The Uniform Oriented-Matroid Bi-Grassmannian and Goncharov Flags

Hypersimplicial PL bundles, Thom–Bier residues, Hodge transfer, and the p_1/p_2 program

Nikolai Mnëv*

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AI† generated major Thom–Bier / hypersimplicial revision (v5)

Abstract

This working note studies the finite labelled deletion–contraction system of uniform oriented matroids and its relation to hypersimplicial geometry, Hodge transfer, and characteristic classes. The present revision incorporates the geometric breakthrough of the companion paper on the hypersimplicial PL bundle: the UOM bi-Grassmannian has a hypersimplicial CW realization carrying a canonical stable PL sphere bundle. Hence stable rational UOM Pontryagin classes exist independently of any proposed local sign formula.

This changes the characteristic-class program substantially. At fixed rank the cone on the covector sphere supplies a relative Thom complex, canonically the suspension of the augmented sphere complex. The Hodge descent underlying the local Euler cocycle is therefore naturally interpreted as Thom descent. For rank $2k$, the critical identity is $p_k = e_{2k}^2$; on the Thom space the correctly graded representative is the Thom cube $U^3 = p_k U$, not the Thom square. We construct a candidate inverse-Thom mechanism from deletion–contraction residues.

The universal incidence algebra admits an unexpectedly explicit solution. Contraction is residue along a coordinate pseudosphere; on

*PDMI RAS

†GPT Sol

the Thom line it is a truncated Koszul complex with Hodge propagator $|A|^{-1}\iota_1$. The full scalar minor complex is the Bier sphere of the rank-r simplex skeleton, equivalently the facet nerve of the simple generic slice $[0,1]^n \cap \{\sum x_i = \lambda\}$ for $r < \lambda < r+1$. After eliminating contractions, the remaining complex is an S_n-equivariant Bier–Pieri resolution of the sign representation with explicit two-term Specht blocks and closed Hodge eigenvalues. In the critical degree $n = 4k+1$, the Bier sphere is balanced between $2k$ deletions and $2k$ contractions, while the unique unused label is exactly the occurrence/rank-changing coordinate.

These results reorganize the p_1 and p_2 calculations. The five-label p_1 seed has the classical coefficient $\varepsilon_2/48$ and the circle Thom–Bier representative differs from the Euler square by an explicit sawtooth Chern–Simons coboundary. The geometric p_2 program starts instead from the rank-four Thom cocycle: only four monomial types in the selected component of $\mathbb{U}_4^3$ can contribute. The older 512/1024 sign formulas are retained as compressed gauges for the primitive Newton/logarithmic class and are no longer used to define p_2. Throughout, proved identities, reported computations, conjectural comparisons, and programmatic analogies are marked separately.

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Part I

The UOM bi-Grassmannian and hypersimplicial flag geometry

1 Purpose, scope, and status conventions

Revision-v5 notice. Part VIII contains the current geometric organization of the Pontryagin program. Where an earlier section treats $\varepsilon_2/512$ or $-\varepsilon_2/1024$ as a foundational p_2 seed, that language should now be read historically: those expressions belong to a compressed primitive/logarithmic gauge. The geometric class itself is defined independently by the hypersimplicial PL bundle and is compared to the Thom–Bier construction in Part VIII.

The present revision also incorporates the separate working note on $\mathbb{F}_1/A_2$/cluster structures. The merge is mathematical rather than merely editorial: the five-point A_2 foundation turns out to control the endpoint character of p_1, and by a parity/norm identity the same atom controls the rank-two endpoint of the p_2 program and, conjecturally, of all Pontryagin diagonals.

The main object of this note is the *uniform oriented-matroid bi-Grassmannian*. It should be distinguished from all of the following:

(i) Goncharov's algebraic bi-Grassmannian of generic Grassmannians and configuration spaces;

(ii) the MacPhersonian/combinatorial Grassmannian, which is a poset built from weak maps or strong-map images;

(iii) the flag MacPhersonian and related flag-matroid constructions;

(iv) hyperline-sequence encodings of an individual oriented matroid;

(v) the unrelated notion of a bigrassmannian permutation in Coxeter combinatorics.

The object introduced here is an incidence object whose two directions are *deletion* and *contraction*. On a fixed diagonal it is finite, and no weak-map compactification is inserted.

The terminology in this note emphasizes a particular finite, uniform, labeled deletion–contraction model. The relationship with existing matroid-minor complexes requires a systematic comparison before any priority or novelty claim is made.

We use the following status labels throughout.

- Proved/formal means that a complete argument is given here or reduces directly to standard oriented-matroid identities.
- Reported computation records an experiment asserted in the preceding draft. Unless a reproducible certificate or an explicitly identified rerun is supplied, its numerical output is evidence to be checked, not a theorem established by this revision.
- Conjectural means a mathematically precise statement for which no proof is presently claimed.
- Heuristic/programmatic means a structural analogy or research direction.

2 The classical algebraic bi-Grassmannian

2.1 Generic configurations and the two elementary maps

Let F be a field and let

$$\mathrm{Conf}^*_m(q)$$

denote generic configurations of m vectors in a q-dimensional vector space, modulo the natural linear equivalence appropriate to the chosen model. The adjective *generic* means that the expected minors do not vanish; for the present comparison it is enough to think of an ordered vector configuration in general position.

There are two canonical operations on a generic configuration

$$(\ell_0, \dots, \ell_m) \in \mathrm{Conf}^*_{m+1}(q).$$

First one may forget a vector:

$$f_i(\ell_0, \dots, \ell_m) = (\ell_0, \dots, \widehat{\ell_i}, \dots, \ell_m) \in \mathrm{Conf}^*_m(q).$$

Second one may quotient by the line spanned by one vector and project all the others:

$$p_i(\ell_0, \dots, \ell_m) = (\bar{\ell}_0, \dots, \widehat{\bar{\ell}_i}, \dots, \bar{\ell}_m) \in \mathrm{Conf}^*_m(q-1).$$

Goncharov identifies these configuration spaces with generic loci in Grassmannians. If $G_m(q)$ is the variety of q-planes in an m-dimensional coordinate vector space which are in generic position with respect to the coordinate hyperplanes, then

$$G_m(q) \cong \mathrm{Conf}^*_m(q).$$

The two operations organize the $G_m(q)$ into the classical bi-Grassmannian.

Passing to free abelian groups and alternating sums gives a bicomplex. In one common convention

$$f = \sum_i (-1)^i f_i, \qquad p = \sum_i (-1)^i p_i,$$

with positions counted from zero and inherited orders on the remaining labels, the two raw alternating operators anticommute. Thus

$$D = f + p, \qquad D^2 = 0.$$

Conventions in which the two displayed operators commute require a totalization twist; no additional twist is applied to the anticommuting convention used here. We omit maps leaving the positive-rank, positive-corank range.

2.2 Hypersimplices

For $p, q \geq 0$ with $p + q = m - 1$, the hypersimplex is

$$\Delta^{p,q} = \left\{ (x_0, \ldots, x_m) \in [0,1]^{m+1} : \sum_{i=0}^{m} x_i = q + 1 \right\}.$$

Its codimension-one faces occur in two families:

$$x_i = 0 \quad \text{of type } \Delta^{p-1,q} \quad (p > 0), \qquad x_i = 1 \quad \text{of type } \Delta^{p,q-1} \quad (q > 0),$$

where the displayed types refer to positive-dimensional faces. For $p = 0$ the polytope is a simplex and only the $x_i = 1$ family gives its facets in dimensions at least two; for $q = 0$ only the $x_i = 0$ family does. In dimension one both endpoint descriptions require the separate vertex convention below.

The classical dictionary is

$$\boxed{\Delta^{p,q} \longleftrightarrow G_{p+q+2}(q+1).}$$

One can see this through the coordinate-torus action: the moment polytope of the generic torus orbit closure in the Grassmannian is the corresponding hypersimplex. Thus the two families of hypersimplex facets match the two canonical Grassmannian maps.

2.3 Decorated flags and the motivic direction

Let V_N be N-dimensional. A decorated flag consists of a complete flag

$$0 = F_0 \subset F_1 \subset \cdots \subset F_N = V_N, \qquad \dim F_i = i,$$

together with a nonzero vector in each one-dimensional quotient F_i/F_{i-1}. Goncharov considers generic configurations of decorated flags and a canonical hypersimplicial N-decomposition of a simplex. Each hypersimplex in that decomposition determines a generic vector configuration, hence a point of the matching Grassmannian. Summing over all hypersimplices yields a chain map

$$\text{decorated-flag complex} \longrightarrow \text{bi-Grassmannian complex.}$$

Internal hypersimplex facets cancel, leaving only the boundary of the original simplex.

