Small Simplicial Models of $\mathbb{CP}^n$ & a Finite Projective-Space Filtration in the Circular-Permutation Model of $K(\mathbb{Z}, 2)$

Perfect matchings, derangements, triangulated realizations, and the Barratt–Eccles E_∞ structure

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Abstract

Let SC_* be the simplicial set of circular permutations. Its realization is a model of $K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty$, and the unique nondegenerate 2-simplex supports the binary representative of the universal first Chern class. We construct an intrinsic increasing filtration by finite simplicial subsets

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots \subset SC, \qquad \bigcup_n \mathcal{X}_n = SC,$$

and prove

$$|\mathcal{X}_n| \simeq \mathbb{CP}^n$$

for every $n \geq 0$.

The filtration is defined by a unique A/B normal form for every nondegenerate circular permutation. The resulting rank-n, degree-k simplices are naturally in bijection with derangements of k letters having $k - n$ cycles. The relative normalized chain complex $N_*(\mathcal{X}_n, \mathcal{X}_{n-1})$ admits a two-stage filtration whose vertical fibers are polygon complexes; their fundamental cycles are exactly the degree-two operator T which previously produced the perfect-matching cycles

$$Z_n = \sum_{M \in PM(2n)} (-1)^{cr(M)} \sigma_M.$$

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This yields the integral recurrence

$$H_k(\mathcal{X}_n, \mathcal{X}_{n-1}; \mathbb{Z}) \cong H_{k-2}(\mathcal{X}_{n-1}, \mathcal{X}_{n-2}; \mathbb{Z})$$

and hence the homology and cohomology rings of $\mathbb{CP}^n$. The universal Chern class then supplies the homotopy equivalence with $\mathbb{CP}^n$.

We compare the canonical small models $\mathcal{X}_n$ with the Kühnel–Arnoux–Marin 9-vertex model of $\mathbb{CP}^2$, the Bagchi–Datta 18-vertex triangulation of $\mathbb{CP}^3$, the all-dimensional simplicial-cell decompositions of Datta–Spreer, and the very small Sergeraert–Kenzo simplicial-set models of the homotopy types of $\mathbb{CP}^n$. Explicit simplicial maps $\mathbb{CP}^2_9 \to \mathrm{SC}$ and $\mathbb{CP}^3_{18} \to \mathrm{SC}$ representing the hyperplane class are recorded; the latter extends the former. In contrast, the available Kenzo models in dimensions $2, 3, 4$ have no nonzero binary 2-class. This makes precise the sense in which $\mathcal{X}_n$ is a projective-space model endowed with additional bundle data, not merely a small model of the homotopy type.

Finally, we explain why the Barratt–Eccles operad is a serious, rather than merely analogical, structure for the continuation of the program. By Berger–Fresse, the normalized cochains of every simplicial set carry a functorial Barratt–Eccles E_∞-algebra structure. Hence the cochains of SC, all $\mathcal{X}_n$, and the finite triangulated models above come equipped with canonical coherent cup products, cup-i operations and Steenrod operations. The prismatic decomposition of the Barratt–Eccles operad and its quotient to the surjection operad provide a natural coherence machine for the shuffle/carry phenomena occurring in the projective-space constructions. A precise identification of the A/B filtration or the operator T with distinguished Barratt–Eccles cells remains open and is stated as a separate problem.

Contents

1 Purpose, context, and proof status

The simplicial set SC_* was introduced in [14] as a quotient of the symmetric crossed simplicial group by the right cyclic action. Its n-simplices are oriented circular permutations of $n+1$ labels, and

$$|SC_*| \simeq K(\mathbb{Z}, 2) \simeq BU(1) \simeq \mathbb{CP}^\infty.$$

The paper [12] identifies a particularly small integral representative of the universal first Chern class: it takes only the values 0 and 1 on 2-simplices. The question motivating the present note is whether finite projective spaces can be seen directly inside this circular-permutation model, and how the answer interacts with small triangulations, local Chern formulas, and higher coherent cochain operations.

The main new result of this revision is that the finite sub-simplicial sets previously generated experimentally by perfect-matching cycles admit an intrinsic description and are homotopy equivalent to projective spaces in every dimension.

Companion note on binary representability. The present note is constructive: it builds projective-space models equipped with a preferred map to SC. The complementary question—which integral classes on an *arbitrary* fixed simplicial presentation admit 0/1 representatives—is studied in the companion working note *Binary 2-cocycles, triangulated circle bundles and ice: which integral classes are binary?* [13]. In particular that note contains the detailed Sergeraert–Kenzo obstruction discussed below.

Theorem 1.1 (projective filtration; main theorem). *There is a canonical rank function* ρ *on the nondegenerate simplices of* SC *such that*

$$\mathcal{X}_n^{\mathrm{nd}} := \{\sigma \in \mathrm{SC}^{\mathrm{nd}} : \rho(\sigma) \le n\}$$

is closed under faces and determines a finite simplicial subset $\mathcal{X}_n \subset \mathrm{SC}$*. These subsets form an exhaustive filtration*

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathrm{SC}, \qquad \bigcup_{n \ge 0} \mathcal{X}_n = \mathrm{SC},$$

and for every $n \geq 0$

$$|\mathcal{X}_n| \simeq \mathbb{CP}^n.$$

Moreover,

$$H_k(\mathcal{X}_n, \mathcal{X}_{n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 2n, \\ 0, & k \neq 2n, \end{cases}$$

and the restriction $c \in H^2(\mathcal{X}_n; \mathbb{Z})$ *of the universal binary Chern class satisfies*

$$H^*(\mathcal{X}_n; \mathbb{Z}) \cong \mathbb{Z}[c]/(c^{n+1}).$$

The proof occupies Sections 3–9. It is elementary once the correct normal form is isolated. The perfect-matching cycles and their Pfaffian signs are then recovered as the top relative generators, rather than inserted as an external ansatz.

We distinguish three levels of assertions throughout the note.

(A) The A/B normal form, derangement bijection, relative spectral sequence, projective-space homotopy type, and perfect-matching formula are proved here.

(B) The $\mathbb{CP}^2_9 \to \mathrm{SC}$ and $\mathbb{CP}^3_{18} \to \mathrm{SC}$ maps are finite exact computations with independent certificates. They are stated as computational propositions, not used in the proof of Theorem 1.1.

(C) The Barratt–Eccles action on normalized cochains is a theorem of Berger–Fresse [6]. The proposed identification of our operator T, the A/B rank, or the Arnoux–Marin carry with specific cells of the Barratt–Eccles/surjection operads is an open structural problem.

