THE UNIVERSALITY THEOREMS ON THE CLASSIFICATION PROBLEM OF CONFIGURATION VARIETIES AND CONVEX POLYTOPES VARIETIES

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The results which we present here form the part of guiding by A.M. Vershic topological investigations of combinatorially defined configuration spaces (see his article in this volume). In this paper we shall outline the proof of coincidence of the two classes of variety: first - the spaces of point configurations in $P_{\mathbb{R}}^2$ having certain oriented combinatorial type, the second - all semi-algebraic varieties over rational numbers. (The oriented combinatorial type of point configurations is a dual object to the well-known combinatorial type of hyperplane arrangements (see [G2]).) As a corollary we obtain the similar fact concerning the spaces of convex polytopes of a fixed combinatorial type. The complete proof of these results is contained in the authors thesis [M] .

1°. Introduction

By a projective point configuration we mean an ordered finite set of (not necessarily different) points of the projective space $P_{\mathbb{R}}^d$. It is natural to identify the space of all projective configurations of n points in $P_{\mathbb{R}}^d$ with $(P_{\mathbb{R}}^d)^n$. Two configurations $X, Y \in (P_{\mathbb{R}}^d)^n$ are said to be <u>combinatorially</u> <u>equivalent</u> provided for arbitrary subset $S \subset \overline{1:n}$ the subconfigurations $\{x_i\}_{i\in S} \subset X$ and $\{y_i\}_{i\in S} \subset Y$ generate the projective subspaces of equal dimensions. So we obtain the partition $Tp(n,d)$ of the space $(P^d_{\mathbb{R}})^n$ into the combinatorial equivalence classes. A combinatorial type of n-point d-configurations is a stratum of $Tp(n,d)$ (an equivalence class of such a partition considering as a variety is called stratum). The combinatorial types of the projective point configurations are naturally subdivised on the oriented combinatorial types. Two point configurations are orientedly combinatorial equivalent if its dual ordered hyperplane configurations can be translated one to another by the homeomorphism of $P^d_{\mathbb{R}}$. So we obtain the partition $Tpo(n,d) \succ Tp(n,d)$ of the space $(P^d_{\mathbb{R}})^n$ into the oriented combinatorial types.

Now we assume that in $P^d_{\mathbb{R}}$ a certain projective basis ($d+2$ points in general position) is chosen . Basis configuration is a point configuration which have the fixed basis as the subconfiguration of its first $d+2$ points. On the space $bc(n,d) \subset (P^d_{\mathbb{R}})^n$ of all basis n-point d-configurations $(n \geqslant d+2)$ the partitions $Tp(n,d)$ and $Tpo(n,d)$ induce the partitions $Tb(n,d)$ and $Tbo(n,d)$ into basis combinatorial types and basis oriented combinatorial types respectively. One can consider a basis (oriented) combinatorial type as a factorspace of the corresponding (oriented) combinatorial type by the free action of $PGL_d(\mathbb{R})$. With the help of the non-homogeneous coordinates on $P^d_{\mathbb{R}}$ connected with the fixed basis we can to identify a basis combinatorial type $\alpha \in Tb(n,d)$ with a subvariety of the space of $\mathbb{R}^{d+1}$-vector n-tuples. Orientation of $\mathbb{R}^{d+1}$ induce the orientation of $(d+1)$-tuples of points of the configurations belonging to α. Fixing the orientations of all the $(d+1)$-tuples we obtain precisely the partition of α into the basis oriented combinatorial types. Two convex polytopes with odered vertices are said to be combinatorial equivalent if the order-preserving correspondence between the vertices induces the isomorphism of the face-lattices. Combinatorial equivalence determines the partition $Tpol(n,d)$

of the space $\mathrm{Pol}(n,d) \subset (\mathbb{R}^d)^n$ of all convex d-polytopes with n odered vertices. A combinatorial type of convex polytopes is a stratum of $\mathrm{TPol}(n,d)$. Combinatorial types of generic configurations and combinatorial types of generic (i.e. simplicial) polytopes we call generic. The topological structure of generic combinatorial types is of the extreme interest.

One can easily establish that every basis oriented combinatorial type is a primary semi-algebraic, defined over $\mathbb{Q}$ (i.e. determined by polynomial equalities and strict unequalities with rational coefficients) subset of a principal affine set in $bc(n,d)$. The same is true in the case of polytopes. As we shall see the converse (modulo sertain stabilization) may be proved:

THEOREM A.

1) For every natural k, d $(d \geqslant 2)$, every primary semi-algebraic subset M of $\mathbb{R}^k$, defined over $\mathbb{Q}$ there exist a natural n and basis oriented combinatorial type ζ of projective n-point d-configurations which is stable equivalent to . (Two semi-algebraic varieties A, B are stable equivalent if there is a piecewise biregular homeomorphism between A and $B \times \mathbb{R}^i$ for a certain natural i)

2) If M is an open subset of $\mathbb{R}^k$ then the type ζ may be chosen to be generic.

The similar fact concerning polytopes is a consequence of the Theorem A and the Gale's duality:

THEOREM B.

1) For every natural k, m $(m \geqslant 4)$, every primary semi-algebraic subset M of $\mathbb{R}^k$ defined over $\mathbb{Q}$ their exist a natural d and a combinatorial type of d-polytopes with $d+m$ vertices ξ which is stable equivalent to $M \times GL_d(\mathbb{R})$

2) If M is an open subset of $\mathbb{R}^k$ than the type ξ may be chosen to be simplicial.

It should be mentioned that every combinatorial type of 3-polytopes ([S.R.]) and the combinatorial types of d-polytopes with $d+3$ odered vertices are topologically trivial (i.e. stable equivalent to $GL_3(\mathbb{R})$ and $GL_d(\mathbb{R})$ respectively). The combinatorial type of -polytopes with 10 vertices constructed in [B.E.K.] is the minimal known example of disconnected (modulo $GL_4(\mathbb{R})$) polytope combinatorial type. Every generic basis oriented combinatorial type of n-point 2-configurations is trivial when $n \leq 7$ (see the paper by S.Finashin in this volume). The author has constructed the example of disconnected type for $n = 19$ (see p.6).

