A Hypersimplicial PL Bundle over the Uniform Oriented-Matroid Bicomplex

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Abstract

We construct a geometric realization of the deletion–contraction bicomplex of labelled uniform oriented matroids and a canonical stable piecewise-linear sphere bundle over it. A cell indexed by a rank-r oriented matroid on E is the hypersimplex $\Delta(r, E)$. Its two positive-dimensional facet families are deletion and contraction. Since distinct facets may have the same minor, the realization is generally nonregular; a labelled face-occurrence category retains the missing incidence multiplicities and provides its derived subdivision.

The fiber over a rank-r cell is the $(N - r)$-fold suspension of the intrinsic PL covector sphere. Deletion is covector aggregation. A contraction retains the contraction equator and aggregates the two complementary hemispheres. For several contractions, different orders are not equal as assembly maps. They are instead the vertices of a canonical permutohedral diagram of simultaneous orthant completions. The whole family of permutohedra provides the higher coherences required to obtain a classifying map to the category of PL sphere assemblies. Using the fragmentation theorem for combinatorial fiber bundles, this map classifies a PL S^{N-1}-bundle. Stabilization by one suspension direction and fiberwise coning produce a stable PL microbundle.

Compatible oriented-matroid labels on Goncharov's hypersimplicial decomposition of a simplex therefore pull back this universal bundle. On the realizable locus the construction is PL-concordant to the sphere bundle coming from the corresponding linear quotient and normal exact sequences. In particular the stable bundle has canonical rational Pontryagin classes. Their cellular degrees force p_j to live on oriented matroids with $4j + 1$

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labels, placing the degree-four and degree-eight comparison problems on five and nine labels respectively. The existence theorem is independent of any proposed explicit sign or Hodge formula; identifying such formulas with the universal PL classes is a separate comparison problem.

Contents

1 Introduction

The starting point of this paper is geometric. The two elementary operations on a vector configuration that occur in the classical bi-Grassmannian are forgetting a vector and passing to the quotient by a vector. On the level of oriented matroids these become deletion and contraction. On the level of the moment polytope of the Grassmannian they are the two facet families of a hypersimplex. The question addressed here is whether this parallel is only a chain-level analogy or whether the resulting deletion–contraction complex carries an actual family of PL spheres, and hence actual PL characteristic classes.

The answer is affirmative. The construction is built from three standard pieces of topology which fit together unusually well in the oriented-matroid setting.

(1) The reduced covector poset of a rank-r oriented matroid is the face poset of a shellable regular cell decomposition of S^{r-1} [1, Thm. 4.3.3]; this is the intrinsic PL covector sphere.

(2) Deletion coarsens this sphere. Contraction cuts out an equator and turns its two complementary sides into the two hemispheres of a suspension. After enough suspensions, both operations act between spheres of one fixed dimension.

(3) The category of PL sphere ball complexes and combinatorial assemblies has classifying space homotopy equivalent to the classifying space of PL sphere bundles [10]. Consequently, a coherent system of deletion and contraction assemblies gives a genuine PL bundle.

The only nontrivial coherence issue is contraction order. Contracting two independent elements in different orders gives the same oriented-matroid minor but not literally the same intermediate ball complex. The relevant higher homotopies are not ad hoc: they form the face diagrams of permutohedra. Ordered set partitions record stages of simultaneous contraction, vertices record total orders, and coarsening an ordered partition gives the required assembly map. This is the central geometric observation of the paper.

A second technical issue is equally important. Different facets of a hypersimplex can attach to the same lower oriented-matroid cell. Thus the hypersimplicial realization is naturally a Δ-complex-like CW object, not in general a regular CW complex. We therefore keep *face occurrences*, not only the partial order of cells. The nerve of this labelled occurrence category is the correct derived subdivision and is the source on which the coherent sphere diagram is defined.

The construction is motivated by Goncharov's hypersimplicial description of maps from configurations of decorated flags to the bi-Grassmannian [6, Sec. 4.2]. Goncharov associates a Grassmannian configuration to every hypersimplex in the canonical N-decomposition of a simplex, and his compatibility lemma identifies the two hypersimplex boundary families with the two bi-Grassmannian differentials. Replacing the signs of Plücker coordinates by uniform oriented matroids leaves exactly the incidence skeleton used here. The present theorem supplies a PL sphere bundle on that skeleton, including nonrealizable oriented matroids.

The paper has four main outputs. First, Theorem 2.1 constructs the universal stable PL bundle. Second, Theorem 2.2 shows that a compatible UOM Goncharov flag pulls it back geometrically, not merely on chains. Third, Proposition 6.3 identifies the construction on realizable data with the expected linear quotient/normal sphere bundle up to PL concordance. Finally, Section 8 defines the resulting rational Pontryagin classes and isolates the separate problem of comparing them with explicit local formulas.

The distinction in the last sentence is essential. The bundle theorem gives a geometric target class before any local formula is chosen. Thus a five-label sign formula for p_1, or a nine-label Hodge construction for p_2, should be tested against a class that already exists rather than used as its definition.

Conventions and scope

All ground sets are finite, labelled, and linearly ordered. We work primarily with uniform oriented matroids because the positive-dimensional faces of a hypersimplex stay in the uniform minor category. Several local assembly lemmas hold more generally; this is indicated when useful. PL bundles are considered up to concordance. The word "canonical" therefore means canonical homotopy class of classifying map, not a preferred point-set triangulation of the total space.

2 Main results and conventions

All oriented matroids in this note are labelled by finite ordered ground sets. This avoids quotienting face maps by automorphism groups. Write $P(M) = \mathcal{L}(M) \setminus \{0\}$ for the poset of nonzero covectors of an oriented matroid M. The order is coordinatewise specialization, with $0 < +$ and $0 < -$.

