underline{\mathrm{Map}}(X, K(A,t)), \qquad \mathcal{M}^t_A(X)_r = Z^t(X \times \Delta[r]; A),$$

is a degreewise finite simplicial abelian group, hence a Kan complex. Its homotopy groups, with the zero cocycle as base point, are

$$\pi_i \mathcal{M}_A^t(X) = H^{t-i}(X; A) \quad (0 \le i \le t), \qquad \pi_i \mathcal{M}_A^t(X) = 0 \quad (i > t).$$

The formula for $i = 0$ is an identification of abelian groups of components. Translation identifies the homotopy groups at every other component with those at zero.

Proof. Cocycle representability gives the displayed equality in every parameter degree, compatibly with all simplicial operators. The group law is pointwise addition. A coarse finite bound is

$$|\mathcal{M}_A^t(X)_r| \le |A|^{|X_t|\binom{r+t+1}{t+1}}, \tag{98}$$

since a cochain is determined on the t-simplices and $|\Delta[r]_t| = \binom{r+t+1}{t+1}$. The cocycle equations involve only the finite set $(X \times \Delta[r])_{t+1}$.

For $i \ge 1$, the homotopy group is the relative cohomology group

$$H^t(X \times S^i, X \times *; A) = \widetilde{H}^t(X_+ \wedge S^i; A) = H^{t-i}(X; A).$$

Here one can use simplicial spheres and cohomology representability; no Kan assumption on X is needed. The assertion for components is the usual classification by H^t. Equivalently, the Alexander–Whitney map compares this mapping space with the Dold–Kan construction on the good nonnegative truncation of the mapping complex from normalized chains of X to $A[t]$; its degree-zero group is $Z^t(X; A)$. □

The mapping-space description is classical; see also [42, Example 3.1.7.11]. The finite bound uses all t-simplices of X, since a degenerate simplex of X can contribute a nondegenerate simplex of the product.

For pointed maps replace the cocycles by $Z^t(X \times \Delta[r], * \times \Delta[r]; A)$ and the cohomology groups by $H^{t-i}(X, *; A) = \widetilde{H}^{t-i}(X; A)$. For $t = 0$ the unpointed mapping space is the constant simplicial group of functions $\pi_0 X \to A$; there are no higher homotopy groups.

Corollary 20.2 (A simplicial category of operations). *Take as objects finite products of spaces $K(A, n)$ with A finite and $n \ge 0$, including the empty product. For any two such objects U, V, the full simplicial mapping space $\underline{\mathrm{Map}}(U, V)$ is a finite product of the spaces in Proposition 20.1. It is therefore degreewise finite and Kan. These mapping spaces have the usual strictly associative and unital simplicial composition: in degree r, compose two parameterized maps using the same $\Delta[r]$ parameter.*

Thus finite-coefficient operations and their higher homotopies belong to a single simplicial category with degreewise finite mapping spaces. Its component sets classify the corresponding unstable cohomology operations by representability. This is a formal consequence of classical cocycle models, not a new identification of the Steenrod algebra.

For example, a natural cocycle-valued expression $O : Z^n(-;\mathbb{F}_p) \to Z^m(-;\mathbb{F}_p)$ defines a strict map

$$\widehat{O} : \mathcal{K}(\mathbb{F}_p, n) \longrightarrow \mathcal{K}(\mathbb{F}_p, m).$$

Apply the expression to the tautological cocycle, or equivalently apply it on each simplex; naturality is exactly the simplicial-map condition. The same argument works for several inputs. It applies to the cocycle representatives supplied by the natural cochain operations of Kaufmann–Medina-Mardones [39, Theorem 24]. An arbitrary raw cup-i expression need not be closed, so the cocycle-valued hypothesis cannot be omitted. Nor must $\widehat{O}$ be a homomorphism of simplicial abelian groups.

The next subsections give explicit edges from cochain identities and compute the obstruction to filling prescribed boundaries.

Transport to finite integral targets. For $t \geq 1$, the strict finite Bockstein representative gives a postcomposition map

$$(b_p)_* : \mathcal{M}^t_{\mathbb{F}_p}(X) \longrightarrow \underline{\mathrm{Map}}(X, D_0^{(t)}). \tag{99}$$

Precisely, for any simplicial parameter set P, a map $P \to \mathcal{M}^t_{\mathbb{F}_p}(X)$ is a map $X \times P \to \mathcal{K}(\mathbb{F}_p, t)$; composition with b_p gives $X \times P \to D_0^{(t)}$. Every face, degeneracy, endpoint, and specified parameter subcomplex is preserved literally. The construction is also compatible with precomposition in X. Consequently every already specified coherent family of mod-p operations with this common final target has an equally coherent family of strict finite-target Bockstein representatives. If X and P are degreewise finite, so is their product.

For $n \geq 2$ and a toric input $D_0^{(n-1)}$ with canonical cocycle c_n, the individual maps have the concrete form

$$D_0^{(n-1)} \xrightarrow{\widehat{\rho_p c_n}} \mathcal{K}(\mathbb{F}_p, n) \xrightarrow{\widehat{O}} \mathcal{K}(\mathbb{F}_p, t) \xrightarrow{b_p} D_0^{(t)}.$$

Their higher parameter families are obtained by the same precomposition and postcomposition, rather than by choosing a new representative separately for every relation.

There are three limitations. First, the whole target mapping space $\underline{\mathrm{Map}}(X, D_0^{(t)})$ has not been shown degreewise finite; the finite source in (99) is what supplies the finite parameter space. Second, b_p is not asserted to preserve addition or to be a Kan fibration. Third, postcomposition at a common final target does not automatically replace every object of an operadic or compositional diagram by a toric model. Such a replacement requires compatible comparison maps at the intermediate objects. The cochain operad action, the finite mapping-space organization, and coherent addition on the toric models are related constructions; they are not yet proved to be one geometrically defined operad action on those models.

20.2 From cochain identities to finite simplicial homotopies

A cochain identity gives more than equality of the represented cohomology operations: it gives an explicit simplicial homotopy. Let A be an abelian coefficient group, let $t \geq 1$, and suppose that $q_0, q_1 \in Z^t(X; A)$ and $H \in C^{t-1}(X; A)$ satisfy

$$\delta H = q_1 - q_0.$$

Let $\alpha \in C^0(\Delta[1]; \mathbb{Z})$ take the values 0 and 1 at the two vertices. Pull the cochains on X and $\Delta[1]$ back to their product, and define

$$\widetilde{q} = \mathrm{pr}_X^* q_0 + \delta\big(\mathrm{pr}_X^* H \smile \mathrm{pr}_{\Delta[1]}^* \alpha\big). \tag{100}$$

Here the cup product uses the ordinary action of $\mathbb{Z}$ on A. The cochain $\widetilde{q}$ is closed, and its restrictions to the two endpoints are q_0 and q_1. Thus its classifying map is a strict simplicial homotopy

$$X \times \Delta[1] \longrightarrow \mathcal{K}(A, t).$$

The construction is natural in (q_0, q_1, H) and uses only normalized faces and the Alexander–Whitney cup product. The case $t = 0$ is simpler: equality of degree-zero cohomology classes already gives equality of the corresponding cocycles.

Proposition 20.3 (Finite universal homotopies for relations). *Fix a prime p, input degrees $n_1, \dots, n_s$, and an output degree $t \geq 1$. Let two natural cocycle formulas with these input and output degrees represent equal mod-p cohomology operations. Then a finite linear computation constructs a natural simplicial homotopy between their classifying maps. This applies to each specified Adem or Cartan relation once cochain representatives of its terms have been fixed.*

Proof. Use the universal input space

$$W = \prod_{i=1}^{s} \mathcal{K}(\mathbb{F}_p, n_i), \qquad |W_d| = p^{\sum_i \binom{d}{n_i}},$$

and let z_i be its universal normalized cocycles. Evaluating the two formulas on these cocycles gives $q_0, q_1 \in Z^t(W; \mathbb{F}_p)$. Their classes agree by the assumed universal relation, so $q_1 - q_0$ is a coboundary. Both groups in the equation

$$\delta H = q_1 - q_0, \qquad H \in C^{t-1}(W; \mathbb{F}_p)$$

are finite-dimensional: enumerate the nondegenerate simplices in degrees $t-1$ and t, form the normalized face matrix, and solve by Gaussian elimination. Existence of a solution follows from the relation; the linear computation supplies a particular finite table of values for H.

For any simplicial set X and normalized cocycles c_i of the specified degrees, their classifying maps give a strict map $f : X \to W$. The cochain f^*H satisfies the same relation on X, and (100) supplies the required homotopy. Thus a choice made once on the universal finite degree data pulls back naturally to all inputs. □

This is a finite-table algorithm for each fixed relation and set of degrees. It does not promise a small table or a short face polynomial. The explicit May–Steenrod formulas supply the cocycle evaluations; universal linear algebra supplies a homotopy when a cohomological relation is known. One can use a supplied explicit homotopy, such as a cochain-level Adem formula [9], instead of performing this linear solve. For the May–Steenrod framework, [39, Lemmas 12–13] proves the cohomological Adem and Cartan relations, the latter with its stated comultiplicative hypothesis. Here the input is the known relation for the classical operations on spaces.

20.3 A worked relation: the square of the Bockstein

There is a short carry formula for one relation at every prime. For $z \in Z^n(X; \mathbb{F}_p)$, let C be its integral lift with values in $\{0, \dots, p-1\}$, and put

$$B = \frac{\delta C}{p} \in Z^{n+1}(X; \mathbb{Z}), \qquad w = \rho_p B.$$

Thus w is our cochain representative of the mod-p Bockstein $\overline{\beta}[z]$. Let W be the residue-valued integral lift of w. Since $W - B$ is divisible by p, the formula

$$H_p(z) = \rho_p \left(\frac{W - B}{p} \right) \in C^{n+1}(X; \mathbb{F}_p) \tag{101}$$

defines a normalized natural cochain. Using $\delta B = 0$ gives the exact cochain identity

$$\delta H_p(z) = \rho_p\left(\frac{\delta W}{p}\right) = \overline{\beta}_{\mathrm{coch}}\big(\overline{\beta}_{\mathrm{coch}}(z)\big).$$

Applying (100) with $q_0 = 0$ gives an explicit simplicial nullhomotopy into $\mathcal{K}(\mathbb{F}_p, n+2)$. For $p = 2$, the cohomology Bockstein is Sq^1, so this is a cochain realization of the Adem relation $Sq^1Sq^1 = 0$. The formula specifies the Bockstein cochain convention; it does not require a pointwise identification with a different chosen formula for Sq^1.

The accompanying `verify_uniform_relations.py` also finds the primitive independently by Gaussian elimination on the universal model. Its complete normalized matrices have the following sizes; the rows are in degree $n + 2$ and the columns in degree $n + 1$.

n	p	Rows	Columns	Rank
2	2	41	4	3
2	3	636	20	18
3	2	958	11	10

In each case it verifies both primitives, their difference as a cocycle, and the endpoint and normalization equations for the resulting prism homotopy. These are complete computations in the indicated degrees, supplementary to the all-degree proof.

20.4 Prescribed higher coherence diagrams

Separate choices of homotopies need not fit into a prescribed higher coherence diagram. There is a precise finite test.

Proposition 20.4 (Boundary coherence obstruction). *Let A be a finite abelian group, let X be degreewise finite, and let $t \geq 0$ and $r \geq 1$. A prescribed simplicial map*

$$X \times \partial\Delta[r] \longrightarrow \mathcal{K}(A, t)$$

extends to $X \times \Delta[r]$ if and only if its connecting obstruction in

$$H^{t+1}\big(X \times \Delta[r], X \times \partial\Delta[r]; A\big) \cong H^{t+1-r}(X; A)$$

vanishes. The obstruction and, when it vanishes, a filler can be computed from finite normalized cochain matrices.

Proof. Write $Y = X \times \Delta[r]$ and $L = X \times \partial\Delta[r]$. The prescribed map is exactly a cocycle $u \in Z^t(L; A)$. Extend its values arbitrarily to a cochain $\widehat{u} \in C^t(Y; A)$, for example by assigning zero to the nondegenerate t-simplices outside L. The cochain $\delta\widehat{u}$ is a relative cocycle. Its relative class is independent of the extension and is zero precisely when there is $v \in C^t(Y, L; A)$ with $\delta v = \delta\widehat{u}$. Then $\widehat{u} - v$ is a cocycle extension and defines the filler.

The obstruction-group identification follows from the relative Eilenberg–Zilber equivalence for the simplex-boundary pair, followed by its degree-r suspension isomorphism. Every group and matrix needed in degrees t and $t+1$ is finite because X is degreewise finite. Over $\mathbb{F}_p$ use Gaussian elimination; for a finite abelian group use its cyclic decomposition and linear congruences. □

For $r = 1$, this is the class of $q_1 - q_0$ in $H^t(X; A)$, with the usual endpoint orientation. For $r > t+1$ the obstruction group is zero, so every prescribed boundary extends. In intermediate dimensions, vanishing is a real condition. For example, take X to be a point, $t = 1$, and $r = 2$. A map from the triangular boundary to $K(A, 1)$ assigns values a_{01}, a_{12}, a_{02} to its three edges. It extends exactly when $a_{12} - a_{02} + a_{01} = 0$.

When a filler exists, its homotopy classes relative to the prescribed boundary form a torsor for $H^{t-r}(X; A)$: subtract a fixed filler and apply relative cocycle representability. We interpret negative cohomology groups as zero. In particular, different homotopies for a fixed relation differ up to homotopy by $H^{t-1}(X; A)$. This records the choices of nullhomotopies that enter secondary constructions and subsequent lifting problems.

In the mapping space of Proposition 20.1, horn fillers always exist, while extension of a full prescribed boundary is governed by the proposition. Thus one may either retain coherences supplied by an operadic construction or build a chosen finite diagram by successive obstruction tests. Solving each Adem or Cartan relation separately does not by itself prove that all the resulting homotopies are mutually compatible.

20.5 Finite encoding of the inherited cochain action

There is a uniform source of coherence for the cochain operations. Normalized cochains of every simplicial set carry a natural action of the Barratt–Eccles E_∞ operad [5]. In the constructions of [39, Sections 6.1–6.4, Proposition 23 and Theorem 24], one can use the Barratt–Eccles operad, or the compatible surjection-operad action, together with the specified May–Steenrod operations. This applies directly to the toric models and to the finite coefficient cocycle models, without a Kan assumption.

Proposition 20.5. *Let Z be a degreewise finite simplicial set, let $N \geq 0$, and fix a prime p. Then*

$$A_N = C^*(\mathrm{sk}_N\, Z; \mathbb{F}_p)$$

is a finite-dimensional algebra over the Barratt–Eccles chain operad. For each bound on arity, its inherited action is specified by finitely many multilinear tables over $\mathbb{F}_p$. Every specified operadic identity among these operations is an exact finite calculation once all terms in that identity are included.

Proof. The normalized cochain complex A_N is concentrated in degrees $0, \dots, N$, with a finite basis in each degree. The degree-d part of the arity-r Barratt–Eccles operad has a basis of tuples $(\sigma_0, \dots, \sigma_d)$ of permutations in Σ_r with consecutive entries distinct; in particular this basis is finite. In cohomological grading its action has components

$$\mathcal{E}(r)_d \otimes A_N^{i_1} \otimes \cdots \otimes A_N^{i_r} \longrightarrow A_N^{i_1 + \cdots + i_r - d}.$$

These are maps between finite-dimensional vector spaces. They vanish for $d > rN$, so bounded arity leaves only finitely many potentially nonzero tables. The differential, equivariance and composition identities hold because these tables are restrictions of an existing operad action. Checking any specified identity, including the differential terms it involves, is therefore finite. □

These finite algebras are compatible as the skeleton grows. Naturality makes the restrictions $A_{N+1} \to A_N$ strict morphisms of algebras over the chosen operad, and there is a natural isomorphism

$$C^*(Z; \mathbb{F}_p) \cong \varprojlim_N A_N \tag{102}$$

in cochain complexes with that operad action. Indeed, $A_N^q = C^q(Z; \mathbb{F}_p)$ once $N \geq q$, and the coboundary out of degree q agrees once $N \geq q + 1$. For an operation with input degrees $i_1, \dots, i_r$ and output degree j, its table stabilizes for $N \geq \max(i_1, \dots, i_r, j)$, again by naturality. Thus the compatible finite algebras recover the full cochain action. The inverse limit is taken degreewise; no assertion about compactness of the ungraded direct sum is intended.

The coherence here is inherited, rather than reconstructed from cohomological relations. An arity or degree bound on the stored tables need not

itself define a suboperad: compositions can leave the chosen bounds. Likewise, solving one equation $\delta H = R$ for the defect of an Adem or Cartan identity does not ensure that arbitrarily prescribed further homotopies can be filled. On a degreewise finite universal source each such equation is a finite linear system, but its solvability still requires the obstruction class to vanish. The existing operad action supplies the coherences it specifies; finiteness alone supplies no additional vanishing.

Naturality also means that every simplicial map induces a strict morphism of these cochain algebras. In particular, the canonical weak equivalence

$$\chi_c : D_0^{(k)} \longrightarrow K(\mathbb{Z}, k+1)$$

induces an E_∞-algebra quasi-isomorphism on full $\mathbb{F}_p$-cochains. No quasi-isomorphism after arbitrary skeletal truncation is asserted. Thus the finite models inherit the usual cochain operations and their operadic organization. Section 22 expresses the action on incidence subflags and makes its compatibility with the carry comparisons explicit. Sections 22.2 and 22.4 also give local generators and recursions through the one-input part of the prop $\mathcal{M}$. These use the ordered incidence simplex and its face maps. The compatibility with subset-sum carrier maps follows from their simplicial naturality.

21 Coherent addition and the spectrum

The use of every nonempty input subset in (1) has a specific purpose: it makes all pointed-set fibre sums into strict carrier functors. In particular, addition is represented by the span

$$D_0^{(k)} \times D_0^{(k)} \xleftarrow{\simeq} X_0^{(k)}(2_+) \xrightarrow{\text{fold}} D_0^{(k)}.$$

All higher arities and their compatibility maps belong to the same finite diagram. This is the coherent abelian structure encoded by a very special Γ-space. Here our convention uses covariant functors from finite pointed sets; some sources call this indexing category Γ^{op}.

Definition 21.1. A reduced pointed-set diagram $X : \mathrm{Fin}_* \to \mathrm{sSet}_*$ is *special* if its maps $X(S) \to X(1_+)^{S^\circ}$, induced by the individual projections, are weak equivalences. It is *very special* if in addition the monoid $\pi_0|X(1_+)|$ induced by the fold is a group.

For the construction here, the projection maps on the covering lattices are the coordinate projections $\mathbb{Z}^{S^\circ} \to \mathbb{Z}$. The contractible-cover argument

therefore shows that the Segal maps induce isomorphisms on every homotopy group. Also $|X_0^{(k)}(1_+)|$ is connected. Thus this diagram is very special; no choice of an inverse Segal map is part of the definition.

21.1 Addition and the integral carry

The canonical cocycles are additive up to an explicit coboundary on each finite resolved product. Fix a finite pointed set S and a nonempty subset $U \subseteq S^\circ$. Write

$$\mu_U : X_0^{(k)}(S) \longrightarrow D_0^{(k)}$$

for the map summing the inputs in U, and let pr_s be the individual coordinate projections.