Obviously, it is sufficient to prove the Theorems A and B for the cases of point 2-configurations and d-polytopes with $d+4$ vertices, respectively. According to which the term configuration further means a configuration of points of $P^2_{\mathbb{R}}$.

2°. Before we proceed to the proof of the Theorem A it is necessary to introduce some new objects.

2.1. <u>A computation of rational map. Partition of the map's domain of definition generated by a computation of this map.</u>

Let $\mathbb{F}$ be a field and let $\mathcal{A}$ be a subset of $\mathbb{F}$. Consider the sets of words $\mathfrak{A}_i$, $i=1,2,\ldots$ over the alphabet $\mathcal{A} \cup \{(,)\} \cup \mathcal{F}$ where $\mathcal{F} = \{+,-,\times,:\}$, $\mathfrak{A}_1 = \mathcal{A}$, $\ldots$ $\mathfrak{A}_i = \{(A \circ B) \mid A, B \in \mathfrak{A}_{i-1}, \circ \in \mathcal{F}\}, \ldots$. Put $\mathfrak{A}(\mathcal{A}) = \lim \mathfrak{A}_i$.

The set of words $\mathfrak{A}(\mathcal{A})$ with the natural action of the operations from $\mathcal{F}$ is called free algebra of words (see [B]). Let $\sigma_{\mathcal{A},\mathbb{F}} : \mathfrak{A}(\mathcal{A}) \to \mathbb{F} \cup \{\infty\}$ be the map of "removing the parenthesis" and let f be an element of $\mathbb{F}$. A <u>computation</u> of f in $\mathfrak{A}(\mathcal{A})$ is an arbitrary word from $\sigma^{-1}_{\mathcal{A},\mathbb{F}}(f) \in \mathfrak{A}(\mathcal{A})$. For $A \in \mathfrak{A}(\mathcal{A})$ denote by $SW(A)$ the set of all subwords of A belonging to $\mathfrak{A}(\mathcal{A})$, $SF(A) = \sigma_{\mathcal{A},\mathbb{F}}(SW(A)) \subset \mathbb{F}$. Let $\mathfrak{X} = (X_1, \ldots, X_k)$ be a k-tuple of independent variables, $\mathcal{A} \subset \mathbb{F}(\mathfrak{X})$. A <u>computation</u>

<u>of rational vector-function</u> $f=(f_1,\dots,f_m)\in \mathbb{F}^m(\mathfrak{X})$ in $\mathcal{O}\!\ell(\mathcal{A})$ is a collection of words $\Psi=(\psi_1,\dots,\psi_m)\in \mathcal{O}\!\ell^m(\mathcal{A})$ where ψ_i is a computation of f_i for $i\in\overline{1:m}$. We denote by $SW(\Psi)$ the set $\bigcup_{i=1}^{m} SW(\psi_i)\subset \mathcal{O}\!\ell(\mathcal{A})$ and by $SF(\Psi)$ the set $\bigcup_{i=1}^{m} SF(\psi_i)=$ $=\sigma_{\mathcal{A},\mathbb{F}(\mathfrak{X})}(SW(\Psi))\subset\mathbb{F}$. The computation Ψ of f is called <u>formula</u> if the map $\sigma_{\mathcal{A},\mathbb{F}(\mathfrak{X})}: SW(\Psi)\longrightarrow SF(\Psi)$ is a bijection.

Let $\mathbb{F}\subset\mathbb{R}$. In this case let $reg(f,\Psi)=\{x\in\mathbb{R}^k \mid$ all the functions from $SF(\Psi)$ are regular at $x\}$. Consider the equivalence relation $\underset{\Psi}{\sim}$ on $reg(f,\Psi)$: $x\underset{\Psi}{\sim}y$ if and only if $sign(v(x)-u(x))=sign(v(y)-u(y))$ for all pairs $u,v\in SF(\Psi)$, $u\neq v$. The partition of $reg(f,\Psi)$ determined by $\underset{\Psi}{\sim}$ we denote by $\Sigma(f,\Psi)$.

2.2. <u>The biregular imbedding $P^{(f,\Psi)}: reg(f,\Psi)\times\mathbb{R}^{\overline{SW}(\Psi)}\longrightarrow$ (the space of basis configurations)</u>.

Let f be a vector-function from $\mathbb{Q}^m(X)$, $X=(x_1,\dots,x_k)$, and let Ψ be a computation of f in $\mathcal{O}\!\ell(0,1,X)$. Put $\overline{SW}(\Psi)=SW(\Psi)\setminus\{0,1,X\}$. Let $x=(x_1,\dots,x_k)\in reg(f,\Psi)\subset\mathbb{R}^k$, $b=\{b_A\}_{A\in\overline{SW}(\Psi)}\in(\mathbb{R}^*)^{\overline{SW}(\Psi)}$, where $\mathbb{R}^*=\mathbb{R}\setminus\{0\}$. Consider the basis configuration $P(x,b)$ presented by Fig. 1. Here $\{p_\alpha, p_0, p_E, p_\infty\}$ is a fixed projective basis. On the line $L_{\mathbb{R}}=p_0\cdot p_\infty$ the points $p_{x_1}(x),\dots,p_{x_k}(x)$ are marked such that $R(p_{x_i}(x),p_1;p_0,p_\infty)=x_i$ for $i\in\overline{1:k}$ where $R(\ ,\ ;\ ,\ )$ is a cross-ratio, $p_1=L_{\mathbb{R}}\cap(p_\alpha\cdot p_E)$. On the line $L_0=p_0\cdot p_\alpha$ the points $\{p_{(0,A)}(b)\}_{A\in\overline{SW}(\Psi)}$ are marked such that $R(p_{(0,A)}(b),p_{(0,1)};p_0,p_\alpha)=$ $=b_A$ for $A\in\overline{SW}(\Psi)$, where $p_{(0,1)}=$ $=L_0\cap(p_\infty\cdot p_E)$. Consider the constructions of sum, difference, product and quotent of