For integers $p, q \geq 0$, set

$$\mathfrak{G}^{\mathrm{UOM}}_{p,q} = \mathrm{UOM}(q+1, p+q+2).$$

Deletion decreases p and contraction decreases q. On the free abelian groups generated by these sets, use the positive-dimensional operators

$$d[M] = \sum_{e \in E} (-1)^{\mathrm{pos}_E(e)} [M \setminus e], \qquad c[M] = \sum_{e \in E} (-1)^{\mathrm{pos}_E(e)} [M/e],$$

with the conventions

$$d = 0 \quad \text{on } p = 0, \qquad c = 0 \quad \text{on } q = 0.$$

These are exactly the codimension-one hypersimplex faces. Vertices at the two extreme ranks are omitted in the positive-dimensional complex. They may be adjoined separately when an augmented cellular complex is useful, but they are not additional terms of the cellular boundary displayed below.

2.1 Oriented-matroid notation

An oriented matroid M on a ground set E will be used through its covector set $\mathcal{L}(M) \subset \{0, +, -\}^E$. Its rank is denoted $\operatorname{rk} M$. Deletion and contraction are characterized on covectors by

$$\mathcal{L}(M \setminus e) = \{X|_{E \setminus e} : X \in \mathcal{L}(M)\}, \qquad \mathcal{L}(M/e) = \{X|_{E \setminus e} : X \in \mathcal{L}(M),\ X_e = 0\}. \tag{1.1}$$

For disjoint subsets $C, D \subset E$ we write $M/C \setminus D$; deletion and contraction on disjoint labels commute canonically. We use the standard partial order on sign vectors, generated by $0 < +$ and $0 < -$. The zero covector is the unique minimum.

The underlying matroid of M is *uniform* of rank r when every r-subset of E is a basis. We write $\mathrm{UOM}(r, E)$ for labelled uniform oriented matroids of rank r on E, and $\mathrm{UOM}(r, n)$ when $E = [n]$ with its standard order. Reorientations are not divided out: the sign information is part of the data.

The minor formulas above and the topological representation theorem are standard; see Folkman–Lawrence [5] and the monograph [1, Chs. 3–5]. A convenient modern proof of the topological representation theorem is [2].

Theorem 2.1 (Hypersimplicial realization and universal PL bundle). *There is a hypersimplicial CW space $\mathscr{B}^{\mathrm{UOM}}$ with the following properties.*

(i) *After orienting its rank-*r *cells by* $(-1)^{r-1}$ *times the standard hypersimplex orientation, there is a natural isomorphism*

$$C_m^{\mathrm{cell}}\big(\mathscr{B}^{\mathrm{UOM}}, \mathscr{B}^{\mathrm{UOM}}_{(0)}; \mathbb{Z}\big) \cong \bigoplus_{p+q=m-1} \mathbb{Z}[\mathfrak{G}^{\mathrm{UOM}}_{p,q}],$$

under which the cellular boundary is $d + c$.

(ii) *For every* $N \geq 1$, *the subspace* $\mathscr{B}^{\mathrm{UOM}}_{\leq N}$ *spanned by oriented matroids of rank at most* N *has a canonically determined concordance class of PL bundles*

$$S^{N-1} \longrightarrow \mathcal{E}^{\mathrm{UOM}}_N \longrightarrow \mathscr{B}^{\mathrm{UOM}}_{\leq N}.$$

Over the cell labelled by a rank-r *oriented matroid* M, *its fiberwise ball complex is represented by*

$$\mathscr{S}_N(M) = \mathrm{sd}\, P(M) * S^0_1 * \cdots * S^0_{N-r}.$$

(iii) *The construction is natural for labelled deletion and contraction and is compatible with stabilization:*

$$\mathcal{E}^{\mathrm{UOM}}_{N+1}\big|_{\mathscr{B}^{\mathrm{UOM}}_{\leq N}} \simeq \Sigma_{\mathrm{fib}} \mathcal{E}^{\mathrm{UOM}}_N.$$

After fiberwise coning, the associated PL microbundles satisfy

$$\xi^{\mathrm{UOM}}_{N+1} \simeq \xi^{\mathrm{UOM}}_N \oplus \varepsilon^1.$$

The word "canonical" in part (ii) refers to the concordance class, or equivalently to the homotopy class of its classifying map. A point-set bundle requires choices of subdivisions and geometric realizations; changing those choices does not change the concordance class.

Theorem 2.2 (Goncharov-flag pullback). *Let* $\mathcal{F}$ *be a compatible UOM Goncharov flag on the hypersimplicial* N*-decomposition of an* m*-simplex. There is a cellular map*

$$g_{\mathcal{F}} : \Delta^m(N) \longrightarrow \mathscr{B}^{\mathrm{UOM}}_{\leq N}$$

which sends each hypersimplex to the cell labelled by its UOM. The map on cellular chains sends the fundamental chain to the flag sum

$$(g_{\mathcal{F}})_*[\Delta^m(N)] = \sum_{\Delta^{p,q}_a} [M^{p,q}_a].$$

Consequently the flag-to-bi-Grassmannian chain map is induced by a cellular map, and the PL sphere bundle of the flag is $g^*_{\mathcal{F}} \mathcal{E}^{\mathrm{UOM}}_N$.

3 The hypersimplicial realization

Let E be a finite ordered set and $0 \le r \le |E|$. The labelled hypersimplex is

$$\Delta(r, E) = \left\{ (x_e)_{e \in E} \in [0,1]^E : \sum_{e \in E} x_e = r \right\}.$$

When $0 < r < |E|$, it has dimension $|E| - 1$. In the bi-Grassmannian notation used by Goncharov we put

$$r = q + 1, \qquad |E| = p + q + 2,$$

and write $\Delta^{p,q} = \Delta(q+1, [p+q+2])$. Thus $\dim \Delta^{p,q} = p + q + 1$. This convention matches the Grassmannian $G_{p+q+2}(q+1)$ and Goncharov's hypersimplicial decomposition [6, Sec. 4.2].