2 The simplicial set of circular permutations

2.1 Circular orders as simplices

For $n \geq 0$, let SC_n be the set of oriented circular orders on

$$[n] = \{0, 1, \ldots, n\}.$$

Equivalently,

$$\mathrm{SC}_n \cong \Sigma_{n+1}/C_{n+1},$$

where the cyclic group acts on the right. We represent a circular order by the unique word beginning with 0,

$$\sigma = (0, \sigma_1, \dots, \sigma_n).$$

The face d_i deletes the bead i and standardizes the remaining labels increasingly. The degeneracy s_i shifts every label $> i$ upward by one and inserts the new bead $i+1$ immediately after i. These operations make SC_* a simplicial set [14].

It is worth emphasizing a point which is occasionally obscured by geometric language. The simplicial set SC has a single 0-simplex. Nevertheless every simplex $x \in SC_n$ has a characteristic or Yoneda map

$$\Delta[n] \longrightarrow SC,$$

and its abstract vertices $0, \dots, n$ are circularly ordered by x. Thus a simplicial map

$$F : K \longrightarrow SC$$

from an arbitrary simplicial set K is equivalently a circular order on the abstract vertices of every characteristic simplex of K, compatible with all faces and all degeneracies. We shall call such a map a *cyclic branching.*

2.2 Degeneracy and derangements

A circular permutation $\sigma \in SC_n$ is degenerate exactly when, for some $0 \le i < n$, the bead $i+1$ immediately follows the bead i in the circular order. Consequently the nondegenerate n-simplices are counted by the derangement number.

Proposition 2.1. *The number of nondegenerate n-simplices of* SC *is*

$$!n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)!.$$

The first normalized ranks are

$$1, 0, 1, 2, 9, 44, 265, 1854, 14833, \dots.$$

Proof. There are $n!$ circular orders on $n+1$ labeled beads. For each $i \in \{0, \dots, n-1\}$ impose the adjacency $i \to i+1$. If j chosen adjacencies are imposed, merge the corresponding successions into blocks; there remain $n+1-j$ circular blocks and hence $(n-j)!$ circular orders. Inclusion–exclusion gives the formula. □

The unique nondegenerate 2-simplex is

$$a = (0, 2, 1).$$

The binary Chern cocycle of [12] is normalized by

$$c(a) = 1,$$

and vanishes on the degenerate 2-simplices. It represents the universal class $c_1 \in H^2(|SC|; \mathbb{Z})$.

3 The A and B insertions

The central combinatorics is controlled by two elementary ways of adding the new largest bead.

Definition 3.1 (A-insertion). Let $\tau \in SC_m$ and $0 \le i \le m$. Define $A_i(\tau) \in SC_{m+1}$ by inserting the new largest bead $m + 1$ immediately after the bead i.

Thus $A_m(\tau) = s_m\tau$ is degenerate.

Definition 3.2 (B-insertion). Let $\tau \in SC_m$ and $0 \le i \le m$. First apply the simplicial degeneracy s_i, so that a new bead $i + 1$ is inserted immediately after i and all old labels $> i$ are shifted upward. Then insert the new largest bead $m + 2$ between i and this new bead $i + 1$. Locally,

$$\cdots i \cdots \quad \longmapsto \quad \cdots i,\ m + 2,\ i + 1 \cdots .$$

The result is denoted $B_i(\tau) \in SC_{m+2}$.

The B-insertions are the degree-two moves which generated the perfect-matching cycle in the earlier version of this note.

3.1 Face identities

For a bead i of a circular word τ, write $\mathrm{pred}_\tau(i)$ for its predecessor. The following formulas will be used repeatedly.

Lemma 3.3 (B-faces). *For $\tau \in SC_m$ and $0 \le i \le m$,*

$$\begin{aligned} d_j B_i(\tau) &= B_{i-1}(d_j\tau), & j < i, \\ d_i B_i(\tau) &= A_{\mathrm{pred}_\tau(i)}(\tau), & \\ d_{i+1} B_i(\tau) &= A_i(\tau), & \\ d_j B_i(\tau) &= B_i(d_{j-1}\tau), & i+2 \le j \le m+1, \\ d_{m+2} B_i(\tau) &= s_i\tau. & \end{aligned}$$

The last term is degenerate.

Proof. Delete the indicated bead before or after the local insertion and then standardize. Only the two faces which delete the two beads adjacent to the new maximum require comment. Deleting the inserted $i+1$ leaves the maximum after i, giving $A_i\tau$. Deleting the old bead i leaves the maximum after the former predecessor of i, giving $A_{\mathrm{pred}_\tau(i)}\tau$. The final face deletes the new maximum and recovers $s_i\tau$. □

There are analogous, simpler identities for A_i. Every nondegenerate face of $A_i\tau$ other than the face deleting the new maximum is an A-insertion of a face of τ (with the evident shift of the insertion index); the face deleting the new maximum is τ. In particular a face of an A- or B-insertion reduces the "reduced degree" used below by at most one.

3.2 The degree-two anti-chain operator

Define

$$T_m(\tau) = \sum_{i=0}^{m} (-1)^i B_i(\tau) \in N_{m+2}(SC; \mathbb{Z}). \tag{1}$$

Theorem 3.4 (anti-chain identity). *On normalized chains,*

$$\partial T = -T\partial.$$

Proof. Expand $\partial T_m(\tau)$ and use Lemma 3.3. The two exceptional faces contribute

$$\sum_{i=0}^{m} \left(A_{\mathrm{pred}_\tau(i)}\tau - A_i\tau\right) = 0,$$

because $i \mapsto \mathrm{pred}_\tau(i)$ is a permutation of the beads and $A_m\tau$ is normalized to zero. The last faces $s_i\tau$ are degenerate. Reindexing all remaining ordinary faces produces exactly $-T_{m-1}(\partial\tau)$. □

4 Unique A/B normal form and the intrinsic rank

We now reverse the insertions. This is the step which turns the earlier experimental filtration into an intrinsic one.

Theorem 4.1 (unique A/B reduction). *Let $\sigma \in SC_k$ be nondegenerate and $k > 0$. Let p and q be respectively the predecessor and successor of the maximal bead k in the circular word σ. Then $p \leq k-2$ and precisely one of the following alternatives occurs.*

(i) If $q \neq p+1$, then $\tau = d_k\sigma$ is nondegenerate and

$$\sigma = A_p(\tau).$$

(ii) If $q = p+1$, delete the bead k and then the bead $p+1$, and standardize. The resulting $\tau \in SC_{k-2}$ is nondegenerate and

$$\sigma = B_p(\tau).$$

Both the alternative and the predecessor p are unique.