For a degree-k simplex, choose any lift of its flag and denote its first open cell by F_0. Define

$$b_U = \left\lfloor \sum_{s\in U} \theta^s_{0,\dots,k}\Big|_{F_0} \right\rfloor - \sum_{s\in U} \left\lfloor \theta^s_{0,\dots,k}|_{F_0} \right\rfloor. \tag{103}$$

The subset-sum walls make both terms constant on F_0. An integral translation changes them by the same integer, so b_U is independent of the lift. It is normalized, since all the full face characters vanish on a degenerate k-simplex. Using the canonical first-cell lift also gives the bound

$$0 \le b_U \le |U| - 1.$$

Thus b_U records the finite carry when the fractional coordinates are added.

Proposition 21.2. *For the canonical cocycle of Theorem 4.1,*

$$\delta b_U = \mu_U^* c_{k+1} - \sum_{s\in U} \mathrm{pr}_s^*\, c_{k+1}. \tag{104}$$

Proof. On the lifted cover, the cochain in (103) is the pullback of a along the actual sum map minus the sum of its coordinate pullbacks. Apply δ and use $\delta a = p^* c_{k+1}$. The resulting identity descends because the cover projection is surjective in every degree. □

The carries are compatible with successive sums. More precisely, partition U into nonempty disjoint subsets $U_1, \dots, U_r$ and let $\gamma : S \to r_+$ send U_j to j and all remaining inputs to the basepoint. Then there is an exact cochain identity

$$b_U = \gamma^* b_{\{1,\dots,r\}} + \sum_{j=1}^{r} b_{U_j}. \tag{105}$$

Indeed, set $A_j = \sum_{s \in U_j} \theta^s_{0,\dots,k}$ in a common lifted first cell; the terms $\sum_j \lfloor A_j \rfloor$ cancel from the right-hand side. Singleton carries vanish, and permutation invariance follows from the same formula. Equations (104) and (105) give explicit compatible cochain homotopies for addition on the finite resolved products.

21.2 A finite prespectrum for integral cohomology

Here is the resulting finite spectrum construction explicitly. Set $X = X^{(1)}_0$, take the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$, and let $S^r = (S^1)^{\wedge r}$, including $S^0 = 1_+$. Prolongation is degreewise evaluation:

$$(E_r)_n = X((S^r)_n)_n. \tag{106}$$

The pointed set $(S^r)_n$ is finite, so every set in (106) is finite. For $s \in (S^1)_n$, the pointed insertion $u \mapsto s \wedge u$ induces the assembly map

$$S^1 \wedge E_r \longrightarrow E_{r+1}.$$

Naturality makes these ordinary simplicial structure maps.

Segal's delooping theorem gives equivalences $|E_r| \simeq \Omega|E_{r+1}|$; hence

$$|E_r| \simeq K(\mathbb{Z}, r+2), \qquad E \simeq \Sigma^2 H\mathbb{Z}.$$

The conventions for prolongation, assembly, and the very-special condition are spelled out in [70, pp. 331–332]; the delooping theorem is [71, Proposition 1.4]. Define a prespectrum F by

$$F_0 = F_1 = *, \qquad F_n = E_{n-2} \quad (n \geq 2),$$

with the induced structure maps and basepoint maps at the bottom. Its only nonzero stable homotopy group is $\pi_0 F = \mathbb{Z}$. Consequently F represents $H\mathbb{Z}$. Every level is degreewise finite, and the realized adjoint maps $|F_n| \to \Omega|F_{n+1}|$ are weak equivalences for $n \geq 2$.

This last qualification is necessary. An unshifted Ω-spectrum for $H\mathbb{Z}$ has zeroth space with component set $\mathbb{Z}$, whereas a degreewise finite simplicial set has only finitely many components. We do not claim a degreewise finite unshifted Ω-spectrum. Likewise, the construction does not supply an ordinary simplicial multiplication $D_0^{(k)} \times D_0^{(k)} \to D_0^{(k)}$ with a homotopy unit; Appendix F proves the obstruction and explains why it leaves genuinely crossed products available. Theorem 8.12 excludes literal crossed simplicial groups on finite levels in these higher degrees. Section 8.10 describes the broader geometric structure that retains exact addition. Section 9 gives its

composition calculus and global finite simplicial representatives for all integer operations. Their refinement arrows are weak equivalences; the fixed carrier fold cannot become a topological group law under any product homeomorphism, by Corollary 8.11. An exact crossed construction would require a replacement fold or a different presentation. This obstruction does not affect the coherent addition or the spectrum construction above. Section 11 supplies a continuous abelian law on the original one-input realization itself (Corollary 11.39). It comes from a homeomorphism with the affine group realization. This is an explicit weighted-centre construction for $k = 1$ and an existence theorem using local topology for $k \geq 2$. It does not replace the simplicial coherent addition by the fixed carrier fold as a strict group law. The spectrum structure maps above remain ordinary simplicial maps, independently of this choice of realized law.

22 Incidence geometry and coherent operations

The additive and cochain constructions can be related explicitly. Ordered incidence subflags realize the higher diagonals, while the carry gives coherent comparison maps for all chains of pointed-set maps. A further cup-two calculation describes how Sq^2 respects the crossed addition, including its associator.

22.1 Higher diagonals as ordered incidence subflags

Fix $k \geq 1$ and abbreviate $X(S) = X_0^{(k)}(S)$. For an n-simplex represented by a lifted incidence flag $x = (F_0 \leq \cdots \leq F_n)$ and an increasing sequence $I = (i_0 < \cdots < i_a)$, define

$$x[I] = \big(\alpha_I^* F_{i_0} \longrightarrow \cdots \longrightarrow \alpha_I^* F_{i_a}\big), \qquad \alpha_I(j) = i_j.$$

Here α_I^* means the carrier functor for restriction of vertex coordinates. Arrows are the images of the corresponding composite incidences. Changing the common lattice lift changes none of the orbit data. This is precisely the simplicial face $\alpha_I^* x$ of the diagonal nerve: one must restrict the vertex coordinates as well as select the flag positions.

Proposition 22.1. *The standard natural Barratt–Eccles higher diagonals admit finite ordered-subflag formulas on every $X(S)$. They commute strictly with every pointed-set carrier map. Consequently, for a commutative coefficient ring R,*

$$S \longmapsto C^*(X(S); R)$$

is a strict contravariant diagram of algebras over the chosen Barratt–Eccles chain operad, indexed by finite pointed sets. Over a finite field, every fixed skeletal and arity bound has finite operation tables, compatible with the carrier maps.

Proof. Choose the natural chain action of Berger–Fresse [5, Theorem 2.1.1 and Section 2.2]. For an operad element e of arity a and chain degree d, expand its higher diagonal on the standard simplex as a finite sum

$$\Delta_e[0,\dots,n] = \sum_{I_1,\dots,I_a} \epsilon_e(I_1,\dots,I_a)[I_1]\otimes\cdots\otimes[I_a].$$

Then the rule on incidence flags is exactly

$$\Delta_e(x) = \sum_{I_1,\dots,I_a} \epsilon_e(I_1,\dots,I_a)x[I_1]\otimes\cdots\otimes x[I_a]. \tag{107}$$

A degenerate tensor factor is zero in normalized chains. The standard construction ensures that this rule also vanishes on degenerate input chains and satisfies the operad identities. For a pointed map $f : S \to T$, commutation of the two carrier directions gives

$$X(f)(x[I]) = (X(f)x)[I].$$

Thus every summand of (107) commutes with $X(f)$, including restrictions whose first vertex is not zero. Dualizing proves the stated strict algebra compatibility. Finiteness follows from degreewise finiteness of $X(S)$ and the finite rank of each fixed operad component: on the N-skeleton an arity-a operation of degree $d > aN$ has only negative possible output degrees and therefore vanishes. □

For clarity, the first three binary formulas are particularly small. In the chain convention of [5, Section 2.2.8],

$$\begin{aligned}
D_0(x) &= \sum_{i=0}^{n} x[0,\dots,i]\otimes x[i,\dots,n],\\
D_1(x) &= \sum_{0\le i<j\le n} (-1)^{(n-j)(j-i)+i} x[0,\dots,i,j,\dots,n]\otimes x[i,\dots,j],\\
D_2(x) &= \sum_{0\le i<j<\ell\le n} (-1)^{(\ell-j)(j-i+1)+i+j} x[0,\dots,i,j,\dots,\ell]\otimes x[i,\dots,j,\ell,\dots,n].
\end{aligned}$$

Writing τ for the graded tensor transposition, these obey

$$\partial D_1 + D_1\partial = \tau D_0 - D_0, \qquad \partial D_2 - D_2\partial = \tau D_1 + D_1.$$

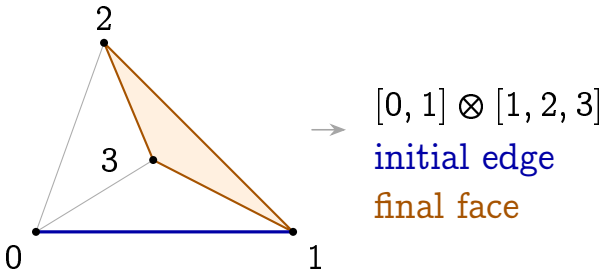


Figure 30: One summand of the Alexander–Whitney diagonal on a three-simplex: the initial edge and final face meet at vertex 1. For an incidence simplex these vertices are the ordered entries of its flag; each selected subflag also uses the corresponding restriction of vertex coordinates. Summing over all cut vertices gives the full diagonal.

These are formulas for flags of the original finite model; they do not require a new subdivision or a natural choice of barycentres. The diagonal used here is the ordinary diagonal $X(S) \to X(S) \times X(S)$ underlying cochain products. It is not a proposed diagonal into the refined addition object $X(S \vee S)$.

For comparison with the earlier cup-one convention, if $|a| = p$ and $|b| = q$, the ordinary product of cochain values gives

$$(a \smile_1 b)(x) = \sum_{i=0}^{p-1} (-1)^{(p-i)(q+1)} a\big(x[0, \dots, i, i+q, \dots, p+q-1]\big) \cdot b\big(x[i, \dots, i+q]\big).$$

This is $(-1)^{p+q}$ times ordinary evaluation of $a \otimes b$ on the displayed D_1; the sentence records the sign conversion rather than suppressing it. In particular, for the integral three-cocycle c,

$$(c \smile_1 c)(x) = c_{0345} c_{0123} + c_{0145} c_{1234} + c_{0125} c_{2345}.$$

Each c_I is evaluated by the canonical integer-part rule on the actual restricted incidence flag $x[I]$.

22.2 Generating the diagonals inside a flag

An explicit coherent diagonal structure can also be generated geometrically from three operations on the ordered index simplex of a single input flag. This is the classical counit–diagonal–join construction of [45, 47, 39]; the following formulation explains exactly how to implement it on the finite incidence model.

Put $C_n = N_*(\Delta[n];\mathbb{Z})$. For a nonempty ordered subset $I \subseteq [n]$, write $[I]$ for its simplex. Define

$$\epsilon[I] = \begin{cases} 1, & |I| = 1, \\ 0, & |I| > 1, \end{cases} \tag{108}$$

$$\Delta[v_0,\dots,v_q] = \sum_{a=0}^{q} [v_0,\dots,v_a] \otimes [v_a,\dots,v_q], \tag{109}$$

$$[I] * [J] = \begin{cases} (-1)^{|I|-1+\mathrm{inv}(I,J)}[I \cup J], & I \cap J = \varnothing, \\ 0, & I \cap J \neq \varnothing. \end{cases} \tag{110}$$

The union is sorted, and $\mathrm{inv}(I,J)$ counts pairs $(i,j) \in I \times J$ with $i > j$. The join has degree one. For a homogeneous chain map expression f of degree d, use $\mathrm{d}f = \partial f - (-1)^d f \partial$ and Koszul signs for tensor maps and permutations. Direct boundary cancellation gives

$$\mathrm{d}\epsilon = \mathrm{d}\Delta = 0, \qquad \mathrm{d}(*) = \epsilon \otimes 1 - 1 \otimes \epsilon. \tag{111}$$

Also $(\epsilon \otimes 1)\Delta = (1 \otimes \epsilon)\Delta = 1$ and $\epsilon * = 0$.

A prop allows operations with several inputs and several outputs, with tensor and compositional products. Let $\mathcal{M}$ be the differential graded prop generated by these three operations, their indicated differentials, and the last three relations. Its one-input part $U(\mathcal{M})(r) = \mathcal{M}(1,r)$ is an E_∞ operad [45]. Thus one-input wiring diagrams built from these generators organize a full coherent diagonal structure, not only the binary operations.

Proposition 22.2 (Local generation on flags). *For a simplicial set Z, an element $w \in \mathcal{M}(1,r)$, and a simplex $z : \Delta[n] \to Z$, put*

$$D_w(z) = (z_*)^{\otimes r} w_{C_n}[0,\dots,n]. \tag{112}$$

This defines a natural $U(\mathcal{M})$-coalgebra on normalized chains of Z. For the toric model, every output factor is an actual restricted incidence subflag $z[I]$. All pointed-set carrier maps strictly preserve this structure.

Proof. The generators commute with every monotone vertex map on normalized simplex chains. For the join, an injective map on $I \cup J$ preserves the ordering sign. If two indices in this union are identified, either an input face degenerates or the two image faces intersect; both sides then vanish. Intersecting input faces also give zero on both sides. The counit and Alexander–Whitney formulas have the same naturality.

Every diagram therefore commutes with these vertex maps. If z factors through a noninjective surjection of index simplices, its top chain maps to

zero; hence (112) vanishes. This proves normalization. Postcomposition by a simplicial map proves naturality in Z.

For operad composition, evaluate a second diagram on each output face of the first. Naturality for face inclusions identifies this with the composite diagram evaluated in the original C_n. The differential, relations and composition identities on simplex chains consequently descend to $N_*(Z;\mathbb{Z})$. The carrier maps are simplicial, giving the last assertion. This is the local construction of [39, Proposition 23]. □

The joins in this algorithm are joins of *positions* in one abstract simplex. Distinct positions remain distinct even if their cell names in the quotient category coincide. Evaluate the entire diagram on these indices, then push its output faces to Z, retaining composite arrows and vertex restrictions. This preserves parallel incidences and applies even when Z has only one vertex. A global join of arbitrary chains is unnecessary: on the discrete simplicial set with two vertices v, w, such a join would require $\partial(v * w) = w - v$ although $N_1 = 0$.

Computationally, a wire carries an integer combination of nonempty subsets of $[n]$. A split makes prefix–suffix cuts; a join tests disjointness and orders a union; the counit keeps only vertices and replaces each by the scalar one. Permutations use Koszul signs. Every finite diagram is therefore evaluated by finitely many operations on one flag, without enumerating the other cells of the model.

22.3 An explicit recursion for every binary diagonal

The integral recursion recorded in [47, Section 3.3] is particularly short. With τ the graded transposition, set

$$D_0 = \Delta, \qquad D_i = (* \otimes 1)(1 \otimes \tau D_{i-1})\Delta \quad (i \geq 1). \tag{113}$$

First split the flag into a front and a back. Apply the preceding diagonal to the back, swap its outputs, and join the front to the first output. An overlap makes that term zero. All integral signs are fixed by (110) and the tensor rule. The first two terms give exactly the signed formulas above.

Proposition 22.3. *The generated diagonals satisfy, in every degree,*

$$\partial D_i - (-1)^i D_i \partial = a_i D_{i-1}, \qquad a_i = \begin{cases} \tau - 1, & i \text{ odd}, \\ \tau + 1, & i \text{ even}. \end{cases} \tag{114}$$

They are natural under every simplicial map. On an n-simplex, $D_i = 0$ for $i > n$. Modulo two, the nonzero terms on the abstract simplex are exactly the parity-subset cup-i formula, with $\binom{n+1}{i+1}$ terms for $0 \leq i \leq n$.

Here is an independent boundary-filling construction, including the normalization argument. For the first vertex v of the current simplex, let $P_v = \eta_v \epsilon$ and let h_v prepend v to a face not containing it, and be zero on faces containing it. On this simplex

$$\partial h_v + h_v \partial = 1 - P_v, \qquad H_v = h_v \otimes 1 + P_v \otimes h_v, \qquad \partial H_v + H_v \partial = 1 - P_v \otimes P_v.$$

Starting with $\widetilde{D}_0 = \Delta$, define, inductively on i and then simplex dimension,

$$\widetilde{D}_i(\sigma) = H_v\big(a_i \widetilde{D}_{i-1}(\sigma) + (-1)^i \widetilde{D}_i(\partial\sigma)\big). \tag{115}$$

The expression in parentheses is a cycle: apply the preceding chain identities and $a_i a_{i-1} = 0$; the remaining boundary terms cancel. It has positive total degree except when $(\dim\sigma, i) = (0, 1)$, in which case it is $(\tau - 1)(v \otimes v) = 0$. The tensor contraction consequently proves (114) for $\widetilde{D}_i$.

Any monotone vertex map intertwines h_v with the cone at the image of v. Induction proves naturality if the image simplex is nondegenerate. If it degenerates, the preceding diagonal term maps to zero by induction on i, and the boundary term maps to zero by induction on dimension and $\alpha_* \partial\sigma = \partial \alpha_* \sigma = 0$. Thus the construction descends to normalized chains, including all degeneracies. Both factors remain faces of σ, so their total degree is at most $2\dim\sigma$; this proves vanishing for $i > \dim\sigma$.

The join recursion (113) satisfies the same signed chain identity by the generator boundary relation (111); this is the classical Steenrod recursion [47, Section 3.3]. Every term of $D_i(\sigma)$ for $i > 0$ has a positive-dimensional first factor containing v, because that factor is joined to a prefix. Hence $H_v D_i(\sigma) = 0$ and $(P_v \otimes P_v) D_i(\sigma) = 0$. Apply the tensor contraction to $D_i(\sigma)$ and then its chain identity to obtain exactly (115). Induction proves $D_i = \widetilde{D}_i$. Thus coning the required boundary and repeated splitting and joining give the same integral formula.

For the mod-two description, select cuts $s_0 < \cdots < s_i$ in $\{0, \dots, n\}$. The consecutive closed intervals

$$[0, s_0],\ [s_0, s_1],\ \dots,\ [s_{i-1}, s_i],\ [s_i, n]$$

go alternately to the first and second tensor factors. Intervals going to one factor are disjoint. Expanding the recursion gives each such pair once, and its shared vertices recover the cuts. This proves the term count. For the complementary list $U = \{u_1 < \cdots < u_{n-i}\}$, the first factor deletes precisely those u_j with $u_j \equiv j \pmod 2$; the second deletes the others. This is the parity-subset formula of [46, Definition 7]. After evaluation on Z, terms can degenerate or cancel, so the count becomes an upper bound.

In particular, the three, four and one terms of the degree-three $\smile_1$, $\smile_2$, $\smile_3$ expressions used below are generated by this recursion. The split and join take place within the ordered incidence simplex; the subset-sum walls ensure that its carrier maps commute with them. This identifies the relevant geometric input without identifying a join with torus addition.

22.4 A direct contraction and cyclic higher diagonals

The contractions reviewed in [47, Sections 3.7–3.8] give a direct construction in every arity. We record the signs and the comparison here.

Use homological grading, with

$$df = \partial f - (-1)^{|f|} f\partial, \qquad (f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

Let $C = N_*(\Delta[n];\mathbb{Z})$ with the counit, diagonal and signed join above; in particular, $d(*) = \epsilon \otimes \mathrm{id} - \mathrm{id} \otimes \epsilon$. Set $E_r = \mathrm{Hom}(C, C^{\otimes r})$ for $r \geq 0$, where $C^{\otimes 0} = \mathbb{Z}$. Define chain maps

$$\begin{aligned} \iota &: E_r \longrightarrow E_{r+1}, \qquad \iota\eta = (\mathrm{id} \otimes \eta)\Delta, \\ \pi &: E_r \longrightarrow E_{r-1}, \qquad \pi\eta = (\epsilon \otimes \mathrm{id}^{\otimes r-1})\eta \quad (r \geq 1), \end{aligned}$$

and a degree-one operator on E_r, for $r \geq 1$, by

$$\sigma_r \eta = (* \otimes \mathrm{id}^{\otimes r-1})(\mathrm{id} \otimes \eta)\Delta.$$

The tensor signs in these formulas are part of the definitions.