The faces of $\Delta(r, E)$ are indexed by disjoint sets $C, D \subset E$:

$$F_{C,D} = x_c = 1 \ (c \in C), \ x_d = 0 \ (d \in D).$$

If nonempty, this face is canonically

$$F_{C,D} \cong \Delta(r - |C|, E \setminus (C \cup D)). \tag{2.1}$$

The face is nonempty exactly when

$$0 \le r - |C| \le |E \setminus (C \cup D)|. \tag{2.2}$$

It is positive-dimensional exactly when both inequalities are strict. Accordingly the codimension-one positive-dimensional facets are

$$x_e = 0: \ \Delta^{p-1,q} \quad (p > 0), \qquad x_e = 1: \ \Delta^{p,q-1} \quad (q > 0). \tag{2.3}$$

This is the geometric source of the deletion/contraction truncation at the two extreme rows.

Example 3.1 (The extreme simplex). *The hypersimplex $\Delta(2,[3]) = \Delta^{0,1}$ is a triangle. Its edges are $x_i = 1$ and correspond to contractions from rank 2 to rank 1. The sets $x_i = 0$ are vertices, not facets. Thus the positive-dimensional differential has no deletion term at $p = 0$.*

Example 3.2 (The central octahedron). *The hypersimplex $\Delta(2,[4]) = \Delta^{1,1}$ is the regular octahedron. It has four facets $x_i = 0$ and four facets $x_i = 1$. Both families are triangles, and they give deletion and contraction on equal footing. This is the first cell on which the two directions of the bicomplex are simultaneously visible.*

Definition 3.3 (UOM hypersimplicial realization). *For every labelled uniform oriented matroid* M *of rank* r *on* E*, take a copy* Δ_{M} *of* $\Delta(\mathsf{r}, \mathsf{E})$*. Attach the face* $\mathsf{F}_{\mathsf{C},\mathsf{D}} \subset \Delta_{\mathsf{M}}$ *to the cell labelled by*

$$\mathsf{M}/\mathsf{C} \setminus \mathsf{D}$$

through the affine identification in (2.1)*. The resulting colimit is denoted by* $\mathscr{B}^{\mathrm{UOM}}$*.*

The minor operations in this formula are performed on disjoint sets. Their standard commutation identities imply that the attaching maps agree on all intersections. Interiors of the characteristic hypersimplices are never identified. Distinct boundary faces may attach to the same lower cell, so the result need not be a regular CW complex. It is nevertheless a CW complex with specified polyhedral characteristic maps, equivalently the realization of a hypersimplicial object with possible repeated face maps.

The repeated face maps must be retained in all later classifying constructions. For this purpose define the *labelled incidence category* $\mathcal{I}^{\mathrm{UOM}}$ as follows. Its objects are the labelled UOM cells M. A nonidentity morphism is a *geometric face occurrence*

$$\alpha : \mathsf{M} \longrightarrow \mathsf{K}. \tag{2.5}$$

Concretely it is a nonempty face $\mathsf{F} \subset \Delta_{\mathsf{M}}$, together with the canonical affine identification of F with the characteristic hypersimplex of the minor K. Away from vertices, F has a unique description $\mathsf{F} = \mathsf{F}_{\mathsf{C},\mathsf{D}}$ and then $\mathsf{K} = \mathsf{M}/\mathsf{C} \setminus \mathsf{D}$. At a vertex several pairs (C, D) can describe the same geometric face; these descriptions are identified. Conversely, two genuinely different faces are kept as different morphisms even if their minor labels K coincide. Composition is passage to a subface, or, in (C, D) notation whenever it is unique, union of contraction and deletion sets after passing to the remaining labels. Since the ground set strictly decreases on a nonidentity morphism, $\mathcal{I}^{\mathrm{UOM}}$ is an acyclic category.

This is the same bookkeeping device that one uses for a Δ-complex or a semisimplicial set with repeated characteristic faces: the target cell does not by itself determine the incidence occurrence.

Proposition 3.4 (Occurrence category and subdivision). *There is a canonical PL homeomorphism*

$$|\mathrm{N}\mathcal{I}^{\mathrm{UOM}}| \;\cong\; \mathrm{sd}\,\mathscr{B}^{\mathrm{UOM}}. \tag{2.2b}$$

In particular the occurrence category, rather than the ordinary poset of unlabelled cells, is the correct incidence object for the possibly nonregular hypersimplicial realization.

Proof. Barycentrically subdivide every characteristic hypersimplex before taking the colimit. A simplex in the subdivision of Δ_M is a strict flag of nonempty faces of Δ_M. Recording successively which coordinates have been fixed to 1 and to 0, with degenerate descriptions of the same vertex identified, gives exactly a composable chain of geometric face occurrences in $\mathcal{I}^{\mathrm{UOM}}$. Conversely such a chain determines a unique flag of geometric faces in the characteristic hypersimplex. The quotient identifications on subdivided faces are precisely the composition rule in $\mathcal{I}^{\mathrm{UOM}}$, including multiplicities when two different faces have the same minor. The resulting simplicial realizations are therefore canonically isomorphic. □

The CW realization itself includes its zero-dimensional strata. The UOM bicomplex used in characteristic-class calculations is the relative cellular complex modulo the zero-skeleton; equivalently it is the positive-dimensional truncation with $d = 0$ on $p = 0$ and $c = 0$ on $q = 0$. The bundle construction is made on the full realization and then restricted or evaluated relatively as needed.

3.1 The cellular differential

Orient the affine hyperplane $\sum x_e = r$ by requiring its positive normal $(1, \ldots, 1)$ to precede the oriented tangent basis. With the induced hypersimplex orientation, for $p, q > 0$ one has

$$\partial[\Delta_M] = \sum_{e \in E} (-1)^{\mathrm{pos}_E(e)} \left([\Delta_{M \setminus e}] - [\Delta_{M/e}]\right). \tag{2.2}$$

At $p = 0$ the $x_e = 0$ descriptions are not facets and the deletion sum is omitted; at $q = 0$ the $x_e = 1$ descriptions are not facets and the contraction sum is omitted. Thus (2.2) is to be read with exactly the truncation conventions stated above. This is the familiar distinction between the two hypersimplex facet families.

