Binary 2-cocycles, triangulated circle bundles and ice: which integral classes are binary?

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AI-generated working notes

September 25, 2026

Abstract

Let X be a finite ordered simplicial complex or semi-simplicial set. More generally, for a finite simplicial set we use normalized cochains and ask for values in $\{0, 1\}$ on the nondegenerate 2-simplices. An integral 2-cocycle is *binary* if it has this property. By a theorem of Mnëv, the binary classes are exactly the first Chern classes of circle bundles admitting a minimal semi-simplicial triangulation over the chosen base; equivalently, they are the classes pulled back from the circular-permutation simplicial set $SC_* \simeq K(\mathbb{Z}, 2)$ by simplicial maps. We study the finite set $\mathrm{Bin}(X) \subset H^2(X; \mathbb{Z})$ and emphasize that binary representability is extra simplicial structure, not a homotopy invariant of the underlying space.

The results fall into four groups.

- *Proven structure.* There is a combinatorial Milnor–Wood inequality, $|\langle \alpha, h\rangle| \leq \frac{1}{2}\|h\|_\Delta$. An exact description of the real relaxation says the relevant polytope is the $\frac{1}{2}$-ball of the dual norm. Every torsion class is binary, and $\mathrm{Bin}(X)$ is closed under taking n-th roots. We give integrality criteria, and show that the extreme classes are binary for orientable 3-pseudomanifolds.
- *Obstructions.* We give examples with gaps in the free part and with torsion translates missing. The central example is an explicit simplicial 3-complex X with $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$ but $\mathrm{Bin}(X) = \{0\}$. A concrete instance with 968 tetrahedra is verified by computer.

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- *Small projective-space models.* Sergeraert's Kenzo constructions give very small finite simplicial-set models $Y_n \simeq \mathbb{CP}^n$ for $n \leq 6$. From the available face tables we prove for $n = 2, 3, 4$ that every integral 2-cocycle takes the values $(a, 2a, \ldots, na)$ on the n nondegenerate 2-simplices. Consequently $\mathrm{Bin}(Y_n) = \{0\}$ for $n = 2, 3, 4$: these very small homotopy models do not support a minimally triangulated tautological circle bundle. This is contrasted with the companion models $\mathcal{X}_n \subset \mathrm{SC}_*$, which satisfy $|\mathcal{X}_n| \simeq \mathbb{CP}^n$ and carry the universal binary generator by construction.
- *Closed orientable 3-manifolds.* Binary cocycles are exactly the ice configurations (two-in/two-out states) on the dual graph, and Chern classes are flux sectors. A locking lemma, a string-ladder algorithm and computer experiments lead to conjectures. They also give a counterexample to the naive torsion conjecture, in a flat 3-manifold.

Contents

1 Introduction

1.1 The question

Let X be a finite simplicial complex with a total order on its vertices; more generally, a finite semi-simplicial set. Every 2-simplex then carries a canonical orientation $[v_0 < v_1 < v_2]$, so it makes sense to ask whether an integral cocycle takes values in $\{0, 1\}$. Mnëv [19] showed that such *binary* cocycles are exactly the Chern cocycles of *minimal* semi-simplicial triangulations of circle bundles over X. Combining this with the spindle-contraction argument of [19], a circle bundle over $|X|$ can be semi-simplicially triangulated over X if and only if its Chern class is represented by a binary cocycle. In [21], binary cocycles are identified with simplicial maps $X \to \mathrm{SC}_*$ into a simplicial set of circular permutations with $|\mathrm{SC}_*| \simeq K(\mathbb{Z}, 2)$. The binary cocycle is the pull-back of the *parity cocycle* c_{01}, which is the Huntington transitivity axiom for cyclic orders [10].

Which classes $\alpha \in H^2(X;\mathbb{Z})$ *are represented by binary cocycles?*

We write $\mathrm{Bin}(X) \subset H^2(X;\mathbb{Z})$ for this set. It is finite, and it depends on the triangulation, not only on the homotopy type of $|X|$. It exhausts H^2 under iterated barycentric subdivision (Theorem 7.3).

Companion projective-space note. A complementary positive construction is developed in *Small Simplicial Models of* $\mathbb{CP}^n$ *& a Finite Projective-Space Filtration in the Circular-Permutation Model of* $K(\mathbb{Z},2)$ [20]: finite simplicial subsets

$$\mathcal{X}_n \subset SC_*, \qquad |\mathcal{X}_n| \simeq \mathbb{CP}^n,$$

carry the distinguished binary generator obtained by restricting the universal cocycle c_{01}.

1.2 Summary of results

Statements labelled *Theorem*, *Proposition* or *Lemma* are proved here, or quoted with a reference. Statements labelled *Conjecture* are supported only by the computations of §10.

(1) **Combinatorial Milnor–Wood and the real relaxation** (§3). Let $\|\cdot\|_\Delta$ be the combinatorial ℓ^1-norm on $H_2(X;\mathbb{R})$. Every binary class satisfies $|\langle\alpha, h\rangle| \leq \frac{1}{2}\|h\|_\Delta$. A real class has a $[0,1]$-valued representative if and only if its dual norm is $\leq \frac{1}{2}$. So the *real* obstruction is exactly membership in the half-ball $P(X) = \frac{1}{2}B^*$.

(2) **Torsion and roots** (§4). Every torsion class is binary, via the "carry" cocycle. More generally, if α is binary and $n\beta = \alpha$, then β is binary (Theorem 4.3).

(3) **Integrality** (§5). If δ^1 is totally unimodular, then $\mathrm{Bin}(X)$ is exactly the set of lattice points of $P(X)$. If δ^2 is totally unimodular, for example for an orientable 3-pseudomanifold, then $\mathrm{conv}\,\mathrm{Bin}(X) = P(X)$, so the Milnor–Wood bound is sharp (Theorem 5.3).

(4) **Obstructions and the main example** (§6). There are examples with gaps in the free part and with missing torsion translates. The main example (Theorem 6.8) is a simplicial 3-complex X with $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Bin}(X) = \{0\}$. In it, a 4-triangle sphere represents 3 times the generator of H_2. Dimension 3 is the smallest possible, and torsion-freeness is necessary. An explicit instance is verified by computer (Example 6.11).

(5) **The SC_* viewpoint** (§7). All obstructions come from exactly eight non-fillable 3-horns of SC_*. Every class becomes binary after enough barycentric subdivisions.

(6) **Small projective models and loss of binarity** (§8). Sergeraert's Kenzo models $Y_n \simeq \mathbb{CP}^n$ are dramatically smaller than ordinary triangulations. For the available complete face data in dimensions $n = 2, 3, 4$ we prove $\mathrm{Bin}(Y_n) = \{0\}$. Thus a very small simplicial homotopy model of $\mathbb{CP}^n$ need not support the minimally triangulated tautological bundle. We compare these negative examples with the companion models $\mathcal{X}_n \subset \mathrm{SC}_*$, where the hyperplane class is binary by construction.

(7) **Ice** (§9). For a closed oriented 3-manifold, binary cocycles correspond to Eulerian subdigraphs of a canonical two-in/two-out orientation of the dual graph, that is, to ice states. Chern classes correspond to flux sectors. We prove a *locking lemma* for extreme sectors and introduce *strings*.

(8) **Evidence and conjectures** (§§10–11). In all tested closed orientable 3-manifolds the free part has no gaps. However, in the half-turn flat manifold $T^2 \times_{-1} S^1$ the torsion part locks at the extreme sectors.

2 Setting and first observations

2.1 Conventions

Unless explicitly stated otherwise, X is a finite ordered simplicial complex, or more generally a finite semi-simplicial set. We write X_k for its set of k-simplices. A k-simplex is written $\sigma = [v_0 < \cdots < v_k]$, and $d_i\sigma$ is the face opposite v_i. Cochains are functions $c\colon X_k \to A$, and

$$(\delta c)(\sigma) = \sum_{i=0}^{k+1} (-1)^i c(d_i\sigma).$$

Chains and cycles use the same orientations, and $\langle c, z\rangle = \sum_\sigma c(\sigma) z_\sigma$. For $h \in H_2$ we write z for a cycle representing it.

Definition 2.1. A *binary cocycle* is a cocycle $c \in Z^2(X;\mathbb{Z})$ with $c(X_2) \subseteq \{0,1\}$. Its class is a *binary class*, and $\mathrm{Bin}(X) \subseteq H^2(X;\mathbb{Z})$ is the set of binary classes.

Remark 2.2 (Finite simplicial sets). For a genuine finite simplicial set we use normalized cochains. Thus a binary cocycle vanishes on degenerate 2-simplices and has value 0 or 1 on every nondegenerate 2-simplex. Proposition 7.1 and the interpretation by minimally triangulated circle bundles remain valid in this form. The linear-programming and total-unimodularity arguments in Sections 3–6, however, are formulated for ordered complexes or semi-simplicial sets and are not silently extended to arbitrary degeneracy identifications.

Remark 2.3. Since $c(-\sigma) = -c(\sigma)$, "values in $\{0,1\}$" only makes sense after an orientation of every triangle has been fixed. We always use the orientation induced by the vertex order, or by the semi-simplicial structure. If orientations are chosen freely per triangle, the question becomes the one about $\{-1,0,1\}$-valued cocycles; everything below adapts.

Lemma 2.4 (Tetrahedral rule). *Let σ be a 3-simplex with faces $f_i = d_i\sigma$, and let $c \in \{0,1\}^{X_2}$. Then $(\delta c)(\sigma) = 0$ if and only if $c(f_0)+c(f_2) = c(f_1)+c(f_3)$. Exactly 6 of the 16 patterns satisfy this: none of the faces, all four faces, or exactly one of $\{f_0, f_2\}$ together with exactly one of $\{f_1, f_3\}$. In comparison, the mod-2 cocycle condition also allows $\{f_0, f_2\}$ and $\{f_1, f_3\}$.*

Lemma 2.5 (Constants). *Let 1_k be the constant cochain 1 on X_k.*

(i) $\delta 1_1 = 1_2$ and $\delta 1_2 = 0$.