Proposition 22.4. *One has $\pi\iota = \mathrm{id}$ and*

$$d\sigma_r + \sigma_r d = \mathrm{id} - \iota\pi.$$

Consequently, the degree-one operator

$$h_r = \sum_{j=0}^{r-1} \iota^j \sigma_{r-j} \pi^j \qquad (r \geq 1)$$

satisfies

$$dh_r + h_r d = \mathrm{id} - \iota^r \pi^r.$$

These are identities on the full Hom complexes, including maps of negative degree.

Proof. The counit identity gives $\pi\iota\eta = \eta$. The differential is a derivation of composition. Since the leftmost map in σ_r has degree one, its application gives

$$\begin{aligned} d(\sigma_r\eta) + \sigma_r(d\eta) &= \big((\epsilon \otimes \mathrm{id} - \mathrm{id} \otimes \epsilon) \otimes \mathrm{id}^{\otimes r-1}\big)(\mathrm{id} \otimes \eta)\Delta \\ &= \eta - \iota\pi\eta. \end{aligned}$$

No degree-dependent sign remains in the first term: the counit kills the terms whose first tensor factor has nonzero degree. Since ι and π are chain maps, summing the resulting identities telescopes to

$$\sum_{j=0}^{r-1} \big(\iota^j\pi^j - \iota^{j+1}\pi^{j+1}\big) = \mathrm{id} - \iota^r\pi^r.$$

□

In particular, under the displayed convention for $d(*)$, the sign is $\mathrm{id} - \iota\pi$, rather than its negative. The sum ends at $r-1$: there is no join operator σ_0. One can equivalently set $\sigma_0 = 0$ and include a final zero term. Set $h_0 = 0$ if a zero-output notation is desired.

For the one-input part of the prop $\mathcal{M}$, this is a contraction to a copy of $\mathbb{Z}$. More precisely, use its standard identification $\mathcal{M}(1,0) = \mathbb{Z}\epsilon$ to define

$$(\epsilon^{\otimes r})w = a_r(w)\epsilon, \qquad \delta_r = \iota^r\epsilon.$$

Here a_r is a degree-zero chain map to $\mathbb{Z}$, and

$$\delta_0 = \epsilon, \qquad \delta_1 = \mathrm{id}, \qquad \delta_{r+1} = (\mathrm{id} \otimes \delta_r)\Delta.$$

Erasing every output reduces a degree-zero diagram to ϵ; a positive-degree diagram contains a join and vanishes under erasure, by $\epsilon* = 0$. Thus the contraction identity on $\mathcal{M}(1,r)$ is

$$dh_r + h_rd = \mathrm{id} - \delta_r a_r.$$

This argument also proves the asserted identification of $\mathcal{M}(1,0)$. The same identity holds after evaluating these diagrams on C. Coassociativity of Alexander–Whitney identifies δ_r with the usual r-fold diagonal on C; no coassociativity relation in the abstract prop is needed. On arbitrary elements of E_r, the residual remains $\iota^r\pi^r$ and need not have this scalar form.

For $r \geq 2$, let ρ act by the graded cyclic permutation

$$\rho(c_1 \otimes \cdots \otimes c_r) = (-1)^{|c_r|(|c_1|+\cdots+|c_{r-1}|)} c_r \otimes c_1 \otimes \cdots \otimes c_{r-1}.$$

Put $T = \rho - 1$ and $N = \sum_{j=0}^{r-1} \rho^j$. Use this same generator convention for the periodic free resolution $W(r)$, with $de_0 = 0$, $de_{2m+1} = Te_{2m}$ for $m \geq 0$, and $de_{2m} = Ne_{2m-1}$ for $m \geq 1$. Define finite integral diagram combinations by

$$\psi_0 = \delta_r, \qquad \psi_{2m+1} = h_r T \psi_{2m} \quad (m \geq 0), \qquad \psi_{2m} = h_r N \psi_{2m-1} \quad (m \geq 1).$$

Extending $e_i \mapsto \psi_i$ $\mathbb{Z}[C_r]$-linearly gives an equivariant chain map $W(r) \to \mathcal{M}(1, r)$. It is a quasi-isomorphism: both complexes resolve $\mathbb{Z}$, and ψ_0 represents its generator. Together with the local operad action, this is a May–Steenrod structure in the sense of [39, Definition 5]. Indeed, the argument of each h_r is a cycle, inductively, because $TN = NT = 0$. Its augmentation vanishes: at the first step $a_r(T\psi_0) = 0$, and at every later step its degree is positive. The contraction identity therefore supplies precisely the required differential. The contraction h_r itself need not be equivariant. Each application uses a finite sum and one additional join, so $|\psi_i| = i$ and the construction uses no denominators.

The arity-one convention $\rho = 1$, $T = 0$, $N = 1$ gives $\psi_0 = \mathrm{id}$ and $\psi_i = 0$ for $i > 0$; arity zero is not covered by the cyclic resolution construction. On a vertex all positive-degree evaluated maps vanish. As a sign check, in arity two on the ordered edge $e = [0, 1]$, the construction gives $\psi_1(e) = e \otimes e$, and its boundary is $(\rho - 1)\Delta(e)$.

This provides a direct choice of higher diagonals inside $U(\mathcal{M})$ and then, by local evaluation, on simplicial chains. It does not require a surjection-to-prop map to preserve operad composition. It also does not by itself prove pointwise equality with the chosen KMM surjection-word formulas: such an identification requires a comparison intertwining the two contractions (or a separate explicit calculation). Agreement of the resolution differential alone does not establish equality of the chain-level formulas. The map SL is used only in its stated role as a morphism of $\mathbb{S}$-modules [39, Section 6.3]; the operad action itself comes from $U(\mathcal{M})$.

There is, however, an explicit comparison up to equivariant chain homotopy. Let ψ'_i be another such family in $\mathcal{M}(1, r)$ with $a_r(\psi'_0) = a_r(\psi_0) = 1$. Write $\alpha_i = T$ for odd i, $\alpha_i = N$ for positive even i, and $\alpha_0 = 0$. Set $L_{-1} = 0$ and define

$$L_i = h_r\big(\psi_i - \psi'_i - \alpha_i L_{i-1}\big) \qquad (i \geq 0).$$

Then $|L_i| = i + 1$ and

$$dL_i + \alpha_i L_{i-1} = \psi_i - \psi'_i.$$

To verify this, denote the argument of h_r by z_i. The preceding identity in degree $i - 1$ gives $dz_i = \alpha_i \alpha_{i-1} L_{i-2} = 0$ when $i \geq 2$; the cases $i = 0, 1$

follow directly. Also $a_r(z_i) = 0$, by equal augmentations when $i = 0$ and by positive degree otherwise. Thus $dh_r z_i = z_i$, proving the induction. Extend $L(e_i) = L_i$ equivariantly; this gives $dL + Ld = \psi - \psi'$.

The same recursion applies directly to evaluated local families in E_r when $\psi_0' = \psi_0$ is the iterated Alexander–Whitney diagonal. Here $z_0 = L_0 = 0$, and for $i > 0$ one has $\pi^r z_i = 0$ because z_i has positive degree and C is nonnegatively graded. Consequently this comparison does not require an abstract prop lift of a chosen KMM family, provided that family is an equivariant chain map on simplex chains. Naturality on the index simplices makes the resulting homotopy descend by the same local evaluation as the operations themselves.

For $r = p$ prime and a homogeneous cocycle c over $\mathbb{F}_p$, the functional $c^{\otimes p}$ is invariant under ρ: for odd p its cyclic Koszul sign is $(-1)^{|c|^2(p-1)} = 1$, and for $p = 2$ signs disappear. It therefore kills both T and N over $\mathbb{F}_p$. Pairing the homotopy identity with $c^{\otimes p}$ shows that the two resulting cochains differ by a coboundary. With the same conventional normalization, their primary Steenrod operations agree, although their selected cochain formulas need not be equal.

Finally, the direct diagram recursion agrees exactly with a tensor-cone construction in every arity. For the first vertex v of an input simplex, use $P_v = \eta_v \epsilon$, the usual vertex cone h_v, and

$$H_v^{(r)} = \sum_{k=0}^{r-1} P_v^{\otimes k} \otimes h_v \otimes \mathrm{id}^{\otimes r-1-k}, \qquad \partial H_v^{(r)} + H_v^{(r)} \partial = \mathrm{id} - P_v^{\otimes r}.$$

Every nonzero tensor term of $h_r \eta(\sigma)$ has the following property: its first positive-dimensional factor contains v, and every preceding factor is the vertex v. Indeed, the j initial factors in a term of $\iota^j \sigma_{r-j} \pi^j \eta(\sigma)$ are the consecutive Alexander–Whitney prefixes. If they all have degree zero, all their cuts occur at v, and the next factor is a nonzero join containing v and having positive dimension. If one of the initial prefixes has positive degree, the first such prefix already has the asserted property. Consequently,

$$H_v^{(r)} h_r \eta(\sigma) = 0, \qquad P_v^{\otimes r} h_r \eta(\sigma) = 0.$$

Apply the tensor contraction to $\psi_i(\sigma)$ for $i > 0$ and use its chain identity. This gives

$$\psi_i(\sigma) = H_v^{(r)} \big(\alpha_i \psi_{i-1}(\sigma) + (-1)^i \psi_i(\partial \sigma) \big).$$

Starting with the same iterated Alexander–Whitney diagonal, induction on i and then simplex dimension uniquely determines this recursion. Thus the finite split–join diagrams and the tensor-cone boundary filling produce exactly the same integral diagonals in all arities and all degrees.

22.5 All higher comparisons from the carry

The partition identity for carries supplies a complete family of coherent comparisons, rather than just the binary additivity homotopy. Fix $n = k+1 \geq 2$, abbreviate $X(S) = X_0^{(k)}(S)$, and let

$$c^S = (\mathrm{pr}_s^* c_n)_{s \in S^\circ} \in Z^n(X(S); \mathbb{Z}^{S^\circ}).$$

For a pointed map $f : S \to T$, write $f_X : X(S) \to X(T)$ for its carrier map and $f_* : \mathbb{Z}^{S^\circ} \to \mathbb{Z}^{T^\circ}$ for the fibre-sum homomorphism. Define the vector carry b_f by

$$(b_f)_t = b_{f^{-1}(t)}, \qquad t \in T^\circ,$$

with an empty-fibre carry interpreted as zero. Equations (104)–(105) give

$$\delta b_f = f_X^* c^T - f_* c^S, \tag{116}$$

$$b_{gf} = f_X^* b_g + g_* b_f, \qquad b_{\mathrm{id}} = 0. \tag{117}$$

The second equation is an exact equality of cochains.

Theorem 22.5 (Coherent comparison with the cocycle diagram). *The maps classified by c^S extend to an explicit homotopy-coherent natural transformation*

$$X(S) \longrightarrow K(\mathbb{Z}^{S^\circ}, n)$$

of pointed-set diagrams. Its component at every S is a weak equivalence. For every chain of pointed maps of length r, the required r-parameter comparison is a closed cochain given by the formula below. It uses only the canonical cocycle and the bounded carries; no further filler choices are needed.

Proof. Consider a chain

$$S_0 \xrightarrow{f_1} S_1 \longrightarrow \cdots \xrightarrow{f_r} S_r.$$

Let $f_{ij} : S_i \to S_j$ be its composite, including $f_{ii} = \mathrm{id}$, and put $\mu_i = (f_{0i})_X$. On $X(S_0)$, with common coefficients $\mathbb{Z}^{S_r^\circ}$, define

$$q_i = (f_{ir})_* \mu_i^* c^{S_i},$$
$$H_{ij} = (f_{jr})_* \mu_i^* b_{f_{ij}} \quad (i \leq j).$$

The carry equations say exactly that

$$\delta H_{ij} = q_j - q_i, \qquad H_{i\ell} = H_{ij} + H_{j\ell}, \qquad H_{ii} = 0.$$

Let $\alpha_i \in C^0(\Delta[r];\mathbb{Z})$ be 1 at vertex i and 0 at the other vertices. With the evident pullbacks to the product, set

$$C_{f_1,\dots,f_r} = q_0 + \delta\left(\sum_{i=1}^{r} H_{0i} \smile \alpha_i\right) \in Z^n(X(S_0)\times\Delta[r];\mathbb{Z}^{S_r^\circ}). \tag{118}$$

For $r=0$ this means c^{S_0}. It is closed, normalized, and its value at parameter vertex i is q_i.

We verify the coherence identities explicitly. An internal face omits one parameter vertex and composes the corresponding two pointed maps. The last face is $(f_r)_*$ applied to the formula for the initial chain. On the first face, $H_{0i} = H_{01} + H_{1i}$ and $\sum_{i=1}^{r}\alpha_i = 1$. Thus the term δH_{01} changes the base cocycle q_0 to q_1, leaving the formula for the tail chain pulled back along $(f_1)_X$. Inserting an identity repeats one q_i and one H_{0i}; the pullback of a vertex cochain under the corresponding degeneracy is the sum of the two repeated vertex cochains. This proves the degeneracy identities as well.

Here these face and degeneracy identities are the definition of the homotopy-coherent transformation: an r-chain is assigned a map from $X(S_0)\times\Delta[r]$ to the cocycle model at S_r, with the stated composition rules on its boundary. The component weak equivalences were proved in Theorem 4.1 and its coordinatewise extension. □

For instance, a chain $S \xrightarrow{f} T \xrightarrow{g} U$ gives a triangle whose three edges compare the direct sum, the first partial sum, and the final sum. Its filling is

$$q_0 + \delta(H_{01}\smile\alpha_1 + H_{02}\smile\alpha_2).$$

This is the explicit associativity comparison. Longer chains are filled by the same formula in every dimension. These are specified coherent boundaries, so their fillability does not contradict the possible obstructions for arbitrary boundaries in Proposition 20.4.

After reduction modulo p, every parameterized cocycle above has a degreewise finite cocycle target. Applying any natural cocycle-valued operation to such a parameterized cocycle preserves its face restrictions. This transports the specified comparisons of input representatives through the operation. There is simultaneous compatibility with the cochain operad. Writing Φ for the simplicial map classified by (118), naturality gives a strict morphism of algebras over the chosen operad

$$\Phi^*: C^*(\mathcal{K}(\mathbb{Z}^{S_r^\circ}, n);\mathbb{F}_p) \longrightarrow C^*(X(S_0)\times\Delta[r];\mathbb{F}_p).$$

It commutes with every operad operation and with the parameter faces and degeneracies just proved. Thus the carry comparisons and the cochain

action are compatible as specified families, not just as induced operations on cohomology.

Comparing the operation of a sum with the sum of the operations also requires its additive cochain identities; the carry formula alone does not make a nonlinear cocycle formula strictly additive. The next calculation provides that additional correction explicitly for Sq^2.

22.6 An explicit operation compatible with a crossed fold

The finite model's fold map adds fundamental cohomology classes, but the preferred cocycles can differ by a coboundary. For Sq^2 on a degree-three class, this discrepancy and its first coherence diagrams have short formulas. All cochains in this subsection are normalized and have coefficients in $\mathbb{F}_2$.

Use the cup-i convention of [46, Definition 7 and Theorem 10]. It agrees with our cup-one insertion convention modulo two and satisfies

$$\delta(a \smile_i b) = \delta a \smile_i b + a \smile_i \delta b + a \smile_{i-1} b + b \smile_{i-1} a, \qquad \smile_{-1} = 0. \tag{119}$$

For a degree-three cocycle c, set

$$q(c) = c \smile_1 c \in Z^5(X; \mathbb{F}_2).$$

Its cohomology class is $Sq^2[c]$. This is a natural cocycle formula on every simplicial set, so it is a strict map to $K(\mathbb{F}_2, 5)$ even when its source is not Kan.

Proposition 22.6 (Additivity and a change of representative). *For $c, d \in Z^3(X; \mathbb{F}_2)$ and $b \in C^2(X; \mathbb{F}_2)$, put*

$$A(c, d) = c \smile_2 d, \tag{120}$$
$$C(c, b) = c \smile_2 \delta b + b \smile_1 \delta b + b \smile b. \tag{121}$$

Then these degree-four cochains satisfy

$$\delta A(c, d) = q(c + d) + q(c) + q(d), \tag{122}$$
$$\delta C(c, b) = q(c + \delta b) + q(c). \tag{123}$$

Consequently, if a fold's preferred cocycle is $a_{12} = a_1 + a_2 + \delta b$, the cochain

$$F(a_1, a_2; b) = A(a_1, a_2) + C(a_1 + a_2, b) \tag{124}$$

satisfies $\delta F = q(a_{12}) + q(a_1) + q(a_2)$.

Proof. Expand $q(c+d)$ and apply (119) with $i=2$. For the second identity, the three summands of C have coboundaries

$$\begin{aligned} &c \smile_1 \delta b + \delta b \smile_1 c, \\ &\delta b \smile_1 \delta b + b \smile \delta b + \delta b \smile b, \\ &\delta b \smile b + b \smile \delta b. \end{aligned}$$

The ordinary cup terms cancel. The remaining terms are precisely $q(c+\delta b)+q(c)$. The fold statement is their sum. □

For readers wanting a face formula, this convention gives

$$\begin{aligned} A(c,d)_{01234} = {} & c_{0134}d_{1234} + c_{0123}d_{1234} \\ & + c_{0234}d_{0124} + c_{0123}d_{0134}. \end{aligned}$$

Thus the four terms can be evaluated directly on a finite flag of cells, using the already constructed cocycle and its carry cochain. For the fold, take X to be the two-input value of the crossed Γ-model and pull a_1, a_2, a_{12} back along its projections and fold. No ordinary simplicial product description of that value is needed.

The equality $\delta F = q(a_{12}) + q(a_1) + q(a_2)$ supplies an actual homotopy. With $\alpha \in C^0(\Delta[1]; \mathbb{F}_2)$ taking endpoint values $0, 1$, the normalized cocycle

$$q(a_1) + q(a_2) + \delta(F \smile \alpha) \quad \text{on } X \times \Delta[1]$$

classifies a strict simplicial homotopy into $K(\mathbb{F}_2, 5)$. Pullbacks to the product are suppressed in this expression. All sets involved in any one simplicial degree remain finite.

The first coherence equations

The additivity primitive itself has strict unit and associativity identities, because it is bilinear:

$$A(c,0) = A(0,c) = 0, \tag{125}$$

$$A(c,d) + A(c+d,f) = A(d,f) + A(c,d+f). \tag{126}$$

Its symmetry defect has the degree-three primitive

$$A(c,d) + A(d,c) = \delta(c \smile_3 d). \tag{127}$$

For this cup convention, $(c \smile_3 d)_{0123} = c_{0123}d_{0123}$.

Changes of representative also admit a composition correction. For $b, e \in C^2(X;\mathbb{F}_2)$, define

$$L(b,e) = b \smile_2 \delta e + e \smile_1 b \in C^3(X;\mathbb{F}_2).$$

Then

$$\delta L(b,e) = C(c,b) + C(c+\delta b, e) + C(c, b+e). \tag{128}$$

The compatibility between changing both inputs and adding them is given by

$$J(c,d;b,e) = d \smile_3 \delta b + L(b,e),$$

with

$$\begin{aligned}\delta J(c,d;b,e) = {} & C(c+d, b+e) + A(c+\delta b, d+\delta e) \\ & + A(c,d) + C(c,b) + C(d,e).\end{aligned} \tag{129}$$

Both equations follow by expanding the right hand side and applying (119). They are identities of natural cochains, not merely statements that obstruction classes vanish.