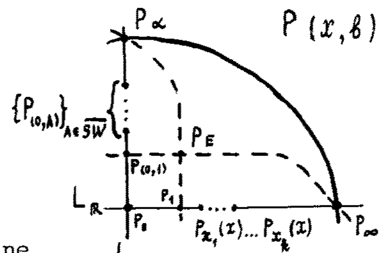

Fig.1

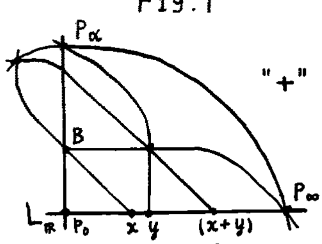

Fig.2a

$L_{\mathbb{R}}$ -points defined by Fig.2. By iterating of these constructions according to increasing of the Ψ -subwords we can supplement the configuration $P(x,b)$ to the configuration $P^{(f,\Psi)}(x,b)$ which has points $p_{\psi_i}(x)\in L_{\mathbb{R}}$ such that $R(p_{\psi_i}(x),p_1;p_0,p_\infty)=f_i(x)$ for $i\in\overline{1:m}$. At the step corresponding to a subword $A\in\overline{SW}(\Psi)$ the point $p_A(x)\in L_{\mathbb{R}}$, $R(p_A(x),p_1;p_0,p_\infty)=(\sigma_{Q(x)}(A))(x)$ is constructed. As a point with the index b at the primary construction, corresponding to $A\in\overline{SW}(\Psi)$ the point $p_{(0,A)}(b)$ is chosen. So we obtain the biregular imbedding $P^{(f,\Psi)}$: $reg(f,\Psi)\times(\mathbb{R}^*)^{\overline{SW}(\Psi)}\longrightarrow bc(S^{\Psi},2)$, where S^{Ψ} is a naturally defined set of indeces,

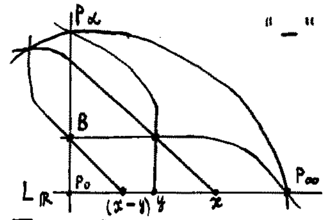

Fig.2b

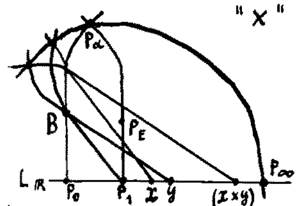

Fig.2c

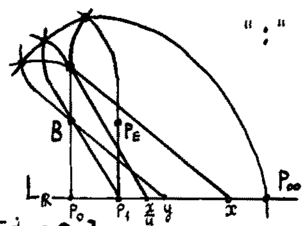

Fig.2d

$$card(S^{\Psi})=card(SW(\Psi))+4card(SW^{+}(\Psi))+5card(SW^{\times}(\Psi))+3,$$
$$SW^{+}(\Psi)=\{A\in SW(\Psi)\mid A=(A_1\pm A_2)\},$$
$$SW^{\times}(\Psi)=\{A\in SW(\Psi)\mid A=(A_1\overset{\times}{:}A_2)\}$$

The image of $P^{(f,\Psi)}$ is a union of entire basis combinatorial types.

2.3. Free basis configurations

Let C be a basis configuration, B is a fixed projective basis, $B\subset C$, $card(B)=4$. Denote by $\mathcal{L}_p(C)$ the set of all projective lines which are incident with more then two points of C . Let $x\in C$. Put $V_c(x)=card(\{\ell\in\mathcal{L}_p(C)\mid x\in\ell\})$.
The point x is said to be <u>proper</u> if $V_c(x)\geqslant 3$, otherwise is <u>unproper</u> point of C . In the latter case C is said to be a <u>free extension</u> of the subconfiguration $C\setminus\{x\}$. The configuration

C is said to be free if it is possible to order the points of $C\setminus B$ $(C\setminus B=(c_1,\dots,c_k))$ in such a way that the configuration $B\cup$ $\cup\{c_1,\dots,c_j\}$ is a free extension of $B\cup\{c_1,\dots,c_{j-1}\}$ for $j\in 1:k$. Generic configurations and the configurations presented by Fig. 1,2 are examples of free configurations. A free basis oriented combinatorial type of configurations is an oriented combinatorial type of free basis configurations.

3°. The Theorem A is a corollary of the following three Lemmas which connect the objects defined in p.2.

LEMMA 1.

1) For every primary semi-algebraic variety M, defined over $\mathbb{Q}$ there exist a natural k, m, a regular vector-function $f\in\mathbb{Q}^m[X_1,\dots,X_k]$ and its formula $\Psi\in\mathcal{O}\!\mathcal{L}^m(0,1,X_1,\dots,X_k)$ such that the partition $\sum(f,\Psi)$ has a stratum γ stable equivalent to M.

2) If M is open (i.e. is defined by strict inequalities only) then the stratum γ can be chosen to be open.

LEMMA 2.

1) For every rational vector-function $f\in\mathbb{Q}^m(X_1,\dots,X_k)$, its computation Ψ in $\mathcal{O}\!\mathcal{L}(0,1,X_1,\dots,X_k)$ and a stratum $\gamma\in\sum(f,\Psi)$ there exists a basis oriented combinatorial type $\beta\subset \operatorname{Im} P^{(f,\Psi)}$ stable equivalent to γ.

2) If the stratum γ is open in $\mathbb{R}^k$ and Ψ is a formula than the type β can be chosen to be free.

LEMMA 3. For every free basis oriented combinatorial type β of configurations there exists a generic basis oriented combinatorial type which is stable equivalent to β.

4°. PROOF OF THE LEMMA 2.