The second decisive feature is that the algebraic bi-Grassmannian is not merely an incidence gadget. In low weights there are maps from its total complex toward Bloch/polylogarithmic or motivic complexes; this is part of Goncharov's program relating configurations, polylogarithms, motivic cohomology and algebraic K-theory. In later formulations the same mechanism gives explicit cocycles for motivic/Deligne Chern classes.

This motivic target is *not* presently available in the oriented-matroid setting. The comparison below is therefore structural rather than an assertion that the oriented-matroid complex computes motivic cohomology.

2.4 Gabrielov's hypersimplicial characteristic chains

Gabrielov's earlier construction of combinatorial Pontryagin classes also uses hypersimplicial complexes. There, GL-invariant chains in spaces of linear maps represent characteristic data, and hypersimplicial families control the change from one collection of sections to another. The key methodological lesson is shared with Goncharov's later construction:

> A single geometric object is resolved into a compatible family indexed by hypersimplices, and characteristic information is obtained only after the internal faces have cancelled.

The oriented-matroid construction below should be viewed against this background.

3 Oriented-matroid preliminaries

3.1 Uniform oriented matroids and chirotopes

Let E be a finite ordered set. We write $[n] = \{1, \dots, n\}$ and let $\mathrm{pos}_E(e)$ denote the zero-based position of e. A uniform oriented matroid of rank r on E may be encoded by a chirotope

$$\chi : E^r \longrightarrow \{0, \pm 1\},$$

which vanishes exactly on tuples with repeated entries, which is alternating and satisfies the Grassmann–Plücker/chirotope axiom, modulo the global replacement $\chi \mapsto -\chi$. We write

$$\mathrm{UOM}(r, E), \qquad \mathrm{UOM}(r, n) := \mathrm{UOM}(r, [n])$$

for the set of labeled uniform oriented matroids.

For $e \in E$ there are two basic minors:

$$M \setminus e \in \mathrm{UOM}(r, E - e), \qquad M/e \in \mathrm{UOM}(r - 1, E - e),$$

in the ranges $r < |E|$ and $r > 0$, respectively. The rank-zero and full-rank minors are uniform as well, but fall outside the positive-rank, positive-corank bi-Grassmannian used below. For chirotope calculations we put contracted labels first:

$$\chi_{M/e}(i_1, \dots, i_{r-1}) = \chi_M(e, i_1, \dots, i_{r-1});$$

multiple contracted labels are ordered increasingly. Reordering them changes only a global chirotope sign.

For distinct e, f one has canonical commutation isomorphisms

$$(M \setminus e) \setminus f = (M \setminus f) \setminus e,$$

$$(M/e)/f = (M/f)/e,$$

$$(M \setminus e)/f = (M/f) \setminus e.$$

All sign issues below arise only from ordering determinant lines or totalizing alternating sums; the underlying minors commute.

3.2 Rank two as signed cyclic order

A rank-two oriented matroid is encoded by a signed circular order, modulo the global orientation convention, or by a rank-two hyperline sequence. Projectivization forgets individual lift signs; because χ and $-\chi$ define the same oriented matroid, the resulting order of projective labels is determined only up to reversal (a dihedral order). An oriented circular order requires an additional orientation choice. This is the first point at which the connection with Connes' cyclic category and the deletion-only rank-two program becomes visible.

The classical theory of hyperline sequences packages a general oriented matroid in terms of its rank-two contraction data. We will use this later to explain why the new bi-Grassmannian is new as a *global deletion–contraction organization*, even though rank-two contraction encodings of individual oriented matroids are classical.

4 Definition of the uniform oriented-matroid bi-Grassmannian

4.1 Bigraded object

Definition 4.1 (Uniform oriented-matroid bi-Grassmannian). *For* $p, q \geq 0$ *set*

$$\mathfrak{B}^{\mathrm{OM}}_{p,q} := \mathrm{UOM}(q+1, p+q+2).$$

The collection

$$\mathfrak{B}^{\mathrm{OM}} = \{\mathfrak{B}^{\mathrm{OM}}_{p,q}\}_{p,q\geq 0}$$

together with all labeled deletion and contraction maps is called the uniform oriented-matroid bi-Grassmannian*.*

The indexing is chosen so that it matches the classical hypersimplex convention:

$$\boxed{\Delta^{p,q} \longleftrightarrow G_{p+q+2}(q+1) \longleftrightarrow \mathrm{UOM}(q+1, p+q+2).}$$

For $M \in \mathfrak{B}^{\mathrm{OM}}_{p,q}$ on the ordered set $[p+q+2]$, deletion of i gives

$$d_i M := M \setminus i \in \mathfrak{B}^{\mathrm{OM}}_{p-1,q} \quad (p \geq 1),$$

and contraction gives

$$c_i M := M/i \in \mathfrak{B}^{\mathrm{OM}}_{p,q-1} \quad (q \geq 1).$$

Hence the two hypersimplex facet families become exactly

$$x_i = 0 \longleftrightarrow d_i, \qquad x_i = 1 \longleftrightarrow c_i.$$

4.2 Fixed diagonals

Fix

$$m = p + q + 1.$$

Then the diagonal consists of oriented matroids on a fixed ground-set size $m + 1$:

$$\boxed{\mathfrak{B}_m^{\mathrm{OM}} = \bigsqcup_{r=1}^{m} \mathrm{UOM}(r, m+1).}$$

Here $r = q + 1$. Unlike the algebraic Grassmannians, every term is finite.

For the degree relevant to the working p_2 program, $m = 8$ and

$$\mathfrak{B}_8^{\mathrm{OM}} = \mathrm{UOM}(1, 9) \sqcup \mathrm{UOM}(2, 9) \sqcup \cdots \sqcup \mathrm{UOM}(8, 9).$$

On nine labels duality exchanges ranks 4 and 5; there is no self-dual central rank. A self-dual central type occurs on an even number of labels, for example rank 5 on ten labels.

4.3 Chain groups and total differential

Let $C_{p,q}^{\mathrm{OM}} = \mathbb{Q}[\mathfrak{B}_{p,q}^{\mathrm{OM}}]$ for $p, q \geq 0$, and put all groups with a negative index equal to zero. Relabel each remaining ordered ground set increasingly. Define

$$dM = \sum_{e \in E} (-1)^{\mathrm{pos}_E(e)} M \setminus e, \qquad cM = \sum_{e \in E} (-1)^{\mathrm{pos}_E(e)} M/e,$$

with $d = 0$ on $p = 0$ and $c = 0$ on $q = 0$. We also write $\partial = c$ when discussing the bi-Grassmannian; the simplex boundary in the shadow construction will be specified separately. For distinct labels in positions $i < j$, removing i changes the position of j to $j - 1$. Pairing the two orders of removal therefore gives

$$d^2 = c^2 = 0, \qquad dc + cd = 0, \qquad D = d + c, \quad D^2 = 0.$$

There is no further totalization twist in this convention. Degree is the hypersimplex dimension $m = p + q + 1$; the complex ends in degree one, and it is not the full cellular chain complex including vertices.

To compare with geometric incidence signs, orient the tangent space of $\sum x_e = r$ by requiring the positive normal $(1, \dots, 1)$ first in the ambient coordinate orientation. In this orientation the positive-dimensional boundary is $d - c$. Multiplying the chosen orientation in rank r by $(-1)^{r-1}$ changes it to $d+c$. All flag sums below use these rank-dependent orientations. Vertices are omitted, consistently with the truncation.

Status: proved/formal. The identities follow from the paired-removal calculation. They hold over $\mathbb{Z}$ as well as over $\mathbb{Q}$ and use no realization-space topology.