Proposition 22.7 (Associator for the carry-corrected fold). *Let c, d, f be degree-three cocycles, and choose degree-two carries $b_{12}, b_{(12)3}, b_{23}, b_{1(23)}$ satisfying the strict carry identity*

$$b_{12} + b_{(12)3} = b_{23} + b_{1(23)}. \tag{130}$$

Put $c_{12} = c + d + \delta b_{12}$ and $c_{23} = d + f + \delta b_{23}$. The two parenthesizations of the fold have a degree-three comparison cochain

$$K = f \smile_3 \delta b_{12} + L(b_{12}, b_{(12)3}) + L(b_{23}, b_{1(23)}),$$

satisfying

$$\begin{aligned}\delta K = {} & F(c,d;b_{12}) + F(c_{12}, f; b_{(12)3}) \\ & + F(d,f;b_{23}) + F(c, c_{23}; b_{1(23)}).\end{aligned} \tag{131}$$

Proof. Let $S = c + d + f$ and $B = b_{12} + b_{(12)3} = b_{23} + b_{1(23)}$. Equations (129) and (128) rewrite the left-parenthesized path, modulo the coboundary of $f \smile_3 \delta b_{12} + L(b_{12}, b_{(12)3})$, as

$$A(c,d) + A(c+d, f) + C(S, B).$$

They rewrite the right-parenthesized path, modulo the coboundary of $L(b_{23}, b_{1(23)})$, as $A(d,f) + A(c, d+f) + C(S,B)$. These expressions agree by (126). □

Strict associativity in (126) is a statement about A. The full corrected expression F instead has the explicit associator (131). On $X_0^{(2)}(3_+)$, take the coordinate reductions for c, d, f and pull back the geometric carries for each partial fold. Then (130) is exactly (105), reduced modulo two. Hence F and K apply to the actual crossed addition. For other choices of carries, equality of the final cocycles alone would only make the two carry sums differ by a cocycle and would not suffice.

From these equations to finite diagrams

Here is an explicit two-dimensional version of the prism formula. Suppose $q_i \in Z^5(X;\mathbb{F}_2)$, $H_{ij} \in C^4(X;\mathbb{F}_2)$ and $J \in C^3(X;\mathbb{F}_2)$ satisfy

$$\delta H_{ij} = q_i + q_j, \qquad \delta J = H_{01} + H_{12} + H_{02}.$$

Let ϵ_i, ϵ_{ij} and ϵ_{012} be the normalized indicator cochains of the vertices, edges and triangle of $\Delta[2]$. Then

$$\sum_i q_i \smile \epsilon_i + \sum_{i<j} H_{ij} \smile \epsilon_{ij} + J \smile \epsilon_{012} \tag{132}$$

is a normalized degree-five cocycle on $X \times \Delta[2]$. Its classifying map fills the three prescribed edge homotopies. The assertion follows directly by applying δ: all edge and triangle coefficients cancel using the two displayed equations. Applying this construction to the identities above, and subdividing a square into two triangles when required, produces actual finite simplicial coherence diagrams.

For arbitrary fixed operations and relations, the finite universal solver remains available. Here the universal input for the fold calculation is

$$\mathcal{K}(\mathbb{F}_2,3)^2 \times \mathcal{E}(\mathbb{F}_2,3), \qquad \mathcal{E}(\mathbb{F}_2,3)_n = C^2(\Delta[n];\mathbb{F}_2),$$

so the input cocycles and the nonclosed carry are represented strictly. Its degree-n cardinality is $2^{2\binom{n}{3}+\binom{n+1}{3}}$. The formulas above provide short solutions without enumerating that entire universal cochain matrix. They establish the specified first and second coherences; they do not by themselves construct all higher coherences of an E_∞ operation.

Exact checks. The accompanying verifier `verify_additive_operations.py` uses only the Python standard library. It checks all displayed identities as polynomials in independent mod-two simplex coordinates, the cup-i differential in 62 degree/index cases, and normalization under every noninjective monotone surjection in each output degree. It also constructs and

solves a 92×25 coefficient matrix of rank 15 for a normalized bilinear additivity primitive: the 25 candidate terms are $c(d_i\sigma)d(d_j\sigma)$ on a four-simplex. This is a complete computation in the stated face-polynomial ansatz, not an enumeration of all cochains on the universal space. The chosen geometric cup-two primitive has the four terms above. No random sampling or toric-cell enumeration is used.

22.7 What additive compatibility determines

Proposition 22.8 (Primitive classes and additive compatibility). *Write $D = D_0^{(2)}$, and let $\mu, \mathrm{pr}_1, \mathrm{pr}_2 : X_0^{(2)}(2_+) \to D$ be the fold and coordinate maps. Call a cohomology class v primitive if*

$$\mu^* v = \mathrm{pr}_1^* v + \mathrm{pr}_2^* v.$$

The canonical integral class $[c] \in H^3(D;\mathbb{Z})$ is primitive. Every natural additive cohomology operation sends primitive classes to primitive classes. In particular, $Sq^2\rho_2[c]$ and $\beta_p\mathcal{P}^1\rho_p[c]$, for odd primes p, are primitive.

Proof. Primitivity of $[c]$ follows from the carry identity

$$\delta b_{\{1,2\}} = \mu^* c - \mathrm{pr}_1^* c - \mathrm{pr}_2^* c.$$

For an additive natural operation Θ, apply Θ to the defining equality for v and commute it with the three pullbacks. This gives the defining equality for $\Theta(v)$ with its target coefficients. □

Primitivity by itself does not determine the chosen operation. The classes 0, $\kappa_3 = \beta_3\mathcal{P}^1\rho_3[c]$, and $-\kappa_3$ are distinct primitive degree-eight classes, with the latter two both of order three. Nor does primitivity mean indecomposability: the nonzero integral square $[c]^2$ is primitive. The two mixed terms in $(\mathrm{pr}_1^*[c] + \mathrm{pr}_2^*[c])^2$ cancel by graded commutativity, since $|c| = 3$. These statements use the fold equality directly and do not require an integral Kunneth tensor-product identification.

This distinguishes an additive compatibility condition from a chosen cochain operation. It is not an impossibility result about the full simplicial Γ-object: that object already includes $X(1_+)$ and all its ordered face maps, to which the natural cochain operad action applies [5, 39]. The local generation in Sections 22.2–22.4 makes this construction explicit: split and join positions inside an ordered incidence flag, then take its actual faces. Thus the full incidence presentation supports a constructive answer. Primitivity alone neither specifies these generators nor selects their signs and chain representatives.

23 Consequences, relation to earlier work, and scope

The cochains T and V in the first two extensions record specified nullhomotopies of the classifying operations on their total spaces. More generally, the Bockstein construction retains a primitive on every Postnikov extension. This makes it possible to carry representative cochains into subsequent stages using finite sets in each required degree. The same finite cochain groups support evaluation of classical cup-i formulas and finite integer linear algebra. Finiteness alone does not imply that these computations are small.

The standard cocycle models, universal cochain fibrations, twisted Cartesian products, and classification by cohomology are classical [43]. The passage from a very special Γ-space to a connective spectrum is likewise classical [71, 70]. Mnev's circular-order model gives the motivating finite integral $K(\mathbb{Z}, 2)$ [51]. The construction studied here uses finite orbit-incidence categories to extend the finite presentation to higher integral degrees and to compatible pointed-set diagrams.

The homeomorphism arguments use classical $\mathbb{R}^\infty$-manifold recognition and classification [68, 37]. The horn quotients first supply a contractible factor without losing finiteness in any degree. The affine realization is recognized by separating points with high-degree cocycles. For the original realization, crossed reordering [40] moves compact families away from bounded row degree whenever they miss the canonical point. This proves the manifold property away from that point.

The anchored join supplies the final local step for $k \geq 2$. Anchor deletion is a quasifibration over every open subset; its proof combines finite PL row maps, homotopy base change, and edgewise subdivision with continuous factorization of compact families. A fixed tag avoids the entire zero fibre. Comparing the two local quasifibration sequences makes the punctured basepoint star weakly contractible and hence its link contractible. For $k = 1$, centres depending on the outer realization weights give a direct homeomorphism compatible with all identifications. Thus the original one-input realization is homeomorphic to the affine group realization in every degree parameter. These arguments use the original carrier maps. The resulting continuous abelian law has a finite simplicial description by coherent refinements up to homotopy; the higher-degree homeomorphism proof does not provide an explicit formula for its strict multiplication.

The universal lattice torsor also has total space $\mathbb{R}^\infty$, and its structure group is an $\mathbb{R}^\infty$-manifold of type $K(\mathbb{Z}, k)$. This is a separate geometric

presentation; it is not the argument used to avoid the full zero fibre of anchor deletion.

The reduced-power formulas used in Section 14 are classical operations in the explicit presentation of Kaufmann–Medina-Mardones [39]. The note applies those formulas to the finite models and retains integral witnesses for the successive lifting equations. The operation itself is not claimed as new.

Section 15 gives a more direct description using ternary incidence labels and ordered face insertions. The twelve-term alternative follows from classical cup-one identities. Its comparison with the published formula is proved here for reductions of integral degree-three cocycles. The secondary class in the square fibre is identified integrally with the negative of the next invariant. No priority claim is made for the alternative expression.

Section 20 gives a uniform organization beyond these examples. Finite-coefficient cocycle formulas define maps in a simplicial category whose mapping spaces are degreewise finite and Kan. Each universal relation can be turned into a natural homotopy by a finite linear solve. Higher prescribed diagrams are governed by explicit cohomology obstructions, and coherences already supplied by the cochain operad action are retained on finite skeleta. These facts combine classical representability and operad actions [43, 5, 39] with the finite presentations.

Section 22 connects the two structures more explicitly. The standard higher diagonals become formulas on ordered incidence subflags and commute with all carrier maps. The exact carry composition gives an all-dimensional coherent comparison with the standard cocycle diagram, compatible with the cochain operad action. For Sq^2, the note gives the fold correction, its symmetry and representative-change corrections, and the associator for the geometric carries. These last formulas establish the specified first and second coherences, rather than an independent construction of every higher additive coherence for every operation. The local counit–diagonal–join construction now gives an explicit generation procedure for higher diagonals on each ordered flag, with an independent binary cone proof and an all-arity cyclic recursion. This applies the published construction of [45, 47, 39] to the incidence model; it does not claim a new construction of the Steenrod operations. The subset-sum arrangements supply the compatible carriers and carries, while the ordered simplex supplies the split and join.

The algorithmic Postnikov literature provides an important comparison. [11, Section 2] explicitly allows infinitely many simplices in a fixed degree and represents the relevant maps by algorithms; its Section 3.7 uses the classical cochain resolution. The finiteness proved here is literal degreewise

finiteness of the simplicial sets. This comparison does not establish historical priority, and no claim of priority is made.

The torsion closure theorem concerns principal extensions by $K(\mathbb{Z}, t)$ with constant coefficients. It applies whenever an integer $m \geq 2$ annihilates the k-invariant, and it can be iterated for any finite sequence of such extensions. For the particular infinite tower of Theorem 18.1, increasing degrees give literal stabilization in every simplicial degree. This additional property proves finiteness of the limit; finiteness of the stages alone would not suffice. The proof does not cover integral classes of infinite order or nontrivial local coefficient actions. Section 15.3 does construct strict maps to finite integral targets for Bockstein classes, after a factorization is chosen. This does not supply canonical finite representatives for arbitrary maps. In particular, strict maps to a non-Kan target do not by themselves make ordinary pullbacks into homotopy pullbacks.

A further question is whether these presentations can be reduced to finite local axioms analogous to those for circular order. An eventual horn-filling bound for the toric models has not been established here; their homotopy types alone do not supply one.

Finally, the main results use the proved toric models $D_0^{(k)}$. The binary three-cocycle candidate in Appendix D is potentially smaller, but its global equivalence to $K(\mathbb{Z}, 3)$ remains a separate question.

24 Rational homotopy and Sullivan models

The finite presentations have a direct rational interpretation. For a principal Postnikov extension, the rational image of its attaching class becomes the differential of a new Sullivan generator. In the prime-switch family this differential can be removed, and there is an explicit rational equivalence obtained by forgetting finite labels. In the Heisenberg family the quadratic differential survives. Several such relations can also produce a nonzero Massey product.

The passage to Sullivan algebras is classical [6, 76]. We describe what it gives for the concrete integral presentations in this note; no novelty claim is made for the general dictionary.

24.1 Polynomial forms on the finite simplicial presentation

For a simplicial set K, put

$$\Omega_{\mathrm{poly}}(\Delta^d) = \frac{\mathbb{Q}[t_0, \dots, t_d] \otimes \Lambda(dt_0, \dots, dt_d)}{(\sum t_i - 1,\ \sum dt_i)}, \qquad d(t_i) = dt_i.$$

Here $|t_i| = 0$ and $|dt_i| = 1$. The commutative differential graded algebra $A_{PL}(K)$ consists of families $\omega_\sigma \in \Omega_{\mathrm{poly}}(\Delta^d)$, one for each $\sigma \in K_d$, such that

$$\omega_{\alpha^*\sigma} = \alpha^*\omega_\sigma \qquad (\alpha : [j] \longrightarrow [d]).$$

Products and differentials are evaluated simplex by simplex. No Kan condition is required. Its cohomology is $H^*(|K|;\mathbb{Q})$, and for a simply connected realization of finite rational type its minimal Sullivan model determines the rational homotopy type [24, Section 2 and Theorem 2.1]. A minimal Sullivan algebra is a free graded-commutative algebra $(\Lambda V, d)$ whose generators can be ordered so that each differential uses earlier generators and has no linear part. An even generator contributes a polynomial variable; an odd generator contributes an exterior variable.

Degreewise finiteness gives finite-dimensional normalized cochain groups $C^d(K;\mathbb{Q})$. It does not make $A_{PL}(K)$ finite-dimensional: polynomial degree is unbounded, even on a single simplex. Integration compares polynomial forms with normalized cochains as complexes; it does not identify their products strictly. In particular the Alexander–Whitney cochain algebra should not simply be declared commutative.

There is nevertheless a useful finite-input assertion. If K is degreewise finite, $|K|$ is simply connected, and $N \geq 2$, the finite skeleton $\mathrm{sk}_{N+1} K$ determines the rational Postnikov N-type and the minimal model through generator degree N. Indeed, its realization includes into $|K|$ by attaching cells of dimensions at least $N + 2$, so the inclusion induces isomorphisms on π_i for $i \leq N$. Apply the rational Postnikov–Sullivan dictionary. This supplies a finite input for each bounded stage, without asserting a small model or a uniform complexity bound.

For the presentations of Theorem 19.2, the common degeneracies identify their normalized cochain vector spaces. In these coordinates the difference of cochain differentials is exactly

$$\delta_q - \delta_0 = (d_0^{(q)})^* - (d_0^{(0)})^*.$$

The right side is first formed on unnormalized cochains; its difference preserves the normalized subspaces, although an individual face pullback need not do so. Thus the attaching information remains visible before passing to a smaller commutative model. This identity alone does not identify the resulting Sullivan models.

24.2 A Postnikov class becomes a differential

Let B be simply connected of finite rational type and let

$$P = \mathrm{hofib}\big(\kappa : B \longrightarrow K(\mathbb{Z}, n+1)\big), \qquad n \geq 2.$$

Choose a Sullivan model (M_B, d) and a cocycle $\omega \in M_B^{n+1}$ representing $\kappa \otimes \mathbb{Q}$. The principal-fibration model is

$$(M_B \otimes \Lambda(z_n), D), \qquad D|_{M_B} = d, \qquad Dz_n = \omega. \tag{133}$$

It is a relative Sullivan model, but it need not already be minimal. This is the elementary-principal-fibration construction of [24, Section 14, Example 1]; compare [76, Theorems 7.2 and 10.1]. If $\omega = d\eta$, replacing z_n by $z_n - \eta$ makes the new differential zero. In particular every torsion integral attaching class vanishes rationally. The primitive and the resulting splitting need not be canonical.

24.3 Forgetting finite labels is a rational equivalence

Retain the prime-switch family P_S of Theorem 19.4. Put

$$W = X \times \prod_{p \text{ prime}} Y_p, \qquad X = D_0^{(2)}, \quad Y_p = D_0^{(2p)}.$$

Both W and P_S have only finitely many nontrivial factors in each simplicial degree.

Proposition 24.1. *For every set of primes S, the strict simplicial map*

$$F_S : P_S \longrightarrow W, \qquad (x, (y_p, b_p)_p) \longmapsto (x, (y_p)_p)$$

realizes to a rational homotopy equivalence. All $|P_S|$ have the same minimal Sullivan model

$$\big(\Lambda_{\mathbb{Q}}(u_3, v_{2p+1} \mid p \text{ prime}), 0\big). \tag{134}$$

In particular all are formal over $\mathbb{Q}$.

Proof. The cochain-resolution projection to (x, y_p) is simplicial, so forgetting the b_p gives the asserted map. In a finite prime range it is a Kan fibration with fibre $\prod_p K(\mathbb{Z}/p, 2p)$: it is a pullback of the product of the finite cochain-resolution fibrations. This does not assert that its base or total space is Kan. The coefficient exact sequence shows that F_S induces the identity on π_3 and multiplication by p, up to the chosen orientation, on π_{2p+1}. These are isomorphisms after tensoring with $\mathbb{Q}$.

The tower stabilizes in each simplicial degree. Consequently each fixed homotopy group is computed in a finite prime range, by the same skeleton argument as in Theorem 19.4. The assertion follows for the infinite tower

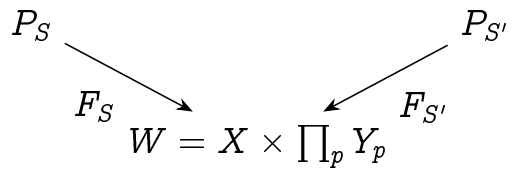


Figure 31: Different integral gluings admit a common rational comparison through an explicitly degreewise finite target. The arrows are actual simplicial maps.

as well. This argument does not require rationalization to commute with an arbitrary inverse limit.

Finite products of the Y_p and X have minimal models freely generated by the corresponding odd-degree classes, with zero differential. Their increasing degrees give (134); only finitely many generators or monomials occur in each bounded degree. Pullback by F_S is a quasi-isomorphism on polynomial forms and gives the same minimal model for P_S. A zero-differential model is formal. □

Thus the continuum of integral homotopy types constructed above lies in one rational homotopy type. The finite labels and their first-face corrections retain information which the rational model cannot distinguish. This also parallels Rector's examples [64]: their loop spaces have the homotopy type of S^3, so their deloopings rationalize to $K(\mathbb{Q}, 4)$. The common rational type does not imply a common integral type.

24.4 The Heisenberg quadratic survives rationalization

Let P_E be the Heisenberg homotopy fibre of Corollary 7.19. If Λ has rank $2g$, write u_i, v_i for the degree-two coordinates on $K(\Lambda \oplus \Lambda, 2)$, and put

$$q_E = \sum_{i,j} e_{ij} u_i v_j.$$

Its minimal Sullivan model is

$$M_E = \big(\mathbb{Q}[u_1, \dots, u_{2g}, v_1, \dots, v_{2g}] \otimes \Lambda(z_3), d\big), \qquad dz_3 = q_E, \tag{135}$$

with $du_i = dv_i = 0$. This follows from (133); decomposability of q_E makes the displayed algebra minimal. Its quadratic differential records the rational Whitehead pairing already computed integrally in that corollary. These

degree-two and degree-three generators concern the delooped space, not the degree-one model of a Heisenberg nilmanifold.