(ii) The map $c \mapsto 1_2 - c$ is an involution on binary cocycles with $[1_2 - c] = -[c]$. Hence $\mathrm{Bin}(X) = -\mathrm{Bin}(X)$.

(iii) Every real 2-cycle z satisfies $\sum_f z_f = 0$. Consequently $\sum_f z_f^+ = \sum_f z_f^- = \frac{1}{2}\|z\|_1$.

Proof. For (i), on $[v_0v_1v_2]$ we get $1-1+1 = 1$, and on a 3-simplex $1-1+1-1 = 0$. Part (ii) follows. For (iii), $\sum_f z_f = \langle 1_2, z\rangle = \langle \delta 1_1, z\rangle = \langle 1_1, \partial z\rangle = 0$. $\square$

Part (iii) generalises Mnëv's observation [19, Lemma 9] that on an ordered, oriented, triangulated surface exactly half of the triangles are positively oriented.

Lemma 2.6 (Reduction to low skeleta). *(i) The groups* $H^2(X;\mathbb{Z})$ *and* $H^2(X^{(3)};\mathbb{Z})$ *coincide, and* $\mathrm{Bin}(X) = \mathrm{Bin}(X^{(3)})$*. So dimension 3 is the general case.*

(ii) Restriction gives an injection $H^2(X) \hookrightarrow H^2(X^{(2)}) = C^2/B^2$*, under which* $\mathrm{Bin}(X)$ *becomes* $\mathrm{Bin}(X^{(2)}) \cap H^2(X)$*. Explicitly,*

$$\mathrm{Bin}(X) = \{[1_S] : S \subseteq X_2,\ \ 1_S \in Z^2(X)\}.$$

Proof. Degree-2 cocycles and coboundaries only involve simplices of dimension ≤ 3. For (ii), B^2 is the same for X and $X^{(2)}$, and $[1_S] \in H^2(X)$ if and only if $1_S \in Z^2 + B^2 = Z^2$. □

Proposition 2.7 (Integer-programming form). *Let* $a \in Z^2(X;\mathbb{Z})$ *represent* α*. Then* $\alpha \in \mathrm{Bin}(X)$ *if and only if there is* $b \in C^1(X;\mathbb{Z})$ *with* $0 \leq a + \delta b \leq 1$ *on every triangle.*

Proposition 2.8 (Dimension ≤ 2). *If* $\dim X \leq 2$*, then* $\mathrm{Bin}(X)$ *generates* $H^2(X;\mathbb{Z})$*. In particular, if* $H^2(X;\mathbb{Z}) \neq 0$ *then* $\mathrm{Bin}(X) \neq \{0\}$*.*

Proof. Every cochain is a cocycle, and $H^2 = C^2/B^2$ is generated by the classes of the indicators $1_{\{f\}}$ of single triangles, which are binary. □

3 The real relaxation and a combinatorial Milnor–Wood inequality

Definition 3.1. For $h \in H_2(X;\mathbb{R})$ define

$$\|h\|_\Delta = \min\Big\{ \textstyle\sum_f |z_f| :\ z \in Z_2(X;\mathbb{R}),\ [z] = h \Big\}.$$

This is a norm on $H_2(X;\mathbb{R})$: it is the quotient of the ℓ^1-norm on Z_2 by the subspace B_2. It depends on the triangulation but not on the vertex order. The dual norm on $H^2(X;\mathbb{R}) = \mathrm{Hom}(H_2(X;\mathbb{R}),\mathbb{R})$ is $\|\alpha\|^* = \sup_{h\neq 0} |\alpha(h)|/\|h\|_\Delta$. We write B^* for its unit ball.

Lemma 3.2 (Duality). $\|\alpha\|^* = \min\{\|r\|_\infty :\ r \in Z^2(X;\mathbb{R}),\ [r] = \alpha\}$.

Proof. $\|\alpha\|^*$ is the norm of the functional $z \mapsto \langle a, z\rangle$ on (Z_2, ℓ^1). By Hahn–Banach it extends to a cochain r on (C_2, ℓ^1) with $\|r\|_\infty = \|\alpha\|^*$. Then $r - a$ vanishes on $Z_2 = \ker \partial_2$, so $r - a \in (\ker \partial_2)^\perp = \operatorname{im} \delta^1 = B^2$. Thus r is a cocycle cohomologous to a. The reverse inequality is immediate. □

Theorem 3.3 (Combinatorial Milnor–Wood inequality). *If* $\alpha \in \mathrm{Bin}(X)$, *then* $|\langle \alpha, h\rangle| \leq \frac{1}{2}\|h\|_\Delta$ *for all* $h \in H_2(X; \mathbb{R})$.

Proof. Let c be a binary representative. By Lemma 2.5, $c - \frac{1}{2}1_2$ is a real cocycle cohomologous to c with values $\pm\frac{1}{2}$. Hence $|\langle c, z\rangle| \leq \frac{1}{2}\|z\|_1$ for every cycle z with $[z] = h$. Now minimise over z. □

This is the same shift by $\frac{1}{2}$ that turns the Mnëv–Sharygin local formula [22] into the binary cocycle. It also parallels the classical proof of the Milnor–Wood inequality through the bounded Euler class [17, 29, 7]; there Gromov's simplicial norm replaces $\|\cdot\|_\Delta$.

Theorem 3.4 (Exact real criterion). *For* $\alpha \in H^2(X; \mathbb{R})$ *the following are equivalent.*

(i) α *is represented by a real cocycle with values in* $[0, 1]$.

(ii) $\|\alpha\|^* \leq \frac{1}{2}$.

(iii) $\langle \alpha, z\rangle \leq \frac{1}{2}\|z\|_1$ *for every elementary (support-minimal) real 2-cycle* z. *There are only finitely many of these up to positive scaling.*

Proof. (i)⇔(ii): r takes values in $[0, 1]$ if and only if $r - \frac{1}{2}1_2$ takes values in $[-\frac{1}{2}, \frac{1}{2}]$, and the two are cohomologous; now apply Lemma 3.2. (ii)⇔(iii): apply (ii) to $\pm z$ and use the conformal decomposition of Rockafellar [24]. Every $z \in Z_2$ is a positive combination $\sum \lambda_j e_j$ of elementary cycles e_j with the same sign pattern as z. Both $\langle \alpha, \cdot\rangle$ and $\|\cdot\|_1$ are additive on such decompositions. □

Definition 3.5. $P(X) := \frac{1}{2}B^* = \{\alpha \in H^2(X; \mathbb{R}) : \|\alpha\|^* \leq \frac{1}{2}\}$. This is a compact, centrally symmetric, rational polytope. By Theorem 3.4 it is also the image of the *cocycle polytope* $C(X) := Z^2(X; \mathbb{R}) \cap [0, 1]^{X_2}$ in $H^2(X; \mathbb{R})$.

Corollary 3.6 (Counting). *Let* $\rho\colon H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{R})$ *be the natural map and put* $\Lambda = \rho(H^2(X; \mathbb{Z}))$. *Then* $\rho(\mathrm{Bin}(X)) \subseteq \Lambda \cap P(X)$, *and*

$$|\mathrm{Tors} H_1(X; \mathbb{Z})| \ \leq\ \#\mathrm{Bin}(X) \ \leq\ |\mathrm{Tors} H_1(X; \mathbb{Z})| \cdot \#\big(\Lambda \cap P(X)\big).$$

For any basis $h_1, \dots, h_b$ *of* $H_2(X;\mathbb{Z})/\mathrm{tors}$ *this gives* $\#\mathrm{Bin}(X) \leq |\mathrm{Tors} H_1| \prod_i (2\lfloor \frac{1}{2}\|h_i\|$
$1)$. *The lower bound uses Proposition 4.1.*

Remark 3.7. For a closed oriented triangulated surface T, the fundamental cycle is the only cycle in its class, so $\|[T]\|_\Delta = \#T_2$. Theorem 3.3 then recovers the bound $|c| \leq \frac{1}{2}\#T_2$ of [19, Thm. 2]. Mnëv shows that all these values occur, so $\#\mathrm{Bin}(T) = \#T_2 + 1$.

4 Torsion, and closure under roots

Proposition 4.1 (Carry cocycles). *Every torsion class of* $H^2(X;\mathbb{Z})$ *is binary, for every triangulation and every vertex order.*

Proof. Let $n\alpha = 0$. By exactness of $H^1(X;\mathbb{Z}/n) \xrightarrow{\beta} H^2(X;\mathbb{Z}) \xrightarrow{\cdot n} H^2(X;\mathbb{Z})$ we have $\alpha = \beta(x)$ for some x. Represent x by a cocycle and lift its values to $u \in C^1(X;\mathbb{Z})$ with values in $\{0, \dots, n-1\}$. Then

$$(\delta u)[v_0v_1v_2] = u[v_1v_2] - u[v_0v_2] + u[v_0v_1] \in [-(n-1), 2(n-1)] \cap n\mathbb{Z} = \{0, n\}.$$

So $\delta u/n$ is a binary cocycle representing $\beta(x) = \alpha$. □

Geometrically, the carry cocycle is the pull-back of c_{01} along the simplicial map from the nerve of $U(1)^\delta$ to SC_* that sends $(g_1, \dots, g_k)$ to the cyclic order of the partial sums $0, g_1, g_1 + g_2, \dots$ on $\mathbb{R}/\mathbb{Z}$, with ties broken by index. Flat bundles are binary without subdivision, and their Chern classes are exactly the torsion classes.