4.1. Put $\theta(f,\Psi)=\operatorname{Im} P^{(f,\Psi)}\subset bc(S^\Psi)$ (see p. 2.2).Consider the projection $\Pi:\theta(f,\Psi)\to\mathbb{R}^k$, $\Pi=\Pi_1\circ(P^{(f,\Psi)})^{-1}$, where

Π_1 - projection of $\mathbb{R}^k \times (\mathbb{R}^*)^{\overline{SW}(\Psi)}$ on the first factor. Let $\mathcal{X}$ be an oriented combinatorial type of basis configurations, $\mathcal{X} \subset \Theta(f,\Psi)$. Let $p^1, p^2 \in \mathcal{X}$ and let $\Pi(p^i) = x^i$ for $i=1,2$. By the definition of $p^{(f,\Psi)}$ the set of points $p^i \cap L_{\mathbb{R}} = \{p^i_A\}_{A \in \overline{SW}(\Psi)} \cup \cup \{p_\infty\}$ is such that $R(p^i_A, p_1; p_0, p_\infty) = (\sigma_{Q(x)}(A))(x^i)$, where $A \in \in SW(\Psi), i=1,2$. Since p_1 and p_2 belong to one oriented combinatorial type the points from $(p^1 \setminus p_\infty) \cap L_{\mathbb{R}}$ and $(p^2 \setminus p_\infty) \cap L_{\mathbb{R}}$ are arranged in the same order on the affine line $L_{\mathbb{R}} \setminus p_\infty$. Hence x^1 and x^2 belong to one stratum of $\sum(f,\Psi)$.

4.2. It is not difficult to show that $\Pi|_{\mathcal{X}} : \mathcal{X} \to \Pi(\mathcal{X})$ is a trivial fibration with a fibre $\mathbb{R}^{\overline{SW}(\Psi)}$.

4.3. Fix any order $\prec$ on the set $\overline{SW}(\Psi)$. Let γ be a stratum of $\sum(f,\Psi)$. Note that independently of $x \in \gamma$ the configuration $p^{(f,\Psi)}(x, \{b^*_A\}_{A \in \overline{SW}(\Psi)})$ lies in the same oriented type $\beta(\gamma, \prec) \subset \Theta(f,\Psi)$ when $b^*_A \gg \sum_{C \prec A} b^*_C$ for arbitrary $A \in \in \overline{SW}(\Psi)$. From the p.4.1 it follows that $\Pi(\beta(\gamma,\prec)) = \gamma$. Hence, by p.4.2 we obtain that $\beta(\gamma,\prec)$ is homeomorphic to $\gamma \times \mathbb{R}^{\overline{SW}(\Psi)}$. This homeomorphism may be chosen piecewise biregular, defined over $\mathbb{Z}$.

4.4. The statement 2) of Lemma 2 may be proved by the induction on the increasing of the Ψ-subwords.

5°. PROOF OF THE LEMMA 3.

Let C be a basis point configuration, $B \subset C$ is the fixed projective basis. (Here we follow the notations of p.3.3). Put $V(C) = = V([C]) = \sum_{x \in C} V_C(x)$, where $[C]$ is the basis oriented combinatorial type containing C. We prove Lemma 3 by the induction on $V(\beta)$. If $V(\beta) = 0$ then β is generic, by the definition. Let $V(\beta) > 0$, $C \in \beta$. In this case among the points of $C \setminus B$ there exists an improper point x. Consider the two situations: 1) $V_C(x) = 1$ and 2) $V_C(x) = 2$.

1): Let $V_C(x)=1$, $\ell \in \mathcal{L}_p(C)$, $x \in \ell$ (see Fig. 3a) Denote by $\mathcal{L}(C)$ the set of all projective lines which are incident with more than one point of C. Denote by $S_C(x)$ the star of the point x in the geometric complex generated on $P^2_{\mathbb{R}}$ by the line configuration $\mathcal{L}(C)$. Choose the line $\ell' \notin \mathcal{L}(C)$ such that ℓ' is in general position with respect to the points of $C\setminus\{x\}$, $\ell' \cap C = \{x\}$. Fix on the open segment $S_C(x) \cap \ell'$ two points a_1, a_2 separated by x (see Fig.3b). Consider the configuration $\tilde{C} = (C\setminus\{x\}) \cup$ $\cup \{a_1, a_2\}$ (Fig. 3c). By the trivial arguments it can be proved that the oriented basis combinatorial type $[\tilde{C}]$ is piecewise-biregular homeomorphic to $[C]\times\mathbb{R}^3 = \beta\times\mathbb{R}^3$, $V([\tilde{C}]) = V(\tilde{C}) = V(C)-1$ and $[\tilde{C}]$ is free.

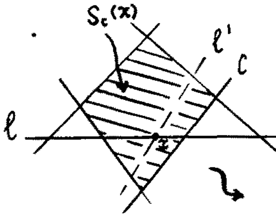

Fig.3a

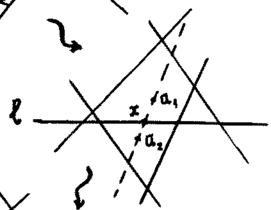

Fig.3b

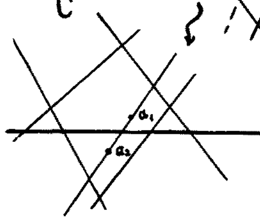

Fig.3c

2): Let $V_C(x)=2$; $\ell_1, \ell_2 \in \mathcal{L}_p(C)$; $x \in \ell_1 \cap \ell_2$ (see Fig.4a)