Proposition 3.5. *After replacing the orientation of every rank-*r *cell by* $(-1)^{r-1}$ *times the orientation in* (2.2), *its cellular boundary is*

$$\partial[\Delta_M]' = d[M] + c[M].$$

Consequently Theorem 2.1(i) *holds.*

Proof. Deletion leaves the rank unchanged, so the rank-dependent signs of a cell and its deletion facet agree. Contraction changes the rank from r to $r-1$; hence the signs differ by -1, cancelling the minus sign in (2.2). The dimension of the cell indexed by $M \in \mathrm{UOM}(q+1, p+q+2)$ is $p+q+1$, which gives the displayed grading. □

This proves more than the algebraic identities $d^2 = c^2 = dc + cd = 0$: those identities are the codimension-two incidence identities of an actual hypersimplicial realization.

4 PL covector spheres and assembly maps

For a rank-r loop-free oriented matroid, the nonzero covector poset

$$P(M) = \mathcal{L}(M) \setminus \{0\}$$

is the face poset of a shellable regular cell decomposition of S^{r-1} [1, Thm. 4.3.3]. We call this the *PL covector sphere* of M. The statement is intrinsic: no realization of M by vectors or by a chosen pseudosphere arrangement is part of the definition. The topological representation theorem identifies the codimension-one zero sets $\{X : X_e = 0\}$ with an arrangement of pseudospheres representing M [5, 2]; we use such a representation only as a convenient proof model for some aggregation maps.

A *finite ball complex* on a compact PL manifold X is a finite regular CW decomposition in which every closed cell is a PL ball and every attaching map is a PL embedding on each face. An *assembly* $K \to L$ of ball complexes of the same underlying PL manifold is a cellular map of face posets obtained by coarsening: every closed cell of L is a union of closed cells of K, and the underlying map of spaces is a PL homeomorphism. Assemblies compose.

Let $\mathcal{R}(X)$ be the category whose objects are finite ball complexes on a PL manifold of type X and whose morphisms are assemblies. In [10], the fragmentation theorem for fiberwise PL homeomorphisms is used to prove

$$B\mathcal{R}(X) \simeq B\,\mathrm{PL}(X), \tag{3.1}$$

where $\mathrm{PL}(X)$ is the simplicial group of PL self-homeomorphisms of X. For us $X = S^n$. The importance of (3.1) is that a coherent combinatorial coloring by sphere ball complexes and assemblies is not merely a model: it classifies an actual PL sphere bundle up to concordance.

4.1 Deletion

Lemma 4.1 (Deletion aggregation). *Let $D \subset E$ and suppose $M \setminus D$ has the same rank as M. Then covector restriction*

$$\rho_D : P(M) \longrightarrow P(M \setminus D), \qquad X \longmapsto X|_{E \setminus D}, \tag{3.2}$$

is an abstract PL assembly. These assemblies compose strictly.

Proof. Choose a topological-representation sphere for M only for the purpose of checking the PL fibers. Erasing the pseudospheres labelled by D leaves an essential arrangement of the same rank, hence a coarsening of the same covector sphere. The induced map on face posets is exactly (3.2). Since the map is intrinsically covector restriction, it is independent of the chosen representation, and compositions are strict. $\square$

Uniformity is not needed in Lemma 4.1; rank preservation is the essential condition.

4.2 One contraction

If L is a regular PL d-sphere complex, let $\Sigma_{\mathrm{cell}} L$ be the ball complex obtained by adjoining two maximal $(d+1)$-balls whose common boundary is L. Its barycentric subdivision satisfies

$$\mathrm{sd}(\Sigma_{\mathrm{cell}} L) \cong \mathrm{sd}\, L * S^0. \tag{3.3}$$

Lemma 4.2 (Hemispherical contraction). *For a nonloop $e \in E$, the map*

$$a_e : P(M) \longrightarrow \Sigma_{\mathrm{cell}} P(M/e), \qquad X \longmapsto \begin{cases} +, & X_e = +, \\ -, & X_e = -, \\ X|_{E \setminus e}, & X_e = 0, \end{cases} \tag{3.4}$$

is an abstract PL assembly.

Proof. In a topological representation used as a proof device, the pseudosphere S_e is the equator whose induced covector decomposition represents M/e. Formula (3.4) retains the equatorial cells and assembles every cell in each complementary open hemisphere into one maximal cell. The inverse image of either new closed cell is a closed PL hemisphere, while inverse images of equatorial cell closures are the corresponding closed cells in S_e. Hence the map is an assembly. $\square$

Example 4.3 (One contraction as suspension). *If M has rank r and e is not a loop, then $P(M/e)$ is an $(r-2)$-sphere. The target of a_e is obtained by adding two maximal $(r-1)$-balls with common equator $P(M/e)$. After barycentric subdivision it is exactly*

$$\mathrm{sd}\, P(M/e) * S^0.$$

Thus contraction decreases the oriented-matroid rank by one while one new normal suspension direction restores the fiber dimension.

5 Simultaneous contractions and permutohedral coherence

The composites of Lemma 4.2 depend on the order of the contracted elements as maps of ball complexes. This does not obstruct the bundle construction: all orders are faces of a canonical permutohedral diagram of assemblies. We now construct this diagram.

5.1 The orthant completion

For a finite set A, let

$$\mathcal{O}_A = \{0, +, -\}^A \setminus \{0\},$$

ordered coordinatewise by $0 < +$ and $0 < -$. Its order complex is the barycentric subdivision of the boundary of the $|A|$-dimensional crosspolytope and hence is a PL $S^{|A|-1}$.

