DIMACS Series in Discrete Mathematics
and Theoretical Computer Science
Volume 6, 1991

The Universality Theorem on the Oriented Matroid Stratification of the Space of Real Matrices

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The problem of describing the class of realization spaces of oriented matroids was posed by A. M. Vershik in the 1970s. This problem was solved in 1984 [**2**, **3**]. The solution was unexpected. By definition, the set of all matrix realizations of rank three oriented matroids is a semialgebraic subset of the space of all $3 \times m$ real matrices. The following phenomenon was established: up to trivial stabilization, any elementary semialgebraic variety can be represented as the realization space of an oriented matroid of rank three.

Any open elementary variety (i.e., a variety defined by strict inequalities) can be represented as a realization space of a uniform, rank three oriented matroid. Moreover, it can be done in such a way that the number of points of the matroid is a linear function of the bit size for describing the polynomial system defining the variety. This means that given an arbitrary system of polynomial equations and inequalities, one can rewrite it in "oriented matroid" form, and in such a way as to preserve the algorithmic complexity class of various elementary problems relating to the geometry of the solutions.

Consider the space $\mathrm{Mx}^{3\times m}$ of all $3 \times m$ real matrices. There is a natural stratification of this space—by signs of maximal minors. Each stratum is a realization space of some rank three oriented matroid on the set $\overline{1 : m}$. The aim of this paper is to announce a universality theorem for this class of stratified varieties. If this theorem is localized to one stratum, we get the universality theorem for the realization spaces of oriented matroids.

We begin with the definition of several notions that may be a little bit unfamiliar in the present context.

We are in the category of elementary semialgebraic varieties defined over $\mathbb{Z}$. The objects of this category are the sets of real solutions of polynomial systems of equations and strict inequalities. We suppose that the coefficients

1991 *Mathematics Subject Classification*. Primary 51A20, 14P10.

of the polynomials are the integers. The morphisms of our category are the maps of the sets which are generated by polynomial vector functions with integer coefficients.

1.

Let N be a variety. Denote by $\mathbb{Z}_+(N)$, the cone of all positive polynomial functions on N. Let ϕ be a finite subset of $\mathbb{Z}_+(N)$, $\phi = \{\varphi_1, \dots, \varphi_k\}$ and consider the variety $\Gamma_+(\phi, N) = \{(x, y) \mid x \in N, 0 < y < \varphi_i(y); i \in \overline{1 : k}\}$. We are interested in the following two properties:

(i) The character of the natural projection $\Gamma_+(\phi, N) \xrightarrow{\pi} N$ (Figure 1a).

(ii) The character of the adjacency of $\Gamma_+(\phi, N)$ and N in the common space (Figure 1b).

FIGURE 1

1.1. Stabilization.

DEFINITION. The map $A \xrightarrow{\psi} B$ of varieties is called a 1-stabilization if it is equivalent to the projection $\Gamma_+(\phi, N) \xrightarrow{\pi} N$ for some N; $\phi \subset Z_+(N)$. (By equivalence we mean here the existence of isomorphisms $A \xrightarrow{\alpha} \Gamma_+(\phi, N)$, $B \xrightarrow{\beta} N$ that make the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \downarrow \alpha & & \downarrow \beta \\ \Gamma_+(\phi, N) & \xrightarrow{\pi} & N \end{array}$$

commutative.)

DEFINITION. The map of varieties is called a *stabilization* if it admits a decomposition into a chain of 1-stabilizations. Obviously a stabilization of varieties preserves the homotopical structure of the variety and the character of its singularities.

1.2. Stable adjacency. Let N be a subvariety of $\mathbb{R}^m$ and let $\mathbb{R}^m = H \oplus \ell$ be some representation of $\mathbb{R}^m$ as a direct sum of a linear hyperplane H and a one-dimensional linear subspace ℓ. Coordinatize ℓ by $t: \ell \to \mathbb{R}$. One can make a correspondence between the data $D =$ (decomposition, coordinatization) and the imbedding $\Gamma_+(N, \phi)_D$ of the variety $\Gamma_+(N, \phi)$

into the space

$$\mathbb{R}^m: \Gamma_+(N, \phi)_D = \{x + y \mid x \in N \ , \ 0 < t(y) < \min_{i \in 1:k} \varphi_i(x)\} .$$

DEFINITION. A subvariety M of the space $\mathbb{R}^m$ is *1-stably adjacent* to the subvariety N if there exists a family of functions $\phi \subset \mathbb{Z}_+[N]$ and $D =$ (decomposition of the space $\mathbb{R}^n$, coordinatization of the one-dimensional element of the decomposition) such that the subvariety M coincides with $\Gamma_+(\phi, N)_D$.