4.4 Realizable comparison map

A generic real vector configuration of n vectors spanning an r-space determines a realizable element of $\mathrm{UOM}(r, n)$ by taking signs of maximal minors. Forgetting one vector becomes deletion; quotienting by the line it spans becomes contraction. Therefore the sign map intertwines the two bi-Grassmannian structures:

$$\begin{tikzcd} G_n(r)(\mathbb{R}),\ \text{generic locus} \arrow[r, "\text{signs of minors}"] \arrow[d, "\text{forget/project}"'] & \mathrm{UOM}_{\mathrm{real}}(r, n) \arrow[d, "\text{delete/contract}"] \\ G_{n-1}(r) \text{ or } G_{n-1}(r-1) \arrow[r] & \mathrm{UOM}(r, n-1) \text{ or } \mathrm{UOM}(r-1, n-1) \end{tikzcd}$$

The oriented-matroid bi-Grassmannian is therefore a finite sign-stratified quotient of the real generic bi-Grassmannian on the realizable locus, enlarged by nonrealizable uniform oriented matroids.

4.5 Why this is not the MacPhersonian

The MacPhersonian $\mathrm{MacP}(r, n)$ is constructed from oriented matroids of fixed rank and ground set using the weak-map specialization order; its topology is intended to model a Grassmannian. By contrast, $\mathfrak{B}_m^{\mathrm{OM}}$ varies the rank and is organized by the two minor operations

$$\text{deletion}, \qquad \text{contraction}.$$

No weak-map partial order is needed. One should therefore resist the temptation to call the new object a compactification of a Grassmannian. It is closer to a finite incidence skeleton of the *two-directional* geometry used by Goncharov.

4.6 Duality

Oriented-matroid duality exchanges deletion and contraction:

$$(M \setminus e)^* = M^*/e, \qquad (M/e)^* = M^* \setminus e.$$

Since

$$\mathrm{UOM}(q+1, p+q+2)^* \cong \mathrm{UOM}(p+1, p+q+2),$$

duality reflects the bi-Grassmannian across the diagonal $p = q$. This is the exact combinatorial counterpart of the duality between complementary Grassmannian dimensions and of the symmetry

$$\Delta^{p,q} \cong \Delta^{q,p}$$

induced by $x_i \mapsto 1 - x_i$.

5 Hypersimplicial interpretation

The utility of the indexing is that the face algebra of a hypersimplex is already the deletion–contraction algebra.

Proposition 5.1 (Hypersimplex/minor dictionary). *Let* $M \in \mathrm{UOM}(q+1, p+q+2)$ *correspond to a cell of type* $\Delta^{p,q}$*. Its positive-dimensional facets* $x_i = 0$ *(when* $p > 0$*) have type* $\Delta^{p-1,q}$ *and labels* $M \setminus i$*, while those* $x_i = 1$ *(when* $q > 0$*) have type* $\Delta^{p,q-1}$ *and labels* M/i*. Vertex faces are governed by the separate convention in §4.3.*

Proof. Deleting i reduces the number of labels by one and leaves the rank $q+1$ unchanged, giving

$$\mathrm{UOM}(q+1, p+q+1) = \mathfrak{B}^{\mathrm{OM}}_{p-1,q}.$$

Contracting i reduces both the number of labels and rank by one, giving

$$\mathrm{UOM}(q, p+q+1) = \mathfrak{B}^{\mathrm{OM}}_{p,q-1}.$$

These are exactly the two hypersimplex facet types. The codimension-two compatibility is the standard commutativity of minors on distinct labels. □

The important point is not merely that the formulas have the right bidegrees. A hypersimplex carries precisely two geometrically distinct boundary operations, and uniform oriented matroids carry precisely the deletion/contraction pair needed to model them.

6 Oriented-matroid Goncharov flags

6.1 Definition

Let $\Delta^m(N)$ be the hypersimplicial N-decomposition of an m-simplex, with hypersimplices

$$\Delta_a^{p,q}, \qquad p+q=m-1.$$

Each such cell has the same $m+1$ vertex labels of the ambient simplex.

Definition 6.1 (Oriented-matroid Goncharov flag). *An* oriented-matroid Goncharov flag *on* $\Delta^m(N)$ *is an assignment*

$$\Delta_a^{p,q} \longmapsto M_a^{p,q} \in \mathrm{UOM}(q+1, m+1)$$

for every hypersimplex, such that whenever two hypersimplices share a facet, the induced minors on that facet agree, with the canonical label and determinant-line identification.

In the standard local picture, if a face is simultaneously the i-contraction face of one cell and the i-deletion face of its neighbor, the condition reads schematically

$$\boxed{M_a^{p,q}/i = M_{a+e_i}^{p+1,q-1} \setminus i.}$$

There is an analogous equation for the opposite adjacency. The exact placement of $a \pm e_i$ depends on the chosen coordinates for the hypersimplicial decomposition; the content is invariant: both cells induce the same oriented matroid on their common face.

6.2 Realizable flags from decorated flags

A generic decorated flag configuration in a real vector space gives, on each hypersimplex, a generic vector configuration in the corresponding quotient. Taking its oriented matroid produces an oriented-matroid Goncharov flag. Since taking signs of maximal minors commutes with forgetting and quotienting, the compatibility equations are automatic.

Thus there is a realizable chain of constructions

$$\text{generic decorated flags} \longrightarrow \text{OM Goncharov flags} \longrightarrow \text{OM bi-Grassmannian}.$$

The first arrow forgets metric/algebraic information but keeps the chirotope up to global sign; the second sums the cells of the hypersimplicial decomposition.

6.3 The flag-to-bi-Grassmannian chain map

Let $[\mathcal{F}]$ denote a compatible OM flag on an oriented simplex. Define

$$\Pi([\mathcal{F}]) = \sum_{\Delta_a^{p,q} \subset \Delta^m(N)} [M_a^{p,q}]$$

with the geometric incidence orientation of each hypersimplex.

Proposition 6.2. *The map Π commutes with the simplicial boundary on flag configurations and the total deletion–contraction differential D on the OM bi-Grassmannian.*

Proof. Every internal codimension-one hypersimplex face occurs twice with opposite geometric incidence signs. Compatibility of the OM labels identifies the two induced minors, so internal contributions cancel. The only surviving terms are the hypersimplices lying in the boundary of the original simplex. This is exactly the same cancellation mechanism as in Goncharov's geometric construction. □

Status: proved/formal. The statement is formal once the OM flag has been defined with determinant-line orientations.

6.4 A resolved flag complex

For calculations it is useful to keep not only the current minor but also its history. A generator is written

$$[M; A \mid C, D],$$

where

- M is a uniform parent oriented matroid;
- A is an ordered set of active labels;
- C is the contraction tail;
- D is the deletion tail;
- the sets are disjoint and exhaust the relevant labels.

The visible minor is

$$\pi[M; A \mid C, D] = (M/C) \setminus D.$$

There are two differentials:

$$\delta_0 : \text{move an active label to } D, \qquad \delta_1 : \text{move an active label to } C,$$

with coefficient $(-1)^{\mathrm{pos}_A(e)}$ for moving e. Set $\delta_0 = 0$ when the visible minor has corank one, and $\delta_1 = 0$ when it has rank one, consistently with §4.3. Then

$$\delta_0^2 = \delta_1^2 = 0, \qquad \delta_0\delta_1 + \delta_1\delta_0 = 0,$$

and

$$\pi(\delta_0 + \delta_1) = (d + \partial)\pi.$$

For a parent of rank r, a partition describes the slice

$$x_c = 1 \ (c \in C), \quad x_d = 0 \ (d \in D), \quad \sum_{a \in A} x_a = r - |C|.$$

It is nonempty only if $0 \le r - |C| \le |A|$. Positive-dimensional faces have $0 < r - |C| < |A|$; degenerate descriptions of a vertex must be identified. Arbitrary partitions without these restrictions form a cube-face indexing system, not the hypersimplex face lattice.

The full cellular complex of this convex hypersimplex has $H_0 \cong \mathbb{Q}$ and no higher homology; its augmented complex is acyclic. The positive-dimensional truncation used above is not in general acyclic. Thus one may use the augmented geometric complex as an auxiliary resolution, but acyclicity does not follow for the displayed truncated flag model.

7 Comparison with adjacent theories

7.1 Classical algebraic geometry

The classical object contains actual varieties $G_m(q)$, algebraic coordinate functions, torus actions, Plücker coordinates, and generic configurations over a field. The OM object retains only the signs of real Plücker coordinates on realizable points, and then formally adds nonrealizable chirotopes.

Thus the relation is similar to passing from a real algebraic arrangement to its oriented matroid, but done simultaneously on every

Grassmannian in the bi-Grassmannian and compatibly with both elementary maps.

