$K(\mathbb{Z}, 2)$ out of circular permutations

Nikolai Mnëv*
mnev@pdmi.ras.ru

April 7, 2024

Abstract

We discuss $\mathbf{SC}$, a simplicial homotopy model of $K(\mathbb{Z}, 2)$ constructed from circular permutations. In any dimension, the number of simplices in the model is finite. The complex $\mathbf{SC}$ manifests naturally as a simplicial set representing "minimally" triangulated circle bundles over simplicial bases. On the other hand, the homotopy $|\mathbf{SC}| \approx B(U(1)) \approx K(\mathbb{Z}, 2)$ appears to be a canonical fact from the foundations of crossed simplicial groups theory.

1. Introduction

The note essentially continues the note [Mnë20]. In that note ([Mnë20, §§ 3.6, 3.7]), we identify circular permutations of $k+1$ ordered elements with "minimal" semi-simplicial triangulations of trivial circle bundles over ordered base k-simplices. Any semi-simplicial triangulation of a circle bundle is non-canonically combinatorially concordant to a minimal triangulation (i.e., having minimal triangulations over

*PDMI RAS; Chebyshev Laboratory, SPbSU. Research is supported by the Russian Science Foundation grant 19-71-30002.

all the simplices of the same base complex), and the simplicial set $\mathbf{SC}$ of circular permutations naturally represents minimally triangulated circle bundles over semi-simplicial complexes. Such triangulations functorially (by Kan's second derived subdivision Sd_2) have a structure of a classical simplicial PL triangulation. But the *minimal* triangulations exist only in the semi-simplicial category.

Our simple observation is that the complex $\mathbf{SC}$ shows up canonically as the simplicial right coset complex of the cyclic crossed simplicial subgroup $\mathbf{C}$ in symmetric crossed simplicial group $\mathbf{S}$, providing the sequence

$$\mathbf{C} \to \mathbf{S} \xrightarrow{\circlearrowleft} \mathbf{SC} \tag{1}$$

Here we discuss a proof of a natural conjecture that $\mathbf{SC}$ is a homotopy model of $\mathsf{K}(\mathbb{Z}, 2)$. To the author's limited knowledge, $\mathbf{SC}$ is the first simplicial model of $\mathsf{K}(\mathbb{Z}, 2)$ having a finite number of simplices in every dimension. This fact probably makes the simplicial set $\mathbf{SC}$ interesting. The situation is a non-direct relative of the well-known subject of triangulating $\mathbb{C}\mathrm{P}^n$. See [MY91, AM91] and the new results [DS24]. There are interesting computer experiments [Ser10]. The connections of these achievements with our construction have to be investigated. Probably the connection is by minimal triangulation of the tautological Hopf bundle $\mathsf{U}(1) \to S^{2n+1} \to \mathbb{C}\mathrm{P}^n$. The fact

$$|\mathbf{SC}| \approx \mathsf{K}(\mathbb{Z}, 2)$$

can be deduced from the very basics of crossed simplicial groups theory ([FT87, Kra87, FL91, Lod98]...).

Theorem 1.

$$|\mathbf{SC}| \approx \mathsf{K}(\mathbb{Z}, 2).$$

Plan of the proof. The miracle of geometric realization of crossed simplicial groups makes $|\mathbf{C}| = \mathsf{U}(1)$, $|\mathbf{S}|$ a contractible topological group, and $|\mathbf{SC}|$ a coset space of the subgroup $\mathrm{U}(1)$ in $|\mathbf{S}|$. The geometric realization of the sequence (1) becomes a principal fibration. Therefore $|\mathbf{SC}| \approx B\mathrm{U}(1) \approx K(\mathbb{Z}, 2)$. □

In this note, we decrypt the presented standard plan of the proof of Theorem 1 using quotes (and copy-pasts) from classic and more modern references. The author is deeply grateful to Boris Tsygan and André Henriques for the enlightening comments in social networks.

References

[AM91] Pierre Arnoux and Alexis Marin. The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. II. *Mem. Fac. Sci., Kyushu Univ., Ser. A*, 45(2):167–244, 1991.

[DS24] Basudeb Datta and Jonathan Spreer. Simplicial cell decompositions of $\mathbb{CP}^n$, 2024.

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. Crossed simplicial groups and their associated homology. *Trans. Am. Math. Soc.*, 326(1):57–87, 1991.

[FT87] B. L. Feigin and B. L. Tsygan. Additive K-theory. K-theory, arithmetic and geometry, Semin., Moscow Univ. 1984-86, Lect. Notes Math. 1289, 67-209 (1987)., 1987.

[Kra87] R. Krasauskas. Skew-simplicial groups. *Lith. Math. J.*, 27(1):47–54, 1987.

[Lod98] Jean-Louis Loday. *Cyclic homology. 2nd ed.* Berlin: Springer, 2nd ed. edition, 1998.

[Mnë20] N. Mnëv. Minimal triangulations of circle bundles, circular permutations, and the binary Chern cocycle. *J. Math. Sci., New York*, 247(5):696–710, 2020.

[MY91] Bernard Morin and Masaaki Yoshida. The Kühnel triangulation of the complex projective plane from the view point of complex crystallography. I. *Mem. Fac. Sci., Kyushu Univ., Ser. A*, 45(1):55–142, 1991.

[Ser10] Francis Sergeraert. Triangulations of complex projective spaces. In *Contribuciones científicas en honor de Mirian Andrés Gómez.*, pages 507–519. Logroño: Universidad de La Rioja, Servicio de Publicaciones, 2010.