Definition 5.1. *For a regular PL sphere complex* L*, define* $B_A(L)$ *to be the abstract ball complex whose face poset is the ordinal sum*

$$P(L) \oplus \mathcal{O}_A. \tag{4.1}$$

Thus every cell of L *lies below every nonzero* A*-sign cell.*

Lemma 5.2 (Orthant sphere)**.** *If* L *is a regular PL* d*-sphere complex, then* $B_A(L)$ *is a regular PL* $(d + |A|)$*-sphere complex and*

$$\Delta P(B_A(L)) = \Delta P(L) * \Delta \mathcal{O}_A. \tag{4.2}$$

Proof. The equality (4.2) is the standard identity between ordinal sums and joins of order complexes. It gives a PL sphere $S^d * S^{|A|-1} \cong S^{d+|A|}$. It remains to check the principal lower ideals. Lower ideals of cells belonging to L are unchanged. If $s \in \mathcal{O}_A$ has support of size t, then $\mathcal{O}_{A,\leq s}$ is the poset of nonempty faces of a $(t-1)$-simplex. Therefore the lower ideal of s realizes as $S^d * B^{t-1}$, a PL ball. The strict lower ideal realizes as $S^d * S^{t-2}$, its boundary sphere. Thus (4.1) is an abstract regular PL sphere complex. □

Lemma 5.3 (Simultaneous contraction aggregation)**.** *Let* $C, D \subset E$ *be disjoint. Assume* C *is independent in* M *and that*

$$K = (M/C) \setminus D$$

has rank $\mathrm{rk}(M) - |C|$. *Then*

$$a_{C,D} : P(M) \longrightarrow B_C(P(K)), \qquad X \longmapsto \begin{cases} X|_C, & X|_C \neq 0, \\ X|_{E\setminus(C\cup D)}, & X|_C = 0, \end{cases} \tag{4.3}$$

is an abstract PL assembly.

Proof. Extend C to a basis B of M. Deleting $E \setminus B$ preserves rank, and the B-subarrangement is the pseudosphere representation of the free rank-$\mathrm{rk}(M)$ oriented matroid. It is PL equivalent to the coordinate arrangement. Consequently the pseudospheres labelled by C are simultaneously coordinate, their common zero set is an unknotted sphere, and every closed C-orthant is a PL ball.

For $s \in \mathcal{O}_C$, the inverse image of its principal lower ideal under (4.3) is precisely the closed orthant in which coordinates outside $\mathrm{supp}(s)$ vanish and those in $\mathrm{supp}(s)$ have the signs prescribed by s. It is a PL ball of the dimension prescribed in Lemma 5.2. When the target cell belongs to $P(K)$, the inverse lower ideal is a closed cell of the induced arrangement on the common zero sphere, with the D-walls erased. It is again a PL ball. This is the inverse-lower-ideal criterion for an abstract assembly. □

Uniformity enters the construction precisely here: every contraction set appearing in a nonempty hypersimplex face is independent.

5.2 Ordered partitions

Let C be finite and let

$$\pi = (C_1|C_2|\cdots|C_s)$$

be an ordered partition of C into nonempty blocks. Define

$$B_\pi(L) = B_{C_1}\big(B_{C_2}(\cdots B_{C_s}(L)\cdots)\big). \tag{4.4}$$

A cell outside L records the first block whose sign vector is nonzero and forgets all later signs.

Lemma 5.4 (Permutohedral assembly diagram). *Suppose π' is obtained from π by splitting one or more blocks, while retaining their order. There is a canonical assembly*

$$B_\pi(L) \longrightarrow B_{\pi'}(L). \tag{4.5}$$

These maps compose strictly. Consequently

$$\pi \longmapsto B_\pi(L)$$

is a functor from the opposite face poset of the permutohedron $\mathrm{Perm}(C)$ *to* $\mathcal{R}(S^{\dim L+|C|})$.

Proof. It is enough to split a block $A \sqcup B$ into $A|B$. A nonzero sign vector (s_A, s_B) is sent to s_A when $s_A \neq 0$; if $s_A = 0$, it is sent to the B-sign cell s_B. Cells in L remain unchanged. Thus the map forgets the B-sign precisely on the region where the earlier A-block is already nonzero. Its inverse cell closures are coordinate orthant balls, so it is an assembly. Describing the map by the first nonzero block shows that a sequence of block splittings is independent of the order in which the splittings are performed. This proves strict composition.

The faces of the permutohedron are indexed by ordered partitions; passing to a refined ordered partition corresponds to passing to a subface. Hence the direction in (4.5) is the opposite face-poset direction. □

For the one-block partition (C), formula (4.4) is the symmetric orthant completion $B_C(L)$. For a vertex $(c_{\sigma(1)}|\cdots|c_{\sigma(k)})$, it is the globular complex obtained by contracting in the order σ. Thus Lemma 5.4 fills all changes of contraction order. Its two-dimensional faces encode the commutation and braid relations, and its higher faces supply the higher coherence relations. We use the standard description of permutohedron faces by ordered set partitions; see, for example, [14, Sec. 0.10].

Example 5.5 (Two and three contractions). *For* $C = \{e, f\}$, $\mathrm{Perm}(C)$ *is an interval. Its endpoints are the two ordered contraction composites* $e|f$ *and* $f|e$*; the interior one-block face* (ef) *is the simultaneous orthant completion. Thus the interval is the homotopy between the two orders.*

For $|C| = 3$, $\mathrm{Perm}(C)$ *is a hexagon. Its six vertices are the six total orders. Its edges merge one adjacent pair into a two-element block, and the 2-cell fills the complete compatibility among these pairwise comparisons. This is the first genuinely higher coherence cell.*

6 Construction of the universal PL bundle

Fix N. If M has rank $r \leq N$, define

$$\mathscr{S}_N(M) = \mathrm{sd}\, P(M) * S^0_1 * \cdots * S^0_{N-r}. \tag{5.1}$$

For the rank-zero empty covector complex, use the convention $P(M) = S^{-1}$, so (5.1) is still S^{N-1}.