What is lost includes cross-ratios, regulator values, rational functions, and all continuous moduli within a realization stratum. What is retained includes orientation signs, minors, duality, genericity, and the complete hypersimplicial face algebra.

7.2 Motivic geometry

Goncharov's motivic program uses configuration complexes because polylogarithmic functional equations and motivic coproducts are naturally controlled by forgetting and projection. In low weights the bi-Grassmannian maps to Bloch/polylogarithmic complexes and yields explicit motivic/Deligne characteristic cocycles.

The OM bi-Grassmannian presently has no known analogue of the Bloch group, motivic Lie coalgebra, or regulator. Therefore phrases such as "motivic oriented matroid" or "field of characteristic one" should be understood only as prompts for structure, not as established mathematics.

Nevertheless there is a concrete reason to take the analogy seriously: the same hypersimplex is governing the same two abstract operations, and the OM construction supplies a finite sign-level realization of the entire incidence pattern. It is therefore reasonable to ask whether some quotient, completion, or enrichment of the OM bi-Grassmannian supports analogues of polylogarithmic functional relations.

7.3 MacPhersonians

The MacPhersonian studies all oriented matroids of a fixed rank on a fixed ground set and uses weak maps/specialization. Its geometric realization is intended as a combinatorial model of a Grassmannian. The OM bi-Grassmannian instead studies all ranks on one fixed diagonal and uses minors.

The distinction is structural:

	MacPhersonian	OM bi-Grassmannian
rank	fixed	varies along diagonal
ground set	fixed	fixed on a diagonal
arrows	weak maps	deletion and contraction
goal	Grassmannian topology	hypersimplicial incidence/characteristic desc

The two constructions should interact, but neither subsumes the other.

7.4 Hyperline sequences

Classical hyperline-sequence theory says that an oriented matroid can be recovered from compatible rank-two contraction data. This fact will become crucial for the $\mathfrak{p}_2$ shadow discussion. It does not, however, organize all ranks and all ground-set sizes into a single two-directional complex. In that sense hyperline sequences supply local coordinates on objects of the OM bi-Grassmannian, not the bi-Grassmannian structure itself.

7.5 Rank two and the cyclic category

At rank two, projectivized uniform oriented matroids determine circular orders modulo reversal. After choosing an orientation, deletion relates to the injective part Λ_{inj} of the cyclic category; without this choice the natural symmetry is dihedral. Higher rows are organized by rank-two contraction data.

Part II

Foundations, $\mathcal{A}_2$ locality, and cluster/tropical enhancements

8 Matroids over coefficient objects

8.1 Krasner, sign and tropical coefficients

Baker and Bowler introduced matroids over hyperfields and, more generally, over tracts, unifying ordinary matroids, oriented matroids, valuated matroids and linear subspaces [20]. The three coefficient objects relevant here are:

- the Krasner hyperfield $\mathbb{K} = \{0, 1\}$, whose matroids are ordinary matroids;
- the sign hyperfield $\mathbb{S} = \{0, +, -\}$, whose matroids are oriented matroids;
- the tropical hyperfield $\mathbb{T}$, whose matroids are valuated matroids.

There is a canonical coefficient map

$$\mathbb{R} \xrightarrow{\mathrm{sgn}} \mathbb{S} \longrightarrow \mathbb{K}.$$

Taking Plücker coordinates and then signs sends a real point of the open Grassmannian to its oriented matroid.

For a finite ordered ground set E, a rank-r Grassmann–Plücker function over a tract/pasture F will be denoted by

$$\varphi : E^r \longrightarrow F.$$

Over $\mathbb{S}$ this is the chirotope. In the uniform case no Plücker coordinate vanishes.

8.2 Matroid moduli, universal pastures, and foundations

Baker–Lorscheid construct an ordered-blue moduli space $\mathrm{Mat}(r, E)$ for *strong* matroids over idylls; its residue object at a classical matroid M is the universal idyll k_M [21]. The foundation theory used here instead concerns *weak* representations, governed by the universal pasture k_M^w. For every pasture F,

$$\mathrm{Hom}(k_M^w, F) \cong \{\text{weak } F\text{-matroid structures with underlying matroid } M\}.$$

A matroid structure here already identifies Grassmann–Plücker functions that differ by a single global unit. Ground-set rescaling is a further quotient.

The *foundation* F_M is the subpasture generated by universal cross-ratios and represents that further quotient:

$$\mathrm{Hom}(F_M, F) \cong \{\text{weak } F\text{-representations of } M\}/\text{rescaling}. \tag{1}$$

Its group of units is canonically the inner Tutte group [21, 22]. Weak and strong matroids agree over fields, $\mathbb{K}$, $\mathbb{S}$, and $\mathbb{T}$; this distinction therefore does not change those specializations, but matters for arbitrary coefficient objects.

Remark 8.1. *For $F = \mathbb{S}$, rescaling by units $\{+, -\}$ is exactly reorientation. Thus $\mathrm{Hom}(F_M, \mathbb{S})$ records reorientation classes of orientations of M, rather than a chosen chirotope representative.*

9 The uniform fiber and an "absolute" Grassmannian interpretation

Let $U_{r,E}$ denote the uniform matroid of rank r on E. Define the uniform-support functor

$$\mathcal{U}_{r,E}(F) := \{\text{weak } F\text{-matroid structures } N \text{ on } E : \underline{N} = U_{r,E}\}, \tag{2}$$

where $\underline{N}$ denotes push-forward to $\mathbb{K}$.

Proposition 9.1 (Formal consequence of the universal pasture). *There is a natural identification*

$$\mathcal{U}_{r,E}(F) \cong \mathrm{Hom}(k^{w}_{U_{r,E}}, F).$$

In particular,

$$\mathcal{U}_{r,E}(\mathbb{K}) = \{U_{r,E}\}, \qquad \mathcal{U}_{r,E}(\mathbb{S}) = \mathrm{UOM}(r, E),$$

and the real points are the generic real Grassmannian configurations with all Plücker coordinates nonzero.

Thus the UOM space is best viewed not as a collection of different "absolute points," but as the space of sign-valued lifts of the single uniform Krasner point. This suggests the slogan

generic real Grassmannian $\longrightarrow$ UOM sign geometry $\longrightarrow$ uniform combinatorial ske

Warning 9.2. *It would be misleading to write* $\mathrm{UOM}(r, n) = \mathrm{Gr}(r, n)(\mathbb{F}_1)$ *without specifying a theory of* $\mathbb{F}_1$*-geometry. In Tits-type counting, for example,* $\mathrm{Gr}(r, n)(\mathbb{F}_1)$ *has* $\binom{n}{r}$ *points, while there are generally many more UOMs. The precise statement available here is the functorial lift picture over* $\mathbb{K}$ *and* $\mathbb{S}$*.*

10 A coefficient-independent enhancement

10.1 Indexing

We use the convention adapted to Goncharov's bigrassmannian:

$$\mathfrak{G}^{\mathrm{UOM}}_{p,q} := \mathrm{UOM}(q + 1, p + q + 2), \qquad p, q \geq 0. \tag{3}$$

This is dual to the convention in which the rank is indexed by $p+1$. The two are exchanged by matroid duality.

For $e \in E$ define

$$D_e M = M \setminus e, \qquad C_e M = M/e.$$

Uniformity is preserved:

$$U_{r,n} \setminus e = U_{r,n-1}, \qquad U_{r,n}/e = U_{r-1,n-1}.$$

Thus

$$D_e : \mathfrak{G}_{p,q} \to \mathfrak{G}_{p-1,q} \quad (p \geq 1), \qquad C_e : \mathfrak{G}_{p,q} \to \mathfrak{G}_{p,q-1} \quad (q \geq 1).$$

The remaining boundary minors have full rank or rank zero and lie outside the displayed first quadrant unless it is augmented.

These are exactly the two operations in Goncharov's bigrassmannian: forgetting a vector and projecting modulo a vector [5, 6]. They are also the two kinds of facets of the hypersimplex

$$\Delta(r,n) = \{x \in [0,1]^n : \sum_i x_i = r\} :$$

$$x_e = 0 \quad \leftrightarrow \quad \text{deletion}, \qquad x_e = 1 \quad \leftrightarrow \quad \text{contraction}.$$

10.2 Coefficient independence

Deletion and contraction of Grassmann–Plücker functions commute with every coefficient morphism $F \to F'$. Consequently the bigrassmannian can be formed over any tract/pasture:

$$F \longmapsto \mathfrak{G}_{\bullet,\bullet}(F).$$

For the coefficient maps above this gives

$$\mathfrak{G}(\mathbb{R}) \longrightarrow \mathfrak{G}(\mathbb{S}) \longrightarrow \mathfrak{G}(\mathbb{K}).$$

The middle object is our UOM bigrassmannian.