Proof. Since σ is nondegenerate, k cannot immediately follow $k-1$, hence $p \neq k-1$. If deleting k does not create a new forbidden succession, the result is nondegenerate and reinserting k after p is exactly A_p. The only new forbidden succession which deletion of k can create is $p, p+1$, and this occurs precisely when $q = p+1$. In that case deleting this newly adjacent bead $p+1$ removes the unique new degeneracy, and the inverse operation is exactly B_p. Every statement is read from the two neighbours of the unique maximal bead, so uniqueness is immediate. □

Iterating Theorem 4.1 reduces every nondegenerate simplex to the unique vertex.

Definition 4.2 (rank). Set $\rho((0)) = 0$. Recursively define

$$\rho(A_i\tau) = \rho(\tau) + 1, \qquad \rho(B_i\tau) = \rho(\tau) + 1.$$

Equivalently, $\rho(\sigma)$ is the length of the unique A/B normal form of σ.

If the normal form contains a letters A and b letters B, then

$$a + b = \rho(\sigma), \qquad a + 2b = \dim \sigma.$$

Thus for a rank-n, degree-k simplex,

$$\boxed{b = k - n, \qquad a = 2n - k.} \tag{2}$$

In particular

$$n \le k \le 2n.$$

For $k > 0$ the lower equality is not attained by a nondegenerate simplex unless the normal form constraints permit it; the formula itself is the useful invariant.

Lemma 4.3 (rank does not increase on faces). *If σ is nondegenerate and $d_j\sigma$ is nondegenerate, then*

$$\rho(d_j\sigma) \le \rho(\sigma).$$

Proof. Induct on $\rho(\sigma)$ and use the A- and B-face identities. An ordinary face of $A_i\tau$ or $B_i\tau$ is an A- or B-insertion of a face of τ, after a possible index shift; the exceptional faces of $B_i\tau$ are A-insertions of τ itself, and the final face is degenerate. The exceptional face of $A_i\tau$ which deletes the new maximum is τ. In no case can more than one new A/B letter survive above the normal form of a face of τ. The induction hypothesis gives the result. □

5 Derangements refined by cycle number

The total number of nondegenerate simplices is a derangement number by Proposition 2.1. The rank refines this to the number of cycles in the derangement.

Let $D(k, j)$ denote the number of derangements of $[k] = \{1, \ldots, k\}$ having exactly j cycles.

Theorem 5.1 (rank-preserving derangement bijection). *There is a canonical bijection*

$$\Phi_k : SC_k^{nd} \longrightarrow \{\textit{derangements of } [k]\}$$

such that

$$\rho(\sigma) = n \iff \Phi_k(\sigma) \textit{ has } k - n \textit{ cycles.}$$

Consequently

$$\boxed{\#\{\sigma \in SC_k^{nd} : \rho(\sigma) = n\} = D(k, k - n).} \tag{3}$$

Proof. Proceed recursively along the unique normal form. Suppose first $\sigma = A_p(\tau)$, where τ has degree $k-1$. Put $x = p+1$. Starting from the cycle notation for $\Phi_{k-1}(\tau)$, insert the new letter k immediately after x in its cycle. This preserves the number of cycles.

Suppose instead $\sigma = B_p(\tau)$, with τ of degree $k-2$. Again put $x = p+1$. Relabel $\Phi_{k-2}(\tau)$ increasingly from $[k-2]$ onto $[k-1] \setminus \{x\}$ and adjoin the 2-cycle $(x\ k)$. This increases the number of cycles by one.

The construction is invertible: inspect the cycle containing k. If it is a 2-cycle, remove it and recover a unique B-step. If it has length at least three, delete k from that cycle and recover a unique A-step together with its predecessor. Thus Φ_k is a bijection.

If the normal form has b letters B, then the derangement has exactly b cycles. By (2), $b = k - n$. □

The standard recurrence now appears without computation:

$$D(k,j) = (k-1)\big(D(k-1,j) + D(k-2,j-1)\big). \tag{4}$$

Indeed the cycle containing the largest letter either has length at least three, or is a 2-cycle.

5.1 Relation with the complex of injective words

The cycle refinement in Theorem 5.1 is strongly reminiscent of a known Hodge decomposition. Let I_k be the complex of injective words on $[k]$, with boundary given by the alternating sum of deletion of letters. Farmer, Björner–Wachs, Reiner–Webb and others studied its topology and symmetric-group representation. Hanlon–Hersh [8] constructed a Hodge decomposition using Eulerian idempotents and proved, in particular, that the dimension of the j-th Hodge summand of the top homology equals the number of derangements with exactly j cycles. They also prove that the Eulerian idempotents intertwine naturally with the simplicial boundary.

Our rank-n diagonal is precisely

$$j = k - n.$$

Under a codimension-one face $k \mapsto k-1$, the same diagonal becomes $j \mapsto j-1$, exactly the shift appearing in the Hanlon–Hersh boundary identities. This is more than an equality of total dimensions, but we do not presently claim an isomorphism of chain complexes. Constructing such a comparison would explain the rank filtration representation-theoretically.

Miller–Wilson [11] obtain a new description of the homology of the complex of injective words in the course of their theory of secondary representation stability. Their secondary stabilization is generated by adding pairs of points, a feature which is visibly parallel to the degree-two B-insertion. This parallel is structural motivation; an explicit identification with B_i is an open problem.

6 The finite filtration $\mathcal{X}_n$

Definition 6.1. Let $\mathcal{X}_n$ be the simplicial subset of SC whose nondegenerate simplices have rank at most n:

$$\mathcal{X}_n^{\mathrm{nd}} = \{\sigma \in \mathrm{SC}^{\mathrm{nd}} : \rho(\sigma) \leq n\}.$$

Set $\mathcal{X}_0$ equal to the unique vertex.

Lemma 4.3 shows that this is closed under faces; degeneracies are included by definition of the generated simplicial subset. Equation (2) shows that $\mathcal{X}_n$ has no nondegenerate simplex above degree $2n$, hence is finite.

Corollary 6.2 (face numbers). *For* $k \geq 1$*,*

$$f_k^{\mathrm{nd}}(\mathcal{X}_n) = \sum_{j \geq \max(1, k-n)} D(k, j),$$

and

$$f_k^{\mathrm{nd}}(\mathcal{X}_n, \mathcal{X}_{n-1}) = D(k, k-n).$$

At the top,

$$f_{2n}^{\mathrm{nd}}(\mathcal{X}_n) = D(2n, n) = (2n-1)!!.$$

The last identity holds because a derangement of $2n$ letters with n cycles must be a perfect matching, i.e. a product of n disjoint transpositions.

For reference, the nondegenerate f-vectors through $n = 6$ are

n	$(f_0^{nd}, \dots, f_{2n}^{nd})$
1	$(1, 0, 1)$
2	$(1, 0, 1, 2, 3)$
3	$(1, 0, 1, 2, 9, 20, 15)$
4	$(1, 0, 1, 2, 9, 44, 145, 210, 105)$
5	$(1, 0, 1, 2, 9, 44, 265, 1134, 2485, 2520, 945)$
6	$(1, 0, 1, 2, 9, 44, 265, 1854, 9793, 28952, 45045, 34650, 10395)$

6.1 Equivalence with the old perfect-matching definition

Earlier versions of this note defined $\mathcal{X}_n$ as the smallest simplicial subset generated by the support of the top cycle Z_n. The definitions agree.