Lemma 4.1, followed by barycentric subdivision and join with the fixed S^0 factors, defines the deletion aggregation

$$\delta_e : \mathscr{S}_N(M) \longrightarrow \mathscr{S}_N(M \setminus e). \tag{5.2}$$

Lemma 4.2 and (3.3) define

$$\gamma_e : \mathscr{S}_N(M) \longrightarrow \mathscr{S}_N(M/e), \tag{5.3}$$

where the extra S^0 is the new normal direction.

Deletion–deletion squares and mixed deletion–contraction squares commute strictly, because they are restrictions of sign vectors on distinct labels. Two contraction orders need not give the same assembly in (5.3). To state the resulting coherence without losing repeated face occurrences, we work over the occurrence category $\mathcal{I}^{UOM}_{\leq N}$ of Proposition 3.4.

Fix, once and for all, the displayed order of the formal suspension factors in $\mathscr{S}_N(M)$. For a face occurrence

$$\alpha = (C, D) : M \longrightarrow K = M/C \setminus D,$$

choose an ordering $C = (c_1, \ldots, c_k)$. Iterating the one-element contraction assemblies in this order, and then deleting D, gives an assembly

$$\Gamma_{\alpha,(c_1,\ldots,c_k)} : \mathscr{S}_N(M) \longrightarrow \mathscr{S}_N(K). \tag{5.4}$$

Different orders give different representatives of the same face occurrence. Lemma 5.4 supplies the higher comparison data between all of them: for every ordered partition π of C, the sphere $B_\pi(P(K))$ (joined with the unchanged old normal factors) is an object of $\mathcal{R}(S^{N-1})$; refinement of ordered partitions gives the assembly maps (4.5). At a vertex $(c_{\sigma(1)}|\cdots|c_{\sigma(k)})$ the resulting composite is precisely (5.4) for the order σ. The one-block face gives the symmetric simultaneous-contraction assembly.

Proposition 6.1 (Coherent classifying-space map). *The object assignment $M \mapsto \mathscr{S}_N(M)$, the deletion assemblies, the one-element contraction assemblies, and the permutohedral diagrams above determine, canonically up to homotopy, a map*

$$B\Phi_N : B\mathcal{I}^{UOM}_{\leq N} \longrightarrow B\mathcal{R}(S^{N-1}). \tag{5.5}$$

Its homotopy class is independent of the auxiliary choices of contraction orders and of the numbering of the formal normal factors.

Proof. A permutohedral assembly diagram is an ordinary functor from the opposite face poset of a permutohedron to $\mathcal{R}(S^{N-1})$. After taking classifying spaces it therefore gives an actual continuous map from the barycentric subdivision of that permutohedron to $B\mathcal{R}(S^{N-1})$. Its vertices are the different ordered contraction composites, its edges are homotopies between adjacent orders, and its higher faces are the required higher homotopies.

For a composable chain of face occurrences, collect all contraction labels and remember the stage at which each label is contracted. This gives an ordered partition of the total contraction set. Forgetting an intermediate object in the chain merges adjacent blocks; refining a chain splits blocks. Thus the boundary restrictions required of the higher homotopies are exactly the face restrictions in the ordered-partition posets of the permutohedra. Lemma 5.4 says that the corresponding assembly maps compose strictly under refinement. Deletions commute strictly with each other and with the contraction construction because they act only on the deepest covector sphere $P((M/C) \setminus D)$. The permutohedral classifying-space maps therefore satisfy all coherence identities.

A homotopy-coherent diagram on a small category may be encoded as a functor from a standard simplicial/topological resolution of that category; Vogt's construction and the comparison theorem of Cordier–Porter show that such coherent diagrams model strict diagrams up to level homotopy equivalence [13, 4]. The preceding permutohedral maps provide exactly the required structure maps on the factorization parameters: deleting an intermediate stage merges blocks, while refining a factorization splits them. Hence the data determine a homotopy-coherent map from the occurrence category to $B\mathcal{R}(S^{N-1})$, and realization gives (5.5). Equivalently one may use the Vogt/Boardman–Vogt style resolution of the source category and realize the same permutohedral factorization diagrams. We emphasize that the target ordinary category is used through its classifying space: we are not declaring distinct contraction composites to be equal morphisms in $\mathcal{R}(S^{N-1})$.

Changing the initial order of C, or renumbering the formal suspension factors, changes only the chosen vertex in a contractible permutohedron (and a combinatorial automorphism of a crosspolytope factor). The full permutohedral diagram gives a coherent homotopy between the resulting maps. $\square$

This is the point at which the occurrence category is essential: replacing it by the ordinary poset of cells would forget repeated face maps.

Using Proposition 3.4 and the canonical homeomorphism between a CW complex and its barycentric subdivision, we obtain a homotopy class

$$\kappa_N : \mathscr{B}^{\mathrm{UOM}}_{\leq N} \longrightarrow B\mathcal{R}(S^{N-1}) \simeq B\,\mathrm{PL}(S^{N-1}). \tag{5.6}$$

By Mnëv's equivalence (3.1), this homotopy class classifies a well-defined concordance class of PL S^{N-1}-bundles. This proves Theorem 2.1(ii).

Adding one more fixed S^0 factor to (5.1) suspends every fiber and every aggregation, including the ordered-partition diagrams. This proves Theorem 2.1(iii).

To relate this sphere bundle to the stable PL tangent-bundle formalism, cone each fiber. A PL self-homeomorphism of S^{N-1} cones to a PL self-homeomorphism of D^N fixing the cone point; restricting to the interior gives an $\mathbb{R}^N$-bundle with zero section. By the Kuiper–Lashof microbundle/bundle theorem, PL $\mathbb{R}^N$-bundles with zero section and rank-N PL microbundles have the same classification up to equivalence [8, 9]. We denote the resulting microbundle by ξ_N^{UOM}. Suspending the sphere bundle before coning adds one trivial radial coordinate, hence

$$\xi_{N+1}^{\mathrm{UOM}} \simeq \xi_N^{\mathrm{UOM}} \oplus \varepsilon^1.$$

Remark 6.2 (Orientation local system). *Permuting two suspension directions reverses the orientation of their normal two-frame. Therefore the natural orientation system of $\mathcal{E}_N^{\mathrm{UOM}}$ is the tensor product of the covector-sphere orientation line and the determinant line of the accumulated contraction directions. An oriented version of Theorem 2.1 requires a coherent trivialization of this determinant line. This is precisely the kind of normal-sign datum retained in the determinant-line formulation of UOM Goncharov flags.*

6.1 Comparison with linear realizations

The preceding construction is intrinsic to oriented matroids. On the realizable locus it has the expected linear meaning.