DEFINITION. A subvariety M of the space $\mathbb{R}^m$ is *stably adjacent* to the subvariety N if there exists a chain of subvarieties: $N = K_0 \subset \mathbb{R}^m$, $K_1 \subset \mathbb{R}^m, \dots, M = K_\ell \subset \mathbb{R}^m$ such that K_i is 1-stably adjacent to K_{i-1} for $i = 1, \dots, \ell$.

Obviously any decomposition from the last definition generates the stabilization $M \to N$. Denote by $\dot{\mathbb{R}}^\ell_+$ the set

$$\{x = (x_1 \cdots x_\ell) \mid x_i \in \mathbb{R} \ , \ x_i > 0 \ ; \ i \in \overline{1:\ell}\} .$$

PROPOSITION. *If a subvariety $M \subset \mathbb{R}^m$ is stably adjacent to the subvariety $N \subset \mathbb{R}^m$ then there exists* (a) *a linear subspace $H \subset \mathbb{R}^m$ such that $N \subset H$, and* (b) *a neighborhood T of M such that the triple $(N \subset H \ , \ M \cap T \ , \ T)$ is isomorphic to the triple $(N \subset H \ , \ N + \dot{\mathbb{R}}^\ell_+ \ , \ H \oplus \mathbb{R}^\ell)$ for some $\ell \in \mathbb{N}$.*

2. Elementary semialgebraic partitions and stratifications

Consider the finite set of indices $\Phi = \{+1, 0, -1\}$ and some subset U of the set Φ^k, $k \in \mathbb{N}$.

DEFINITION. A *partition* of the variety N is the map $N \to U$ of type $(\operatorname{sign} f_1, \dots, \operatorname{sign} f_k)$, where $f_i \in \mathbb{Z}[N]$ for $i \in \overline{1:k}$.

The element $(\epsilon_1 \cdots \epsilon_k) \in \Phi^k$ is called uniform if $\epsilon_i \neq 0$ for all $i \in \overline{1:k}$.

DEFINITION. A *stratification* of the variety N is a finite, numbered covering of N by some disjoint set of subvarieties (i.e., a stratification is a map $I \xrightarrow{\sigma} 2^N$, where I is a finite set, $\sigma(i)$ is a subvariety of N, $\sigma(j) \cap \sigma(i) = \varnothing$ when $i \neq j$, $i, j \in I$, $\bigcup_{i \in I} \sigma(i) = N$). An element $\sigma(i)$ of the stratification is called a *stratum*. Each partition $N \xrightarrow{p} U$ corresponds to the stratification

$$U \xrightarrow{p^\sigma} 2^N \quad \text{where } p^\sigma(j) = p^{-1}(j) \text{ for } j \in U .$$

The strata which correspond to the uniform elements of U are called *open*. These strata are really open subsets of N in the strong topology.

3. The main example

Let m be a natural number. Consider the set

$$\Lambda(m) = \{(i, j, k) \mid i < j < k \ , \ i, j, k \in \overline{1:m}\} .$$

An oriented matroid is a function $\Lambda(m) \to \Phi$ which satisfies some requirements as given, for example, in [**1**]. So one can identify the set of all oriented matroids on $\{\overline{1:m}\}$ with some subset $\mathcal{Md}(m)$ of the set $\Phi^{\Lambda(m)}$. Every $3 \times m$ matrix x with real coefficients defines the oriented matroid

$$\mathrm{Md}(x) \in \mathcal{Md}(m) : \mathrm{Md}(x)_{i,j,k} = \operatorname{sign} \Delta_{i,j,k}(x) ,$$

where $\Delta_{i,j,k}(x)$ is the determinant of the maximal minor of x with the columns (i, j, k). So we have the partition

$$\mathrm{Mx}^{3\times m} \xrightarrow{\mathrm{Md}} \mathcal{Md}(m) \subset \Phi^{\Lambda(m)} .$$

The stratification $\mathcal{Md}(m) \xrightarrow{\mathrm{Md}^{\sigma}} 2^{\mathrm{Mx}^{3\times m}}$ associates to each oriented matroid the space of all its matrix (vector) representations. A uniform matroid χ is associated with an open stratum $\mathrm{Md}^{\sigma}(\chi)$.