Proposition 6.3. *Every rank-n nondegenerate simplex is a face of a rank-n simplex of degree $2n$. Equivalently, the rank filtration $\mathcal{X}_n$ is the simplicial subset generated by its* B*-only top simplices.*

Proof. An A-step may be promoted to a B-step because

$$A_i(\tau) = d_{i+1}B_i(\tau).$$

The ordinary face identities in Lemma 3.3, together with the analogous A-face identities, let this promotion be lifted through all outer insertions. Induct on the number $a = 2n - k$ of A letters in the normal form. Replacing one A by a B increases the dimension by one and leaves the rank unchanged; after a promotions one obtains a B-only simplex in degree $2n$ having the original simplex as an iterated face. □

The top simplices are therefore naturally indexed by perfect matchings on $[2n]$. This recovers the old cycle construction and explains why perfect matchings appeared.

7 Perfect-matching cycles and Pfaffian signs

Let $\mathrm{PM}(2n)$ denote the perfect matchings on $[2n]$. Draw a matching by arcs above the line and let $\mathrm{cr}(M)$ be the number of crossing pairs of arcs. The recursive B-construction gives a top simplex $\sigma_M \in \mathcal{X}_n$ for every matching.

Define

$$Z_0 = (0), \qquad Z_n = T_{2n-2}Z_{n-1}.$$

By Theorem 3.4, every Z_n is a cycle.

Theorem 7.1 (perfect-matching/Pfaffian formula). *For every* $n \geq 1$,

$$\boxed{Z_n = \sum_{M \in PM(2n)} (-1)^{\mathrm{cr}(M)} \sigma_M.}$$

All coefficients are ± 1 *and*

$$\|Z_n\|_1 = (2n-1)!!.$$

Proof. At the final B_i insertion the new chord joins the inserted position to the new maximum. Modulo two, the number of old chords crossed by it is congruent to the insertion index i. Thus the factor $(-1)^i$ in T changes the coefficient exactly by the Pfaffian crossing sign. Induction gives the formula. $\square$

The first two cycles are

$$Z_1 = (0, 2, 1)$$

and

$$\boxed{Z_2 = (0, 2, 4, 3, 1) - (0, 3, 1, 4, 2) + (0, 4, 1, 3, 2).}$$

8 The relative polygon complex and integral homology

This section gives the key homological argument.

Let

$$C_*^{(n)} = N_*(\mathcal{X}_n, \mathcal{X}_{n-1}; \mathbb{Z}).$$

Its basis consists exactly of the rank-n nondegenerate simplices. By the first step of the unique normal form, every basis vector is uniquely either

$$A_i\tau \qquad \text{or} \qquad B_i\tau,$$

where τ has rank $n-1$. If $\tau \in SC_m$, the nondegenerate A-terms are $A_0\tau, \ldots, A_{m-1}\tau$, while the B-terms are $B_0\tau, \ldots, B_m\tau$.

Filter $C^{(n)}$ by the reduced degree $m = \dim \tau$. The boundary preserves m only through the two exceptional faces in Lemma 3.3. Modulo lower filtration,

$$\partial_v B_i\tau = (-1)^i \big(A_{\mathrm{pred}_\tau(i)}\tau - A_i\tau\big).$$

Put

$$b_i = (-1)^i B_i\tau, \qquad a_i = A_i\tau,$$

and set $a_m = 0$ because $A_m\tau = s_m\tau$ is degenerate. Then

$$\partial_v b_i = a_{\mathrm{pred}_\tau(i)} - a_i. \tag{5}$$

Since $i \mapsto \mathrm{pred}_\tau(i)$ is a single cyclic permutation of the $m+1$ beads, (5) is the incidence matrix of an $(m+1)$-gon with one vertex collapsed to the basepoint.

Lemma 8.1 (polygon fiber). *For fixed $\tau \in SC_m^{nd}$, the vertical two-term complex generated by the $B_i\tau$ and $A_i\tau$ has homology $\mathbb{Z}$ in the B-degree and zero otherwise. Its canonical generator is*

$$T_m(\tau) = \sum_{i=0}^{m} (-1)^i B_i(\tau).$$

Proof. The incidence map of a connected polygon with one vertex pinned has rank m, hence zero cokernel and a rank-one kernel. In the rescaled basis the kernel is generated by $\sum_i b_i$, which is precisely $T_m(\tau)$ in the original basis. □

After taking vertical homology, the first differential is determined by Theorem 3.4:

$$\partial T(\tau) = -T(\partial\tau).$$

Thus the E^1 complex is the rank-$(n-1)$ relative complex shifted by two degrees. All remaining boundary terms lower the reduced-degree filtration by exactly one, so there are no higher filtration drops and consequently no higher differentials.

Theorem 8.2 (relative recurrence). *For every $n \geq 2$,*

$$\boxed{H_k(\mathcal{X}_n, \mathcal{X}_{n-1}; \mathbb{Z}) \cong H_{k-2}(\mathcal{X}_{n-1}, \mathcal{X}_{n-2}; \mathbb{Z}).}$$

Under this isomorphism the relative class of $Z_n = TZ_{n-1}$ corresponds to the relative class of Z_{n-1}.

Taking $\mathcal{X}_0$ to be a point, $\mathcal{X}_1$ has one nondegenerate 2-cell and no nondegenerate 1-cells, so

$$H_k(\mathcal{X}_1, \mathcal{X}_0; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 2, \\ 0, & k \neq 2. \end{cases}$$

Induction gives:

Corollary 8.3. *For every* $n \geq 1$,

$$H_k(\mathcal{X}_n, \mathcal{X}_{n-1}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 2n, \\ 0, & k \neq 2n. \end{cases}$$

The relative generator is represented by Z_n.