At the level of free abelian groups, alternating sums of deletion and contraction give the usual Grassmannian-type double complex after the standard sign convention. The basic minor identities are

$$D_i D_j = D_{j-1} D_i, \qquad C_i C_j = C_{j-1} C_i, \qquad D_i C_j = C_{j-1} D_i \quad (i < j),$$

with the expected shifted variants.

11 Foundations and the A_2-locality theorem

11.1 The fundamental presentation

Baker–Lorscheid–Zhang prove that the foundation F_M of an arbitrary matroid M is the colimit of the foundations of all embedded minors of M isomorphic to one of

$$U_4^2,\ U_5^2,\ U_5^3,\ C_5,\ C_5^*,\ U_4^2 \oplus U_2^1,\ F_7,\ F_7^*. \tag{4}$$

Moreover this list is minimal in the appropriate universal sense [23].

Theorem 11.1 (A_2-locality of uniform foundations). *Let* $U_{r,n}$ *be a uniform matroid. Let* $\mathcal{D}_{r,n}$ *be the diagram of all embedded minors of* $U_{r,n}$ *isomorphic to*

$$U_4^2, \qquad U_5^2, \qquad U_5^3,$$

together with the minor embeddings among them. Then

$$F_{U_{r,n}} \cong \operatorname*{colim}_{N \in \mathcal{D}_{r,n}} F_N. \tag{5}$$

Proof. Every minor of a uniform matroid is uniform. Among the eight isomorphism types in (4), the only uniform ones are U_4^2, U_5^2, U_5^3. Therefore all other nodes are absent from the fundamental diagram of $U_{r,n}$, and the Baker–Lorscheid–Zhang colimit theorem reduces exactly to (5). The colimit is taken in the category of pastures, with each embedded minor retained separately. An empty diagram has colimit $\mathbb{F}_1^{\pm}$. □

This is the rigorous basis for the phrase "uniform geometry is A_2-local." To justify the notation A_2 we now identify the five-point foundation.

11.2 The four-point and five-point foundations

Baker–Lorscheid–Zhang give the explicit presentations

$$F_{U_4^2} \cong \mathbb{F}_1^{\pm}(x, y) /\!\!/ \langle x + y = 1 \rangle, \tag{6}$$

and

$$F_{U_5^2} \cong F_{U_5^3} \cong \mathbb{F}_1^{\pm}(x_1, \ldots, x_5) /\!\!/ \langle x_i + x_{i-1}x_{i+1} = 1 \mid i \in \mathbb{Z}/5 \rangle\,. \tag{7}$$

See Proposition 3.1 and Corollary 3.2 of [23]. They denote the latter pasture by $\mathbb{V}$ and identify it with the 2-regular partial field.

Over an ordinary field, a realization of U_5^2 modulo row operations and column rescaling is a configuration of five labelled distinct points of $\mathbb{P}^1$ modulo PGL_2. Hence its realization moduli is

$$M_{0,5}.$$

Brown's dihedral coordinates on $M_{0,n}$ satisfy

$$u_{ij} + \prod_{kl \text{ crossing } ij} u_{kl} = 1$$

[24]. For a pentagon this becomes precisely the five cyclic relations

$$u_i + u_{i-1}u_{i+1} = 1.$$

Thus the same pentagon relation appears independently as the foundation of U_5^2 and as the dihedral-coordinate algebra of $M_{0,5}$.

The positive real component has the familiar two-dimensional associahedral compactification: its closure is a pentagon. This is the rank-2, type-A_2 cluster/Ptolemy geometry.

Corollary 11.2 (Pentagon descent). *For every pasture* P*,*

$$\mathrm{Hom}(F_{U_{r,n}}, P) \cong \lim_{N \in \mathcal{D}_{r,n}} \mathrm{Hom}(F_N, P). \tag{8}$$

Thus a rescaling-class P*-representation of* $U_{r,n}$ *is equivalent to a compatible family of representations on the four- and five-point nodes of the full embedded-minor diagram. For* $n \geq 5$ *and* $2 \leq r \leq n-2$*, every four-point node belongs to a five-point node, so this is five-point descent with four-point compatibility. For* $U_{2,4}$ *the diagram consists of its single four-point node; when* $r \in \{0, 1, n-1, n\}$ *it is empty.*

Proof. Apply $\mathrm{Hom}(-, P)$ to the colimit (5). Representable contravariant functors convert colimits to limits. □

12 Oriented matroids as compatible oriented pentagons

Set $P = \mathbb{S}$ in (8). Since $\mathrm{Hom}(F_M, \mathbb{S})$ is the set of reorientation classes of orientations of M, we obtain:

Proposition 12.1. *A reorientation class of uniform oriented matroids with underlying matroid $U_{r,n}$ is exactly a compatible family of sign points on the U_4^2, U_5^2, and U_5^3 nodes of the fundamental embedded-minor diagram. In the five-point range specified in theorem 11.2, the four-point sign points are their common restrictions.*

Every rank-2 oriented matroid is realizable, and duality gives the same conclusion for orientations of U_5^3. Consequently every local sign configuration in this descent problem admits real realizations. Nevertheless a compatible *real* family may fail to exist globally.

Corollary 12.2 (Nonrealizability as a gluing obstruction). *Let $\chi : F_{U_{r,n}} \to \mathbb{S}$ be a UOM sign point. Its restrictions to every node of $\mathcal{D}_{r,n}$ are individually liftable to $\mathbb{R}$. The UOM is realizable over $\mathbb{R}$ if and only if these lifts can be chosen compatibly on every arrow of $\mathcal{D}_{r,n}$. In the five-point range this is precisely compatibility on all four-point overlaps.*

This reformulates nonrealizability as a descent obstruction among individually elementary and realizable A_2 patches. In this form, Mnëv universality can be read as saying that arbitrarily complicated realization spaces can arise solely from the compatibility equations among five-point pieces; the local geometry itself is only the pentagon.

13 The case $U_{3,6}$: twelve pentagons and thirty overlaps

13.1 The fundamental diagram

For $U_{3,6}$ there are precisely six deletion minors

$$D_j = U_{3,6} \setminus j \cong U_5^3, \qquad j = 1, \dots, 6,$$

and six contraction minors

$$C_i = U_{3,6}/i \cong U_5^2, \qquad i = 1, \dots, 6.$$

For each ordered pair $i \neq j$ there is a four-point minor

$$E_{ij} = U_{3,6}/i \setminus j \cong U_4^2,$$

which lies simultaneously under C_i and D_j. Hence there are 12 five-point pieces and $6 \cdot 5 = 30$ four-point overlaps.

The incidence graph between the six C_i and six D_j is

$$K_{6,6} \setminus \{C_i D_i : i = 1, \dots, 6\}.$$

Schematically:

$$F_{U_{3,6}} \cong \operatorname{colim} \left[\begin{array}{c} 12 \text{ copies of } F_{U_5^2} \cong F_{U_5^3} \\ \text{glued through } 30 \text{ copies of } F_{U_4^2} \end{array} \right].$$

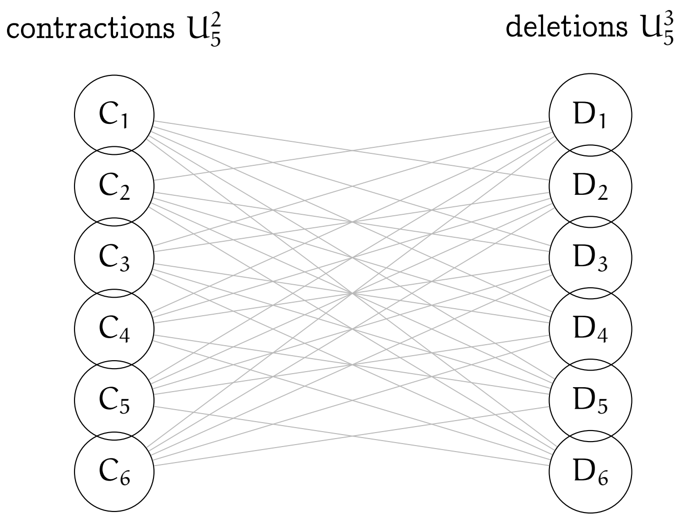

Figure 1: The $U_{3,6}$ fundamental incidence graph: $K_{6,6}$ minus the diagonal matching. Each edge represents a U_4^2 overlap.