Proposition 6.3 (Realizable linear comparison). *Let M be represented by nonzero vectors $(v_e)_{e\in E}$ spanning a real r-space V. Then the covector sphere $|P(M)|$ is naturally the cell decomposition of the unit sphere in V^* cut out by the hyperplanes $v_e^\perp$. Under this identification:*

(a) *a rank-preserving deletion erases the corresponding hyperplane walls and is the identity PL sphere with a coarsened cell structure;*

(b) *for an independent set C, contraction is the equatorial inclusion associated with the exact sequence*

$$0 \longrightarrow (V/\langle v_c : c \in C\rangle)^* \longrightarrow V^* \longrightarrow \langle v_c : c \in C\rangle^* \longrightarrow 0, \tag{5.8}$$

and the simultaneous orthant completion is PL concordant to adjoining the normal sphere of the last vector space in (5.8);

(c) *after stabilization by $\mathbb{R}^{N-r}$, the homotopy-coherent diagram of Proposition 6.1 is concordant to the sphere diagram of the stabilized dual spaces and their quotient-normal exact sequences.*

Consequently, whenever a realizable UOM Goncharov flag comes from an actual family of linear quotient configurations (for example from a decorated flag), the pullback $g_{\mathcal{F}}^{*}\mathcal{E}_{N}^{\mathrm{UOM}}$ is PL-concordant to the sphere bundle of the corresponding stabilized dual quotient bundle.

Proof. A covector of the vector configuration is the sign vector $\big(\operatorname{sign} f(v_e)\big)_{e\in E}$ of a nonzero functional $f \in V^*$. Hence radial projection identifies the covector decomposition with the indicated hyperplane decomposition of $S(V^*)$. Deletion simply forgets walls. If C is independent, the common zero set of the coordinates in C is the unit sphere in the annihilator $(V/\langle C\rangle)^*$, and the complementary sign regions are the orthants of the quotient normal space $\langle C\rangle^*$. Choosing an inner product and a splitting of (5.8) identifies this with the orthant completion of Section 4. The space of inner products and linear splittings is contractible, so different choices give PL-concordant sphere diagrams. Stabilization adds the fixed trivial summand $\mathbb{R}^{N-r}$. Compatibility with chains of quotients is exactly the flag compatibility encoded by the permutohedral coherence cells. □

Remark 6.4. *Proposition 6.3 is a comparison statement, not a claim that the underlying vector spaces V form a bundle over the whole UOM realization. The latter statement is false without additional family data. The proposition says that wherever such linear family data exist, the intrinsic UOM bundle has the same stable PL concordance class.*

7 Goncharov flags

Following Goncharov [6, Sec. 4.2], realize the dilated m-simplex as

$$\Delta^m(N) = \{(x_0, \dots, x_m) \in \mathbb{R}^{m+1}_{\geq 0} : x_0 + \cdots + x_m = N\}.$$

Cut it by all integral coordinate hyperplanes $x_i = a$, $a \in \mathbb{Z}$. The resulting polyhedral decomposition is the *hypersimplicial* N*-decomposition.* Its full-dimensional cells are

$$\Delta_a^{p,q} = \{x \in \Delta^m(N) : a_i \leq x_i \leq a_i + 1 \text{ for all } i\}, \tag{6.1}$$

where $a = (a_0, \dots, a_m) \in \mathbb{Z}^{m+1}_{\geq 0}$ and

$$p + q = m - 1, \qquad \sum_i a_i = N - (q+1).$$

After the translation $y_i = x_i - a_i$, the cell $\Delta_a^{p,q}$ is canonically the hypersimplex $\Delta^{p,q}$.

A *compatible UOM Goncharov flag* on this decomposition is an assignment of a labelled uniform oriented matroid of the matching rank to every full-dimensional hypersimplex, together with the requirement that the two induced minors coincide on every shared facet. Thus we assign Suppose a UOM Goncharov flag assigns

$$\Delta_a^{p,q} \longmapsto M_a^{p,q} \in \mathrm{UOM}(q+1, m+1)$$

and the assignments agree on common facets. For example, across an internal facet the condition is

$$M_a^{p,q}/i = M_{a+e_i}^{p+1,q-1} \setminus i. \tag{6.2}$$

Each cell $\Delta_a^{p,q}$ is affinely the hypersimplex attached to $M_a^{p,q}$ in $\mathscr{B}^{\mathrm{UOM}}$. Equation (6.1) and its deletion analogues say exactly that these affine maps coincide on shared facets. They therefore glue to a cellular map

$$g_{\mathcal{F}} : \Delta^m(N) \longrightarrow \mathscr{B}_{\leq N}^{\mathrm{UOM}}.$$

On cellular chains, the image of the oriented fundamental chain is the sum of the labelled hypersimplices with their incidence orientations. This proves Theorem 2.2.

For comparison, a decorated flag in an N-dimensional vector space is a complete flag together with a nonzero vector in every one-dimensional successive quotient. Goncharov's construction associates to a generic configuration of $m+1$ decorated flags a Grassmannian configuration for every cell $\Delta_a^{p,q}$ of the decomposition: one first quotients by the sum of the initial flag pieces prescribed by a, and then takes the next decorated vectors. The key compatibility statement is that the two families of hypersimplex facets agree with the two bi-Grassmannian boundary operators [6, Sec. 4.2]. Passing to signs of maximal minors therefore gives a compatible UOM Goncharov flag in the sense above.