9 The Chern pairing and the homotopy type of $\mathcal{X}_n$

We use the Alexander–Whitney cup product. If $\sigma \in SC_{2n}$, then

$$c^n(\sigma) = \prod_{r=0}^{n-1} c\big(\sigma|_{\{2r,2r+1,2r+2\}}\big).$$

Lemma 9.1. *If* $\tau \in SC_{2r}$ *and* $0 \leq i \leq 2r$, *then*

$$c^{r+1}(B_i\tau) = \begin{cases} 0, & i < 2r, \\ c^r(\tau), & i = 2r. \end{cases}$$

Proof. In the Alexander–Whitney product the last factor is the restriction to the final three labels $\{2r, 2r+1, 2r+2\}$. For $i = 2r$, the local order is $(2r, 2r+2, 2r+1)$ and standardizes to the unique nondegenerate triangle $(0, 2, 1)$. For $i < 2r$ the final triangle is degenerate or has binary Chern value zero. The front face in the terminal case is τ. □

Theorem 9.2 (primitive pairing). *For all* $n \geq 1$,

$$\boxed{\langle c^n, Z_n \rangle = 1.}$$

Proof. Expand Z_n by the iterated T formula. Lemma 9.1 kills every term except the sequence of terminal choices $i = 0, 2, 4, \dots, 2n-2$. All these indices are even, so its coefficient is $+1$. Repeated application of the lemma gives value 1. □

The long exact sequence of the pair and Corollary 8.3 imply

$$H_k(\mathcal{X}_n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise}. \end{cases}$$

For $r \leq n$, the class Z_r remains a generator of $H_{2r}(\mathcal{X}_n; \mathbb{Z})$, and Theorem 9.2 shows that c^r is its dual generator. Hence

$$H^*(\mathcal{X}_n; \mathbb{Z}) \cong \mathbb{Z}[c]/(c^{n+1}).$$

Theorem 9.3 (finite projective-space model). *For every* $n \geq 0$*,*

$$|\mathcal{X}_n| \simeq \mathbb{CP}^n.$$

Proof. Compose the inclusion $|\mathcal{X}_n| \to |\mathrm{SC}|$ with a fixed homotopy equivalence $|\mathrm{SC}| \simeq \mathbb{CP}^\infty$. The CW dimension of $|\mathcal{X}_n|$ is $2n$, so cellular approximation deforms this map into the $2n$-skeleton $\mathbb{CP}^n \subset \mathbb{CP}^\infty$. Let

$$g_n : |\mathcal{X}_n| \longrightarrow \mathbb{CP}^n$$

be the resulting map. It satisfies $g_n^*(h) = c$, where h is the hyperplane class. The preceding calculation shows that g_n is an integral cohomology and hence homology isomorphism. The space $|\mathcal{X}_n|$ has one 0-cell and no nondegenerate 1-cells, so it is simply connected; so is $\mathbb{CP}^n$. The homological Whitehead theorem gives the homotopy equivalence. □

Corollary 9.4 (exhaustion of $K(\mathbb{Z}, 2)$). *The projective models form an exhaustive filtration*

$$\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots \subset \mathrm{SC}, \qquad \bigcup_{n \geq 0} \mathcal{X}_n = \mathrm{SC}.$$

Proof. Every nondegenerate simplex has a finite A/B normal form and hence finite rank. □

Remark 9.5. The theorem identifies each individual stage with $\mathbb{CP}^n$. A filtered homotopy equivalence which simultaneously identifies all inclusions $\mathcal{X}_{n-1} \hookrightarrow \mathcal{X}_n$ with the standard inclusions $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ should exist, but this stronger filtered statement is not needed above and is left separate from the proved theorem.

10 The divided-power viewpoint

The space $\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$ is an abelian H-space. With respect to its Pontryagin product, integral homology is the divided-power algebra on the fundamental class in degree two. If $\beta_n \in H_{2n}(\mathbb{CP}^\infty; \mathbb{Z})$ is dual to c^n, then

$$\beta_i * \beta_j = \binom{i+j}{i} \beta_{i+j}.$$

Theorem 9.2 says precisely that Z_n is an explicit normalized-chain representative of β_n.

The operator T should therefore be regarded as an integral chain-level *divided-power raising mechanism*: it sends a chosen representative of β_{n-1} to one of β_n without dividing by the integer n which appears in ordinary Pontryagin multiplication by β_1.

Warning 10.1. This does *not* mean that T is a unary operation supplied by an E_∞ operad. A contractible arity-one part cannot contain such a new degree-two unary homology operation. Any operadic explanation of T must involve the binary multiplication together with coherent choices, a bar/divided-power construction, or a related coalgebraic structure.

This distinction becomes important in Section 13.

11 Classical and modern finite models of $\mathbb{CP}^n$

The models $\mathcal{X}_n$ are finite simplicial *sets*; their characteristic simplices may have repeated geometric vertices after realization and they are not ordinary PL triangulations. It is therefore useful to compare them with genuinely triangulated or regular simplicial-cell models.

11.1 $\mathbb{CP}^2_9$

Kühnel and Banchoff constructed the famous 9-vertex triangulation $\mathbb{CP}^2_9$ with

$$f = (9, 36, 84, 90, 36)$$

[9]. The complex-crystallographic structure was developed further by Arnoux–Marin [1].

An exact finite calculation produces a cyclic branching

$$F_2 : \mathbb{CP}^2_9 \longrightarrow \mathrm{SC}$$

such that

$$F_2^* c = h \in H^2(\mathbb{CP}^2; \mathbb{Z}).$$

The 36 top 4-simplices have only 13 distinct images in SC_4; among these, only two are nondegenerate. Counting with multiplicity, five of the 36 facets map to nondegenerate 4-simplices and 31 to degenerate ones. The two nondegenerate images are

$$(0, 2, 4, 1, 3), \qquad (0, 3, 1, 4, 2).$$

After orienting the fundamental cycle and normalizing away degenerate simplices, its pushforward is

$$-(0,2,4,1,3) + 2(0,3,1,4,2),$$

which has ℓ^1-norm 3, the same as the canonical perfect-matching cycle Z_2. Thus a geometric triangulation can fold dramatically in SC while retaining the primitive Chern pairing.

11.2 $\mathbb{CP}^3_{18}$

Bagchi and Datta constructed an 18-vertex triangulation $\mathbb{CP}^3_{18}$ with

$$f = (18, 153, 783, 2110, 3021, 2177, 622),$$

containing Kühnel's $\mathbb{CP}^2_9$ as the induced subcomplex on nine of its vertices [2]. The complementary nine-vertex subcomplex is a 6-ball.

A complete exact certificate gives a binary integral cocycle

$$u \in Z^2(\mathbb{CP}^3_{18}; \mathbb{Z}), \qquad u(\tau) \in \{0, 1\},$$

with support size 362, restricting exactly to the Arnoux–Marin-compatible cocycle on the induced $\mathbb{CP}^2_9$ and pairing to $+1$ with its generating 2-sphere. The binary Chern theorem reconstructs a compatible circular order on every characteristic simplex, hence a simplicial map

$$F_3 : \mathbb{CP}^3_{18} \longrightarrow \mathrm{SC}, \qquad F_3^* c = h.$$

All 622 top simplices pass the circular-order reconstruction test.