Theorem 4.2 (Mnëv [19, Thm. 1]). *Let* X *be a finite semi-simplicial set. An oriented circle bundle over* $|X|$ *admits a semi-simplicial triangulation over* X *if and only if its first Chern class lies in* $\mathrm{Bin}(X)$.

Theorem 4.3 (Closure under roots). *If* $\alpha \in \mathrm{Bin}(X)$, $n \geq 1$ *and* $\beta \in H^2(X;\mathbb{Z})$ *satisfy* $n\beta = \alpha$, *then* $\beta \in \mathrm{Bin}(X)$.

Proof. Let ξ_α be a minimally triangulated bundle with $c_1 = \alpha$; it exists by Theorem 4.2. Let ξ_β be any bundle with $c_1 = \beta$. Since $\xi_\beta^{\otimes n} \cong \xi_\alpha$, the fibrewise map $z \mapsto z^n$ is an n-fold covering $E(\xi_\beta) \to E(\xi_\alpha)$ of total spaces, commuting with the projections.

Pull the semi-simplicial structure of $E(\xi_\alpha)$ back along this covering. Every simplex is simply connected, so it lifts in exactly n ways, and the lifts form a semi-simplicial set E_β with $|E_\beta| \cong E(\xi_\beta)$. The composite $E_\beta \to E_\alpha \to X$ is simplicial. Over a simplex whose necklace (circular permutation) is ω, the lifted necklace is the n-fold repetition ω^n. So ξ_β is semi-simplicially triangulated over X, and $\beta \in \mathrm{Bin}(X)$ by Theorem 4.2. ◻

Corollary 4.4. *(i) Taking $\alpha = 0$ recovers Proposition 4.1.*

(ii) If $\alpha \in \mathrm{Bin}(X)$ is divisible by n, then the whole coset $\{\beta : n\beta = \alpha\}$, a torsor under the n-torsion subgroup, lies in $\mathrm{Bin}(X)$.

(iii) If $b_2(X) = 1$ and $k\,g + \tau \in \mathrm{Bin}(X)$ for a generator g of the free part, then $(k/d)g + \tau' \in \mathrm{Bin}(X)$ for every divisor $d \mid k$ and every τ' with $d\tau' = \tau$.

Remark 4.5. $\mathrm{Bin}(X)$ is *not* closed under adding torsion classes in general; see Example 6.3 and §9.9. The natural combinatorial attempt to "tensor a triangulated bundle with a flat bundle" twists the n-fold repeated necklaces ω^n by rotations. That produces the n-th roots of ξ_α twisted by flat bundles, not ξ_α itself; Theorem 4.3 is what this construction really proves.

5 Integrality: when the real relaxation is exact

Theorem 5.1. *Suppose the coboundary matrix $\delta^1 \colon C^1 \to C^2$ is totally unimodular (TU). Then $H^2(X;\mathbb{Z})$ is torsion-free, and*

$$\mathrm{Bin}(X) = \{\alpha \in H^2(X;\mathbb{Z}) : \ \|\rho(\alpha)\|^* \le \tfrac{1}{2}\}.$$

Proof. A TU matrix has all elementary divisors in $\{0, 1\}$, so C^2/B^2 is torsion-free, and hence so is $H^2 = Z^2/B^2 \subseteq C^2/B^2$. By Theorem 3.4 the polyhedron $\{b : -a \le \delta b \le 1_2 - a\}$ is non-empty whenever $\|\rho(\alpha)\|^* \le \frac{1}{2}$. By Hoffman–Kruskal [9] it then contains an integer point, and Proposition 2.7 applies. ◻

Dey, Hirani and Krishnamoorthy [5] characterise total unimodularity of ∂_2 (equivalently of δ^1): it holds if and only if $H_1(L, L_0;\mathbb{Z})$ is torsion-free for all pure subcomplexes $L_0 \subset L$ of dimensions 1 and 2.

Proposition 5.2. *Suppose $H_1(X;\mathbb{Z})$ is torsion-free and $Z_2(X;\mathbb{R})$ is a unimodular subspace, meaning every elementary 2-cycle is a multiple of a $\{0, \pm 1\}$-vector. Then the conclusion of Theorem 5.1 holds.*

Sketch. Since Z_2 is unimodular, it is the row space of a TU matrix A (see [27, Ch. 19–21] and the theory of regular matroids), and therefore $B^2_{\mathbb{R}} = Z_2^{\perp} = \ker A$. The polytope $\{x : Ax = Aa,\ 0 \le x \le 1\} = (a + B^2_{\mathbb{R}}) \cap [0,1]^{X_2}$ is therefore integral. Any integral point x satisfies $x - a \in B^2_{\mathbb{R}} \cap \mathbb{Z}^{X_2}$. This equals $\delta(C^1(X;\mathbb{Z}))$ because H^2, and hence C^2/B^2, is torsion-free. ☐

Theorem 5.3 (Theorem A: extreme classes are binary). *Suppose* $\delta^2 = \partial_3^{\mathsf{T}}$ *is totally unimodular. Then:*

(i) *The cocycle polytope* $C(X)$ *is integral, so its vertices are binary cocycles.*

(ii) $\operatorname{conv} \rho(\operatorname{Bin}(X)) = P(X)$.

(iii) *For every* $h \in H_2(X;\mathbb{R})$, $\max_{\alpha \in \operatorname{Bin}(X)} \langle \alpha, h \rangle = \frac{1}{2}\|h\|_{\Delta}$. *So the combinatorial Milnor–Wood inequality is sharp.*

Proof. $C(X) = \{x : \delta^2 x = 0,\ 0 \le x \le 1\}$ is defined by a TU system with integral right-hand sides, so by [9] it is integral. Then $P(X) = \pi(C(X)) = \operatorname{conv} \pi(\operatorname{vert} C(X))$, where π is the projection to $H^2(X;\mathbb{R})$. Part (iii) follows by duality. ☐

Proposition 5.4. ∂_3 *is TU in each of the following cases:* X *is an orientable* 3*-pseudomanifold, possibly with boundary (every triangle lies in at most two tetrahedra and the tetrahedra can be oriented coherently); or* X *is a* 3*-complex embedded in* $\mathbb{R}^3$ *[5].*

Proof of the first case. Choose signs $s_\sigma = \pm 1$ so that whenever $f = d_i\sigma = d_j\sigma'$, we have $s_\sigma(-1)^i = -s_{\sigma'}(-1)^j$. After multiplying column σ of ∂_3 by s_σ, every row has at most two non-zero entries, and when there are two they are $+1$ and -1. Such a matrix is a submatrix of the transposed node–arc incidence matrix of a directed graph, hence TU. Rescaling columns by ± 1 preserves total unimodularity. ☐

Remark 5.5 (Limits). (a) Total unimodularity of ∂_3 does not imply that every lattice point of $P(X)$ is binary. Example 6.1 has $\partial_3 = 0$ and still has gaps.

(b) For triangulations of T^3, the subspace $B^2_{\mathbb{R}}$ is typically *not* unimodular. Take three pairwise disjoint dual loops in the classes $(1,1,0)$, $(1,-1,0)$ and $(1,0,0)$. Then $C_1 + C_2 - 2C_3$ is a support-minimal null-homologous dual 1-cycle, which corresponds to an elementary coboundary with an entry equal to 2. So Proposition 5.2 does not explain the absence of gaps observed in §10.

6 Obstructions, and the main example

6.1 Gaps in the free part

Example 6.1 (Gaps in the free part). Let $Y_{2,3}$ be the semi-simplicial set with one vertex x, a loop e, and two disks: a disk D_2 made of triangles T_1, T_2 and a disk D_3 made of triangles U_1, U_2, U_3. Each disk is a cone on a boundary that wraps 2, respectively 3, times around e. Concretely, $T_i = [x, x, c]$ with $d_2 T_i = e$, $d_1 T_i = r_i$, $d_0 T_i = r_{i+1}$, so $\partial \sum_i T_i = 2e$, and similarly $\partial \sum_j U_j = 3e$.

Then $H_1 = 0$ and $H_2 = \mathbb{Z}z$ with $z = 3\sum T_i - 2\sum U_j$, so $H^2 \cong \mathbb{Z}$ via $\alpha \mapsto \langle \alpha, z\rangle$. Every cochain is a cocycle. A binary cochain that is 1 on a of the T's and b of the U's has class $3a - 2b$. By enumeration,

$$\rho(\mathrm{Bin}) = \{-6, -4, -3, -2, -1, 0, 1, 2, 3, 4, 6\}, \qquad P = [-6, 6],$$

so ± 5 are missing.

Lemma 6.2 (Arithmetic gaps). *Let* $\dim X = 2$ *and suppose* $H^2(X; \mathbb{Z}) \cong \mathbb{Z}$ *via* $\alpha \mapsto \langle \alpha, z\rangle$ *for a cycle* z *all of whose non-zero coefficients satisfy* $|z_f| \geq 2$. *Put* $M = \sum_f z_f^+$. *Then* $M \in \mathrm{Bin}(X)$, *but* $M - 1 \notin \mathrm{Bin}(X)$, *although* $M - 1 \in P(X)$.