Proposition 6.3 identifies the pullback of the intrinsic UOM bundle for such a flag, up to stable PL concordance, with the sphere bundle supplied by those linear quotient data. Thus the construction extends the geometric bundle from realizable decorated flags to all compatible UOM flags without postulating a vector bundle on the nonrealizable locus.

8 Characteristic-class consequences

Let BPL_N denote a standard classifying space for rank-N PL microbundles and let $\mathrm{BPL} = \mathrm{colim}_N \, \mathrm{BPL}_N$. Coning the sphere bundle constructed above yields a classifying map

$$\bar{\kappa}_N : \mathscr{B}^{\mathrm{UOM}}_{\le N} \longrightarrow \mathrm{BPL}_N. \tag{7.1}$$

The stabilization of Theorem 2.1 is compatible with the standard maps $\mathrm{BPL}_N \to \mathrm{BPL}_{N+1}$.

The forgetful map BO $\to$ BPL is a rational cohomology equivalence: its homotopy fiber PL/O has finite homotopy groups. Hence there are unique universal rational PL Pontryagin classes

$$p_j^{\mathrm{PL}} \in H^{4j}(\mathrm{BPL};\mathbb{Q})$$

whose pullbacks to BO are the ordinary rational Pontryagin classes. This is one standard formulation of PL invariance of rational Pontryagin classes; see Kirby–Siebenmann [7] and the survey [12, Secs. 18,22]. We therefore define

$$p_j^{\mathrm{UOM}} = \bar{\kappa}_N^* p_j^{\mathrm{PL}} \in H^{4j}(\mathscr{B}^{\mathrm{UOM}}_{\le N};\mathbb{Q}), \qquad N \gg j. \tag{7.2}$$

The stabilization makes these classes independent of N once the rank is large enough and gives stable classes on the filtered union.

The cellular description of Section 2 has a useful numerical consequence. A cellular representative of p_j^{UOM} in degree $4j$ is evaluated on UOMs with

$$|E| - 1 = 4j, \qquad \text{that is,} \qquad |E| = 4j + 1.$$

Thus p_1^{UOM} is naturally a five-label cocycle and p_2^{UOM} is naturally a nine-label cocycle. This supplies a geometric comparison target for the five-point sign character and the nine-point p_2 constructions.

Warning 8.1. *Theorem 2.1 proves the existence of the canonical PL characteristic classes* (7.1). *It does not prove that a previously proposed explicit sign or Hodge formula represents them. Such an identification requires evaluating a universal PL Pontryagin cocycle on the aggregation diagrams, or proving a separate uniqueness theorem.*

Theorem 2.2 implies

$$p_j(g_{\mathcal{F}}^* \xi_N^{\mathrm{UOM}}) = g_{\mathcal{F}}^* p_j^{\mathrm{UOM}}.$$

Hence the UOM flag complex is not merely a resolution of the chain-level deletion–contraction identities: it is a geometric domain on which the universal UOM PL characteristic classes pull back.

9 Scope and remaining comparison problem

The construction separates three statements which should not be conflated.

1. The hypersimplicial realization and its universal stable PL bundle are combinatorial consequences of the intrinsic PL covector sphere, minor functoriality, the labelled occurrence category, and permutohedral coherence.

2. A common parent oriented matroid supplies a simultaneous concrete pseudosphere model, but is not needed for the concordance class of the bundle.

3. The equality of a specific local UOM formula with a Pontryagin class is a further comparison theorem. The present construction specifies exactly which class must be computed.

The most direct next calculation is degree four. Choose a cellular representative of the universal class p_1^{PL} on a combinatorial model of $B\mathcal{R}(S^{N-1})$, pull it back through κ_N, and evaluate it on the five-label UOM cells. The comparison problem is whether this cocycle is cohomologous, with the expected normalization, to the known alternating five-point sign character. This is now a theorem-identification problem: p_1^{UOM} already exists independently of the answer.

Degree eight is the analogous nine-label problem. One must compare p_2^{UOM} with the proposed marked-current/Hodge construction and prove that all auxiliary Hodge choices alter the representative only by a coboundary. In particular, a vanishing or nonvanishing residual in a chosen filtration has meaning only after it is compared with the geometrically defined class.

More generally, cellular degree forces the stable class p_j^{UOM} to be tested first on $4j+1$ labels. Thus the geometric theorem predicts the sequence

$$5, 9, 13, 17, \ldots$$

without reference to any explicit formula.

10 Concluding perspective

The construction may be summarized by the diagram

$$\begin{tikzcd}
\text{uniform oriented matroids} \arrow[r] \arrow[d, "\text{delete/contract}"] & \text{PL covector spheres} \arrow[d, "\text{assemblies + permutohedral coherence}"] \\
\mathscr{B}^{\mathrm{UOM}}_{\leq N} \arrow[r, "\kappa_N"] & B\mathcal{R}(S^{N-1}) \simeq BPL(S^{N-1}) \arrow[d, "\text{cone}"] \\
 & BPL_N.
\end{tikzcd}$$

The upper row is entirely combinatorial. The lower row converts that combinatorics into a genuine PL bundle and therefore into characteristic classes. Realizable configurations are a comparison locus, not a prerequisite for the construction.

This point is useful for the wider characteristic-class program. It separates the geometric existence of the universal class from the search for small local formulas. The latter can now be attacked by Hodge transfer, symmetry, rank–nullity compression, or other algebraic devices without risking a circular definition of the characteristic class.

Remark 10.1 (What has been repaired). *The global bundle argument uses three points which are easy to lose if one works only at chain level: (i) extreme-rank nonfacets are removed by the positive-dimensional truncation; (ii) repeated hypersimplex faces are retained as distinct morphisms in* $\mathcal{I}^{\mathrm{UOM}}$*; and (iii) different contraction orders are not declared equal, but are connected by the full permutohedral homotopy-coherence diagram before rectification. These are the only extra global ingredients needed beyond the local deletion and contraction assemblies.*

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