The top-dimensional image contains 275 distinct SC_6-simplices, of which 71 are nondegenerate and 204 degenerate. With multiplicity, 140 of the 622 facets map to nondegenerate simplices. After orientations and cancellation, the normalized pushforward fundamental chain uses 54 distinct nondegenerate 6-simplices and has ℓ^1-norm 66. This is much larger than the canonical 15-term cycle Z_3 in $\mathcal{X}_3$.

Thus $\mathcal{X}_2$ and $\mathcal{X}_3$ should be viewed as highly compressed simplicial-set models, while $\mathbb{CP}^2_9$ and $\mathbb{CP}^3_{18}$ are genuine manifold triangulations mapping into them only after substantial folding in SC.

11.3 Datta–Spreer simplicial-cell models

Datta and Spreer use the symmetric-product presentation

$$\mathrm{Sym}^n(S^2) \cong \mathbb{CP}^n$$

to construct, for every $n \geq 2$, an explicit regular simplicial-cell decomposition T_n of $\mathbb{CP}^n$ [7]. The first derived subdivision is an ordinary triangulation. Their top cells satisfy

$$f_{2n}(T_n) = \frac{(2n)!}{n!}.$$

A top cell may be encoded by a perfect matching of the $2n$ monotone-path steps together with one U/L choice for each of the n matched pairs. Therefore

$$\frac{(2n)!}{n!} = 2^n(2n-1)!!.$$

Forgetting the U/L colors leaves exactly the number of top simplices of $\mathcal{X}_n$. This is a clean numerical and combinatorial bridge, but it does not by itself define a simplicial map $T_n \to \mathcal{X}_n$; face coherences are essential.

11.4 Sergeraert–Kenzo models: small homotopy type versus binary structure

Sergeraert used Kenzo effective homology to extract finite simplicial subsets of the minimal Kan model of $K(\mathbb{Z}, 2)$ having the homotopy type of $\mathbb{CP}^n$ for $n \leq 6$ [18, 16]. Write these models as Y_n. Sergeraert stresses that the homeomorphism question is open; the statement is about homotopy type. The models are extremely economical. For example

$$\begin{aligned}
f(Y_2) &= (1, 0, 2, 3, 3), \\
f(Y_3) &= (1, 0, 3, 10, 25, 30, 15), \\
f(Y_4) &= (1, 0, 4, 22, 97, 255, 390, 315, 105).
\end{aligned}$$

The reported models continue through Y_6. Their top-dimensional count agrees exactly with ours:

$$f_{2n}(Y_n) = (2n-1)!! = f_{2n}^{\mathrm{nd}}(\mathcal{X}_n).$$

Sergeraert already singled out this odd double factorial in the Kenzo experiments. For $\mathcal{X}_n$, it is explained by the perfect matchings supporting Z_n.

The agreement stops dramatically in degree two. Let $e_1, \ldots, e_n$ denote the nondegenerate 2-simplices in the Kenzo bar notation and write $u_r = u(e_r)$ for a normalized integral cocycle. The complete Y_2 data printed by Sergeraert and the Y_3, Y_4 data now distributed with SageMath [17] contain tetrahedral relations forcing

$$(u_1, \ldots, u_n) = a(1, 2, \ldots, n), \qquad n = 2, 3, 4.$$

Hence
$$\mathrm{Bin}(Y_n) = \{0\}, \qquad n = 2, 3, 4.$$
So these small simplicial homotopy models do *not* carry a minimally triangulated tautological circle bundle without further subdivision.

This contrast clarifies what extra structure the present models retain. We should regard

$$(\mathcal{X}_n, i_n), \qquad i_n : \mathcal{X}_n \hookrightarrow \mathrm{SC},$$

as the basic object. The inclusion equips $\mathcal{X}_n$ with the canonical binary cocycle
$$c_n = i_n^* c_{01}, \qquad [c_n] = h,$$
and hence with the minimally triangulated tautological circle bundle (up to the sign convention exchanging tautological and dual tautological line bundles). A bare homotopy equivalence $Y \simeq \mathbb{CP}^n$ contains no such information.

For the explicit tetrahedral relations, the comparison table through $n = 6$, and the conjectural continuation to all Kenzo stages, see the [13].

12 Arnoux–Marin cyclic cuts and the carry picture

A second geometric source of circular orders is the complex-crystallographic construction of Arnoux–Marin [1]. The following elementary group observation isolates the part directly related to SC.

Let
$$H_n = \{\sigma \in \Sigma_{n+1} : \sigma(n) = n\}.$$
Every right coset of the cyclic subgroup C_{n+1} contains exactly one element of H_n: rotate a circular word until the bead n is last. Hence

$$H_n \cong \Sigma_{n+1}/C_{n+1} = \mathrm{SC}_n$$

canonically as sets. Arnoux–Marin use the same "last letter fixed" transversal when they pass to a reduced collection of $n!$ prisms in their fundamental region.

Let
$$j_n : \mathrm{SC}_n \longrightarrow \Sigma_{n+1}$$

be this canonical cut. It commutes with all faces except the face deleting the cut bead n. For that face, one must rotate again until $n-1$ is last. Thus there is a unique cyclic rotation $t_n(x) \in C_n$ such that

$$j_{n-1}(d_n x) = d_n j_n(x)\, t_n(x).$$

This rotation is the discrete *carry* of the cut.

In degree two there are only two circular permutations. The positive one requires no recut after deleting the terminal bead, whereas the negative one requires one cyclic rotation. Thus the first nontrivial carry is precisely the binary Chern cocycle

$$c(012)_{\mathrm{cyc}} = 0, \qquad c(210)_{\mathrm{cyc}} = 1.$$

This gives a conceptual explanation for why the Arnoux–Marin cyclic geometry is so well adapted to maps into SC: the Chern class measures the failure of a preferred linear cut of a circular order to be simplicial.

For higher n, writing the last-letter representative as

$$(a_0, \dots, a_{n-1}, n)$$

and setting

$$d_i = [a_i - a_{i-1}]_{n+1}, \qquad a_{-1} = n,$$

produces a positive cyclic-difference word. The residual Arnoux–Marin cyclic symmetry rotates this difference word. Its normalized sum is the cyclic descent number. This Eulerian stratification is suggestive in view of the Eulerian idempotents in the injective-word Hodge decomposition, but no theorem identifying the two structures is claimed here.

13 The Barratt–Eccles operad

The Barratt–Eccles connection is important for the continuation of the program, particularly for Thom classes, higher diagonals and Steenrod operations. We therefore record the construction and separate the established facts from the conjectural bridge to the A/B filtration.