Proof. Moving down from the maximum M changes the value by a sum of numbers $|z_f| \geq 2$, so the value $M - 1$ is never reached. Since $Z_2 = \mathbb{R}z$, we have $\|[z]\|_\Delta = \|z\|_1 = 2M$, so $P(X) = [-M, M]$. □

6.2 Torsion translates

Example 6.3 (Missing torsion translates). Let $Y_{3,6}$ be built like $Y_{2,3}$ but with disks wrapping 3 and 6 times. Then $H_1 = \mathbb{Z}/3$, $H_2 = \mathbb{Z}z$ with $z = 2\sum T_i - \sum U_j$, and $H^2 \cong \mathbb{Z} \oplus \mathbb{Z}/3$. Take $w = \sum T_i$, so that $\partial w = 3e$. The map $\alpha \mapsto (\langle \alpha, z\rangle,\ \langle \alpha, w\rangle \bmod 3)$ is an isomorphism. A binary cochain has class $(2a - b,\ a \bmod 3)$ with $0 \leq a \leq 3$ and $0 \leq b \leq 6$. Exhaustive enumeration gives:

free part	$\pm 6, \pm 5$	$4, 3$	$-3, -4$	$-2, \ldots, 2$
torsion parts attained	$\{0\}$	$\{0, 2\}$	$\{0, 1\}$	$\mathbb{Z}/3$

So $(6, 0) \in \mathrm{Bin}$ but $(6, 1), (6, 2) \notin \mathrm{Bin}$, although all three lie in P. The table is consistent with Theorem 4.3; for instance $(4, 2) \in \mathrm{Bin}$ forces $(2, 1)$ and $(1, 2)$ into Bin. In a simplicial triangulation of the same space the maximal value is attained by a unique binary cocycle ($c = 1$ exactly on the triangles with positive coefficient), so again only one torsion class occurs there.

6.3 The main example: $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ but $\mathrm{Bin}(X) = \{0\}$

Lemma 6.4 (A small sphere carrying a multiple). *Let $S \subseteq X$ be a subcomplex that is a closed oriented triangulated surface with $2m$ triangles. Suppose $[S] = k\,h$ in $H_2(X;\mathbb{Z})$ with $k > m$. Then $\langle \alpha, h\rangle = 0$ for every $\alpha \in \mathrm{Bin}(X)$.*

Proof. The fundamental cycle z_S has coefficients ± 1, and by Lemma 2.5(iii) exactly m of them are $+1$. For a binary cocycle c this gives $\langle c, z_S\rangle \in [-m, m]$. But $\langle c, z_S\rangle = k\langle c, h\rangle \in k\mathbb{Z}$, and $k > m$, so $\langle c, h\rangle = 0$. □

For $S = \partial\Delta^3$ we have $m = 2$, so we need a 4-triangle sphere representing $3h$ (or more).

Construction 6.5. Let Q be a triangulated 3-sphere, oriented, and let $\tau_0, \tau_1, \tau_2, \tau_3$ be 3-simplices of Q. For a simplex τ let $N(\tau)$ be the set of vertices of Q that lie in a closed simplex meeting τ. Assume:

(a) $N(\tau_i) \cap N(\tau_j) = \emptyset$ for $1 \le i < j \le 3$;

(b) τ_0 shares no vertex with τ_1, τ_2, τ_3.

Let $W \subset Q$ be the subcomplex consisting of all simplices except the four open 3-simplices $\tau_0, \dots, \tau_3$. Give each $\partial\tau_i$ the boundary orientation from τ_i. Let $L = \partial[A,B,C,D]$, oriented as the boundary of $+[A,B,C,D]$. For $i = 1,2,3$ choose vertex bijections $\psi_i \colon V(\tau_i) \to \{A,B,C,D\}$ that carry the orientation of τ_i to that of $[ABCD]$. Define

$$X := W/\big(x \sim \psi_i(x),\ x \in \partial\tau_i,\ i = 1,2,3\big), \qquad S := \partial\tau_0 \subset X,$$

and order the vertices of X arbitrarily.

Lemma 6.6. *X is a 3-dimensional simplicial complex. The quotient map $\pi \colon W \to X$ is injective on every simplex. Two distinct simplices of W have the same image exactly when they lie in $\partial\tau_i$ and $\partial\tau_j$ and correspond under $\psi_j^{-1}\psi_i$. The surfaces S and L are subcomplexes of X isomorphic to $\partial\Delta^3$.*

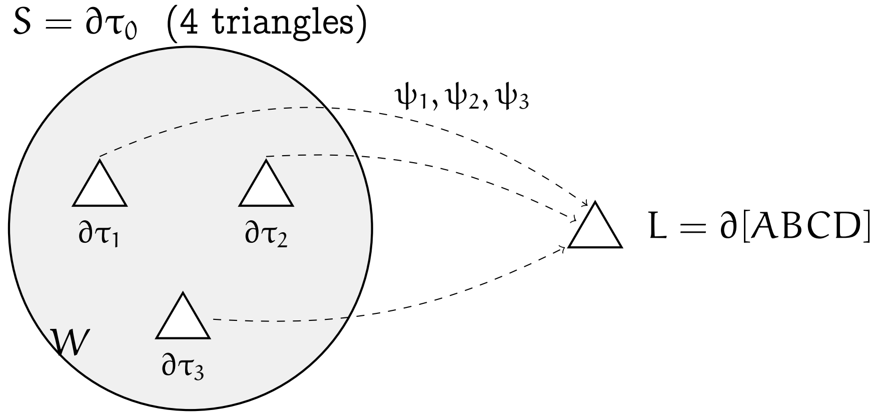

Figure 1: A two-dimensional cartoon of Construction 6.5. W is S^3 with four open tetrahedra removed. The three inner boundary spheres are glued to a single 4-triangle sphere L. In X, as chains, $\partial\,\pi_*[W] = -(S + 3L)$.

Proof. A simplex of W containing identified vertices would have vertices in two different τ_i; its vertices would then lie in $N(\tau_i) \cap N(\tau_j)$, which is excluded by (a). Suppose two distinct simplices σ, σ' have the same image. Then some vertex $u \in \sigma \setminus \sigma'$ is identified with a vertex $u' \in \sigma'$ lying in a different τ_j. If σ also had a vertex w in no $\partial\tau$, then $w \in \sigma'$ too, so $w \in N(\tau_i) \cap N(\tau_j)$, again excluded by (a). Hence $\sigma \subseteq \partial\tau_i$ and $\sigma' \subseteq \partial\tau_j$, and they correspond under the gluing. Condition (b) keeps the four vertices of S distinct in X. □

Proposition 6.7 (Homology of X). *$H_0(X) = \mathbb{Z}$, $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^2$, $H_2(X;\mathbb{Z}) = \mathbb{Z}\,h$ with $h = [L]$, and $H_3(X) = 0$. In particular $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$, detected by evaluation on h. Moreover $[S] = -3h$ with the orientations above; under a different orientation convention for S this reads $[S] = \pm 3h$. Finally, $\pi_1(X)$ is free of rank 2.*

Proof. The relation $[S] = -3h$. Since $[Q] = [W] + \sum_{i=0}^{3} \tau_i$ is a cycle, we have $\partial[W] = -\sum_{i=0}^{3} \partial\tau_i$. Because the ψ_i preserve orientation, $\pi_*(\partial\tau_i) = L$ as chains for $i = 1, 2, 3$. Hence $\partial\,\pi_*[W] = -(S + 3L)$.

The rest. W is S^3 minus four open balls, so $H_1(W) = 0$, $H_3(W) = 0$, and $H_2(W)$ is free on $[\partial\tau_1], [\partial\tau_2], [\partial\tau_3]$. The space X is the pushout of $W \hookleftarrow \coprod_{i=1}^{3} \partial\tau_i \to L$. Up to homotopy this is $W \cup \mathrm{Cyl}(f)$, where f is the fold map and $\mathrm{Cyl}(f) \simeq L \cong S^2$. Mayer–Vietoris gives exact sequences

$$0 \to H_3(X) \to \bigoplus_i H_2(\partial\tau_i) \xrightarrow{\varphi} H_2(W) \oplus H_2(L) \to H_2(X) \to 0,$$

$$0 \to H_1(X) \to \bigoplus_i H_0(\partial\tau_i) \to H_0(W) \oplus H_0(L) \to H_0(X) \to 0,$$

with $\varphi(e_i) = ([\partial\tau_i], \pm[L])$. The map φ is injective, so $H_3(X) = 0$. Its cokernel is $\mathbb{Z}$, generated by the image of $[L]$, and each $[\partial\tau_i]$ maps to $\pm[L]$. The second sequence gives $H_1(X) \cong \ker(\mathbb{Z}^3 \to \mathbb{Z}^2) \cong \mathbb{Z}^2$. The universal coefficient theorem gives $H^2(X;\mathbb{Z}) \cong \mathrm{Hom}(H_2,\mathbb{Z}) \oplus \mathrm{Ext}(H_1,\mathbb{Z}) \cong \mathbb{Z}$. For π_1: W and $\mathrm{Cyl}(f)$ are simply connected and meet in three simply connected components, so van Kampen for graphs of spaces gives the free group of rank $3 - 1 = 2$. □

Theorem 6.8 (Main example). *For the complex X of Construction 6.5, with any vertex order:*

(i) $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$*, but* $\mathrm{Bin}(X) = \{0\}$.

(ii) No non-zero class has even a real *cocycle representative with values in* $[0,1]$*. Indeed* $\|h\|_\Delta \leq \frac{4}{3}$*, so* $P(X) \cap \Lambda = \{0\}$.

(iii) Circle bundles over $|X|$ *are classified by* $n = \langle c_1, h\rangle \in \mathbb{Z}$*, but only the trivial bundle (*$n = 0$*) admits a semi-simplicial triangulation over* X.

Proof. (i) Apply Lemma 6.4 with $m = 2$ and $k = 3$. (ii) By Proposition 6.7, $-\frac{1}{3}z_S$ is a real cycle representing h, so $\|h\|_\Delta \leq \frac{1}{3} \cdot 4$. A class α with $\langle \alpha, h\rangle = n \neq 0$ has $|n| > \frac{1}{2} \cdot \frac{4}{3}$, so by Theorem 3.4 it has no $[0,1]$-valued representative. (iii) By Theorem 4.2 this is equivalent to (i). A direct argument: a triangulated bundle over X restricts to one over $S \cong \partial\Delta^3$ with Chern number $\mp 3n$, and over $\partial\Delta^3$ only $|c| \leq 2$ is possible [18, 19]. □

Remark 6.9 (Why these hypotheses). (a) *Dimension* 3 *is necessary*, by Proposition 2.8. It is also the general case, by Lemma 2.6.