4. The imbedding of partitions

By *imbedding of partition* $N \xrightarrow{\psi} U \subset \Phi^k$ into the partition $M \xrightarrow{\varphi} V \subset \Phi^{\ell}$ we shall understand the pair of imbeddings (α, β) where $N \overset{\alpha}{\hookrightarrow} M$, $U \overset{\beta}{\hookrightarrow} V$, and the square

$$\begin{array}{ccc} N & \xrightarrow{\psi} & U \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ M & \xrightarrow{\varphi} & V \end{array}$$

is cartesian. This means, for example, that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi^{\sigma}} & 2^N \\ \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\bar\alpha} \\ V & \xrightarrow{\varphi^{\sigma}} & 2^M \end{array}$$

is commutative, where $\bar{\alpha}$ is the imbedding of 2^N into 2^M defined by the imbedding $N \overset{\alpha}{\hookrightarrow} M$. So, each stratum of the stratification ψ^{σ} becomes a stratum in the stratification φ^{σ} (Figure 2).

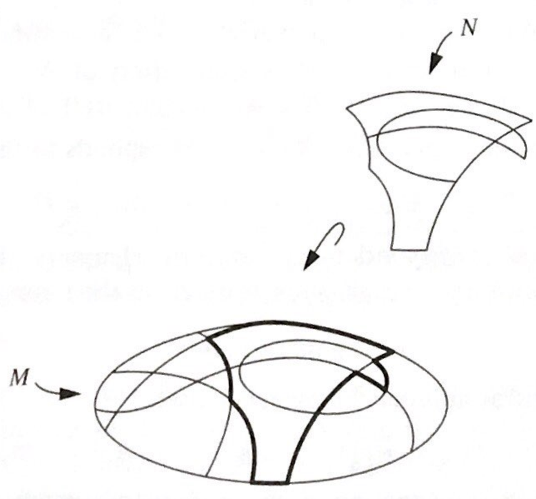

FIGURE 2

5. Partitions defined by partially oriented matroids

Let χ be a partially oriented matroid of rank three on the set $\overline{1:m}$. The space $[\chi]$ of all matrix representations of χ is naturally imbedded into $\mathrm{Mx}^{3\times m}$. Consider the set $\mathrm{Spec}(\chi) \subset \mathcal{Md}(m)$ of all oriented matroids on $\overline{1:m}$ whose orientations agree with χ.

The partition $p(\chi) = \mathrm{Md}|_{[\chi]} : [\chi] \to \mathrm{Spec}(\chi)$ is naturally imbedded into the partition Md. The stratification $\sigma(\chi) = p(\chi)^\sigma \colon \mathrm{Spec}(\chi) \to 2^{[\chi]}$ is a stratification of the set $[\chi]$ by complete oriented matroid type.

6. Stabilization of partitions

DEFINITION. A *stabilization of partitions* $M \xrightarrow{\psi} U$ and $N \xrightarrow{\varphi} V$ is a pair of maps $S = (\alpha, \beta)$ where $M \xrightarrow{\alpha} N$ is a stabilization of varieties, $U \xrightarrow{\beta} V$ is a bijection, and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & U \\ \downarrow \alpha & & \downarrow \beta \\ N & \xrightarrow{\varphi} & V \end{array}$$

is commutative.

The stabilization S defines the map $V \xrightarrow{S^\sigma} 2^M$ with the property that for any $j \in V$, $S^\sigma(j) = \psi^\sigma(\beta^{-1}(j))$ and the map $\alpha|_{S^\sigma(j)} \colon S^\sigma(j) \to \varphi^\sigma(j)$ is a stabilization of strata.