13.1 Definition

For $r \geq 0$, let $E\Sigma_r$ be the homogeneous bar construction of the symmetric group. Its q-simplices are tuples

$$(w_0, \dots, w_q), \qquad w_i \in \Sigma_r,$$

with faces obtained by deleting an entry and degeneracies by repeating an entry. The right diagonal action of Σ_r is free and $E\Sigma_r$ is contractible. The collection

$$\mathcal{E}(r) = E\Sigma_r$$

forms a simplicial operad; operadic composition is induced by the usual block substitution of permutations. This is the Barratt–Eccles operad, originating in the Γ^+ constructions of Barratt–Eccles [3, 4]. Its normalized chains

$$N_*(\mathcal{E})$$

form an E_∞ differential graded operad.

There are two indices here which should never be confused: r is the operadic arity, whereas q is the simplicial degree inside $E\Sigma_r$. In contrast, an element of SC_q is a single circular permutation of $q+1$ labels. Thus there is no formal identity between SC and the Barratt–Eccles operad merely because both are built from symmetric groups.

13.2 Action on cochains: the established connection

The decisive theorem for the present program is due to Berger–Fresse.

Theorem 13.1 (Berger–Fresse [6]). *For every simplicial set X, the normalized cochain complex $N^*(X)$ carries, naturally in X, an algebra structure over the chain Barratt–Eccles operad $N_*(\mathcal{E})$. In particular the ordinary cup product is the degree-zero binary operation, and the higher operadic chains provide coherent homotopies for commutativity and all higher coherences.*

Therefore the following is not conjectural:

$$N^*(SC), \quad N^*(\mathcal{X}_n), \quad N^*(\mathbb{CP}^2_9), \quad N^*(\mathbb{CP}^3_{18})$$

all carry canonical, functorial Barratt–Eccles E_∞ structures, and the pullback maps induced by

$$\mathcal{X}_n \hookrightarrow SC, \qquad F_2 : \mathbb{CP}^2_9 \to SC, \qquad F_3 : \mathbb{CP}^3_{18} \to SC$$

are morphisms of E_∞ cochain algebras.

This gives a canonical answer to the question of how to transport higher cochain operations from the universal circular-permutation model to the finite projective models: no additional choices are required.

13.3 Higher diagonals on chains

For the later cellular and Thom-class applications it is useful that the construction is not only an abstract action after dualizing cochains. Berger–Fresse construct the relevant operations through the Eilenberg–Zilber operad and explicit interval-cut maps. Thus a Barratt–Eccles chain determines natural higher diagonal approximations

$$N_*(X) \longrightarrow N_*(X)^{\otimes r},$$

whose transposes give the operations on normalized cochains. In finite simplicial models such as $\mathcal{X}_n$, $\mathbb{CP}^2_9$ and $\mathbb{CP}^3_{18}$ there is no finiteness issue in passing between the two descriptions.

This chain-side formulation is important for the larger program. A strict cellular diagonal may fail to exist or may be noncanonical, while the Barratt–Eccles structure retains a canonical hierarchy of diagonal approximations and the homotopies between them. Consequently it is a natural language for comparing cellular, Whitney and Hodge constructions without pretending that a preferred strict diagonal exists.

13.4 The surjection operad and prismatic decomposition

Berger–Fresse also relate the Barratt–Eccles operad to the surjection operad. In arity r and degree d, the latter is generated by nondegenerate surjections

$$u : \{1, \ldots, r + d\} \twoheadrightarrow \{1, \ldots, r\}$$

with no equal adjacent values; the differential deletes entries with the standard signs. Its elements act on simplicial cochains by explicit interval-cut operations.

There is a table-reduction morphism

$$TR : N_*(\mathcal{E}) \longrightarrow \mathcal{S}$$

which is compatible with natural filtrations by E_n suboperads [6]. Geometrically, Berger–Fresse prove that the simplicial Barratt–Eccles operad admits a prismatic decomposition indexed by surjections, and the cellular chains of this decomposition are the surjection operad [5].

This prismatic description is particularly relevant to the present project. The Arnoux–Marin and Datta–Spreer constructions repeatedly confront us with products of simplices, staircase/shuffle subdivisions, and the need to compare different local linearizations after cyclic recutting. The Barratt–Eccles prism decomposition is a standard device whose purpose is precisely to package coherent permutation and shuffle homotopies. It is therefore a natural candidate for organizing the face compatibilities which are awkward if treated by individual triangulation tables.

13.5 Cup-i and Steenrod operations

The arity-two part makes the connection with Steenrod theory explicit. Over $\mathbb{F}_2$, the alternating Barratt–Eccles simplex

$$(\mathrm{id}, \tau, \mathrm{id}, \tau, \dots) \in E\Sigma_2, \qquad \tau = (12),$$

with $i+1$ entries represents, under table reduction and the Berger–Fresse action, the classical cup-i operation, up to the standard convention on indexing. Equivalently, the surjection-operad representative is the alternating surjection

$$(1, 2, 1, 2, \dots)$$

of the corresponding degree. Thus for a mod-two cocycle x of degree m,

$$\mathrm{Sq}^k[x] = [x \smile_{m-k} x].$$

The point for us is not merely that Steenrod squares exist abstractly: the Barratt–Eccles and surjection-operad models give explicit finite chain formulas on the same simplicial data used for the binary Chern class.

Consequently, once a combinatorial Thom class U is fixed, the Steenrod–Thom identity can in principle be verified at chain level in the same model. For the underlying real 2-plane bundle of the universal complex line bundle,

$$\mathrm{Sq}(U) = w(L_{\mathbb{R}}) \smile U = (1 + \bar{c}) \smile U,$$

so in particular

$$\mathrm{Sq}^2(U) = \bar{c} \smile U.$$

Here $\bar{c}$ is the mod-two reduction of the binary Chern cocycle. This is the precise entry point for the planned branch concerning combinatorial Thom classes and Steenrod operations.

13.6 What is and is not yet identified

There are three increasingly strong statements one might seek.

BE1. *Proved:* the cochains of SC and $\mathcal{X}_n$ are functorial Barratt–Eccles algebras, hence carry canonical cup-i and Steenrod operations.

BE2. *Strongly motivated:* the prismatic/table-reduction description should organize the shuffle and cyclic-cut coherences in the Arnoux–Marin and Datta–Spreer comparisons.

BE3. *Open:* identify the A/B filtration, the polygon-fiber operator T, or the perfect-matching chains Z_n with a distinguished family of cells, composites, or divided-power expressions in a Barratt–Eccles/surjection-operad construction.

The third statement must not be replaced by a superficial identification. In particular T is a degree-two unary chain operator, whereas the arity-one part of an E_∞ operad is contractible. Any genuine operadic interpretation of T must involve binary operations, bar constructions, divided powers, or a coalgebraic resolution rather than a single unary Barratt–Eccles element.