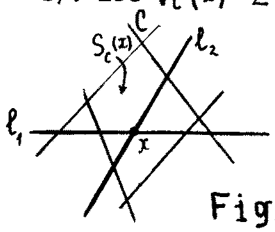

Fig.4a

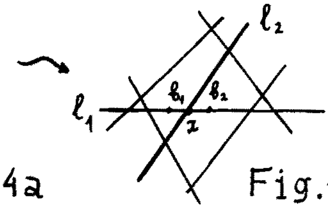

Fig.4b

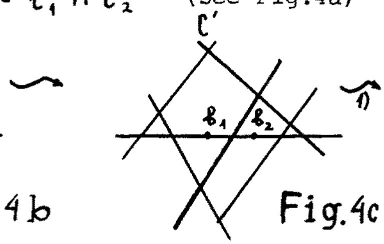

Fig.4c

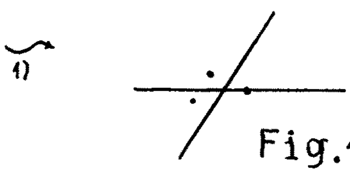

Fig.4d

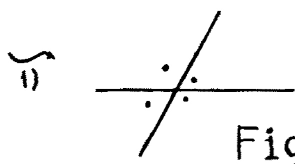

Fig.4e

Fix on the open segment $\ell_1 \cap S_C(x)$ two points b_1, b_2 separated by x (Fig.4b). Consider the configuration $C' = (C\setminus\{x\}) \cup$ $\cup \{b_1, b_2\}$ (Fig.4c). The configuration C' is free, $[C'] \simeq$ $\simeq [C]\times\mathbb{R}^2 = \beta\times\mathbb{R}^2$, $V([C']) = V(C) - V(\beta)$. Since $V_{C'}(b_1) = 1$ and $b_1 \notin B$ we are at the situation of 1). (see

Fig. 4d, 4e).

6°. Examples

6.1. For $f \in \mathbb{Q}^m(X_1,\dots,X_k)$ and its formula Ψ in $\mathcal{O}\!\ell(0,1,X_1,\dots,X_k)$ put $G(\Psi) = \mathrm{card}(SW(\Psi)) + 16\,\mathrm{card}(SW^+(\Psi)) + 20\,\mathrm{card}(SW^\times(\Psi)) + 3$ (see p.2.2 for the definitions SW^+ and $SW^\times$). Because the proof of the Lemma 2.3 has a constructive character it enables us to obtain the following evaluation:

COROLLARY 1.

For every rational vector-function $f \in \mathbb{Q}^m(X_1,\dots,X_k)$ its formula Ψ in $\mathcal{O}\!\ell(0,1,X_1,\dots,X_k)$ and a stratum $\Gamma \in \Sigma(f,\Psi)$ their exists a generic oriented combinatorial type of $G(\Psi)$ -point 2-configurations stable equivalent to Γ.

On the basis of the Lemma 2 and 3 one can easily construct various particular examples of generic oriented combinatorial types with non-trivial topology. The fact is that even very simple couples (vector-function, its formula) generate partitions Σ which have open strata with non-trivial topology.

6.2. An example of a disconnected generic oriented basis combinatorial type of configurations.

Consider the vector-function $g \in \mathbb{Q}^2[\ \ ]$, $g = (X^2 - X, -\frac{1}{4})$ and its formula $\Phi = [\Phi_1, \Phi_2] = [((X \times X) - X), (1 : ((((0-1)-1)-1)-1))] \in \mathcal{O}\!\ell^2(0,1,X)$; $SF(\Phi) = \{0,1,-1,-2,-3,-4,-\frac{1}{4}, X, X^2, (X^2-X)\} \subset \mathbb{Q}[X]$. Consider the stratum η of $\Sigma(g,\Phi)$, where $\eta = \{x \in \mathbb{R} \mid 1 > x > x^2 > 0 > (x^2 - x) > -\frac{1}{4} > -1 > -2 > -3 > -4\} = \{x \in \mathbb{R} \mid 0 > (x^2 - x) > -\frac{1}{4}\} = (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. The stratum η is open and disconnected. Hence by 2) Lemma 2 and by the evaluation of $\mathrm{card}(S^\Phi)$ from p.2.2 there exists a disconnected free oriented basis combinatorial type of 43 -

-point configurations. By Corollary 1 there exists a disconnected generic oriented basis combinatorial type of 144-point configurations. By specialization of the general construction for the particular case of $(\mathcal{G}, \Phi)$ the author had constructed the examples of a disconnected free oriented basis combinatorial type of 16-point configurations and a disconnected generic oriented basis combinatorial type of 19-point configurations (see [M]).

6.3. An example of generic oriented combinatorial type which is homotopically equivalent to S^1

Consider the vector-function $h=(h_1,h_2)=((X^2-\frac{X}{2}+Y^2-Y),(-\frac{5}{16}))\in \mathbb{Q}^2[X,Y]$. The function $h_1(X,Y)$ has unique minimum at the point $x^*=(\frac{1}{4},\frac{1}{2})$, $h_1(x^*)=-\frac{5}{16}=h_2$. According to the definition, for arbitrary formula Ψ of the vector-function h the set of rational functions $SF(\Psi)\subset\mathbb{Q}(X,Y)$ contains the functions h_1 and h_2. Hence, the partition $\sum(h,\Psi)$ contains the partition $\sum_S(h,\Psi)$ of the set $S=\{(x,y)\mid h_1(x,y)>h_2=-\frac{5}{16}\}=$ $=\mathbb{R}^2\setminus\{x^*\}$. Suppose that the formula Ψ has the following property:

(*) $u(x^*)-v(x^*)\neq 0$ for every $u,v\in SF(\Psi)$ such that $\{u,v\}\neq\{h_1,h_2\}$, $u\neq v$.