(b) H^2 *must be torsion-free*, since torsion classes are always binary (Proposition 4.1).

(c) *The multiplicity must be at least* 3. With a 4-triangle sphere and $k = 2$, the value $\langle c, S\rangle = \pm 2$ is allowed. In general a $2m$-triangle sphere needs $k > m$.

(d) *Mechanism.* The real reason is $\|h\|_\Delta < 2$, i.e. the generator of H_2 is “cheaper than one unit of flux”. This is only possible because h is represented rationally, by $S/3$, much more efficiently than integrally.

Remark 6.10 (Variants). (a) *Arbitrary rank.* The wedge of r copies of X at a vertex has $H^2 \cong \mathbb{Z}^r$ and still $\mathrm{Bin} = \{0\}$: binary cocycles on a wedge are tuples of binary cocycles on the pieces.

(b) *A version homotopy equivalent to S^2.* Take the simplicial mapping cylinder of a simplicial map of degree 3 between triangulated 2-spheres, for example $z \mapsto z^3$ from the suspension of a 9-gon to the suspension of a triangle. Then glue a triangulated collar $S^2 \times I$ whose free end is $\partial\Delta^3$.

(c) *Not topological.* After one barycentric subdivision, S has 24 triangles and Lemma 6.4 no longer applies. By Theorem 7.3, every class becomes binary on $\mathrm{Sd}^k X$ for k large. So the phenomenon is genuinely combinatorial.

Example 6.11 (An explicit, computer-verified instance). Let B be the Freudenthal (staircase) triangulation of the box $[0,3] \times [0,3] \times [0,12]$: every unit cube is split into 6 tetrahedra along its main diagonal. Let $Q = B \cup \mathrm{cone}(\partial B)$. Take

$$\begin{aligned} \tau_0 &= \{(0,0,0),(1,0,0),(1,1,0),(1,1,1)\}, \\ \tau_{1,2,3} &= \{(1,1,z),(2,1,z),(2,2,z),(2,2,z{+}1)\} \quad \text{for } z = 2,6,10. \end{aligned}$$

The resulting X has 201 vertices, 1169 edges, 1936 triangles and 968 tetrahedra, so $\chi(X) = 0$. We verified the following by computer (files `X_example.json` and `verify_example.py`):

- X is a simplicial complex, and only the intended faces are identified.
- The Betti numbers are $(b_0, b_1, b_2, b_3) = (1,2,1,0)$ over $\mathbb{Q}$ and over $\mathbb{F}_p$ for $p = 2,3,5,7$.
- $[S] = \pm 3[L]$, witnessed by an explicit integral 3-chain with coefficients ± 1.
- $\max\{\langle c,[L]\rangle : c \in Z^2(X;\mathbb{R}) \cap [0,1]^{X_2}\} = \frac{2}{3}$ exactly. So $\|h\|_\Delta = \frac{4}{3}$, and the integer maximum over binary cocycles is 0.

The results are the same for the natural vertex order and for a random one.

7 The SC_* viewpoint

Proposition 7.1 (Mnëv [19, 21]). *(i)* SC_n *is the set of circular permutations of* $\{0, \dots, n\}$*, so* $\#SC_n = n!$*. Face maps delete a bead; the degeneracy* s_i *inserts a new bead* $i+1$ *immediately after bead* i*.*

(ii) $c_{01} \in C^2(SC_*)$*, the parity of a circular permutation of three beads, is a normalised cocycle. Pulling back gives a bijection* $\mathrm{Hom}(X, SC_*) \cong Z^2(X; \{0,1\})$*,* $f \mapsto f^*c_{01}$*. In particular* SC_* *is the simplicial subset of binary cocycles in the standard Kan model of* $K(\mathbb{Z}, 2)$*, whose* n*-simplices are* $Z^2(\Delta^n; \mathbb{Z})$*.*

(iii) $|SC_*| \simeq K(\mathbb{Z}, 2)$*, and* c_{01} *represents a generator.*

Proposition 7.2 (Where the Kan condition fails). SC_* *is* 3*-coskeletal. Every horn* $\Lambda^n_k \to SC_*$ *has a filler except exactly eight horns* $\Lambda^3_k \to SC_*$ *(*$k = 0, 1, 2, 3$*, two for each* k*): those in which the three given faces force the value of* c_{01} *on the missing face to be* 2 *or* -1*. For example,* Λ^3_3 *with face values* $(1, 0, 1)$ *forces* $1 - 0 + 1 = 2$*.*

Sketch. For $n \leq 2$ there is nothing to check, because $SC_{\leq 1}$ is a point. For $n = 3$, a horn extends if and only if the forced value is 0 or 1 (Lemma 2.4 and [19, Prop. 7]). For $n = 4$, the boundary of the missing 3-face lies in the horn. It satisfies the cocycle condition because $\delta^2 = 0$ on Δ^4, so the face exists, and the 4-simplex then exists uniquely [19, Prop. 6]. For $n \geq 5$ the argument is the same using 3-coskeletality. The explicit 3-skeleton is listed in Appendix A. □

Theorem 7.3 (Exhaustion under subdivision). *Let* $\lambda\colon \mathrm{Sd}X \to X$ *be the last-vertex map. Pulling back along it and identifying* $H^2(\mathrm{Sd}^k X) \cong H^2(X)$ *gives*

$$\mathrm{Bin}(X) \subseteq \mathrm{Bin}(\mathrm{Sd}X) \subseteq \mathrm{Bin}(\mathrm{Sd}^2 X) \subseteq \cdots, \qquad \bigcup_k \mathrm{Bin}(\mathrm{Sd}^k X) = H^2(X; \mathbb{Z}).$$

Since each $\mathrm{Bin}(\mathrm{Sd}^k X)$ *is finite, the chain grows infinitely often whenever* $b_2(X) > 0$*.*

Sketch. The inclusions hold because $f \circ \lambda$ is simplicial whenever $f\colon X \to SC_*$ is. For exhaustion, $SC_* \to \mathrm{Ex}^\infty SC_*$ is a Kan replacement [12], and for finite X we have $\mathrm{Hom}(X, \mathrm{Ex}^\infty SC_*) = \mathrm{colim}_k \mathrm{Hom}(\mathrm{Sd}^k X, SC_*)$. Every class of $[|X|, K(\mathbb{Z}, 2)] = H^2(X; \mathbb{Z})$ is therefore represented by some simplicial map $\mathrm{Sd}^k X \to SC_*$, i.e. by a binary cocycle on $\mathrm{Sd}^k X$. □

Definition 7.4. The *binary depth* of $\alpha \in H^2(X;\mathbb{Z})$ is the least k with $\alpha \in \mathrm{Bin}(\mathrm{Sd}^k X)$. In the main example the generator has depth ≥ 1.

8 Small projective models and loss of binarity

8.1 Why this comparison matters

A finite simplicial set can model the homotopy type of $\mathbb{CP}^n$ with very few nondegenerate simplices. This does *not* imply that the hyperplane generator $h \in H^2(\mathbb{CP}^n;\mathbb{Z})$ is binary on that particular simplicial presentation. The datum relevant to a minimally triangulated tautological circle bundle is not merely a homotopy equivalence $|Y| \simeq \mathbb{CP}^n$, but a simplicial map

$$Y \longrightarrow SC_*$$

representing the generator. In this sense binary representability is extra geometric combinatorial structure attached to a chosen simplicial model.

Sergeraert used the Kenzo effective-homology program to extract finite simplicial subsets of the canonical minimal Kan model of $K(\mathbb{Z},2)$ whose realizations have the homotopy type of $\mathbb{CP}^n$ for $n \leq 6$ [28]. We denote these models by Y_n. Sergeraert explicitly leaves open whether the resulting realizations are homeomorphic to $\mathbb{CP}^n$; here only their homotopy type is relevant. Nor is a global minimality theorem for these finite models asserted in [28]: they are simply strikingly small models produced by the Kenzo reduction.

Their nondegenerate simplex counts are

n	Sergeraert–Kenzo Y_n	circular-permutation model $\mathcal{X}_n$
1	$(1,0,1)$	$(1,0,1)$
2	$(1,0,2,3,3)$	$(1,0,1,2,3)$
3	$(1,0,3,10,25,30,15)$	$(1,0,1,2,9,20,15)$
4	$(1,0,4,22,97,255,390,315,105)$	$(1,0,1,2,9,44,145,210,105)$
5	$(1,0,5,40,271,1197,3381,$ $5985,6405,3780,945)$	$(1,0,1,2,9,44,265,$ $1134,2485,2520,945)$
6	$(1,0,6,65,627,4162,18496,$ $54789,107933,139230,112770,$ $51975,10395)$	$(1,0,1,2,9,44,265,1854,$ $9793,28952,45045,34650,10395)$

In both families the number of top $2n$-simplices is

$$1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n-1)!!.$$

For the $\mathcal{X}_n$ this is the number of perfect matchings supporting the canonical cycle Z_n. Sergeraert already observed the same top-dimensional sequence in the Kenzo output. This agreement strongly suggests that the two constructions compress the same divided-power fundamental class in different simplicial models of $K(\mathbb{Z}, 2)$; no explicit comparison map is claimed here.

8.2 The Kenzo models have no nonzero binary class in dimensions $2, 3, 4$

The low-dimensional Kenzo face tables are especially revealing. Write

$$e_1, \dots, e_n \in (Y_n)_2^{\mathrm{nd}}$$

for the nondegenerate 2-simplices carrying the bar labels $(1), \dots, (n)$, and let $u_r = u(e_r)$ for a normalized integral 2-cocycle u. The complete Y_2 table is printed in [28]; the Y_3 and Y_4 tables are distributed with SageMath. They are used by the routine

$$\texttt{ComplexProjectiveSpace(n)}, \qquad n = 3, 4,$$

from the Kenzo data files [26].