Question 13.2 (Barratt–Eccles realization of the rank filtration). *Is there a natural bar, divided-power, or surjection-operad construction on* $N_*(\mathrm{SC})$ *whose canonical filtration induces the* A/B *rank filtration* $\mathcal{X}_\bullet$ *and whose rank-one polygon fiber has fundamental chain* T*?*

Question 13.3 (operadic explanation of perfect matchings). *Can the Pfaffian cycle*

$$Z_n = \sum_M (-1)^{\mathrm{cr}(M)} \sigma_M$$

be obtained as the pairing sector of an iterated Barratt–Eccles/surjection operation, with the crossing sign arising from the ordinary shuffle/Koszul sign?

These questions are concrete enough to test first for $n = 2, 3$.

14 Why the Barratt–Eccles structure should be used in the other branches

The operadic language is particularly well suited to several later parts of the larger program.

14.1 Higher diagonals and cellular approximations

An ordinary cellular diagonal need not exist for an arbitrary regular cell structure, whereas chain diagonals and higher coherent diagonals do. The Barratt–Eccles action is one canonical organization of these coherences on simplicial cochains. This makes it a natural replacement for trying to choose simultaneously strict cellular diagonals in all local models.

14.2 Thom classes

A combinatorial Thom class should not merely represent a cohomology class; for characteristic-class calculations one wants its higher cochain operations to behave correctly. Working in a Barratt–Eccles algebra fixes, functorially, the coherent cup products needed to formulate the Thom identity and Steenrod operations before passing to cohomology.

14.3 Chern powers and the rational-parity formula

The Alexander–Whitney product already suffices to prove $\langle c^n, Z_n \rangle = 1$. Barratt–Eccles adds the coherent homotopies relating this chosen product to its permutations. This is the correct framework in which to compare the binary Chern cocycle with symmetric rational local formulas such as the Mnëv–Sharygin parity formula [15]; the comparison should not be made by equating their pointwise coefficients without tracking the relevant chain homotopies.

14.4 Products, shuffles, and cyclic recutting

The obstacle encountered in the Arnoux–Marin prism analysis was not local existence of staircase triangulations, but compatibility when different product orders are glued by cyclic symmetry. The surjection operad arises precisely as a cellular/prismatic model for coherent interval cuts and shuffles. A serious next step is therefore to formulate the cyclic recut as an explicit low-dimensional Barratt–Eccles chain and test whether its table reduction gives the carry terms already visible in the binary Chern cocycle.

15 Open directions

Question 15.1 (Comparison with the Kenzo filtration). *Do the Sergeraert–Kenzo fundamental cycles and the perfect-matching cycles* Z_n *become chain-homotopic under a natural comparison between the minimal Kan model of* $K(\mathbb{Z}, 2)$ *and* SC*? The identical top count* $(2n-1)!!$ *suggests that the same divided-power generator is being compressed in two different gauges, while the very different degree-two behavior shows that any such comparison cannot preserve binary representatives strictly.*

The main topological question of the old note is now settled by Theorem 9.3. The remaining questions are more structural.

Question 15.2 (filtered equivalence). *Can one construct a single filtered homotopy equivalence*

$$(\mathcal{X}_0 \subset \mathcal{X}_1 \subset \cdots) \simeq (\mathbb{CP}^0 \subset \mathbb{CP}^1 \subset \cdots)$$

compatible with the inclusions, not merely separate homotopy equivalences at each stage?

Question 15.3 (injective-word chain comparison). *Is there a direct chain map, respecting the diagonal* $j = k-n$*, between the rank-graded relative complex of* SC *and the Hanlon–Hersh Hodge decomposition of injective words? Such a map should explain the derangement bijection and Eulerian idempotents simultaneously.*

Question 15.4 (Datta–Spreer comparison). *Is there a natural chain-level or homotopy-coherent map*

$$T_n \rightsquigarrow \mathcal{X}_n$$

whose top-dimensional indexing forgets the U/L *colors of a Datta–Spreer cell and retains its perfect matching?*

Question 15.5 (sharp ℓ^1 norm). *Is* $\|Z_n\|_1 = (2n-1)!!$ *the minimum* ℓ^1 *norm of a primitive integral* $2n$*-cycle in* $N_*(\mathrm{SC})$*? This is true in the checked* $n = 1, 2$ *situations, but the old argument using the coefficient of the rational-parity formula was too naive: the rational local Chern representative and the Alexander–Whitney power of the binary cocycle must first be compared by an explicit cochain homotopy.*

Question 15.6 (Steenrod/Thom implementation). *Construct an explicit simplicial or cellular Thom model for the universal circle/line bundle over* SC, *restrict it to* $\mathcal{X}_n$, *and implement the Barratt–Eccles or surjection-operad cup-*i *formulas on its Thom cocycle. The first required identity is*

$$\mathrm{Sq}^2(U) = \bar{c}\,U.$$

This should be the entry point to the Steenrod-operation branch of the program.

A Exact low-dimensional checks

A short verification script accompanying this note performs the following exact combinatorial checks through a user-selectable degree.

1. enumerate circular permutations and test the degeneracy criterion;
2. reconstruct the unique A/B normal form;
3. verify rank nonincrease on all nondegenerate faces;
4. count rank-n degree-k simplices and compare with $D(k, k-n)$;
5. construct T and Z_n and verify $\partial Z_n = 0$;
6. close the top perfect-matching support under faces and compare it with the rank sublevel set;
7. reproduce the nondegenerate f-vectors displayed above.

No floating-point arithmetic is involved.

B Reference data

The top numbers for the canonical models are

$$f_{2n}^{\mathrm{nd}}(\mathcal{X}_n) = (2n-1)!! = 1, 3, 15, 105, 945, 10395, \ldots,$$

whereas the Datta–Spreer top-cell numbers are

$$f_{2n}(T_n) = \frac{(2n)!}{n!} = 2, 12, 120, 1680, 30240, 665280, \ldots.$$

Their ratio is exactly 2^n.

C Proof-status checklist

For future revisions it is useful to keep the following distinctions visible.

Statement	Status
Unique A/B normal form and rank	proved
Rank–derangement-cycle bijection	proved
$\mathcal{X}_n$ generated by perfect-matching top layer	proved
Relative polygon spectral sequence	proved
$\lvert\mathcal{X}_n\rvert \simeq \mathbb{CP}^n$	proved
Perfect-matching/Pfaffian formula for Z_n	proved
$\langle c^n, Z_n\rangle = 1$	proved
$\mathbb{CP}^2_9 \to \mathrm{SC}$ representing h	exact finite computation
$\mathbb{CP}^3_{18} \to \mathrm{SC}$ extending the $\mathbb{CP}^2$ map	exact finite computation
Barratt–Eccles action on all normalized cochains	theorem of Berger–Fresse
Identification of T with a Barratt–Eccles cell	open
Filtered equivalence with the standard $\mathbb{CP}^n$ skeleton filtration	open
Sharp ℓ^1 minimality of Z_n	open

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