13.2 A normalized Plücker chart

Take the generic matrix

$$A(x, y, z, w) = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & -x & -z \\ 0 & 0 & 1 & 1 & y & w \end{pmatrix}. \tag{9}$$

After GL_3 and column rescaling, this is a standard four-dimensional chart for generic configurations of six labelled points in $\mathbb{P}^2$. The nonconstant minors are

$$\begin{aligned} &p_{125} = y, && p_{126} = w, && p_{135} = x, && p_{136} = z, \\ &p_{145} = x - y, && p_{146} = z - w, && p_{156} = yz - xw, \\ &p_{245} = 1 - y, && p_{246} = 1 - w, && p_{256} = y - w, \\ &p_{345} = 1 - x, && p_{346} = 1 - z, && p_{356} = x - z, \\ &p_{456} = x + yz + w - xw - y - z. \end{aligned} \tag{10}$$

The remaining six minors in this gauge are equal to 1.

Introduce

$$\begin{gathered} e = x - y, \quad f = z - w, \quad g = yz - xw, \\ h = 1 - y, \quad i = 1 - w, \quad j = y - w, \\ k = 1 - x, \quad \ell = 1 - z, \quad m = x - z, \\ n = x + yz + w - xw - y - z. \end{gathered}$$

Then the short Plücker relations include the exchange network

$$\begin{aligned} x &= y + e, & z &= w + f, & yz &= xw + g, && (11)\\ 1 &= y + h, & 1 &= w + i, & y &= w + j, && (12)\\ 1 &= x + k, & 1 &= z + \ell, & x &= z + m, && (13) \end{aligned}$$

and the useful identity

$$ei = fh + n. \tag{14}$$

Equation (14) follows by direct expansion and is another short Plücker relation in the normalized chart.

13.3 Four UOM types as four orientations of one exchange network

Antolini–Early use four classes of realizable uniform rank-3 chirotopes on six elements, where equivalence allows both reorientation and relabeling [19]. In their lexicographic representatives, the negative triples may be taken as

Type	negative Plücker coordinates	f-vector of $\mathrm{Dr}^{\chi}(3,6)/L$
1	$146, 156, 256$	$(15, 60, 90, 45)$
2	$356, 456$	$(15, 60, 89, 44)$
3	456	$(14, 55, 82, 41)$
4	none	$(16, 66, 98, 48)$

In the chart (9), these signs are exactly the signs of

$$f = p_{146}, \quad g = p_{156}, \quad j = p_{256}, \quad m = p_{356}, \quad n = p_{456}.$$

Thus all four chambers are obtained by orienting the *same* algebraic exchange relations.

For example:

- Type 4: all displayed units are positive, and

$$ei = fh + n.$$

- Type 3: $n < 0$. With $\nu = -n > 0$,

$$fh = ei + \nu.$$

- Type 2: $m, n < 0$. With $\mu = -m > 0$ and $\nu = -n > 0$,

$$z = x + \mu, \qquad fh = ei + \nu.$$

- Type 1: $f, g, j < 0$. Put

$$\phi = -f, \qquad \gamma = -g, \qquad \jmath = -j.$$

  Then
$$w = z + \phi, \qquad xw = yz + \gamma, \qquad w = y + \jmath,$$
  and (14) becomes
$$n = ei + \phi h.$$

This is a concrete realization of the slogan:

> A chirotope does not change the universal Plücker algebra; it chooses which side of every short Plücker relation is positive/subtraction-free.

14 Positive foundation units and Antolini–Early coordinates

A universal cross-ratio is a degree-zero Laurent monomial in Plücker coordinates. Consequently its sign is invariant under reorientation: every ground-set element occurs with total exponent zero. Hence the sign of every foundation unit depends only on the reorientation class $\bar{\chi} : F_M \to \mathbb{S}$.

For a fixed sign point define

$$P_\chi := \{u \in F_M^\times : \bar{\chi}(u) = +\}.$$

For each ternary foundation null relation with three nonzero entries, $\bar{\chi}$ assigns both signs. Moving the terms of one sign to the other side, and replacing a negative unit a by $-a$, rewrites it as

$$u = v + w, \qquad u, v, w \in P_\chi. \tag{15}$$

This rewriting uses the fixed sign point; it does not change it by reorientation.

Construction 14.1 (Positive semiring envelope). *Let* $\mathbb{N}[P_\chi]$ *be the group semiring of the multiplicative group* P_χ*. Define*

$$F_M^{\chi,+} := \mathbb{N}[P_\chi]/\sim,$$

where $\sim$ *is the semiring congruence generated by every equality* $u = v + w$ *obtained from a nonzero ternary null relation as above. The symbols for* P_χ *are distinguished invertible generators; the quotient is not asserted to embed in a field.*

Proposition 14.2 (Real-lift universal property). *For unital semiring morphisms there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{sr}}(F_M^{\chi,+}, \mathbb{R}_{\geq 0}) \cong \{\rho : F_M \to \mathbb{R} : \operatorname{sgn} \rho = \bar{\chi}\}.$$

Proof. A sign-compatible ρ sends P_χ to positive real numbers and respects all defining equalities, hence extends to a semiring morphism. Conversely, every distinguished group element has positive image under a unital morphism to $\mathbb{R}_{\geq 0}$, because its inverse must also map and their product is 1. Each nonzero $a \in F_M$ is uniquely u or $-u$ with $u \in P_\chi$; send it respectively to $f(u)$ or $-f(u)$, and send 0 to 0. This extension is multiplicative and preserves ternary null relations by construction. Binary null relations are automatic. The two operations are inverse. □

The right side is the realization space with prescribed reorientation class, modulo rescaling; it can be empty or disconnected. This elementary envelope specifies the real-lift functor. A cluster structure or a scattering diagram would require additional definitions and results.

Antolini–Early exhibit four positive coordinate systems $y_1, \ldots, y_4$ for the four chambers of $X^\chi(3,6)$ and express each y_i as a sign-corrected Laurent monomial in Plücker coordinates [19]. In particular every y_i has multidegree zero, so each is a foundation unit. For the positive type 4, substitution of (10) gives

$$(y_1, y_2, y_3, y_4) = \left(\frac{eg}{yn}, \frac{gk}{yn}, \frac{yw}{g}, \frac{we}{g}\right). \tag{16}$$

For type 3, with $\nu = -n > 0$,

$$(y_1, y_2, y_3, y_4) = \left(\frac{y}{h}, \frac{g}{ym}, \frac{wh}{j}, \frac{em}{x\nu}\right). \tag{17}$$

Thus at least in these charts the positive coordinates are not external parameters: they are Laurent coordinates in the χ-positive foundation.

Antolini–Early further show that, for each of the four types, the normal fan of the Newton polytope of the product of all irreducible polynomials occurring in their minors is isomorphic to the corresponding chirotropical fan. This motivates comparing Newton fans from positive charts with the positive foundation. Such a comparison still has to address dependence on the chosen coordinates; neither a canonical Newton polytope nor a scattering diagram follows from the semiring envelope alone.

15 Signed tropicalization and chirotropical fibers

Ben Smith develops tropical extensions of tracts. In particular, the signed tropical hyperfield/tract can be regarded as the tropical extension of the sign hyperfield,

$$\mathbb{T}_{\pm} \simeq \mathbb{S}[\mathbb{R}],$$

and matroids over it are oriented-valuated matroids [29].

The coefficient diagram

$$\mathbb{T}_{\pm} \longrightarrow \mathbb{S} \longrightarrow \mathbb{K}$$

therefore refines the earlier sign/underlying-matroid picture. A $\mathbb{T}_{\pm}$-Plücker coordinate retains both a sign and a valuation.

For a fixed UOM sign point $\bar{\chi} : F_{U_{r,n}} \to \mathbb{S}$, consider the fiber

$$\mathcal{T}_{\chi}(r, n) := \{\widetilde{\chi} : F_{U_{r,n}} \to \mathbb{T}_{\pm} : \operatorname{sgn} \widetilde{\chi} = \bar{\chi}\}. \tag{18}$$

Because the foundation already quotients by rescaling, this is naturally a projectivized/lineality-quotiented object.