Proposition 8.1 (Additive low-dimensional Kenzo cocycles). *For $n = 2, 3, 4$, every normalized integral 2-cocycle on Y_n satisfies*

$$(u_1, u_2, \dots, u_n) = a(1, 2, \dots, n)$$

for some $a \in \mathbb{Z}$. Consequently

$$\mathrm{Bin}(Y_n) = \{0\}, \qquad n = 2, 3, 4.$$

In particular the generator of $H^2(Y_n; \mathbb{Z}) \cong \mathbb{Z}$ cannot be represented by a binary cocycle and the tautological circle bundle admits no minimal triangulation over these models without subdivision.

Proof. Only a few 3-simplices are needed. Degenerate 2-faces contribute zero to a normalized cocycle. In Y_2 one of Sergeraert's tetrahedra has ordered face list

$$(0, e_1, e_2, e_1),$$

so the cocycle equation gives $u_2 = 2u_1$.

The Kenzo table for Y_3 contains tetrahedra with face lists

$$(0, e_1, e_2, e_1), \qquad (0, e_2, e_3, e_1),$$

so $u_2 = 2u_1$ and $u_3 = u_2 + u_1 = 3u_1$.

Similarly the Y_4 table contains

$$(0, e_1, e_2, e_1), \qquad (0, e_1, e_3, e_2), \qquad (0, e_3, e_4, e_1),$$

and therefore

$$u_2 = 2u_1, \qquad u_3 = u_1 + u_2 = 3u_1, \qquad u_4 = u_1 + u_3 = 4u_1.$$

Since the e_r exhaust the nondegenerate 2-simplices, this determines every normalized 2-cocycle in degree two. If all u_r lie in $\{0, 1\}$ and $n \geq 2$, then $u_2 = 2u_1$ forces $u_1 = u_2 = 0$, hence all $u_r = 0$. □

Remark 8.2. The proposition is stronger than saying that a particular generator is not binary: there is no nonzero binary class at all. Conceptually, the minimal Kan model stores the universal degree-two class *additively*; its low-dimensional bar simplices impose $u_r = ru_1$. The circular-permutation model stores the same class through cyclic order, where the universal cocycle is binary by construction.

Conjecture 8.3 (Kenzo additivity in all projective stages). *For every Sergeraert–Kenzo stage Y_n with $n \geq 2$, the normalized integral 2-cocycles satisfy $u_r = ru_1$ for $1 \leq r \leq n$. Equivalently,*

$$\mathrm{Bin}(Y_n) = \{0\}.$$

The conjecture is proved above for $n \leq 4$. Sergeraert reports models through $n = 6$, but the complete Y_5, Y_6 face tables are not printed in the report and are not currently distributed by SageMath, so we do not promote those cases to a theorem here.

8.3 The companion positive models $\mathcal{X}_n \subset \mathrm{SC}_*$

The contrast with the projective filtration inside SC_* is useful. The companion working note constructs finite simplicial subsets

$$\mathcal{X}_n \subset \mathrm{SC}_*, \qquad |\mathcal{X}_n| \simeq \mathbb{CP}^n,$$

and the inclusion itself supplies the distinguished binary generator

$$c_n = i_n^* c_{01} \in Z^2(\mathcal{X}_n; \{0, 1\}), \qquad [c_n] = h.$$

Thus $\mathcal{X}_n$ should be regarded not merely as a small homotopy model of $\mathbb{CP}^n$, but as the pair $(\mathcal{X}_n, i_n)$ equipped with a minimally triangulated tautological (or, according to sign convention, dual tautological) circle bundle.

This gives a clean pair of natural examples with the same homotopy type but opposite binary behavior:

$$\boxed{\begin{array}{rcll} Y_n^{\text{Kenzo}} & \simeq & \mathbb{CP}^n, & \text{very small but nonbinary for } n = 2,3,4, \\ \mathcal{X}_n \subset \mathrm{SC}_* & \simeq & \mathbb{CP}^n, & h \in \mathrm{Bin}(\mathcal{X}_n) \text{ canonically.} \end{array}}$$

For the construction, perfect-matching cycles, the proof $|\mathcal{X}_n| \simeq \mathbb{CP}^n$, and the Barratt–Eccles E_∞ structure on their cochains, see the companion projective-space note.

9 Closed oriented 3-manifolds: binary cocycles as ice

9.1 The dual digraph

Let M be a closed oriented 3-manifold with an ordered triangulation, or an ordered semi-simplicial triangulation. For a tetrahedron σ let $s_\sigma = +1$ if the vertex order of σ agrees with the orientation of M, and $s_\sigma = -1$ otherwise. For a triangle $f = d_i\sigma$ put $\varepsilon(f,\sigma) = s_\sigma(-1)^i$. If f is shared by σ and σ', then $\varepsilon(f,\sigma) = -\varepsilon(f,\sigma')$.

The *dual digraph* D has one node per tetrahedron and one arc per triangle f. The arc points from the tetrahedron with $\varepsilon = +1$ to the one with $\varepsilon = -1$.

Lemma 9.1. *At every node σ the out-arcs are $\{d_0\sigma, d_2\sigma\}$ if $s_\sigma = +1$ and $\{d_1\sigma, d_3\sigma\}$ if $s_\sigma = -1$. So D is a two-in/two-out orientation of the 4-regular dual graph, and it is determined by the vertex order.*

Proposition 9.2 (Binary cocycles are ice states). *(i) $c \in \{0,1\}^{M_2}$ is a cocycle if and only if $Y_c = \{f : c(f) = 1\}$ is balanced at every node, i.e. it is an Eulerian subdigraph of D (an arc-disjoint union of directed cycles).*

(ii) Reversing the arcs of Y_c in D gives a two-in/two-out orientation O_c (an ice state*). The map $c \mapsto O_c$ is a bijection between binary cocycles and ice states, and $O_0 = D$.*

(iii) Under Poincaré duality $H^2(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$, the class $[c]$ corresponds to $[Y_c]$, with Y_c read as a dual 1-cycle. Moreover $\langle c, [\Sigma]\rangle = Y_c \cdot \Sigma$ for every surface Σ.

(iv) *The all-arc directed cycle* Y_{1_2} *has class* $0 \in H_1(M;\mathbb{Z})$*, including the torsion part; equivalently the reference orientation* $O_0 = D$ *has trivial topological flux. The field* $B_c := c - \frac{1}{2}1_2 \in \{\pm\frac{1}{2}\}^{M_2}$ *is divergence-free, meaning* $\delta B_c = 0$*, which is the ice rule. Its flux through* Σ *equals* $\langle c, [\Sigma]\rangle$.

Proof. (i) $s_\sigma(\delta c)(\sigma) = \sum_i \varepsilon(d_i\sigma, \sigma)c(d_i\sigma)$, which is the number of chosen out-arcs minus the number of chosen in-arcs. (ii) Reversing k in-arcs and k out-arcs at a node preserves the two-in/two-out property; conversely, two ice states differ on a balanced set. (iii) This is the standard cap-product description of Poincaré duality. (iv) This is Lemma 2.5. □

9.2 Why exactly six local states?

The terminology "ice" is not an analogy imposed after the fact. It is already contained in the tetrahedral cocycle equation. Assume first that the vertex order of $\sigma = [0123]$ agrees with the orientation of M, and abbreviate

$$c_i = c(d_i\sigma) \in \{0, 1\}.$$

Then

$$(\delta c)(\sigma) = 0 \quad \Longleftrightarrow \quad c_0 + c_2 = c_1 + c_3. \tag{1}$$

The common value is $0, 1$, or 2. If it is 0 there is the single pattern 0000; if it is 2 there is the single pattern 1111; and if it is 1 one chooses one of $\{0, 2\}$ and one of $\{1, 3\}$, giving four further patterns. Thus there are precisely

$$1 + 4 + 1 = 6$$

allowed local binary patterns. After reversing the corresponding dual arcs, these are exactly the six orientations of a four-valent vertex with two arrows entering and two leaving. This is the state space of the classical six-vertex model. In the usual square-lattice realization the same rule is called the ice rule; mathematically, the model can be formulated on any four-regular graph as the set of its Eulerian orientations. For introductions emphasizing the combinatorics and integrability see [2, 15, 23].

Our dual graph need not be planar and no Yang–Baxter structure is being assumed. Accordingly, the exact statement needed here is only

$$\boxed{\text{binary 2-cocycles on } M \longleftrightarrow \text{Eulerian orientations of the dual four-regular graph}}$$

The connection with the integrable square-lattice six-vertex model is therefore a specialization, not part of the proof of any topological statement below.

9.3 Defects and the divergence formula

The same calculation identifies precisely what happens when a binary cochain is not a cocycle. At a dual node corresponding to σ, define

$$q_\sigma = s_\sigma(\delta c)(\sigma) \in \{-2, -1, 0, 1, 2\}.$$

If $\mathrm{div}_{O_c}(\sigma)$ denotes "out-degree minus in-degree" in the orientation O_c, then

$$\mathrm{div}_{O_c}(\sigma) = -2q_\sigma. \tag{2}$$

Indeed, the reference orientation has two incoming and two outgoing arcs, and reversing one selected outgoing arc decreases the divergence by 2, while reversing one selected incoming arc increases it by 2. Consequently

$$q_\sigma = 0 \iff \text{2-in/2-out},$$

while $q_\sigma = \pm 1$ gives the 3-in/1-out or 1-in/3-out defects and $q_\sigma = \pm 2$ gives the all-in/all-out defects. In spin-ice language the former are monopoles and the latter are doubly charged defects; see [4, 8]. Thus

$$\boxed{\delta c = 0 \iff \text{there are no ice-rule defects.}}$$

From the simplicial point of view, the same local failure is a non-fillable assignment on the boundary of a tetrahedron, hence a non-fillable 3-horn in the circular-order model SC_*.