Proposition 24.2. *If $E \neq 0$, then P_E is rationally formal, with*

$$H^*(P_E;\mathbb{Q}) = \mathbb{Q}[u_i, v_i]/(q_E).$$

For polarizations on abelian varieties of a fixed complex dimension g, all the spaces P_E have the same unmarked rational homotopy type, although their integral polarization types may differ.

Proof. Put $R = \mathbb{Q}[u_i, v_i]$. The only nonzero differential in $R \oplus Rz_3$ sends fz_3 to fq_E. A nonzero polynomial q_E is a non-zero-divisor in R. Thus odd cohomology vanishes, the even cohomology is $R/(q_E)$, and the projection $M_E \to (R/(q_E), 0)$ is a quasi-isomorphism. This is precisely formality. It is compatible with a nonzero minimal differential and a nonzero Whitehead bracket.

Every nondegenerate alternating form of dimension $2g$ over $\mathbb{Q}$ admits a symplectic basis. Applying the same rational basis change to both sets of degree-two variables identifies the polynomials q_E. The elementary divisors of an integral polarization become invertible over $\mathbb{Q}$, so they do not distinguish these rational models. □

For an elliptic curve with polarization degree $d > 0$,

$$dz_3 = d(u_1v_2 - u_2v_1).$$

Replacing z_3 by z_3/d removes d over $\mathbb{Q}$. The remaining quadratic relation is still nonzero. This separates the integral polarization multiplicity from its rational attaching geometry.

24.5 Finite models for arbitrary quadratic two-stage algebras

The geometric construction extends beyond the single polarization relation. The following statement includes nonformal examples.

Theorem 24.3. *Let $q_1, \dots, q_s \in \mathbb{Z}[x_1, \dots, x_N]$ be homogeneous quadratic polynomials, with $|x_i| = 2$ and $N, s \geq 1$. There is a degreewise finite simplicial set whose realization has homotopy type*

$$P_q = \operatorname{hofib}\big((q_1, \dots, q_s) : K(\mathbb{Z},2)^N \longrightarrow K(\mathbb{Z},4)^s\big).$$

Its only nonzero positive homotopy groups are $\pi_2 = \mathbb{Z}^N$ and $\pi_3 = \mathbb{Z}^s$. Its minimal Sullivan model is

$$\big(\mathbb{Q}[x_1, \dots, x_N] \otimes \Lambda(z_1, \dots, z_s), D\big), \quad |z_\ell| = 3, \quad Dx_i = 0, \quad Dz_\ell = q_\ell. \tag{136}$$

Consequently every finite-dimensional choice of generators in degrees two and three with rational quadratic differential has a rational realization by such a degreewise finite integral simplicial model.

Proof. Choose integral matrices J^{ℓ} with $q_{\ell} = \sum_{i,j} J^{\ell}_{ij} x_i x_j$; for example put the off-diagonal coefficients above the diagonal. We first construct compact semialgebraic geometric input, before applying the finite replacement theorem.

Take the square elliptic curve $A = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ with its real algebraic Weierstrass model and the principal line bundle $L = \mathcal{O}_A(0)$. Put $T = A(\mathbb{R})^0$, let $\iota : T \hookrightarrow A(\mathbb{C})$ be the inclusion, and let $j = [i]$ be its algebraic complex multiplication automorphism. Analytically the two circles $\iota(T)$ and $j\iota(T)$ have primitive cycles α, β with $E_L(\alpha, \beta) = 1$, after choosing the compatible orientation. Analytic uniformization is used here only to identify these cycles. The maps and line bundles in the construction are algebraic or semialgebraic in Weierstrass coordinates.

Let B_L be the normalized biextension of Proposition 7.8. Then

$$\mathcal{M}(s,t) = B_L(\iota(s), j\iota(t)), \qquad s,t \in T,$$

is a semialgebraic biextension on $T \times T$. Its mixed Chern number is $E_L(\alpha, \beta) = 1$. On $G = T^N$ define the s multiplicative lines

$$\mathcal{C}_{\ell}(g,h) = \bigotimes_{i,j} \mathcal{M}(g_i, h_j)^{\otimes J^{\ell}_{ij}}.$$

Negative powers mean duals. The biextension laws give associators and units satisfying their coherence identities. Together these lines define a central categorical extension of G with band $U(1)^s$ after taking phases.

Apply the triangle-label construction in the proof of Proposition 7.14 componentwise. Its nth space is a principal

$$U(1)^{s\binom{n}{2}}\text{-bundle over } G^n.$$

Anchor triangles are independent in each of the s components; tetrahedron equations determine the remaining triangles. All spaces are compact semialgebraic and all operators are semialgebraic. The same closed-degeneracy argument proves properness. The comparison via the contractible edge-cochain groups $C^1(\Delta^n; U(1)^s)$ identifies its realization with the delooping of this central categorical extension.

The mixed Chern matrices are J^{ℓ}. With the sign convention of Proposition 7.12, the categorical-torus classification assigns symmetric forms $J^{\ell} +$

$(J^\ell)^{\mathrm{t}}$ and quadratic classes $\sum_{i,j} J^\ell_{ij} x_i x_j = q_\ell$ [29, Construction 2.3 and Theorem 4.1]. Equivalently its quadratic value is half the value of the symmetric form on the diagonal; there is no extra factor of two in q_ℓ. Thus its delooping is P_q over $BG \simeq K(\mathbb{Z},2)^N$. Theorem 7.15 now supplies the degreewise finite simplicial replacement.

The homotopy exact sequence gives the stated homotopy groups. Formula (133), applied to the s coordinates, gives (136), which is minimal. For rational coefficients multiply each q_ℓ by a positive common denominator and rescale its degree-three generator over $\mathbb{Q}$. This produces the same rational model from integral quadratic polynomials. □

The theorem gives finite presentations up to homotopy. It uses choices of compatible face carriers and does not assert a prescribed common set of toric flags for all quadratic forms. Its algebraic input is classical categorical-torus and Sullivan theory; the proof identifies compact semialgebraic data to which the earlier finite replacement applies.

Example 24.4 (A nonzero Massey product). Take $N = 3$, $s = 2$, and $q_1 = x_1x_2$, $q_2 = x_2x_3$. The resulting finite presentation has minimal model

$$\mathcal{A} = \big(\mathbb{Q}[x_1,x_2,x_3] \otimes \Lambda(a,b), d\big), \qquad da = x_1x_2, \quad db = x_2x_3, \quad |a| = |b| = 3.$$

The degree-five element

$$\mu = x_1 b - a x_3$$

is closed, since $d\mu = x_1x_2x_3 - x_1x_2x_3 = 0$. It is nonzero in the free algebra and cannot be a boundary: every degree-four element is a polynomial in the x_i and has zero differential. Also $H^3(\mathcal{A}) = 0$ because the two displayed quadratic differentials are linearly independent. Hence

$$\langle [x_1], [x_2], [x_3] \rangle = \{[\mu]\} \neq \{0\},$$

with the convention for which the defining representative is $x_1 b - a x_3$. The indeterminacy $[x_1]H^3 + H^3[x_3]$ is zero. In fact the only linear syzygy of x_1x_2 and x_2x_3 is generated by $(x_3, -x_1)$, so $H^5(\mathcal{A}) = \mathbb{Q}\cdot[\mu]$. A nontrivial Massey product obstructs formality; thus this degreewise finite model is rationally nonformal.

There is a useful algebraic distinction here. If the q_ℓ form a regular sequence in the polynomial ring, the Koszul complex (136) projects quasi-isomorphically onto $\mathbb{Q}[x_i]/(q_1,\dots,q_s)$, so the space is formal. The single nonzero Heisenberg relation is such a case. The two relations in Example 24.4 share the factor x_2 and give the displayed Massey class. Failure to be a regular sequence is not itself a nonformality criterion: redundant relations can instead produce closed odd generators.

24.6 Scope of the comparison

The rational model is a smaller algebraic description of the realization. It does not supply an ordinary simplicial addition on the finite presentation, nor does it retain all finite-prime information. Conversely, degreewise finite flags alone do not force a zero Sullivan differential or formality. To model arbitrary rational differentials by these geometric methods one must still construct the corresponding compact coherent presentations; Sullivan's realization theorem by itself does not furnish those particular finite toric presentations. Section 25 supplies a further family of higher differentials from compact sphere actions and identifies an obstruction to extending that route.

25 Higher Sullivan differentials from compact group actions

Odd representation spheres give a concrete extension beyond quadratic differentials. Their Euler classes are products of character weights, so the resulting Sullivan differentials can have arbitrarily large word length. The compact action nerve also retains the first-face description of twisting. For a finite list of actions, even the finite carrier flags can be chosen in common.

These facts use classical sphere-bundle and Borel models [25, Section 15(b), Example 4; Section 15(f), Example 2]. The additional construction here is their comparison with the finite carrier presentations of this note.

25.1 Weights, spheres, and the action nerve

Let $T = (S^1)^N$. For $w = (w_1, \dots, w_N) \in \mathbb{Z}^N$, let $\mathbb{C}_w$ be the character representation

$$(t_1, \dots, t_N) \cdot z = t_1^{w_1} \cdots t_N^{w_N} z.$$

Choose s representations

$$V_\ell = \bigoplus_{j=1}^{r_\ell} \mathbb{C}_{w_{\ell j}}, \qquad r_\ell \geq 2, \qquad M = \prod_{\ell=1}^{s} S(V_\ell).$$

Here $S(V_\ell)$ is the unit sphere, of dimension $2r_\ell - 1$. Denote the Borel construction by $Z_w = ET \times_T M$. No freeness assumption is made. Write $x_i \in H^2(BT; \mathbb{Z})$ for the first Chern class of the associated weight-one line; this specifies the character convention.

Theorem 25.1. *The space Z_w has a degreewise finite simplicial presentation. Its minimal Sullivan model is*

$$\mathcal{M}_w = \big(\mathbb{Q}[x_1,\dots,x_N]\otimes\Lambda(z_1,\dots,z_s),D\big), \qquad |x_i|=2, \quad |z_\ell|=2r_\ell-1, \tag{137}$$

where

$$Dx_i=0, \qquad Dz_\ell=\prod_{j=1}^{r_\ell}\left(\sum_{i=1}^{N} w_{\ell j,i}x_i\right).$$

The space is simply connected, and its integral homotopy groups, independently of the weights, are

$$\pi_2(Z_w)\cong\mathbb{Z}^N, \qquad \pi_i(Z_w)\cong\pi_i(M) \quad (i\geq 3).$$

Proof. The action groupoid has compact levels

$$X_n = T^n\times M.$$

In a convention where t_1 acts first, its faces are

$$\begin{aligned} d_0(t_1,\dots,t_n;m) &= (t_2,\dots,t_n;t_1m),\\ d_i(t_1,\dots,t_n;m) &= (t_1,\dots,t_it_{i+1},\dots,t_n;m), \quad 0<i<n,\\ d_n(t_1,\dots,t_n;m) &= (t_1,\dots,t_{n-1};m). \end{aligned}$$

Degeneracies insert the identity. The action axioms give the simplicial identities. All operations are semialgebraic: on the unit circle negative powers use complex conjugation. Degeneracies are closed embeddings and their finite unions are semialgebraic cofibrations. The realization is the Borel construction. Theorem 7.15 therefore gives its degreewise finite replacement.

For the model calculation, first consider the universal complex rank-r bundle. Its unit sphere bundle is the fibration

$$S^{2r-1}\longrightarrow BU(r-1)\longrightarrow BU(r).$$

The polynomial Chern-class models have a relative model

$$\big(\mathbb{Q}[c_1,\dots,c_r]\otimes\Lambda(z_{2r-1}),D\big), \qquad Dz_{2r-1}=c_r.$$

Indeed, projection killing c_r and z_{2r-1} is a quasi-isomorphism onto $\mathbb{Q}[c_1,\dots,c_{r-1}]$. Pull back along the classifying map of V_ℓ and tensor these independent relative models over $\mathbb{Q}[x_1,\dots,x_N]$. The Sullivan pullback

theorem identifies the result with a model of Z_w [38, Theorem 2.4]. The Whitney product formula gives

$$c_{r_\ell}(V_\ell) = \prod_j c_1(\mathbb{C}_{w_{\ell j}}) = \prod_j \left(\sum_i w_{\ell j,i} x_i \right)$$

[36, Theorems 3.2 and 3.9; Proposition 3.10]. Choose the orientation of z_ℓ to give this sign. Since $r_\ell \geq 2$, each differential is decomposable, and the displayed model is minimal.

Finally M is simply connected and has $\pi_2(M) = 0$. The homotopy exact sequence of $M \to Z_w \to BT$, with $BT = K(\mathbb{Z}^N, 2)$, proves all the homotopy assertions. In particular higher torsion homotopy groups of the spheres remain present integrally. □

25.2 Common finite flags for a finite family of actions

Keep N and $r_1, \dots, r_s$ fixed. The compact spaces $X_n = T^n \times M$ then do not depend on the weights. All positive faces and all degeneracies above are also independent of the weights; only d_0 uses the action. This remains true after a suitably coordinated finite replacement.

Proposition 25.2. *For any finite list of weight systems with the same $N, r_1, \dots, r_s$, their degreewise finite replacements can be chosen to have a common restriction to Δ_0. They consequently have identical ordinary simplex counts, identical nondegenerate simplex counts, and identical CW cell counts in every dimension.*

Proof. Repeat the carrier construction of Theorem 7.15 simultaneously for the finite list. At degree q, triangulate the common X_q compatibly with the inverse images of the previous cells under every face of every listed action. There are only finitely many such semialgebraic subsets. This gives one finite cell poset P_q. Each action has its own first carrier face; its positive carrier faces are literally the same as those of the other actions. The unique-carrier argument proves each set of semisimplicial identities exactly.

The free-degeneracy completion used in that theorem has bisimplicial sets

$$D_{p,q} = \coprod_{\rho : [q] \twoheadrightarrow [r]} N_p P_r.$$

These underlying sets are common to the whole list. For $\alpha : [j] \to [q]$ preserving zero, factor $\rho\alpha$ as a surjection followed by an injection. Since

$(\rho\alpha)(0) = 0$, that injection also preserves zero, and hence uses only positive original faces. Thus the induced operator is independent of the action. The nerve operators are common as well. Taking diagonals proves agreement on Δ_0, including all degeneracies. The same simplices are therefore degenerate in every model, which proves the counting assertions. □

Finiteness of the list is essential to this particular proof: simultaneous semialgebraic triangulation is being applied to a finite family of subsets in each degree. No common triangulation for all integer weights is asserted. The flags here are carriers in products of tori and spheres. They are not asserted to be the earlier integer-wall flags in tori alone.

25.3 Projective space and a visible higher attaching relation

Take $T = S^1$ and $M = S^{2r-1}$, with $r \geq 2$. For the trivial action the Borel space is $\mathbb{CP}^{\infty} \times S^{2r-1}$; for the scalar weight-one action it is homotopy equivalent to $\mathbb{CP}^{r-1}$. The latter action is free, so the Borel construction compares to its orbit space by a fibration with contractible fibre ET. Their minimal models are respectively

$$(\mathbb{Q}[x_2] \otimes \Lambda(z_{2r-1}), 0), \qquad (\mathbb{Q}[x_2] \otimes \Lambda(z_{2r-1}), Dz = x^r).$$

Both have the same graded integral homotopy groups, by Theorem 25.1, and admit the common finite flags of Proposition 25.2. But their rational cohomology in degree $2r$ is $\mathbb{Q}$ and 0, respectively. For $r \geq 3$ both models have zero quadratic differential, so all their binary rational Whitehead products vanish. The word-length-r relation still distinguishes their rational homotopy types.

The model for $\mathbb{CP}^{r-1}$ is classical [25, Section 15(b), Example 5]. The comparison here exhibits that familiar relation as a change of first face in common finite presentations. It does not identify $\mathbb{CP}^{r-1}$ integrally with

$$\operatorname{hofib}\bigl(x^r : K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z},2r)\bigr).$$

The latter is its $(2r-1)$-Postnikov section. Their rational types agree, while the projective space retains the higher torsion homotopy of S^{2r-1}.

25.4 Closed nonformal examples in arbitrarily high degrees

For $k \geq 1$, let T^2 act on $M_k = (S^{2k+1})^3$ using the complex representations

$$V_1 = \mathbb{C}_{(1,0)}^{\oplus(k+1)}, \qquad V_2 = \mathbb{C}_{(1,0)}^{\oplus k} \oplus \mathbb{C}_{(0,1)}, \qquad V_3 = \mathbb{C}_{(0,1)}^{\oplus(k+1)}.$$

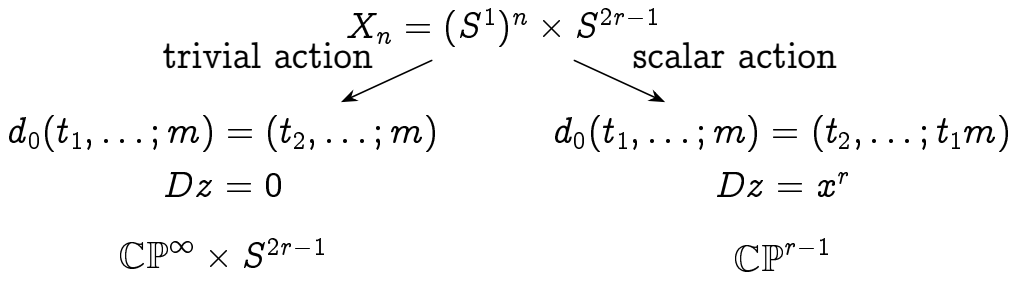


Figure 32: The same compact levels and positive faces support different rational attaching relations. A common finite carrier replacement preserves this first-face distinction.

This action is free: a stabilizer must act trivially on the first and third unit spheres, forcing both circle coordinates to be one. The quotient $Q_k = M_k/T^2$ is therefore a closed smooth manifold of dimension $6k+1$. It is simply connected by the homotopy exact sequence of the principal T^2-bundle, and its Borel space is homotopy equivalent to Q_k. Equivalently, quotient the two outer Hopf factors first. Then Q_k is the unit sphere bundle of $L_x^{\oplus k} \oplus L_y$ over $\mathbb{CP}^k \times \mathbb{CP}^k$, where the associated line conventions give $c_1(L_x) = x$ and $c_1(L_y) = y$. In particular it admits a finite triangulation. The common carrier presentations used to compare it with the trivial action are generally infinite-dimensional; their finiteness is degreewise.

Proposition 25.3. *The manifold Q_k has minimal Sullivan model*

$$\mathcal{A}_k = \big(\mathbb{Q}[x,y] \otimes \Lambda(a,b,c), d\big), \qquad |x| = |y| = 2, \quad |a| = |b| = |c| = 2k+1,$$

with

$$da = x^{k+1}, \qquad db = x^k y, \qquad dc = y^{k+1}.$$

Its triple Massey product

$$\langle [x], [x^k], [y] \rangle = \{[xb - ay]\} \neq \{0\} \quad \text{in } H^{2k+3}(Q_k; \mathbb{Q}) \tag{138}$$

has zero indeterminacy. Thus Q_k is nonformal. For $k \geq 2$, all its binary rational Whitehead products vanish. Moreover, Q_k and $BT^2 \times M_k$ have the same graded integral homotopy groups and admit degreewise finite presentations with common Δ_0 restriction.