Proposition 15.1 (Signed-tropical fiber as a set). *Fix $1 \le r < n$ and a representative chirotope χ of $\bar{\chi}$. Let $\mathrm{Dr}^{\chi}(r, n)$ denote the finite-valued solutions of all signed three-term tropical Plücker relations,* *with the minimum convention. Then there is a natural bijection*

$$\mathcal{T}_\chi(r,n) \cong \mathrm{Dr}^\chi(r,n)/L_{r,n}, \qquad L_{r,n} = \left\{ \left(\sum_{i \in I} a_i \right)_{I \in \binom{[n]}{r}} : a \in \mathbb{R}^n \right\}.$$

For realizable χ*, this is the chirotropical Dressian convention of [19]. The assertion concerns sets; comparison of preferred fan structures is an additional step.*

Proof. By the foundation universal property, its signed-tropical points are rescaling classes of weak signed-tropical Grassmann–Plücker functions. In the prescribed sign fiber, choose a representative with entries (χ_I, w_I). A signed-tropical null relation means that the minimum valuation occurs among terms of both signs. For the three-term Plücker relations these are exactly the defining equations for $w \in \mathrm{Dr}^\chi(r,n)$. Rescaling the ground-set elements adds $\sum_{i \in I} a_i$ to w_I; global valuation shifts are already in $L_{r,n}$. Conversely every such w defines a signed-tropical function, and two functions with the prescribed signs give the same foundation point exactly under these rescalings. □

Remark 15.2. *Antolini–Early formulate* Dr^χ *for realizable uniform chirotopes. The foundation fiber* (18) *makes sense for every UOM sign point, including nonrealizable ones, and therefore suggests a natural extension of the abstract chirotropical Dressian beyond the realizable locus. The map* $\mathbb{S} \to \mathbb{T}_\pm$ *sending each nonzero sign to that sign with valuation zero is a coefficient morphism. Thus every such fiber contains its zero-valuation point, even when the UOM has no real realization. In that case there is no nonempty real realization space whose tropicalization could supply the fiber.*

The local mechanism is transparent. For a short Plücker relation, write after sign orientation

$$A = B + C, \qquad A, B, C > 0.$$

In an ordered valued field whose valuation ring is convex, positive summands cannot cancel at the leading valuation, so

$$\nu(A) = \min\{\nu(B), \nu(C)\}.$$

Equivalently, this equality follows directly from the signed-tropical null condition for $A - B - C$. No nontrivial valuation of the ordinary real numbers is being assumed. Changing the chirotope changes which one of the three monomials plays the distinguished role.

16 What is cluster-like, and what is not

16.1 Local cluster structure

Scott proved that the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ carries a cluster algebra structure [28]. For $\mathrm{Gr}(2,n)$ it is of type A_{n-3}, and the exchange relations are Ptolemy/Plücker relations. Speyer–Williams showed that the positive tropical Grassmannian $\mathrm{Trop}^+ \mathrm{Gr}(2,n)$ is combinatorially the fan dual to the type-A_{n-3} associahedron, while $\mathrm{Trop}^+ \mathrm{Gr}(3,6)$ is closely related to type D_4 [31].

Theorem 11.1 now gives a structural explanation for why rank-two cluster geometry keeps reappearing: uniform foundations are assembled from four-point A_1 and five-point A_2 foundations, with the boundary cases described in theorem 11.2. This does *not* prove that every UOM carries an ordinary cluster algebra. It proves instead that the universal cross-ratio algebra is assembled from elementary A_2 exchange patches.

16.2 Why signs alone do not define deterministic mutation

Suppose a cluster-like exchange has the form

$$xx' = M_1 + M_2.$$

If M_1 and M_2 have opposite signs, their signs alone do not determine the sign of their sum. A full chirotope already records the signs of every Plücker coordinate, but generally supplies no signs for non-Plücker cluster variables. Thus signs of seed variables alone do not define deterministic ordinary cluster mutation. Signed valuations determine the dominant term when the valuations differ; opposite leading signs at equal valuation allow cancellation and remain ambiguous. This motivates $\mathbb{T}_\pm$ as a coefficient object for a signed-tropical mutation picture.

16.3 Weak separation and oriented-matroid separation

Galashin–Postnikov define $\mathcal{M}$-separated collections for an arbitrary oriented matroid $\mathcal{M}$ and prove that maximal-by-size separated collections correspond to fine zonotopal tilings in the realizable case, and to general-position one-element liftings in general [25]. They classify rank-3 pure oriented matroids: purity holds precisely for positroids up to reorientation and relabeling.

For the alternating rank-3 oriented matroid, fixed-cardinality slices of their separation complex recover the weak-separation/plabic combinatorics underlying Grassmannian clusters. This gives a second, discrete cluster-like structure attached to an oriented matroid.

However, one should not identify the separation complex with the chirotropical fan. Already for positive $\mathrm{Gr}(3,6)$ the two structures have different combinatorics: the tropical fan contains information beyond maximal weakly separated Plücker collections, including non-Plücker cluster phenomena and secondary/coherence data. Higher secondary polytopes and regular plabic graphs provide part of the bridge on the positive side [27].

17 A bigrassmannian of oriented foundations

A minor embedding $N \hookrightarrow M$ induces a canonical morphism of foundations

$$F_N \longrightarrow F_M$$

[23]. Hence deletion and contraction refine the UOM bigrassmannian from a set-valued object to a diagram of algebraic coefficient objects.

Define provisionally

$$\mathfrak{F}_{p,q} := \{(F_{U_{q+1,p+q+2}}, \chi) : \chi : F_{U_{q+1,p+q+2}} \to \mathbb{S}\}.$$

Then deletion and contraction pull χ back along the corresponding foundation morphisms. The resulting structure has three shadows:

$$\begin{array}{rcl} \mathbb{S}\text{-points} & : & \text{UOM bigrassmannian modulo reorientation,} \\ \mathbb{R}\text{-lifts} & : & \text{real realization spaces modulo rescaling,} \\ \mathbb{T}_\pm\text{-lifts} & : & \text{signed-tropical/chirotropical fibers.} \end{array}$$

$$
\begin{tikzcd}
 & \substack{F_{U_{r,n}} \\ \text{absolute/projective algebra}} \arrow[dl] \arrow[dr] & \\
\substack{\chi : F_{U_{r,n}} \to \mathbb{S} \\ \text{oriented residue}} & & \substack{\widetilde{\chi} : F_{U_{r,n}} \to \mathbb{T}_{\pm} \\ \text{signed tropical lift}} \arrow[ll, "\text{forget valuation}"'] \\
\substack{\rho : F_{U_{r,n}} \to \mathbb{R} \\ \text{real lift}} \arrow[u, "\text{sgn}"'] & &
\end{tikzcd}
$$

This is the form in which Soulé's old intuition seems most defensible: the sign hyperfield is not literally $\mathbb{F}_1$, but an oriented residue coefficient object of a coefficient-independent Grassmannian geometry.

18 The $U_{3,6}$ chirotropical data and a test for descent

Antolini–Early prove

$$\operatorname{Trop}^{\chi} G(3,n) = \mathrm{Dr}^{\chi}(3,n), \qquad n = 6, 7, 8,$$

for every realizable uniform rank-3 chirotope, and compute all such fans [19]. For $n = 6$ the four f-vectors are listed above. They also prove that each chirotopal configuration space $X^{\chi}(3,6)$ is diffeomorphic to a polytope and verify that the corresponding fan is the normal fan of a Newton polytope constructed from their positive chart.

The A_2-descent theorem suggests a finite reconstruction problem.

Problem 18.1 (Twelve-pentagon reconstruction). *For each of the four rank-3, six-element chirotope types, reconstruct the fan* $\mathrm{Dr}^{\chi}(3,6)/L$ *directly from:*

(i) the twelve local signed A_2 *tropical fans attached to* C_i *and* D_j*;*

(ii) the thirty restriction maps to the A_1 *overlaps* E_{ij}*;*

(iii) the inverse-limit compatibility equations induced by the foundation colimit.

Recover, in particular, the four f*-vectors*

$$(15, 60, 90, 45), \quad (15, 60, 89, 44), \quad (14, 55, 82, 41), \quad (16, 66, 98, 48).$$

A positive answer would turn the statement "the chirotropical fan is glued from signed A_2 fans" into an explicit computational theorem rather than an abstract consequence of representability.