Then by the definition of $\sum(h,\Psi)$ there exists unique open stratum $\nu(x^*,\Psi)\in\sum_S(h,\Psi)$ such that $\nu(x^*,\Psi)\supset(V(x^*)\cap S)=$ $=V(x^*)\setminus\{x^*\}$ for a sertain neighbourhood $V(x^*)$ of point x^*. Obviously $\pi_1(\nu(x^*,\Psi))\neq 0$

Consider the following formula of the vector-function h.

$$\chi=[\chi_1,\chi_2]=[(((X\times X)-(X:2))+(Y\times(Y-1))),$$

$$((\ldots\underbrace{(1+1)+\ldots+1}_{5}):(\ldots(0-\underbrace{1)-\ldots-1}_{16}))]\in\mathcal{O}^2(0,1,X,Y)\ ;$$

$$SF(\chi)=\{-16,-15,\dots,-2,-1,(Y-1),h_2,h_1,(Y^2-1),(-X^2-\tfrac{X}{2}),\\ 0,X^2,\tfrac{X}{2},X,Y,1,2,\dots,5\}$$

One can easily verify by direct calculations the correctness of condition (*) for χ. Hence, $\pi_1(\nu(x^*,\chi))\neq 0$. More detailed analysis enables us to establishe that $\nu(x^*,\chi)$ is homotopically equivalent to S^1 (see Fig.5). By 2) of Lemma 2 and by the evaluation of $card(S^{\chi})$ from p.2.2. there exist a free basis oriented combinatorial type of 146-point configurations which is homotopically equivalent to $\nu(x^*,\chi)$. By Corollary 1 there exists a generic basis oriented combinatorial type of 514-point configurations which is homotopically equivalent to $\nu(x^*,\chi)$

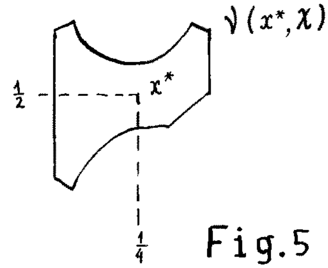

Fig.5

7°. PROOF OF THE LEMMA 1.

We prove the Lemma 1 by the extension of the procedure used in p.6.3. Here we shall outline the proof of statement 2) of Lemma 1. The statement 1) is proved by the analogous but more refined analysis.

7.1. Let $\mathbb{F}$, $\mathbb{G}$ be a fields, $\mathbb{G}\supset\mathbb{F}$ and f be a vector-function, $f\in\mathbb{F}^m(\mathfrak{X})$, where $\mathfrak{X}=(X_1,\dots,X_k)$. We introduce the following notations: $f^0_{\mathbb{G}}=\{x\in\mathbb{G}^k \mid f(x)=0\}$ and in the case of $\mathbb{G}=\mathbb{R}$: $f^+=\{x\in\mathbb{R}^k \mid f(x)>0\}$.

DEFINITION. A computation Ψ of a vector-function $f\in\mathbb{F}^m(\mathfrak{X})$ is said to be <u>non-degenerate at point</u> $x^*\in(f^0_{\mathbb{G}}\cap reg(f,\Psi))$ if $u(x^*)\neq v(x^*)$ for arbitrary $u,v\in SF(\Psi)$ such that $\{u,v\}\not\subset\{f_1,\dots,f_m,0\}$, $u\neq v$.

Let M be the variety being introduced in the statement 2), Lemma 1, let $M=h^+$, where $h=(h_1,\dots,h_m)\in\mathbb{R}^m[X_1,\dots,X_k]$. Without loss of generality (i.e. substituting if it is necessary the variety M by a stable equivalent one) we can assume that the polynomials h_j are homogeneous, $h_j = \sum_{i_1+\ldots+i_k = D(h)} h^j_{i_1,\ldots,i_k} \cdot X^{i_1} \cdot \ldots \cdot X^{i_k}$ for $j \in \overline{1:m}$ and the following assumptions are valid:

(*) $h_1(y) > h_2(y) > \ldots > h_m(y)$ for arbitrary point $y \in h^+$ which is near to origin;

(**) the set $C(h) = \{h^j_{i_1,\ldots,i_k}\}_{i_1+\ldots+i_k = D(h),\, j \in \overline{1:m}} \subset \mathbb{R}$ is algebraic independent over $\mathbb{Q}$.

7.2. PROPOSITION 1.

Let $g \in \mathbb{Q}^m[X_1,\ldots,X_k]$ be a polynomial vector-function and let x^* be a point from $g^0_{\mathbb{R}}$. Suppose that for (g, x^*) the following assumptions are valid:

1) There exist a non-degenerate at x^* formula $\Phi \in \mathcal{U}^m(0,1,X_1,\ldots,X_k)$ of g ;

2) $g_1(z) > \ldots > g_m(z)$ for arbitrary point $z \in g^+$ which is near to x^*.

Then there exist the other vector-function $\tilde{g} \in \mathbb{Q}^{m_1}[Y_1,\ldots,Y_{k_1}]$ and its formula $\tilde{\Phi} \in \mathcal{U}^{m_1}(0,1,Y_1,\ldots,Y_{k_1})$ such that the partition $\Sigma(\tilde{g},\tilde{\Phi})$ contains an open stratum which is stable equivalent to the cone over g^+ with the apex x^*.

For the proof see p. 7.5.

Consider a vector-function $\mathcal{H} \in \mathbb{Z}^m[\mathcal{A},\mathcal{X}]$ where $\mathcal{A} = \{A^j_{i_1,\ldots,i_k}\}_{i_1+\ldots+i_k = D(h),\, j \in \overline{1:m}}$, $\mathcal{X} = (X_1,\ldots,X_k)$, $\mathcal{H}_j(\mathcal{A},\mathcal{X}) = \sum_{i_1+\ldots+i_k = D(h)} A^j_{i_1,\ldots,i_k} \cdot X_1^{i_1} \cdot \ldots \cdot X_k^{i_k}$ for $j \in \overline{1:m}$. The vector-function $\mathcal{H}$ can be regarded as the generic homogeneous polynomial vector-function $\mathbb{R}^k \to \mathbb{R}^m$ of degree $D(h)$. From the assumption (**) on $C(h)$ it follows that $\mathrm{cone}_{(C(h),\mathbb{O})} \mathcal{H} \underset{\Delta}{\approx} \mathrm{cone}_{\mathbb{O}} h^+$ (we denote by $\underset{\Delta}{\approx}$ a stable equivalence of semi-algebraic varieties). For the couple $(\mathcal{H},(C(h),\mathbb{O}))$ the assumption 2) of Proposition 1 is valid (it follows from (*)), while the assumption 1) is not. Consider a vector-function $\mathcal{H}^1$ - the composition of $\mathcal{H}$ and the "generic translation of $\mathbb{R}^k$ ": $\mathcal{H} \in \mathbb{Q}^m[\mathcal{A}, \mathcal{T}, \mathfrak{X}]$ where $\mathcal{T}(T_1, \ldots, T_k)$, $\mathcal{H}_j^1 = \sum_{i_1+\ldots+i_k = D(h)} A^j_{i_1,\ldots,i_k} \cdot (X_1+T_1)^{j_1} \cdot \ldots \cdot (X_k+T_k)^{j_k}$ for $j \in \overline{1:m}$