Proof. The Euler products in Theorem 25.1 give the displayed minimal model. The three monomials da, db, dc are linearly independent, so $H^{2k+1}(\mathcal{A}_k) = 0$. The element $xb - ay$ is nonzero and closed. It cannot be a boundary: every element of degree $2k+2$ is a polynomial in x, y

and has zero differential; products of two odd generators begin in degree $4k+2 > 2k+2$. The two products defining the Massey bracket are da and db, and its indeterminacy is $[x]H^{2k+1} + H^{2k+1}[y] = 0$. This proves (138) and nonformality.

For $k \geq 2$ every nonzero differential has word length $k+1 \geq 3$. The quadratic part of the minimal differential is therefore zero. Its duality with the rational Whitehead pairing proves the stated vanishing [24, Sections 5–6]. Apply Theorem 25.1 and Proposition 25.2 to the free action and the trivial action on the same M_k. This proves the final comparisons. The trivial-action space has a zero-differential model and is formal. □

The first case is a seven-dimensional nonformal manifold; it falls within the sphere-bundle construction of [13, Example 1]. For $k = 2$ the cubic minimal algebra is already [32, Example 2], after renaming generators and taking the even generators in degree two. That example explicitly has abelian rational homotopy Lie algebra. Here it has a concrete thirteen-dimensional sphere-bundle realization and a Massey class in degree seven. No novelty claim is made for these underlying rational types or Massey products. More general sphere-bundle formality criteria are given in [80, Theorem 1.1]. Thus vanishing binary rational Whitehead products and the common finite incidence counts together still fail to determine the rational homotopy type.

25.5 A localization obstruction to ordinary torus actions

The factorization into linear weights is a genuine restriction of one compact realization route. Let $R = H^*(BT;\mathbb{Q}) = \mathbb{Q}[x_1,\dots,x_N]$, with $|x_i| = 2$.

Proposition 25.4. *Suppose a finite T-CW complex F has Borel model, as an algebra over the specified R,*

$$(R \otimes \Lambda(z_{2r-1}), D), \qquad Dz = P \neq 0, \quad r \geq 2.$$

Then the homogeneous polynomial P splits over $\mathbb{Q}$ as a product of linear forms, up to a nonzero scalar. Conversely every such split polynomial is realized by a representation sphere, after rationally rescaling z.

Proof. Since P is a non-zero-divisor in R, the assumed model gives $H_T^*(F;\mathbb{Q}) = R/(P)$. Let S be generated multiplicatively by all nonzero rational degree-two linear forms. The Borel localization theorem gives

$$S^{-1}(R/(P)) \cong S^{-1}R \otimes H^*(F^T;\mathbb{Q})$$

[82, Definition 2.14 and Remark 2.15]. The left side is annihilated by P. The right side is a free module over the domain $S^{-1}R$, so both sides must be zero. Hence $(P) \cap S \neq \varnothing$: P divides a product of linear forms. Unique factorization in R proves the necessity. For the converse, clear denominators in each factor and use its integral coefficients as a character weight. Rescaling z absorbs the resulting nonzero scalar, and Theorem 25.1 applies. □

This is a Sullivan-algebra formulation of classical equivariant Euler-class restrictions. In particular, [34, Section 5, p. 309] recalls factorization of the Euler transgression for a rational cohomology sphere into linear characters, citing Hsiang's Theorem IV.6. We retain the localization proof to make the hypothesis over the specified base algebra R explicit; the obstruction is not proposed as a new theorem of transformation groups.

For example $Dz = x_1^2 + x_2^2$ cannot be the one-odd-generator Borel model of a finite T^2-CW complex over this specified BT^2. Yet Theorem 24.3 does give a degreewise finite model of its integral two-stage space using a categorical torus. Thus this obstruction concerns an ordinary compact group action, and does not obstruct the more general compact categorical presentation. Zoller gives a broader equivariant realization framework requiring algebraic data over the subgroups, together with localization conditions [82, Theorem 5.5].

25.6 Higher base degrees and the boundary of the method

The compact action-nerve argument also applies to compact real algebraic groups, using the usual groupoid nerve. For a noncommutative group and a left action in which t_1 acts first, the interior face merges $t_{i+1}t_i$; the order in the torus display was immaterial because the torus is abelian. For example, use $Sp(1)^3$ and its standard quaternionic coordinate representations. Writing u_i for the degree-four second Chern classes, the two representations $\mathbb{H}_1^{\oplus k} \oplus \mathbb{H}_2$ and $\mathbb{H}_2 \oplus \mathbb{H}_3^{\oplus k}$ give the model

$$\big(\mathbb{Q}[u_1, u_2, u_3] \otimes \Lambda(a, b), d\big), \qquad |u_i| = 4, \quad |a| = |b| = 4k + 3,$$

$$du_i = 0, \qquad da = u_1^k u_2, \qquad db = u_2 u_3^k.$$

The same sphere-bundle pullback proof applies, using complex rank $2k + 2$ and the standard complex rank-two model of a quaternionic line. This supplies higher closed even generators as well as higher word length. It does not justify arbitrary substitutions of degrees.

Abstract integral existence is much older and more general: Sullivan's Theorem 10.2(ii) gives integral forms of finite-type nilpotent rational models; its proof chooses coefficient lattices stage by stage [76, pp. 307 and 310].

For simply connected rational types of finite rational type, finite-type CW representatives exist, and finite-dimensional total rational cohomology permits a finite CW representative [38, Proposition 1.8]. These statements do not preserve a specified integral Postnikov system or construct the compact action nerve and common finite flags above.

The explicit construction in this section treats products of linear weights and the indicated compact-group variants. The quadratic categorical-torus construction of Theorem 24.3 allows arbitrary quadratic sums by a different mechanism. General higher polynomial sums, differentials involving earlier odd generators, and a uniform finite geometric presentation for arbitrary Sullivan models remain outside the proofs given here. This is a question about explicit compatible geometry, not about the classical existence of integral representatives of rational types.

25.7 Historical placement of the finite comparisons

The literature comparison below records the scope of the claims as checked on 15 October 2026. It separates established constructions from the specific finite presentation supplied here. It is not an exhaustive priority determination.

First-face twisting and finite examples. The first-face convention is classical twisted Cartesian-product theory: on $F \times_\tau B$, the twisting function changes d_0, while the positive faces and all degeneracies retain their product formulas [43, Definition 18.3 and Propositions 18.4, 18.7]. Thus the common Δ_0-restriction is already built into that formalism. It is not a new replacement for the theory of fibrations.

There are even classical degreewise finite examples of the same pattern. Take the standard simplicial abelian-group models

$$B = K(\mathbb{F}_2, 2), \qquad F = K(\mathbb{F}_2, 3).$$

They have finite sets of simplices in each degree. The principal F-fibrations with invariants 0 and $u^2 \in H^4(B; \mathbb{F}_2)$ have twisted Cartesian models $F \times_{\tau_0} B$ and $F \times_{\tau_1} B$. Here u is the universal degree-two class and its square is nonzero. Both models are Kan, share all positive faces and degeneracies, and have precisely $\pi_2 = \pi_3 = \mathbb{F}_2$ as their nonzero homotopy groups. Their different Postnikov invariants distinguish their homotopy types. This is a direct application of classical twisted products and the cohomology of Eilenberg–Mac Lane spaces [43, 73], not an additional novelty claim. Consequently, common finite Δ_0-data and equal homotopy groups by themselves

do not distinguish the present work from earlier theory. Rector's examples [64] give another classical comparison for equal cell counts and homotopy groups.

The specific simultaneous carrier statement. Proposition 25.2 keeps the finite carrier flags of the specified compact torus/sphere presentations in common. It also retains their integral homotopy types, which generally have infinite homotopy groups. The construction coordinates compatible semialgebraic triangulations and free degeneracies. These ingredients are established [23, 57]; the proposition is an explicit refinement of them.

Basu–Karisani construct efficient finite simplicial replacements preserving bounded homotopy information [2], and give algorithms for homology of semialgebraic maps and zigzag diagrams [3]. These are close methodological comparisons. The results inspected do not state the particular common-Δ_0 replacement of an entire compact action nerve proved here. Conversely, our proof provides none of their complexity bounds. We have not located the precise simultaneous carrier statement in the sources examined; this leaves priority open and does not justify a claim to a new general twisting principle.

The integral models and the spectrum. The higher finite flag construction should likewise be distinguished from its classical inputs. Compact cocycle tori, the Eilenberg–Mac Lane homotopy types, and the Segal spectrum construction are established. The direct circular-order predecessor is [51]. The specific data constructed here are integer-wall incidence categories, primitive integral cocycles, compatible pointed-set operators, and finite refinement diagrams. We have not identified an earlier source giving this exact combination in all degrees.

This does not establish that the bare existence of every degreewise finite integral model is new. Neither should one dismiss that finiteness as a consequence of a differently defined "finite type": finite-type CW models, finite approximations in each homotopy range, finite-coefficient Kan models, and one simplicial set finite in every degree are different assertions. The general carrier theorem supplies the last assertion here; the explicit toric construction supplies the additional compatible data.

A defensible description of the contribution. The recent rational-homotopy section gives worked geometric realizations of familiar Sullivan models and compares their finite presentations. The more distinctive research claim to investigate is the compatible integral finite presentation, with its cocycles, operations, and geometric interpretation. A suitable formulation is: *we*

give explicit degreewise finite presentations of classical homotopy types, retaining specified cochain data and simultaneous control of simplicial operators. The examples illustrate that construction; their known rational phenomena are not evidence of historical priority.

A Natural characteristic-cell comparisons

The main finite-model proof uses a contractible lattice cover. For the comparisons in the following appendices it is useful also to retain the actual characteristic polytopes. This gives natural comparison spans even though chosen barycentric homeomorphisms need not be natural.

To obtain *natural* comparisons, retain abstract characteristic polytopes. For a face orbit c, let $E(c)$ be the disjoint union of the closures of all its lifts modulo simultaneous translation. An incidence arrow induces the appropriate face embedding. This is a diagram on $\mathcal{C}$, with colimit $\mathbb{T}^d$, and it gives

$$\mathbb{T}^d \xleftarrow{\simeq} W := \operatorname{hocolim}_{c\in\mathcal{C}} E(c) \xrightarrow{\simeq} |N\mathcal{C}|. \tag{139}$$

The right map is an equivalence because each $E(c)$ is contractible. For the left, the category of nonidentity arrows into a chosen target is exactly the proper face poset of its lifted polytope. Its latching space is the boundary. The boundary inclusions are cofibrations, so the finite filtration by dimension identifies homotopy colimit and colimit up to weak equivalence.

These spans are natural for the affine maps on characteristic polytopes, including maps that lower dimension. Duplicating vertices or inserting zero inputs gives a closed polyhedral subcomplex: the image is cut out by the appropriate zero face evaluations. The induced arrangement is the source arrangement. The integral retraction of each degeneracy also makes its image lattice saturated. Thus the source category has exactly the incidence branches of this subcomplex. Degeneracy maps of the nerves and resolutions are closed subcomplex inclusions; their finite latching unions are too. Consequently these simplicial spaces are proper and the levelwise equivalences in (139) survive realization, as in [71].

B Cyclic partitions for the circle model

For $k = 1$ and one input, a cocycle describes labelled points of a circle modulo common rotation. Its cell is a cyclically ordered partition

$$\Pi = (B_1, \dots, B_r) \quad \text{of } \{0, \dots, n\}.$$

Labels in one block coincide. Distinct blocks have the specified cyclic order, with r positive gaps of sum one. The cell dimension is $r-1$ and its abstract closure is Δ^{r-1}.

Every incoming incidence at Π is specified by a nonempty subset J of its r gaps: retain those gaps, collapse the others, and merge the corresponding blocks. Distinct singleton choices give r parallel incidences from the one-block partition to Π. Composition retains subsets of already retained gaps.

A monotone map $\alpha : [m] \to [n]$ replaces each block B by $\alpha^{-1}B$, omits empty blocks, and preserves cyclic order. Every new gap is a sum of consecutive old gaps. On an incidence it is retained precisely when this sum contains a retained gap. If only one block survives, its sole cyclic gap is the sum of all old gaps, namely one. These rules describe the entire simplicial category with finite combinatorial data.

Proposition B.1. *Writing $S(N,r)$ for a Stirling number of the second kind,*

$$|N_q C_n^{(1)}(1_+)| = \sum_{r=1}^{n+1} S(n+1,r)(r-1)!\big((q+1)^r - q^r\big). \tag{140}$$

Proof. There are $S(n+1,r)(r-1)!$ cyclic partitions with r blocks. Fix the terminal object of a nerve chain. Its unique lift to the characteristic simplex identifies the chain with a flag $\varnothing \neq J_0 \subseteq \cdots \subseteq J_q = \{1,\dots,r\}$. Assign each gap its first entrance time in $\{0,\dots,q\}$. At least one gap enters at time zero, giving $(q+1)^r - q^r$ flags. □

n	Objects	Arrows	$\lvert D^{(1)}_{0,n}\rvert$
0	1	1	1
1	2	4	4
2	6	24	54
3	26	196	1544
4	150	2040	75750

The maximal cells are the singleton-block cyclic partitions: exactly the $n!$ circular orders $\mathbf{SC}_n$. They are compatible with faces, but are not a simplicial subobject of the cell objects. A degeneracy puts two labels in one block; the degeneracy of $\mathbf{SC}$ instead remembers an adjacent order of the two labels. The diagonal model is therefore substantially larger than $\mathbf{SC}$.

B.1 A finite bridge with no internal gap cuts

Use the simplicial gap space $M_n = \coprod_{c\in\mathbf{SC}_n} \Delta^n$ of version 04. Deleting a label merges gaps; duplicating a label inserts a zero gap. Every pulled-back zero wall is a consecutive gap sum equal to zero or one, hence a union of ordinary simplex faces. Therefore the bridge simplifies to

$$R^0_n = \coprod_{c\in\mathbf{SC}_n} N_n\,\mathrm{Face}(\Delta^n), \qquad \mathbf{SC} \xleftarrow{\simeq} R^0 \xrightarrow{\simeq} D^{(1)}_0.$$

The right map retains the face-incidence branch of the angle map. The closed-cell resolution compares this span with $\mathbf{SC} \leftarrow M \rightarrow A^{(1)}$. The first map has contractible components. The primitive H_2 calculation from version 04 identifies the second with the generator: the gap lifts of circular orders 021 and 012 have boundary coordinates $(1,0,0)$ and $(0,0,0)$, and the intervening gap path maps once around $A^{(1)}_1 = \mathbb{T}$. Since both realizations are $K(\mathbb{Z},2)$, the map is an equivalence. Here the identification for $\mathbf{SC}$ is [51].

The exact size of the bridge is

$$|R^0_n| = n!\big((n+2)^{n+1} - (n+1)^{n+1}\big).$$

Each gap has a first entrance time $0,\dots,n$ or never; at least one enters at zero. For the direct $k=2$ model the first nontrivial level has $|D^{(2)}_{0,2}| = 6$, compared with 12 in version 04: a circle category has two objects and two nonidentity arrows, giving six two-step nerve chains.

C An interval bar and its explicit comparison

Full symmetric addition uses all subsets of the inputs. The bar construction needs less. In bar degree p and cocycle degree q, start with $(A^{(1)}_q)^p$ and use only walls

$$\sum_{r=a}^{b} a^r_{ij} \in \mathbb{Z}, \qquad 1 \le a \le b \le p, \quad 0 \le i < j \le q. \tag{141}$$

Let $\mathcal{C}^{\mathrm{int}}_{p,q}$ be the finite orbit-incidence category. Bar faces discard an end input or merge adjacent inputs; bar degeneracies insert zero. A target interval pulls back to an interval or the empty set. Vertical cochain operators also preserve the wall family. Thus this is a bisimplicial category, and its nerve has three simplicial directions. Take their diagonal:

$$Y^{\mathrm{int}}_n = N_n\mathcal{C}^{\mathrm{int}}_{n,n}. \tag{142}$$

The natural characteristic-cell spans and proper realization compare this to the ordinary bar of $|A^{(1)}|$. Hence it is a degreewise finite $K(\mathbb{Z},3)$.

There are $p(p+1)/2$ interval families instead of 2^p-1 subset families. This model supports the bar operators, not all maps of pointed sets. Use the full subset model when symmetric Γ-coherence is needed. The full-subset bar refines this one and maps to it by a weak equivalence of finite models.

Theorem C.1. *For $(a^1,\dots,a^n)\in (A^{(1)}_n)^n$, the formula*

$$\Theta_{ijk}=\sum_{r=i+1}^{j} a^r_{jk}, \qquad i<j<k, \tag{143}$$

defines a simplicial torus map to $A^{(2)}$. It induces a strict finite carrier map

$$\Phi : Y^{\mathrm{int}} \longrightarrow D^{(2)}_0$$

which is a weak equivalence.

Proof. In $\delta\Theta$ on $i<j<k<l$, the first two terms cancel the rows $j+1,\dots,k$. The remaining expression is

$$\sum_{r=i+1}^{j}(-a^r_{kl}+a^r_{jl}-a^r_{jk})=0.$$

For $\alpha:[m]\to[n]$, the rth new bar input is the sum of old inputs with indices $\alpha(r-1)+1,\dots,\alpha(r)$. Simultaneous vertical pullback therefore sends the right side of (143) to

$$\sum_{s=\alpha(i)+1}^{\alpha(j)} a^s_{\alpha(j),\alpha(k)},$$

which is the pullback of Θ. Empty intervals and repeated vertices give zero. This proves strict simplicial naturality. Each target wall pulls back to one interval wall from (141), so the finite carrier map exists.

It remains to check its degree on π_3. In source degree two, use winding coordinates

$$(a^1_{01},a^1_{02},a^2_{01},a^2_{02})=(x_{11},x_{12},x_{21},x_{22}).$$

The face maps to the degree-one winding group $\mathbb{Z}$ are

$$d_0=x_{22}-x_{21}, \qquad d_1=x_{12}+x_{22}, \qquad d_2=x_{11}.$$

Thus $N_2 = \ker d_1 \cap \ker d_2$ has $x_{11} = 0$, $x_{22} = -x_{12}$, and its differential is $d_0 = -x_{12} - x_{21}$. Its kernel is generated primitively by

$$g = (0, 1, -1, -1).$$

There are no higher normalized winding groups. Indeed the degree-n winding rank is n^2, and its binomial decomposition is $n^2 = \binom{n}{1} + 2\binom{n}{2}$. The Dold–Kan splitting gives these as the normalized ranks; the groups are free. Hence their second homology is $\mathbb{Z}\langle g \rangle$.

The level-torus spectral sequence identifies this group with π_3 of the source realization. Formula (143) sends g to $a^1_{12} = x_{12} - x_{11} = 1$, the generator of the target winding group in degree two. It therefore induces $+1$ on π_3. Both realizations are $K(\mathbb{Z}, 3)$, so the map is an equivalence. Naturality of (139) transfers this calculation to Φ. □

The generator has a concrete circle representative:

$$(a^1_{01}, a^1_{02}, a^2_{01}, a^2_{02}) = (0, t, -t, -t).$$

Every face is identically zero and $\Theta_{012} = t$. This is the primitive three-dimensional class, with no unknown integer multiplier in the comparison.

D Binary cocycles select entire chambers

This appendix records comparisons from the companion research notes [27]. The strict-primitives and connectivity statements quoted here are not hypotheses of the main theorems.