7. Multiplication of partition by direct factor

Let $N \xrightarrow{\psi} U$ be a partition and let A be a semialgebraic set. Consider the variety $N \times A$ and the partition $\psi^{\times A} \colon N \times A \to U$ which is defined by the commutative diagram

$$\begin{array}{ccc} N \times A & \xrightarrow{\psi^{\times A}} & U \\ \pi \downarrow & \nearrow \psi & \\ N & & \end{array}$$

where π is the projection on the direct factor.

Obviously for any $j \in U$ the stratum $(\psi^{\times A})^\sigma(j)$ is isomorphic to the variety $\psi^\sigma(j) \times A$ (Figure 3).

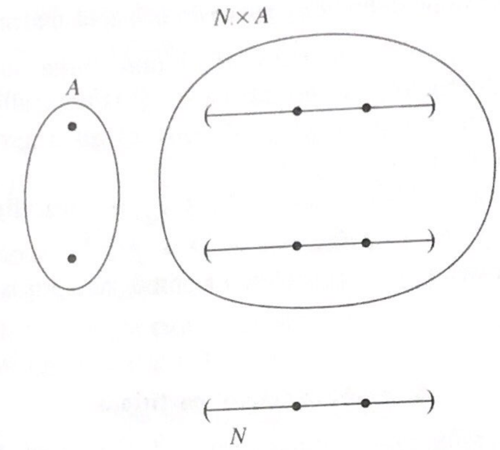

FIGURE 3

8.

An elementary semialgebraic partition of Euclidian space is a partition of type $\mathbb{R}^n \xrightarrow{\psi} \Phi^k$, where k, n are natural numbers, and

$$\psi = (\operatorname{sign} f_1, \dots, \operatorname{sign} f_k), \qquad \{f_1, \dots, f_k\} \subset \mathbb{Z}[\mathbb{R}^n].$$

Obviously, any semialgebraic partition can be imbedded into some partition of Euclidian space.

9.

Now we are able to formulate the universality theorem for oriented-matroid partitions (stratifications) of the spaces of real matrices. On the space $[\chi]$ (of all matrix representations of a rank three, partially oriented matroid χ), there is the natural action of the group $GL_3(\mathbb{R})$. Modulo this action and some stabilization the partition $[\chi] \xrightarrow{p(x)} \operatorname{Spec}(\chi)$ can coincide with any semialgebraic partition of Euclidian space.

THEOREM. *Let $\mathbb{R}^n \xrightarrow{\psi} \Phi^k$ be an elementary semialgebraic partition of Euclidian space.*

(1) *There is a natural number m, a partially oriented matroid χ on the set $\overline{1:m}$, and a pair of maps (α, β), $[\chi] \xrightarrow{\alpha} \mathbb{R}^n \times GL_3(\mathbb{R})$, $\operatorname{Spec}(\chi) \xrightarrow{\beta} \Phi^k$, such that (α, β) is a stabilization of the partitions $\psi^{\times GL_3}$ and $\sigma(\chi)$.*

Note that

(a) *by §6, for any $j \in \Phi^k$, the realization space of the oriented matroid $\beta^{-1}(j)$ is a stratum $\sigma(\chi)(\beta^{-1}(j))$ of the stratification $\sigma(\chi)$ and this stratum is a stabilization of the stratum $(\psi^{\times GL_3})^\sigma(j) = \psi^\sigma(j) \times GL_3$.*

(b) *by* §5, *the partition* $p(\chi)$ *is naturally imbedded into the partition* $\mathrm{Mx}^{3\times m} \xrightarrow{\mathrm{Md}(m)} \mathscr{Md}(m)$.

Moreover,

(2) *The partially oriented matroid* χ *can be chosen in such a way that for any stratum* $\Delta(j^*) = \sigma(\chi)(\beta^{-1}(j^*))$ *which corresponds to a uniform element* $j^* \in \Phi^m$ *there is an open stratum* $\beta(j^*)$ *of the stratification* $\mathrm{Md}^\sigma(m)$ *which is stably adjacent to* $\Delta(j^*)$. *(So,* $\beta(j^*)$ *is by definition the realization space of some uniform rank three oriented matroid.)*

The proof of this result is long and complicated. It will appear separately.

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