Choose a point $t^* = (t_1^*, \ldots, t_k^*)$ such that the set $C(h) \cup \{t_i^*\}_{i=1}^k \subset \mathbb{R}^k$ is algebraic independent over $\mathbb{Q}$. Put $-t^* = (-t_1^*, \ldots, -t_k^*) \in \mathbb{R}^k$. Obviously, $\mathrm{cone}_{(C(h),t^*,-t^*)}(\mathcal{H}^1)^+ \underset{\Delta}{\approx} \mathrm{cone}_{(C(h),\mathbb{O})} \mathcal{H}^+ \underset{\Delta}{\approx} M$. Let $\mathcal{D}$ be the composition of $\mathcal{H}^1$ and the "generic homotety of $\mathbb{R}^m$ ", $\mathcal{D} \in \mathbb{Q}^m[\mathcal{A}, \mathcal{T}, \alpha, \mathfrak{X}]$, $\mathcal{D}_j = \alpha \cdot \mathcal{H}_j^1$ for $j \in \overline{1:m}$. Choose a point $\alpha^* \in \mathbb{R}$, $\alpha^* > 0$ such that the set $C(h) \cup \{t_i^*\}_{i=1}^k \cup \{\alpha^*\} \subset \mathbb{R}$ is algebraically independent over $\mathbb{Q}$. Obviously, $\mathrm{cone}_{(C(h),t^*,\alpha^*,-t^*)} \mathcal{D} \underset{\Delta}{\approx} \mathrm{cone}_{(C(h),t^*,-t^*)} \mathcal{H}^1$. Put $\theta(h) = (C(h), t^*, \alpha^*, -t^*)$. For the couple $(\mathcal{D}, \theta(h))$ the assumption 2) of Proposition 1 is valid (it follows from the construction). We shall complete the proof of statement 2), Lemma 2 by presenting of the non-degenerate at $\theta(h)$ formula of $\mathcal{D}$ in $\mathcal{O}\!\mathcal{L}(0,1,\mathcal{A},\mathcal{T},\alpha,\mathfrak{X})$

7.3. The Horner-type computation Horn of a polynomial vector-function.

Let $\mathbb{F}$ be a field and let $\{X_1, \ldots, X_\ell\}$ be a collection of independent variables. Consider the family of the maps $\{K_t^i\}_{t \in \overline{0:\infty},\, i \in \overline{1:\ell}}$ where $K_i^t : \mathbb{F}[X_1, \ldots, X_\ell] \to \mathbb{F}[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_\ell]$, $K_i^t(f)$ is a coefficient of f for t-degree of x_i.

For $g \in \mathbb{F}$ put $\mathrm{Horn}(g) = g$;

for $g \in \mathbb{F}[X_i, \ldots, X_\ell]$ put

$$\mathrm{Horn}(g) = (\mathrm{Horn}(K_i^0(g)) + X_i \times (\mathrm{Horn}(K_i^1(g)) + \ldots$$

$$\ldots + X_i \times \mathrm{Horn}(K_i^{\deg_{x_i} g}(g))\ldots)) \in \mathcal{O}\!\mathcal{L}(C(g), X_i, \ldots, X_\ell).$$

Let $f = (f_1, \ldots, f_m) \in \mathbb{F}^m[X_1, \ldots, X_k]$

Put $\mathrm{Horn}(f) = (\mathrm{Horn}(f_1), \ldots, \mathrm{Horn}(f_m)) \in \mathcal{O}^m(C(f), X_1, \ldots, X_k)$.

7.4. Consider the vector- function $\mathcal{H}^1$ (p.7.2) as an element of $\mathbb{Q}[\mathcal{A}, \mathcal{T}][\mathcal{X}]$,

$$\mathcal{H}^1_j = \sum_{i_1+\ldots+i_k \leq D(h)} H^j_{i_1,\ldots,i_k}(\mathcal{A}, \mathcal{T}) \cdot X_1^{i_1} \cdot \ldots \cdot X_k^{i_k} \quad \text{for} \quad j \in \overline{1:m}$$

Denote by $(-\mathcal{T})$ the point $(-T_1, \ldots, -T_k) \in (\mathcal{H}^1)^0_{\mathbb{Q}(\mathcal{A},\mathcal{T})}$

PROPOSITION 2. The computation $\mathrm{Horn}(\mathcal{H}^1)$ of the vector-function $\mathcal{H}$ is a non-degenerate at point $(-\mathcal{T})$ formula of $\mathcal{H}^1$ in $\mathcal{O}(\{H^j_{i_1,\ldots,i_k}(\mathcal{A}, \mathcal{T})\}_{\substack{i_1+\ldots+i_k \leq D(h) \\ j \in \overline{1:m}}}, \mathcal{X})$.

One can obtain the proof of Proposition 2 by direct calculation .