Let $\mathrm{B}^{(3)}_n$ be the integral three-cocycles on $\Delta[n]$ whose values lie in $\{0, 1\}$. For $z \in \mathrm{B}^{(3)}_n$, put

$$P_z = \{u \in [0, 1]^{\binom{n+1}{3}} : \delta u = z\}.$$

We use the strict-primitives theorem proved in version 03: there is a primitive with $0 < u_F < 1$ on every triangle. In the anchor coordinates $x_{ij} = u_{0ij}$, the other values are

$$u_{ijk} = x_{jk} - x_{ik} + x_{ij} + z_{0ijk}. \tag{144}$$

Proposition D.1. *P_z is exactly the closure of one full-dimensional chamber of the integer-wall arrangement for $A^{(2)}_n$ on its anchor cover. It requires no internal subdivision for a carrier comparison.*

Proof. Every cube inequality is an integer-strip inequality for one of the full set of face characters, including the anchors. The simultaneous strict inequalities have a solution, are convex, and avoid every wall. They specify precisely the strip occupied by the chamber of that solution, so their intersection is the whole chamber. Interpolation toward a strict solution shows that its closure is all of P_z. □

Define the finite simplicial set

$$R^0_{\mathrm{bin},n} = N_n \coprod_{z \in \mathsf{B}^{(3)}_n} \mathrm{Face}(P_z).$$

Its component and orbit-incidence maps give

$$\mathsf{B}^{(3)} \xleftarrow{\simeq} R^0_{\mathrm{bin}} \longrightarrow D^{(2)}_0. \tag{145}$$

The left map is an equivalence because the P_z are contractible, using their natural closed-cell resolutions and properness. Under a vertex pullback, primitive anchor coordinates change by the cocycle's integer-linear pullback plus one integer translation depending on z. This follows from (144). The translation is constant on the whole component, and the actual primitive pullbacks compose strictly. Passing to lattice orbits removes it on both objects and arrows. Thus the right map in (145) is strictly simplicial.

For binary two-cocycles the same argument yields all $n!$ gap chamber orbits, represented in the unit anchor cube, recovering the bridge to **SC**. For three-cocycles, the binary labels select a proper subset of chambers: a negative carry already occurs in a missing tetrahedral chamber.

D.1 Identifying the canonical class

The following comparison identifies the right map in (145), rather than merely finding a map to some space of type $K(\mathbb{Z}, 3)$. For general $k \geq 1$ define the topological simplicial abelian group

$$E^{(k)}_n = \{u \in C^k(\Delta[n]; \mathbb{R}) : \delta u \text{ is integral}\}.$$

There is a natural span

$$K(\mathbb{Z}, k+1) \xleftarrow{\delta} E^{(k)} \xrightarrow{\mathrm{mod}\ 1} A^{(k)}, \tag{146}$$

where the left object denotes the standard integral-cocycle simplicial set. Both arrows are realization weak equivalences.

For the left arrow, an integral cocycle has an integral cone primitive. Its inverse image is an affine real cocycle space, hence a contractible component.

The spaces are proper: componentwise degeneracies and their finite latching unions are closed polyhedral inclusions.

For the right arrow, reduction modulo one is the pullback of the standard covering of the torus of all cochains, restricted to its cocycle subtorus. Its kernel is $C^k(\Delta[\bullet];\mathbb{Z})$. Every singular simplex lifts to this cover, so levelwise singular complexes give a short exact sequence of bisimplicial abelian groups. The kernel is constant in the singular direction. Its normalized complex in the other direction is

$$\mathbb{Z} \xrightarrow{1} \mathbb{Z} \quad \text{in degrees } k+1, k.$$

Indeed, $\bigcap_{i>0} \ker d_i$ consists of cochains supported on k-faces containing every vertex $1,\dots,n$. Only the two listed degrees occur, and d_0 carries the face $\{1,\dots,k+1\}$ to the unique k-face with coefficient one. Thus the kernel is acyclic, and the map on diagonal singular complexes is a weak equivalence. Proper realization gives the assertion for (146).

The inclusion of the primitive space $\coprod_z P_z \subset E^{(2)}$ commutes with δ and reduction modulo one. Its composite with δ is the component projection $\coprod_z P_z \to \mathsf{B}^{(3)}$ followed by the canonical binary inclusion $j : \mathsf{B}^{(3)} \to K(\mathbb{Z},3)$. Therefore the right map in (145) represents j. The version-03 proof that j is 7-connected applies to this finite comparison. Proving it is an equivalence in every degree remains the missing step for the binary model.

E A half-integral chamber vertex from $\mathbb{RP}^2$

A natural shortcut would be to identify each P_z with the convex hull of its binary primitive points. This is false, first in simplex degree five. The obstruction has a small exact certificate and a topological explanation.

Use anchor-edge order $12, 13, 14, 15, 23, 24, 25, 34, 35, 45$ and set

$$x = (\tfrac12, 0, \tfrac12, 1, 0, 0, \tfrac12, \tfrac12, \tfrac12, 0).$$

For triples on $\{1,\dots,5\}$ in lexicographic order, put

$$(z_{0ijk}) = (0,0,1,0,1,1,0,1,1,1).$$

Define the remaining u by (144). In lexicographic orders on all triangles and all tetrahedra, respectively, this gives

$$\begin{aligned} u &= (\tfrac12, 0, \tfrac12, 1, 0, 0, \tfrac12, \tfrac12, \tfrac12, 0,\\ &\qquad \tfrac12, 0, 1, 0, \tfrac12, \tfrac12, \tfrac12, 1, \tfrac12, 1),\\ z &= \delta u = (0,0,1,0,1,1,0,1,1,1,0,1,1,1,1). \end{aligned}$$

Thus z is binary and $u \in P_z$.

Five active anchor constraints are

$$x_{13} = 0, \quad x_{15} = 1, \quad x_{23} = 0, \quad x_{24} = 0, \quad x_{45} = 0.$$

The five other active constraints are

$$u_{124} = 0, \quad u_{125} = 1, \quad u_{134} = 0, \quad u_{235} = 1, \quad u_{345} = 1.$$

Writing $(a, b, c, d, e) = (x_{12}, x_{14}, x_{25}, x_{34}, x_{35})$, the latter reduce to

$$a = b, \qquad a + c = 1, \qquad d = b, \qquad e = c, \qquad d = e.$$

They force $a = b = c = d = e = 1/2$. The ten active equations uniquely determine the point in its ten-dimensional affine primitive space; their matrix has determinant -2 in the recorded order. Consequently u is a nonintegral vertex, and

$$P_z \neq \operatorname{conv}\bigl(P_z \cap \{0, 1\}^{20}\bigr).$$

The projective plane inside the constraints. The active triangles are exactly

$$013, 015, 023, 024, 045, 124, 125, 134, 235, 345.$$

Every one of the fifteen edges belongs to two triangles; each vertex link is a five-cycle. This is a connected closed triangulated surface with Euler characteristic $6 - 15 + 10 = 1$, hence $\mathbb{RP}^2$. After collapsing the spanning tree of edges $0i$, the active matrix is the transpose of its reduced triangle boundary matrix. Its determinant of absolute value two records $H_1(\mathbb{RP}^2; \mathbb{Z}) = \mathbb{Z}/2$. Modulo the anchor lattice, the fractional vertex becomes a nonzero order-two vertex of the toric arrangement. The general connection between boundary-matrix integrality and torsion is discussed in [16]; the certificate here proves this example directly.

Why degree five is minimal. On $\Delta[4]$, every triangle lies in two tetrahedra. Reorient the five tetrahedron rows by their signs in the boundary of $\Delta[4]$. The matrix of $\delta : C^2 \to C^3$ becomes the oriented vertex-edge incidence matrix of K_5. It is totally unimodular: for a square submatrix, either a column has at most one nonzero entry and determinant induction applies, or every column has its two opposite entries and the row sum is zero. Adding coordinate-bound rows preserves total unimodularity. Thus every vertex of the cube slice for integral z is integral. Lower simplex degrees give submatrices. Degree five is therefore the first possible failure, and the example realizes it.

This does not refute the binary model. The convex hull of binary primitives is still nonempty, contractible, and natural under pullback, so it gives a different thickening of $\mathrm{B}^{(3)}$. The false step is equating that smaller polytope with the full real primitive chamber.

F Ordinary product maps: the obstruction

The distinction between ordinary and crossed products is essential. An ordinary simplicial Segal comparison for these finite models cannot realize a homeomorphism.

Lemma F.1. *If an ordinary simplicial map $f : K \to L$ has bijective realization, then f is a simplicial isomorphism.*

Proof. Every realization point has a unique representative in the interior of a nondegenerate simplex. If f sends a nondegenerate n-simplex to a degenerate one, its interior maps by a dimension-lowering affine map and injectivity fails. Distinct nondegenerate simplices cannot have the same image either. Surjectivity on realization supplies every target nondegenerate simplex. The unique degeneracy decomposition then gives a bijection in every degree, compatible with all operators. □

Theorem F.2. *Let K be degreewise finite with connected realization. If an ordinary simplicial map $\mu : K \times K \to K$ realizes a multiplication with a two-sided homotopy unit, then $\widetilde{H}_*(|K|;\mathbb{Q}) = 0$. Neither associativity nor commutativity is needed.*

Proof. Otherwise let $r > 0$ be the first nonzero rational homology degree. The Künneth decomposition in degree r has just the two factor summands; the unit implies $\mu_*(a, b) = a + b$. Define simplicial powers by $P_1 = \mathrm{id}$ and $P_{m+1}(x) = \mu(P_m(x), x)$. Then $(P_m)_* = m\,\mathrm{id}$ on H_r.

There are only finitely many functions $K_r \to K_r$. Thus two different powers agree in degree r, hence induce the same map on degree-r normalized chains and on H_r. This contradicts their actions by different integers on a nonzero rational vector space. □

Apply the lemma and theorem to the ordinary simplicial Segal map

$$X_0^{(k)}(2_+) \longrightarrow X_0^{(k)}(1_+) \times X_0^{(k)}(1_+).$$

If its realization were a homeomorphism, it would be an isomorphism. Its inverse followed by the fold would give an ordinary unital multiplication

on a degreewise finite $K(\mathbb{Z}, k+1)$, contradicting Theorem F.2. The same reasoning applies to the half-wall construction and to any ordinary simplicial span with a homeomorphic product comparison and homotopy-unital fold.

Why a crossed product escapes. For a crossed simplicial group G, the standard comparison

$$|G \times_t K| \cong |G| \times |K|$$

is a geometric homeomorphism, with simplex coordinates transformed by the crossing; see [51, Sections 2.6.2–2.7]. It is not generally the realization of an ordinary simplicial map to the Cartesian product. In a crossed face formula the index used in the second factor is changed by the first factor. Thus the literal second projection need not commute with faces, and Lemma F.1 does not apply. These results support the search for genuinely crossed data; they do not rule it out.

Keeping actual characteristic polytopes and their affine maps also retains exact products geometrically: forgetting extra cuts is the identity of the underlying torus and the fold is actual addition. Realization preserves finite products in the usual compactly generated setting [19]. This is a finite polyhedral presentation in every degree, but its levels are tori, not finite discrete simplex sets. Passing to carrier nerves gives the latter and loses the exact product comparison.

G Verification records

The further certificates `verify_geometric_cup1.py` and `verify_secondary_cup1.py` check the insertion identities, the twelve-term formula, and the exact integral secondary identity. The latter uses 56 independent cocycle anchors and 56 independent nullcochain anchors on $\Delta[8]$ to cover all solutions of $\delta T = c^2$ there. It imports the polynomial utilities from the accompanying `verify_prime3_operation.py`. The carrier-map certificate `verify_finite_bockstein_map.py` checks 9020 elementary face and degeneracy pullbacks, including 909 instances of d_0, and 4186 canonical-class identities in its stated finite range. The all-degree carrier argument is given in Section 15.3.

The operation certificate `verify_prime3_operation.py` checks the exact identity $\delta Q(c) = 3R(c)$ in all 56 independent integral anchor variables on $\Delta[8]$ and checks $\delta R(c) = 0$ on $\Delta[9]$. It also checks normalization of Q under all 127 surjective monotone maps $[7] \to [m]$ with $m < 7$, and the suspension sign convention. The accompanying JSON record contains the

signed formulas and check results. These are exact polynomial identities, not random tests; the identification with the classical reduced power uses [39].

The uniform relation certificate is `verify_uniform_relations.py`. It constructs normalized universal face matrices for $(n,p) = (2,2),(2,3),(3,2)$ and independently solves the Bockstein-square relation. It verifies the carry primitive, the adjacent identity $\delta^2 = 0$, and 11926 prism endpoint, expansion, and normalization cases in the stated range. The separate `verify_bockstein_square.py` exhaustively checks the carry identity on 17602 simplex cocycles in seven degree/prime cases. Both use only the Python standard library and have accompanying JSON records. These checks do not assert that arbitrary higher boundaries are fillable.

The geometric-coherence certificate `verify_coherent_carry.py` checks the interpolation formula over the integers in cocycle degrees two and three, with parameter dimensions zero through three. It verifies 212850 monotone parameter pullbacks, 12544 closedness identities, 900 endpoint identities and 3185 normalization identities as exact linear polynomials. The coefficient and pointed-set composition compatibilities have the all-degree proof in Theorem 22.5. The separate `verify_additive_operations.py` verifies eight universal mod-two identities, including the full corrected-fold associator, and the finite polynomial solve described in Section 22.6. Neither certificate uses random sampling.

Two further certificates check local flag generation. The integral verifier is `verify_binary_cone.py`. It checks 54 cone–join comparisons through dimension eight, 45 signed resolution identities, the comparisons with the displayed Berger–Fresse D_1, D_2, and 10290 naturality checks covering all monotone maps with source and target dimensions at most five. The mod-two verifier is `verify_generated_diagonals.py`. Here the join recursion agrees with an independent parity-subset implementation in 65 cases through dimension nine and satisfies 65 chain identities. There are 45038 naturality checks over all 6427 monotone maps with source and target dimensions at most six, and 16129 join boundary checks. The certificates retain their precise finite ranges; the propositions supply the all-degree proofs.

The all-arity certificate `verify_arity_diagonals.py` implements the wiring-diagram contraction independently of the tensor cone. In arities two, three and four, through simplex and resolution degree four, it checks 75 signed diagram identities, 75 tensor-cone identities, their 75 exact comparisons, and 25 binary comparisons. It also checks 1815 monotone-map naturality cases with source and target dimensions at most three. This script imports the sibling `verify_binary_cone.py`; each script has a JSON

record.

The reordering certificate `verify_reordering.py` checks 6110 signed boundary identities for all vertex maps between sets of at most four elements, and 8930 signed composition identities for sets of at most three elements. On a three-face chain with tuples of length at most four, it checks 2868 sorting-homotopy endpoints, 2868 realization identifications, and 28680 naturality identities for monotone carrier maps. It also checks 2025 exact normalization-composition identities on a rational grid, including integer walls, and 328 exact barycentric addition identities in the two-triangle circle product, together with its two face-map checks. These bounded checks support the all-degree proofs in Section 8; they do not assert a crossed multiplication.

The finite-refinement certificate `verify_refinement_law.py` checks 3765 triple composition identities and 7530 unit equations for its specified matrices of ranks zero through two. On its rational grid, including wall points, it checks 12321 carry compositions, normalized-chart compositions and lattice-translation invariances, together with 18801 coordinate bounds. Duplication followed by addition retains the coefficient 2 and has 32 checked affine branch values. In arities four and five, all 5 and 14 binary parenthesizations give 19423 exact carry telescoping checks and 28292 rotation comparisons; all 1800 permuted tree families lie in the appropriate all-subset arrangement. The JSON record gives the precise finite ranges. These checks supplement the all-degree proofs in Section 9; they do not certify a higher localization or an exact group law on the carrier realization.

The certificate `verify_nonopen_fold.py` checks the exact six-wall configuration behind Proposition 8.10: the source rank is six with determinant of absolute value two, the target rank is two, both constant diagonal flags are nondegenerate, and precisely the zeroth face of the displayed 4-simplex equals the prescribed target 3-simplex. The proof is also supplemented by exact rank and nondegeneracy checks on the suspended configurations for $1 \leq k \leq 8$. The argument in the text handles all higher degenerate cofaces and extends the configuration to every $k \geq 1$; these conclusions do not depend on extrapolation from a finite computation.

The geometric-deformation certificate `verify_geometric_deformation.py` checks every chain of length one through four with circle degrees in $\{-2, -1, 0, 1, 2\}$ on its specified rational grid. It verifies 13260 carrier compositions, 63070 coherent face equations, 63070 identity-degeneracy equations, 13260 lattice translations, and 26520 endpoints. It also recovers the exact simultaneous-interpolation defect $-1/32$. The refinement-operad certificate `verify_refinement_operad.py` checks 11182 associativity cases, including

10952 with nullary substitutions, 3105 unit/nullary equations, 1318 outer and 533 inner equivariance cases, 742 refinement comparisons, and 16 nerve counts. Its range and exact counts are recorded in JSON. The universal coherence and all-arity operad proofs are given in Section 10; these bounded checks do not assert a homeomorphism with the original diagonal realization.

The certificate `verify_polarization_cochains.py` verifies the exact diagonal primitive and change-of-marking homotopy from Section 7.6. It checks all 36 and 256 respective bilinear coefficients on $\Delta[4]$, together with 80 structural face identities. Its JSON record includes the full coefficient and residual matrices. These checks are universal over the integers, with no random sampling.

The new fibre results have structural, all-degree proofs. The accompanying exact verifiers supply additional local checks:

- `verify_integral_cup1_identity.py` checks $\delta h = -2c \smile c$ as a universal polynomial identity in twenty unrestricted integer anchor variables on $\Delta[6]$, covering all 210 possible quadratic monomials.

- `verify_finite_bockstein_fibre.py` checks three rank equalities for each of 30 degree/prime pairs, 3381 deterministic integer nullcochain cases, and 3431 monotone pullbacks. Its JSON record reports success. These are bounded local checks; they do not enumerate the complete toric models.

Three companion standard-library Python verifiers are also included with exact JSON records. They check the following finite instances; the general theorems above have structural proofs.

- The cyclic-partition category: 32849 associativity triples, 47050 functor-composition checks, 347205 simplicial-composition checks, and the size formula through degree four.

- The interval comparison: 464 cocycle identities, 109884 naturality identities on integer basis vectors, 6566 interval-preimage checks, and the primitive pairing 1. Source degrees run through five and target degrees through four.

- The fractional primitive: every cube constraint and cocycle equation, the exact active determinant -2, and the projective-plane edge incidences and vertex links.

The earlier research notes and their certificates are preserved [27]. The global 7-connectivity statement and strict-primitives theorem are inherited

from version 03; the local fibre H_4 certificate of version 04 is unchanged. No new conclusion about the global π_7 of the binary candidate is inferred from a local fibre or from the fractional vertex.

The finite replacement now has explicit abelian coherence, a smaller bar, and a concrete equivalence to the direct finite $K(\mathbb{Z}, 3)$ model. The open targets are to show that the much smaller binary chamber selection is already equivalent, or to construct a genuine crossed multiplication directly from circular order data with a geometric product homeomorphism. No historical priority or absolute minimality claim is made.

References

[1] P. Arnoux and A. Marin, *The Kühnel triangulation of the complex projective plane from the view point of complex crystallography, Part II*, Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics **45** (1991), no. 2, 167–244. doi:10.2206/kyushumfs.45.167. Sections 12–14, on elliptic configurations and projective quotients.

[2] S. Basu and N. Karisani, *Efficient simplicial replacement of semialgebraic sets*, Forum of Mathematics, Sigma **11** (2023), e41. doi:10.1017/fms.2023.36; arXiv:2009.13365.

[3] S. Basu and N. Karisani, *Computing the homology functor on semi-algebraic maps and diagrams*, Discrete & Computational Geometry **72** (2024), no. 4, 1437–1462. doi:10.1007/s00454-024-00627-z; arXiv:2207.10497.