9.4 The network-flow formulation

For a mathematical audience it is useful to forget the arrows for a moment and retain the corresponding capacitated circulation. Let B_D be the signed node–arc incidence matrix of the oriented dual graph D. Proposition 9.2(i) says that the real relaxation of the binary cocycle equations is exactly

$$\mathcal{C}(D) = \{x \in [0, 1]^{E(D)} : B_D x = 0\}. \tag{3}$$

Thus binary cocycles are the 0/1 integral circulations in D. The total unimodularity statement used earlier for oriented 3-pseudomanifolds is, in these coordinates, the standard integrality of a network-flow polytope: B_D is a directed incidence matrix, hence totally unimodular; compare [27].

There is one topological quotient still to remember. A circulation is a 1-cycle in the dual graph, hence in the 1-skeleton of the dual cell decomposition. Passing from graph cycles to

$$H_1(M;\mathbb{Z})$$

additionally quotients by boundaries of dual 2-cells. Therefore the projection

$$\mathcal{C}(D) \longrightarrow H_1(M;\mathbb{R})$$

is the flux map, and its image is the polytope dual to the combinatorial norm described in Section 3. The set $\mathrm{PD}(\mathrm{Bin}(M))$ is the set of those integral flux sectors which actually contain a $0/1$ circulation. In this language the "no free gaps" conjecture of Section 11 is an integer-lifting statement for the projection of the circulation polytope.

This also gives a useful generating function. The unweighted six-vertex partition function merely counts Eulerian orientations. Our topological refinement is the group-ring-valued sum

$$Z_M = \sum_{c \in Z^2(M;\mathbb{Z}),\ c \in \{0,1\}^{M_2}} [\mathrm{PD}(c)] \ \in\ \mathbb{Z}[H_1(M;\mathbb{Z})]. \tag{4}$$

The support of Z_M is precisely $\mathrm{PD}(\mathrm{Bin}(M))$, while the coefficient of a class counts ice states in that flux sector. Local six-vertex weights can of course be inserted, but all results in this note use the unweighted state space.

9.5 Height functions, winding and flux

On a simply connected planar domain, a divergence-free six-vertex configuration may be encoded by a height function on complementary faces; on a torus the height becomes multivalued and its periods record winding numbers. This is one of the standard bridges between six-vertex configurations, lattice paths and alternating-sign matrices; see, for example, [15, 23]. Our three-manifold situation has no preferred planar height function. The correct replacement is the homology class

$$[Y_c] \in H_1(M;\mathbb{Z}),$$

or equivalently the Chern class $[c] \in H^2(M;\mathbb{Z})$. Thus the familiar winding sector of a periodic ice model becomes literally a Poincaré-dual topological flux sector. For an oriented surface $\Sigma \subset M$,

$$\langle [c], [\Sigma] \rangle = Y_c \cdot \Sigma$$

is the algebraic number of dual flux lines crossing Σ.

9.6 Loop flips as the cycle space

If O_c and $O_{c'}$ are two ice states, orient every edge on which they differ as it is oriented in O_c. At every node the number of such edges entering equals the number leaving. Hence the difference set is an Eulerian directed subgraph and splits into directed cycles. This elementary graph-theoretic fact is the mathematical core of the loop and worm algorithms used in simulations [1]. Flipping one directed cycle preserves the ice rule. Topologically, a contractible or null-homologous flip stays in the same flux sector, while a homologically nontrivial cycle changes the Chern class by its Poincaré dual. Thus local dynamics and topology separate cleanly:

cycle flips generate the state graph, while $H_1(M)$ records its flux sectors.

9.7 Extreme flux, frozen arcs and strings

The combinatorial Milnor–Wood inequality acquires a particularly transparent flow interpretation. Let z be an ℓ^1-minimal 2-cycle representing a surface class h. The pairing $\langle c, h \rangle$ is the signed flux of the dual circulation through z, and

$$|\langle c, h \rangle| \leq \frac{1}{2} \|h\|_\Delta$$

is a capacity bound. Equality forces every triangle in $\operatorname{supp} z$ to take the unique value compatible with the sign of its coefficient. In flow language those arcs are saturated; in ice language they are frozen. The locking lemma below is therefore a topological refinement of the elementary fact that a saturated cut restricts which circulations can be modified without leaving the extremal face of the flow polytope.

To move one unit away from an extreme flux, one needs a directed cycle crossing the chosen surface exactly once in the lowering direction.

This is what we call a *string*. In three-dimensional spin ice, field-driven transitions are likewise described by system-spanning strings, and the corresponding saturation transition is usually called a Kasteleyn transition [11]. We use this terminology only as a guide: the propositions below are statements about directed cycles and intersection numbers in a finite dual graph. The open "string conjecture" asks whether such a single-crossing directed cycle must exist whenever the flux is not already minimal.
The resulting dictionary, with the mathematical statement in the left column and the standard ice terminology on the right, is:

binary cocycles / triangulated bundles	ice (spin ice, Coulomb phase)
$c(f) \in \{0, 1\}$ on a triangle	spin flipped relative to the reference state
$c - \frac{1}{2} \in \{\pm\frac{1}{2}\}$ (Mnëv–Sharygin curvature)	emergent field B [8]
tetrahedral rule, Lemma 2.4	ice rule $\operatorname{div} B = 0$ (six vertex types)
non-fillable horn of SC_*	monopole, three-in/one-out [4]
vertex order	reference ice state in the trivial flux sector
$c' = c + \delta b$	local flux rearrangement by a dual boundary ("curl")
Chern class $[c]$	flux (topological) sector in $H_1(M;\mathbb{Z})$
Milnor–Wood bound $\frac{1}{2}\|h\|_\Delta$	maximal, fully polarised flux
extreme classes (Theorem 5.3)	polarised sectors, Kasteleyn regime [11]
loop moves (Lemma 9.3)	loop and worm algorithms [1]
strings (Definition 9.5)	system-spanning strings [11]
$Sd^k X$, Theorem 7.3	thermodynamic / continuum limit

9.8 Loop moves

Lemma 9.3. *Two ice states differ on an Eulerian subdigraph of either one, which splits into arc-disjoint directed cycles. Flipping a directed cycle C of O_c gives $O_{c'}$ with $[c'] - [c] = [C]$ under Poincaré duality, where C is read in its own direction.*

9.9 Extreme sectors and locking

Theorem 9.4 (Locking lemma). *Let* X *be any finite ordered complex and* $h \in H_2(X;\mathbb{R})$*, and put* $K = \frac{1}{2}\|h\|_\Delta$*. Let* E_h *be the set of binary cocycles* c *with* $\langle c, h\rangle = K$*.*

(i) Let z *be any* ℓ^1*-minimal real cycle representing* h*. Every* $c \in E_h$ *equals* 1 *on* $\operatorname{supp} z^+$ *and* 0 *on* $\operatorname{supp} z^-$*.*

(ii) Let F *be the set of triangles on which all* $c \in E_h$ *agree, and* $\bar{F}$ *the subcomplex they generate. Then for* $c, c' \in E_h$*,*

$$[c'] - [c] \in \operatorname{im}\big(H^2(X, \bar{F}; \mathbb{Z}) \to H^2(X;\mathbb{Z})\big).$$

(iii) If $X = M$ *is a closed oriented* 3*-manifold, this image is* $\operatorname{im}\big(H_1(M \setminus \bar{F}) \to H_1(M)\big)$ *by Lefschetz duality.*

Proof. (i) $\langle c, z\rangle \leq \sum z^+ = \frac{1}{2}\|z\|_1 = K$ by Lemma 2.5, with equality exactly under the stated conditions. (ii) $c' - c$ vanishes on F, so it is a relative cocycle. (iii) is Lefschetz duality. □

In words: in an extreme sector, every minimal surface is fully polarised. The torsion class can only be changed by loops that live in the unfrozen region. In physics terms, saturation freezes the topological sectors that such loops would carry.

9.10 Strings and the string ladder

Definition 9.5. Let $\Sigma \subset M^{(2)}$ be a closed oriented surface subcomplex with fundamental cycle z_Σ (coefficients ± 1). The Σ*-arcs* are the arcs dual to triangles of Σ. In the ice state O_c, a Σ-arc f is *lowering* if flipping it lowers $\langle c, z_\Sigma\rangle$ by 1. A *string through* Σ in O_c is a directed cycle that contains exactly one Σ-arc, and that arc is lowering. Flipping a string lowers $\langle c, z_\Sigma\rangle$ by exactly 1.

String-ladder algorithm. Start from a maximiser c^* of $\langle c, z_\Sigma\rangle$. It is integral by Theorem 5.3, so an LP solver finds it. Then repeat: look for a lowering Σ-arc f and a directed path in O_c from the head of f back to its tail that avoids all Σ-arcs (breadth-first search); flip f together with that path. If the ladder reaches the minimum, every intermediate value along $[\Sigma]$ is binary.