Now we are able to construct a non-degenerate at point $\theta(h)$ formula of in $\mathcal{O}(0,1,\mathcal{A},\mathcal{T},\alpha,\mathcal{X})$. Let $\{\bar{H}^j_{i_1,\ldots,i_k}\}_{\substack{i_1+\ldots+i_k \leq D(h) \\ j \in \overline{1:m}}}$ be a certain formula of the vector-function $\{H^j_{i_1,\ldots,i_k}(\mathcal{A},\mathcal{T})\}_{\substack{i_1+\ldots+i_k \leq D(h) \\ j \in \overline{1:m}}}$ in $\mathcal{O}(0,1,\mathcal{A},\mathcal{T})$. Replace in formula $\mathrm{Horn}(\mathcal{H}^1)$ the subword $H^j_{i_1,\ldots,i_k}(\mathcal{A},\mathcal{T})$ by the word $(\alpha \times \bar{H}^j_{i_1,\ldots,i_k}$ for every $(j, i_1, \ldots, i_k)$. Thus we obtain a formula Ψ of the vector-function in $\mathcal{O}(0,1,\mathcal{A},\mathcal{T},\alpha,\mathcal{X})$. The non-degeneracy of Ψ at point $\theta(h)$ follows immediately from the Proposition 2, linearity and homogenity on $\{H^j_{i_1,\ldots,i_k}\}$ of the polynomials from $SF(\mathrm{Horn}(\mathcal{H}^1))$ and from the algebraic independence over $\mathbb{Q}$ of the set $\{C(h)\} \cup \{t_i^*\}_{i=1}^k \cup \{\alpha^*\}$.

7.5. PROOF OF THE PROPOSITION 1 (p.7.2).

We shall demonstrate how to rearrange the couple (g, Φ) from the statement of Proposition 1 to obtain a new couple $(\tilde{g}, \tilde{\Phi})$ which has the following property: the partition $\Sigma(\tilde{g}, \tilde{\Phi})$ has a stratum which is stable equivalent to $g^+ \cap V(x^*)$, where $V(x^*)$ is a small convex neighbourhood of x^*.

Because of the assumptions 1), 2) of Proposition 2 their exists a collection of rational numbers $\Lambda = \{\lambda_i^j\}_{i \in \overline{1:k},\, j=1,2}$ such that

a) $x^* \in K_\Lambda = \{x \in \mathbb{R}^k \mid \lambda_i^1 < x_i < \lambda_i^2$ for $i \in \overline{1:k}$

b) the primary semi-algebraic set $cone_{x^*}\, g^+$ is piecewise-biregular homeomorphic to $g^+ \cap K_\Lambda$.

c) $sign(u(x) - v(x)) = sign(u(y) - v(y))$ for arbitrary rational functions $u, v \in SF(\Phi)$ and arbitrary points $x, y \in K_\Lambda \cap g^+$.

Let Z be a certain new variable. Consider the vector function $\tilde{g} \in \mathbb{Q}^{m+|\Lambda|}[X_1, \dots, X_k, Z]$, $\tilde{g}(X, Z) = (g, \Lambda)$. Let $abs(\lambda_i^j) = \mu_i^j / \eta_i^j$ where $\mu_i^j, \eta_i^j \in \mathbb{N}$

Consider the following computation $\tilde{\Phi}$ of $\tilde{g}$: $\tilde{\Phi} = (\Phi, \{\chi_i^j\}_{i \in \overline{1:k},\, j=1,2} \in \mathcal{O}\!\iota^{m+|\Lambda|}(0, 1, X, Z)$, where

$$\chi_i^j = \begin{cases} ((\dots\overbrace{(Z+Z)+\dots+Z}^{\mu_i^j}):(\dots\overbrace{(Z+Z)+\dots+Z}^{\eta_i^j})), & \text{when } \lambda_i^j > 0 \\ ((\dots\underbrace{(Z+Z)+\dots+Z}_{\mu_i^j}):(\dots\underbrace{(0-Z)-\dots-Z}_{\eta_i^j})), & \text{when } \lambda_i^j < 0 \end{cases}$$

It is easy to check that $SF(\tilde{\Phi}) = SF(\Phi) \cup \Lambda \cup \{kZ\}_{k=1}^{m_1} \cup \cup \{-kZ\}_{k=1}^{m_2}$ for a certain m_1 and m_2, and that $\tilde{\Phi}$ is a formula when Φ is a formula. Let γ be a stratum of $\Sigma(\tilde{g}, \tilde{\Phi})$ which contains the point (y^*, c^*), where $y^* \in K_\Lambda \cap \cap g^+$, $c^* > max\{v(y^*) \mid v \in SF(\Phi) \cup \Lambda\}$. According to the construction, the stratum γ is biregular isomorphic to

$(K_\Lambda \cap g^+) \times \mathbb{R}^1 \underset{\Delta}{\approx} K_\Lambda \cap g^+ \underset{\Delta}{\approx} cone_{x^*}\, g^+$.

8°. The Theorem B is a simple corollary of the Theorem A and the

Gale duqlity. Consider an ordered configuration $C = \{c_i\}_{i=1}^{n}$ of $\mathbb{R}^3$ -vectors. The Gale face of C is a subconfiguration C' of C such that $\mathbb{0} \in$ relint cone $(C \setminus C')$. Let $G(C)$ be the lattice of Gale faces of the configuration C . If all points of C are Gale vertices than their exists a convex polytope $P \in$ $\in pol(n-4, n)$ such that $G(C)$ is the face-lattice of P . Moreover, there is the canonical biregular isomorphism between $[C]_g / GL_3(\mathbb{R})$ and $[P]_c / AGL_{n-4}(\mathbb{R})$, where $[C_g]$ and $[P]_c$ are the combinatorial type of Gale diagramms and of convex polytopes which contain C and P , respectively; $AGL_{n-4}(\mathbb{R})$ is the group of affine automorphisms of $\mathbb{R}^{n-4}$ (see [G1]). By the construction similar to the Perle's one (see [G1, § 5.5 Theorem 4]) we can to put in correspondence to arbitrary oriented basis combinatorial type α of point 2-configurations the combinatorial type $\gamma(\alpha)$ of $\mathbb{R}^3$ -Gale diagramms such that $\gamma(\alpha)/GL_3(\mathbb{R}) \underset{\Delta}{\approx} \alpha$ If the type α is generic then the type $\gamma(\alpha)$ can be chosen generic too.

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