[4] A. Beauville, *The action of* SL_2 *on abelian varieties*, Journal of the Ramanujan Mathematical Society **25** (2010), no. 3, 253–263. Author's published text; arXiv:0805.1541. Sections 1.6 and 2.1–2.2, on the kernel sign and the modular central extension; Section 2.4, on rational Chow correspondences.

[5] C. Berger and B. Fresse, *Combinatorial operad actions on cochains*, Mathematical Proceedings of the Cambridge Philosophical Society **137** (2004), 135–174. doi:10.1017/S0305004103007138; arXiv:math/0109158.

[6] A. K. Bousfield and V. K. A. M. Gugenheim, *On PL de Rham theory and rational homotopy type*, Memoirs of the American Mathematical Society **8** (1976), no. 179, ix+94 pp. doi:10.1090/memo/0179.

[7] L. Breen, *Notes on* 1- *and* 2-*gerbes*, 2006, arXiv:math/0611317. Sections 4–6.

[8] E. H. Brown, Jr., *Finite computability of Postnikov complexes*, Annals of Mathematics (2) **65** (1957), no. 1, 1–20. doi:10.2307/1969664.

[9] G. Brumfiel, A. Medina-Mardones and J. Morgan, *A cochain level proof of Adem relations in the mod 2 Steenrod algebra*, Journal of Homotopy and Related Structures **16** (2021), 517–562. doi:10.1007/s40062-021-00287-3; arXiv:2006.09354, Section 0.1.1.

[10] J. L. Bryant, *Piecewise linear topology*, in R. J. Daverman and R. B. Sher (eds.), *Handbook of Geometric Topology*, North-Holland, Amsterdam, 2002, 219–259. Chapter text, Section 2, on subdivisions and the PL category.

[11] M. Čadek, M. Krčál, J. Matoušek, L. Vokřínek and U. Wagner, *Polynomial-time computation of homotopy groups and Postnikov systems in fixed dimension*, SIAM Journal on Computing **43** (2014), 1728–1780. doi:10.1137/120899029; arXiv:1211.3093.

[12] B. Casselman, *Compact groups as algebraic groups*, notes, revised 20 August 2015. Author's text. Lemma 1.6 and Theorem 2.5.

[13] G. R. Cavalcanti, *Formality of k-connected spaces in $4k+3$ and $4k+4$ dimensions*, Mathematical Proceedings of the Cambridge Philosophical Society **141** (2006), no. 1, 101–112. doi:10.1017/S0305004106009340; Author's text. Example 1, on nonformal sphere bundles.

[14] G. d'Antonio and E. Delucchi, *A Salvetti complex for toric arrangements and its fundamental group*, arXiv:1101.4111, Definition 7, Proposition 1.2, Lemma 3.3 and Corollary 3.4.

[15] E. Delucchi and K. Knauer, *Finitary affine oriented matroids*, Discrete & Computational Geometry **73** (2025), 208–257. doi:10.1007/s00454-024-00651-z. Sections 1.2 and 5.

[16] T. K. Dey, A. N. Hirani and B. Krishnamoorthy, *Optimal homologous cycles, total unimodularity, and linear programming*, SIAM Journal on Computing **40** (2011), 1026–1044. arXiv:1001.0338.

[17] T. Dimofte, M. Gabella and A. B. Goncharov, *K-decompositions and 3d gauge theories*, Journal of High Energy Physics **2016** (2016), no. 11, article 151. doi:10.1007/JHEP11(2016)151; arXiv:1301.0192. Sections 2.5–2.6 and 4.2.

[18] A. Dold and R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Annals of Mathematics (2) **67** (1958), 239–281. doi:10.2307/1970005; Article text. Satz 2.2 and its corollary (p. 245), and Theorems 6.10 and 7.1.

[19] V. Drinfeld, *On the notion of geometric realization*, arXiv:math/0304064, Section 1.

[20] W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, Journal of Pure and Applied Algebra **18** (1980), no. 1, 17–35. doi:10.1016/0022-4049(80)90113-9; Article text. Section 7, Propositions 7.2–7.3, including the right-fraction dual.

[21] W. G. Dwyer and D. M. Kan, *Simplicial localizations of categories*, Journal of Pure and Applied Algebra **17** (1980), no. 3, 267–284. doi:10.1016/0022-4049(80)90049-3; Article text. Sections 1–2, on retaining higher coherence when weak equivalences are inverted.

[22] T. Dyckerhoff and M. Kapranov, *Crossed simplicial groups and structured surfaces*, in *Stacks and Categories in Geometry, Topology, and Algebra*, Contemporary Mathematics **643** (2015), 37–110. doi:10.1090/conm/643/12896; arXiv:1403.5799. Theorems I.7 and I.33, and Table 2.

[23] J. Ebert and O. Randal-Williams, *Semisimplicial spaces*, Algebraic & Geometric Topology **19** (2019), no. 4, 2099–2150. doi:10.2140/agt.2019.19.2099. Section 1.1, Lemma 1.8, Theorem 2.2, and Section 7.

[24] Y. Félix and S. Halperin, *Rational homotopy theory via Sullivan models: a survey*, ICCM Notices **5** (2017), no. 2, 14–36. arXiv:1708.05245. Sections 2, 5–6, and 14; the elementary-principal-fibration model is Section 14, Example 1, p. 27 of the preprint.

[25] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics 205, Springer, 2001. doi:10.1007/978-1-4613-0105-9. Theorem 9.11; Section 15(b),

Examples 4–5, pp. 202–203; Proposition 15.8; Section 15(f), Example 2, p. 219.

[26] Z. Fiedorowicz and J.-L. Loday, *Crossed simplicial groups and their associated homology*, Transactions of the American Mathematical Society **326** (1991), no. 1, 57–87. doi:10.1090/S0002-9947-1991-0998125-4.

[27] *Finite addition and delooping*, accompanying unpublished research assessments, versions 03–05, 12 October 2026, with exact verification scripts and records. The binary-cocycle connectivity and strict-primitives results cited in the appendices belong to these companion calculations; they are not inputs to the main theorems of this note.

[28] R. Fritsch and R. A. Piccinini, *Cellular Structures in Topology*, Cambridge Studies in Advanced Mathematics, vol. 19, Cambridge University Press, 1990. doi:10.1017/CBO9780511983948. Corollary 4.6.12, on triangulating simplicial realizations.

[29] N. Ganter, *Categorical Tori*, SIGMA **14** (2018), 014, 18 pp. doi:10.3842/SIGMA.2018.014. Constructions 2.1–2.3, Proposition 2.4, and Theorem 4.1; the sign in the latter theorem is used in the present comparison.

[30] A. Garbagnati and A. Sarti, *Kummer surfaces and K3 surfaces with* $(\mathbb{Z}/2\mathbb{Z})^4$ *symplectic action*, Rocky Mountain Journal of Mathematics **46** (2016), no. 4, 1141–1205. doi:10.1216/RMJ-2016-46-4-1141; arXiv:1305.3514. Section 2, on the quotient of an abelian surface and its K3 resolution.

[31] I. M. Gelfand and R. D. MacPherson, *A combinatorial formula for the Pontrjagin classes*, Bulletin of the American Mathematical Society (N.S.) **26** (1992), no. 2, 304–309. doi:10.1090/S0273-0979-1992-00282-3; arXiv:math/9204231.

[32] S. Ghorbal and B. Jessup, *Estimating the rational LS-category of elliptic spaces*, Proceedings of the American Mathematical Society **129** (2001), no. 6, 1833–1842. doi:10.1090/S0002-9939-00-05739-7. Example 2 (p. 7 of the author manuscript) gives the cubic pure minimal algebra used here, with abelian homotopy Lie algebra.

[33] A. B. Goncharov, *Exponential complexes, period morphisms, and characteristic classes*, Annales de la Faculté des Sciences de Toulouse.

Mathématiques (6) **25** (2016), no. 2–3, 619–681. Journal text. Sections 4.1–4.3.

[34] R. Gonzales, *Rational smoothness, cellular decompositions and GKM theory*, Geometry & Topology **18** (2014), no. 1, 291–326. doi:10.2140/gt.2014.18.291; Journal text. Section 5, p. 309, recalls the equivariant Euler factorization and attributes it to Hsiang's Theorem IV.6.

[35] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002. Proposition 0.17, Example 3E.2, Section 4.3 (including Exercise 19), and Section 4.L. Author-hosted text.

[36] A. Hatcher, *Vector Bundles and K-Theory*, version 2.2, November 2017. Author's text. Theorems 3.2 and 3.9, Propositions 3.10 and 3.13.

[37] R. E. Heisey, *Manifolds modelled on the direct limit of lines*, Pacific Journal of Mathematics **102** (1982), no. 1, 47–54. Journal text. Theorem C, on replacing a prescribed homotopy equivalence by a homotopic homeomorphism.

[38] K. Hess, *Rational Homotopy Theory: A Brief Introduction*, preprint, 2006. arXiv:math/0604626, version 2. Proposition 1.8 and Theorem 2.4 in this version.

[39] R. M. Kaufmann and A. M. Medina-Mardones, *Cochain level May–Steenrod operations*, Forum Mathematicum **33** (2021), no. 6, 1507–1526. doi:10.1515/forum-2020-0296; arXiv:2010.02571, Theorem 24 and Example 27.

[40] R. Krasauskas, *Skew-simplicial groups*, Lithuanian Mathematical Journal **27** (1987), no. 1, 47–54. doi:10.1007/BF00972021; English translation.

[41] K.-W. Lan, *Arithmetic compactifications of PEL-type Shimura varieties*, Ph.D. thesis, Harvard University, 2008; author's revision, 14 March 2021. Revised thesis. Propositions 3.2.1.7 and 3.2.3.1, on Poincaré biextensions and cubical structures.

[42] J. Lurie, *Kerodon*, Example 3.1.7.11, Tag 03DY; Section 3.3, the extension functor, especially Theorem 3.3.5.1 and Proposition 3.3.6.9; Proposition 3.6.1.9 and Corollary 3.6.1.10 on finite simplicial sets and filtered colimits. Accessed 12 October 2026.

[43] J. P. May, *Simplicial Objects in Algebraic Topology*, Van Nostrand Mathematical Studies 11, D. Van Nostrand, 1967. Definition 18.3 and Propositions 18.4, 18.7; Sections 23–25, especially Theorems 23.9–23.10 and Lemmas 24.2–24.3. Author-hosted scan.

[44] J. P. May, *The Geometry of Iterated Loop Spaces*, Lecture Notes in Mathematics, vol. 271, Springer, 1972. Author's text. Corollary 11.6, on realization of fibre products.

[45] A. M. Medina-Mardones, *A finitely presented E_∞-prop I: algebraic context*, Higher Structures **4** (2020), no. 2, 1–21. doi:10.21136/HS.2020.08; arXiv:1808.00854.

[46] A. M. Medina-Mardones, *New formulas for cup-i products and fast computation of Steenrod squares*, arXiv:2105.08025v3 (2022), Definition 7 and Theorem 10.

[47] A. M. Medina-Mardones, *The diagonal of cellular spaces and effective algebro-homotopical constructions*, EMS Surveys in Mathematical Sciences **10** (2023), no. 2, 223–241. doi:10.4171/EMSS/71; arXiv:2111.06164, Sections 3.2–3.8.

[48] J. S. Milne, *Jacobian Varieties*, in G. Cornell and J. H. Silverman (eds.), *Arithmetic Geometry*, Springer, 1986, 167–212; revised notes, 12 June 2021. Author's revised text. Sections 5–6, especially Remark 5.6(c).

[49] J. S. Milne, *Abelian Varieties*, course notes, version 2.0 (2008). Author's text. Chapter I, Section 2 and Theorem 7.2, on complex tori and isogenies. Sections 5 and 8–9, on the cube theorem, Poincaré lines, and extensions by $\mathbb{G}_m$.

[50] J. S. Milne, *Elliptic Curves*, second edition, World Scientific, 2021. `https://www.jmilne.org/math/Books/EC2.pdf`. Chapter III, Proposition 3.17 and Example 3.18.

[51] N. Mnev, *$K(\mathbb{Z}, 2)$ out of circular permutations*, arXiv:2406.01625v2, 2024.

[52] S. Mukai, *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya Mathematical Journal **81** (1981), 153–175. doi:10.1017/S002776300001922X. Theorem 2.2 and equation (3.1).

[53] D. Mumford, *On the equations defining abelian varieties. I*, Inventiones Mathematicae **1** (1966), no. 4, 287–354. doi:10.1007/BF01389737.

[54] B. A. Munson and I. Volić, *Cubical Homotopy Theory*, New Mathematical Monographs, vol. 25, Cambridge University Press, 2015. Author's text. Section 2.7, especially Proposition 2.7.7 and Definition 2.7.12–Proposition 2.7.13.

[55] M. K. Murray and D. Stevenson, *Bundle gerbes: stable isomorphism and local theory*, Journal of the London Mathematical Society (2) **62** (2000), 925–937. arXiv:math/9908135.

[56] T. Nikolaus and K. Waldorf, *Lifting problems and transgression for non-abelian gerbes*, Advances in Mathematics **242** (2013), 50–79. doi:10.1016/j.aim.2013.03.022; arXiv:1112.4702. Theorem 4.2.7 and Corollary 4.2.8 of the published version.

[57] T. Ohmoto and M. Shiota, *C^1-triangulations of semialgebraic sets*, Journal of Topology **10** (2017), no. 3, 765–775. doi:10.1112/topo.12024. Theorem 2.2, recalling compatible semialgebraic triangulation.

[58] D. O. Orlov, *Derived categories of coherent sheaves on abelian varieties and equivalences between them*, Izvestiya: Mathematics **66** (2002), no. 3, 569–594. doi:10.1070/IM2002v066n03ABEH000389; arXiv:alg-geom/9712017. Theorem 2.19 and Proposition 2.21.

[59] A. Polishchuk, *A remark on the Fourier–Mukai transform*, Mathematical Research Letters **2** (1995), no. 2, 193–202. Publisher's text. Theorem 3.1, on a coherent action of the modular central extension.

[60] A. Polishchuk, *Symplectic biextensions and a generalization of the Fourier–Mukai transform*, Mathematical Research Letters **3** (1996), no. 6, 813–828. doi:10.4310/MRL.1996.v3.n6.a9. Definition 2.1 and the following remark.

[61] A. Polishchuk, *Abelian Varieties, Theta Functions and the Fourier Transform*, Cambridge Tracts in Mathematics **153**, Cambridge University Press, 2003. doi:10.1017/CBO9780511546532. Section 1.2, Theorem 1.3.

[62] R. D. Porter and A. I. Suciu, *Differential graded algebras, Steenrod cup-one products, binomial operations, and Massey products*, arXiv:2105.10753v2 (2021), equations (3.1), (3.2), and (4.5).

[63] D. Quillen, *The geometric realization of a Kan fibration is a Serre fibration*, Proceedings of the American Mathematical Society **19** (1968), 1499–1500. doi:10.1090/S0002-9939-1968-0238322-1.

[64] D. L. Rector, *Loop structures on the homotopy type of S^3*, in *Symposium on Algebraic Topology*, P. J. Hilton (ed.), Lecture Notes in Mathematics 249, Springer, Berlin–Heidelberg, 1971, 99–105. doi:10.1007/BFb0060900. The theorem on p. 99 and the cell description on p. 100.

[65] C. Rezk, *When are homotopy colimits compatible with homotopy base change?*, research note, 2 February 2014. Author's text. Definition 1.1 and Propositions 2.3(2) and 5.4.

[66] D. M. Roberts and D. Stevenson, *Simplicial principal bundles in parametrized spaces*, New York Journal of Mathematics **22** (2016), 405–440. Journal text, Theorem 2.

[67] S. Saito, *On the geometric realization and subdivisions of dihedral sets*, Algebraic & Geometric Topology **13** (2013), 1071–1087. doi:10.2140/agt.2013.13.1071; arXiv:1012.0380. Theorem 1.5, on edgewise subdivision.

[68] K. Sakai, *Topology of Infinite-Dimensional Manifolds*, Springer Monographs in Mathematics, Springer, Singapore, 2020. doi:10.1007/978-981-15-7575-4. Chapter 6, *Manifolds modeled on direct limits and combinatorial ∞-manifolds*; author's March 2020 chapter preprint. See also the author's Chapter 1 preprint. We use the numbering in these preprints: Theorem 1.12.8, Proposition 6.1.3(2), Corollary 6.4.3, and Theorems 6.4.8, 6.5.1, 6.10.1, 6.11.1, and 6.12.2, and Proposition 6.5.6.

[69] C. J. Schommer-Pries, *Central extensions of smooth 2-groups and a finite-dimensional string 2-group*, Geometry & Topology **15** (2011), no. 2, 609–676. doi:10.2140/gt.2011.15.609.

[70] S. Schwede, *Stable homotopical algebra and Γ-spaces*, Mathematical Proceedings of the Cambridge Philosophical Society **126** (1999), 329–356. Author's version.

[71] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312, Propositions 1.4–1.5. Original article scan.

[72] A. Semidetnov, *The operad associated to a crossed simplicial group*, preprint (2025), arXiv:2512.14498v1. Definitions 2.1–2.4, Theorem 3.6, and Remark 3.7.

[73] J.-P. Serre, *Homologie singulière des espaces fibrés. Applications*, Annals of Mathematics (2) **54** (1951), 425–505. doi:10.2307/1969485.

[74] N. E. Steenrod, *Products of cocycles and extensions of mappings*, Annals of Mathematics (2) **48** (1947), 290–320. doi:10.2307/1969172.

[75] D. Stevenson, *Bundle 2-gerbes*, Proceedings of the London Mathematical Society (3) **88** (2004), no. 2, 405–435. doi:10.1112/S0024611503014357; arXiv:math/0106018.

[76] D. Sullivan, *Infinitesimal computations in topology*, Publications Mathématiques de l'IHÉS **47** (1977), 269–331. doi:10.1007/BF02684341; Primary scan. Theorems 7.1–7.2, 8.1, and 10.1.

[77] *The uniform oriented-matroid bi-Grassmannian: deletion–contraction geometry, sphere bundles, and Pontryagin classes*, accompanying unpublished discussion draft, 9 October 2026, section on configurations, flags, and cyclic atlases; with the corrected technical supplements *The UOM left arrow from BO* and *The UOM right arrow to BPL*, second rewrites. The present note restates the flag definition and proves its comparison statements independently of a general bundle descent claim for coarse flags.

[78] J. Voight, *Quaternion Algebras*, Graduate Texts in Mathematics **288**, Springer, 2021. doi:10.1007/978-3-030-56694-4. Section 43.4.15, on the elementary divisors of a polarization.

[79] J. H. C. Whitehead, *Note on a theorem due to Borsuk*, Bulletin of the American Mathematical Society **54** (1948), 1125–1132. doi:10.1090/S0002-9904-1948-09138-8. Theorem 1, on adjunctions of compact ANRs.

[80] J. Zhou, *Formality of sphere bundles*, Algebraic & Geometric Topology **25** (2025), 2297–2315. doi:10.2140/agt.2025.25.2297; arXiv:2304.09594. Theorem 1.1, on sphere bundles over formal bases.

[81] C. Zhu, *n-Groupoids and Stacky Groupoids*, International Mathematics Research Notices **2009** (2009), no. 21, 4087–4141. doi:10.1093/imrn/rnp080. Section 4.1.

[82] L. Zoller, *Torus equivariant algebraic models and compact realization*, Annales de l'Institut Fourier **75** (2025), no. 1, 67–124. doi:10.5802/aif.3618. Definition 2.14, Remark 2.15, Theorem 5.5, and Example 6.3.