Proposition 9.6 (The first step down). *Suppose z_Σ is ℓ^1-minimal, so $K = \frac{1}{2}\#\Sigma_2$. Then the value $K-1$ is attained by a binary class if and only if some extreme ice state contains a string through Σ.*

Proof. A state with value $K-1$ is polarised on all Σ-arcs except one, say g (Theorem 9.4(i)). Pick any extreme state. It differs from the given state on an Eulerian set that contains g and no other Σ-arc. The directed closed trail through g in that set is a string. Flipping it produces an extreme state, which contains the reversed string. The converse is immediate. $\square$

Proposition 9.7 (Multi-crossing loops always exist). *Keep the hypotheses of Proposition 9.6, and assume $M \setminus \Sigma$ is connected. In an extreme state, write $f \to g$ if some directed path avoiding Σ-arcs runs from the head of f to the tail of g. Then this relation on Σ-arcs is strongly connected. In particular every Σ-arc lies on a directed loop that crosses Σ only in the polarised direction.*

Proof. Let S be a terminal strong component, and U the set of nodes reachable from the heads of arcs in S without using Σ-arcs. The set U has no outgoing non-Σ arcs. Its outgoing Σ-arcs lie in S, and its incoming Σ-arcs include all of S. Balance at the nodes of U forces equality everywhere, so no non-Σ arc enters U. Hence U is a union of components of $M \setminus \Sigma$, so U is everything. Then every Σ-arc enters U, so S consists of all Σ-arcs. $\square$

Whether a *single*-crossing loop exists is exactly the open point; see Conjecture 11.2.

10 Computational evidence

All computations used exact 0/1 integer programming (HiGHS through SciPy 1.17), LP relaxations, and breadth-first search. Torsion parts were detected by pairing with integral 2-chains w satisfying $\partial w = n\gamma$, where γ generates the torsion of H_1, reduced mod n. This map is well defined on cohomology and injective together with the free pairings. "Scan" means that every class in the stated range was tested for membership in Bin.

M, triangulation	tets	orders	result of the exact scan
$S^2 \times S^1 = K \times C_n$; K an octahedron, icosahedron or randomly flipped icosahedron; $n = 3, 4$	72–240	product + 2 random	every $k \in [-\frac{1}{2}\|h\|_\Delta, \frac{1}{2}\|h\|_\Delta]$ binary ($\frac{1}{2}\|h\|_\Delta = 4$ or 10)
T^3, grid C_3^3	162	4	all 1086 lattice points of P (up to sign) binary
T^3, C_3^3 after 80 random bistellar moves	204, 214	random	all 2110 resp. 2230 lattice points binary
$L(3,1)\#(S^2 \times S^1)$, semi-simplicial	118	fixed	all 9×3 classes (k, t) binary
twisted translation mapping tori of T^2 ($\cong T^3$; 2–3 layers)	432–768	2 each	every $k \in [0, K]$ along the fibre binary
half-turn manifold $T^2 \times_{-1} S^1$, 4×4 fibre, 3 layers	288	2	126/132; missing exactly $(\pm 16, t)$ with $t \neq 0$
the same, 5 layers	480	1	$k = 16$: only $t = 0$; $k = 15$: all $t \in (\mathbb{Z}/2)^2$

string ladder: M, triangulation	tets	K	ladder from $+K$ to $-K$ in unit steps
T^3 grids 4^3, 5^3, 6^3 (3 orders each)	384–1296	16, 25, 36	always
T^3 4^3 after 300 random bistellar moves	566–580	16 or 15	always
$S^2 \times S^1$, irregular 80-triangle sphere, 200 moves	1106–1108	36–38	always, even though Σ is not minimal
twisted translation mapping tori	432–768	36, 64	always (the exact scan also finds no gaps)
half-turn manifold	288, 480	16	always

Torsion locking, explained. In the half-turn manifold (4×4 fibre, 3 layers, 576 triangles), exactly 288 triangles are frozen in the extreme sector, including all 96 level triangles. The unfrozen region has 48 independent (fundamental) cycles, and all of them pair trivially with both $\mathbb{Z}/2$ torsion invariants. So Theorem 9.4 accounts for the locking completely.

At $k = K - 1$ there is exactly one string. Because of the half-turn monodromy it can wander sideways, and the parity of its sideways winding is free; so it carries the torsion. By contrast, in $L(3,1)\#(S^2 \times S^1)$ the torsion lives in the lens summand, away from the frozen region, and nothing locks.

11 Conjectures and questions

Conjecture 11.1 (No free gaps for 3-manifolds). *Let M be a closed orientable 3-manifold with any ordered (semi-)simplicial triangulation. Then*

$$\rho(\mathrm{Bin}(M)) = \Lambda \cap P(M),$$

that is, the free parts of the binary classes are exactly the lattice points of the $\frac{1}{2}$-ball of the dual combinatorial norm.

Conjecture 11.2 (Strings). *For every closed oriented surface subcomplex $\Sigma \subset M^{(2)}$ and every ice state whose Σ-flux is above its minimum, there is a string through Σ. This implies Conjecture 11.1 along every surface direction.*

Conjecture 11.3 (Torsion in the interior). *If $\rho(\alpha)$ lies in the interior of $P(M)$, then $\alpha + \mathrm{Tors}H^2(M;\mathbb{Z}) \subseteq \mathrm{Bin}(M)$.*

Question 11.4. *Is the locking lemma sharp on faces? That is, on the relative interior of a face with frozen set F, is $\mathrm{Bin}(M) \cap (\alpha + \mathrm{Tors})$ exactly $[c^*] + \big(\mathrm{Tors} \cap \mathrm{im}\, H_1(M \setminus \bar{F})\big)$?*

What is already known to fail.

- The "full" conjecture $\mathrm{Bin}(M) = \{\alpha : \rho(\alpha) \in P(M)\}$, including torsion, is false: the half-turn manifold is a counterexample, at the vertices of P.
- Conjecture 11.1 fails for non-manifolds (Example 6.1).
- Conjecture 11.3 fails for non-manifolds, even in the interior (Example 6.3).

A possible route to Conjecture 11.1 for fibred M. For mapping tori with layered triangulations, near-polarised states are families of arc-disjoint "upward" strings. This is the three-dimensional Kasteleyn picture [11], and it suggests a free-fermion or non-intersecting-path analysis, layer by layer.

Question 11.5. *(a) How hard is it to decide* $\alpha \in \mathrm{Bin}(X)$ *for general 3-complexes? Proposition 2.7 places it in NP.*

(b) How do the binary depth and the growth of $\#\mathrm{Bin}(\mathrm{Sd}^k X)$ *behave? Note that* $\|\cdot\|_{\Delta,\mathrm{Sd}X} \le 6\|\cdot\|_{\Delta,X}$.

(c) What happens for non-orientable 3-manifolds, where ∂_3 *is not TU and Theorem 5.3 may fail?*

(d) For $X = BG$ *(the nerve of a group), binary maps* $BG \to SC_*$ *are closely related to circular orders on* G*. Which classes are Euler classes of circular orders [3]?*

12 Connections

Spin ice and lattice gauge theory. The dictionary of §9 is exact. Monopoles [4] are the non-fillable horns. The Coulomb phase [8] is the regime of interior sectors, and saturation (the Kasteleyn transition [11]) is the regime of extreme sectors. In compact $U(1)$ lattice gauge theory, binary cocycles are monopole-free Dirac-string configurations with values in $\{0,1\}$. The $\frac{1}{2}$-bound plays the role of Lüscher's admissibility condition [16], and the need for subdivision is the familiar statement that small instantons "fall through" a coarse lattice.
Circle actions. c_{01} is the orientation cocycle of Thurston and Ghys [7]. SC_* is a combinatorial classifying object for circular orders, and Theorem 3.3 is a combinatorial Milnor–Wood inequality [17, 29].
Combinatorial characteristic classes. SC_* is a circle analogue of MacPherson's combinatorial Grassmannian and of the Gelfand–MacPherson formula [6]. A cyclic order of points is the chirotope of points on a conic.
Small homotopy models versus binary models. The Sergeraert–Kenzo examples of Section 8 show that aggressive simplicial compression can destroy binary representability even when the homotopy type and the integral generator of H^2 remain unchanged. The companion filtration $\mathcal{X}_n \subset SC_*$ is designed to retain precisely this extra structure.
Algebraic geometry, by analogy. Consider $Z_X(u) = \sum_c u^{[c]} \in \mathbb{Z}[H^2(X;\mathbb{Z})]$, the sum over binary cocycles. Its Newton polytope is $P(X)$ in the TU case (Theorem 5.3). Conjecture 11.1 asserts that its support is saturated, as for the spectral curves of dimer models [13]. Theorem 4.3 resembles the torsor structure of n-th roots of line bundles. On a

surface, the unique maximal binary cocycle recalls rigidity of maximal representations (Goldman).

A The 3-skeleton of SC_*

Write $\sigma = (2,1,0)_\circlearrowright$ for the unique non-degenerate 2-simplex, so $c_{01}(\sigma) = 1$; the degenerate 2-simplex $(0,1,2)_\circlearrowright$ has $c_{01} = 0$. Circular words are written starting from bead 0. The six 3-simplices, with the values of c_{01} on the faces d_0, d_1, d_2, d_3, are:

3-simplex	status	$(c_{01}(d_0), \dots, c_{01}(d_3))$
$(0,1,2,3)_\circlearrowright$	totally degenerate	0000
$(0,1,3,2)_\circlearrowright = s_0\sigma$	degenerate	1100
$(0,3,1,2)_\circlearrowright = s_1\sigma$	degenerate	0110
$(0,2,3,1)_\circlearrowright = s_2\sigma$	degenerate	0011
$(0,2,1,3)_\circlearrowright$	non-degenerate	1001
$(0,3,2,1)_\circlearrowright$	non-degenerate	1111

These are exactly the six solutions of Lemma 2.4. The 2-skeleton is $\Delta^2/\partial\Delta^2 \simeq S^2$.

B Files and reproducibility

- `X_example.json`: facets of the complex of Example 6.11, together with the vertex sets of S and L.
- `verify_example.py`: checks the homology, the relation $[S] = \pm 3[L]$, the LP value $\frac{2}{3}$, and the integer maximum 0 (optionally under a shuffled vertex order).
- `ice.py`, `binexp.py`, `delta.py`: construction of the dual two-in/two-out digraph, the string ladder, and the scans of §10.

Acknowledgement. These notes were drafted with the assistance of AI systems (Claude, ChatGPT, Lemma); all computations are reproducible with the files above.

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