19 Pentagon descent, coherence, and possible holonomy

A sign point on a U_5^2 piece determines a dihedral ordering of the five labels: a cyclic order modulo reversal. It does not choose an orientation of the resulting pentagon. On a U_4^2 overlap one sees a cross-ratio coordinate, with its familiar sixfold anharmonic symmetry and, on a chosen positive interval, the involution $u \leftrightarrow 1 - u$ among the simplest transition operations.

It is therefore tempting to encode a UOM by nonabelian transition data among local pentagon charts. For $U_{3,6}$ the underlying incidence graph is the graph in figure 1. We emphasize, however, that a literal D_5-valued or S_3-valued Čech cocycle has *not* yet been constructed: the automorphism groups, overlap maps and higher compatibility conditions must be formulated in the category of foundations or ordered blue spaces.

Conjecture 19.1 (Associahedral descent interpretation). *There exists a natural groupoid-valued enhancement of the fundamental-minor diagram of* $U_{r,n}$ *in which:*

(i) each U_5^2/U_5^3 *node is the absolute* A_2 *pentagon object;*

(ii) each U_4^2 *node is the corresponding* A_1 *overlap;*

(iii) a UOM reorientation class is equivalent to a coherent oriented descent datum;

(iv) real realizability is equivalent to the existence of a compatible positive real lift of this descent datum.

The foundation colimit proves the set-valued version of this statement. The conjectural content is the geometric/groupoid enhancement and a useful intrinsic description of its higher coherence.

20 A possible scattering/cluster completion

The local positive relations

$$u = v + w$$

define tropical walls

$$\mathrm{val}(v) = \mathrm{val}(w).$$

Crossing such a wall changes which monomial controls the valuation. This is formally the local mechanism of tropical cluster mutation and scattering diagrams.

For the positive Grassmannian the cluster algebra provides a distinguished completion of the Plücker exchange system, including non-Plücker cluster variables in cases such as $\mathrm{Gr}(3,6)$. For a general UOM there is no reason that the completion should be an ordinary skew-symmetrizable cluster algebra. The oriented foundation suggests a more flexible replacement.

Conjecture 20.1 (Oriented-foundation scattering fan). *To every sign point $\chi : F_{U_{r,n}} \to \mathbb{S}$ one can attach canonically a wall/scattering structure generated by the χ-oriented ternary foundation relations, such that its support/refinement fan agrees with the signed-tropical fiber*

$$\mathcal{T}_\chi(r,n) \simeq \mathrm{Dr}^\chi(r,n)/L$$

whenever the latter is defined in the usual realizable sense.

For positive χ this structure should compare with the usual positive Grassmannian cluster/scattering fan; in other chambers it should give a genuinely sign-dependent deformation.

21 Mathematical interpretation of the coefficient models

The combined picture can be summarized as follows.

1. **Absolute skeleton.** The uniform matroid $U_{r,n}$ is a single $\mathbb{K}$-point in matroid moduli. Its universal pasture and foundation are coefficient-independent algebraic objects.

2. **Oriented residue.** A UOM, modulo reorientation, is a sign point

$$F_{U_{r,n}} \to \mathbb{S}.$$

It orients every universal three-term Plücker relation.

3. **A_1/A_2 locality.** The entire foundation is a colimit of the embedded U_4^2, U_5^2, and U_5^3 foundations. In the five-point range this is pentagon descent along four-point overlaps; $U_{2,4}$ and the regular boundary cases retain the conventions of theorem 11.2.

4. **Global complexity by descent.** Realizability, universality and higher combinatorics live in the compatibility of these local pieces, not in the local pieces themselves.

5. **Tropical thickening.** Passing from $\mathbb{S}$ to $\mathbb{T}_{\pm}$ adds valuation directions. The resulting fibers are oriented-valuated matroid spaces and, for realizable chirotopes, match the equations of chirotropical Dressians.

6. **Cluster shadow.** Ordinary Grassmannian cluster structures are special global completions of the same local Plücker exchange algebra. Positivity is the particularly coherent chamber in which the classical plabic/weak-separation and positive tropical pictures are available. General UOMs retain the local exchange algebra but can have different global chirotropical fans.

This suggests a cautious reformulation of Soulé's remark:

> Uniform oriented matroids behave as oriented sign-valued realizations of an absolute Grassmannian algebra. The absolute algebra is A_2-local: it is glued from five-point pentagon pieces. Tropicalization thickens the sign residue into a polyhedral deformation theory, while ordinary real geometry is the problem of lifting the same descent datum to $\mathbb{R}$.

The statement is not that $\mathbb{S} = \mathbb{F}_1$ or that UOMs are literally $\mathbb{F}_1$-points. Rather, the modern pasture/hyperfield formalism realizes the underlying intuition through a precise web of coefficient changes.

22 Concrete next problems

Problem 22.1 (Foundation-level derivation of the four $U_{3,6}$ charts). *Starting only from a cross-ratio presentation of $F_{U_{3,6}}$ and representatives of the four relabeling orbits of sign morphisms to $\mathbb{S}$, derive Antolini–Early's four positive parameter systems by systematic elimination. Determine to what extent the resulting positive coordinates are canonical up to mutation/Laurent transformation.*

Problem 22.2 (Functorial tropical descent). *Use the set-valued fiber identification in theorem 15.1 and the pasture colimit to write explicit inverse-limit equations for the local A_2/A_1 tropical fibers. Determine how the polyhedral fan structure induced by these equations compares with the preferred chirotropical fan, and whether a common refinement is needed.*

Problem 22.3 (Deletion–contraction of separation complexes). *Relate the Galashin–Postnikov $\mathcal{M}$-separation complexes and their maximal separated collections functorially to deletion and contraction. Compare the resulting seed-decorated bigrassmannian with the foundation/chirotropical bigrassmannian. Determine what additional coherence/secondary data is needed to pass from separated collections to chirotropical cones.*

Problem 22.4 (Nonrealizable chirotropical fibers). *For a nonrealizable UOM χ, study the signed-tropical foundation fiber* (18)*. Determine its dimension, purity, connectivity and minor functoriality, and compare it with realizable chirotropical Dressians. This is a natural arena where the abstract oriented-matroid geometry exists without a classical real lift.*

Problem 22.5 (Higher Goncharov flags). *Construct flag objects over the foundation bigrassmannian before specializing coefficients. Their $\mathbb{S}$-points should give the proposed oriented-matroid analogues of Goncharov flags; their $\mathbb{R}$-points should recover generic real flag configurations; their $\mathbb{T}_{\pm}$-points should produce signed-tropical flag matroids, in line with the tract-valued flag theory of tropical extensions.*

Part III

The p_1 and p_2 deletion–contraction program

23 A first sanity check: the p_1 formula

The classical Gelfand–Gabrielov–Losik local formula for the first Pontryagin class is naturally expressed in signs of determinants. In the uniform rank-two and rank-three cells relevant to the minimal diagonal, let

$$\varepsilon_r(M) := \prod_{|I|=r} \chi_M(I).$$

With standard orientation conventions the sign part of the local p_1 cocycle is precisely this complete chirotope product, and the classical coefficient is $1/48$.

Thus the known first Pontryagin formula already factors through the oriented-matroid sign shadow:

$$\omega_{p_1}(M) = \frac{1}{48}\varepsilon_r(M)$$

on the appropriate rank-two/rank-three pieces, with the extreme pieces zero after totalization.

Status: proved/formal. MacPherson's explicit formulas for the two interior hypersimplex types give this coefficient [35, §4, pp. 497-9–497-10]. They count positive ordered basis determinants; since $\binom{5}{2} = \binom{5}{3} = 10$ is even, the parity of that count is the complete chirotope product. The identification of the full geometric cocycle still uses MacPherson's hypersimplicial and orientation conventions.

This is important conceptually: the OM bi-Grassmannian is not being invented merely because it resembles a classical diagram. A classical characteristic cocycle already lives naturally on its sign-level cells.

24 The p_2 diagonal and the historical rank-two primitive seed

The working p_2 diagonal is

$$\mathfrak{B}_8^{\mathrm{OM}} = \bigsqcup_{r=1}^{8} \mathrm{UOM}(r, 9).$$

In the earlier logarithmic/primitive gauge a convenient rank-two compressed seed was

$$\ell_2(L) = -\frac{1}{1024}\varepsilon_2(L), \qquad \varepsilon_2(L) = \prod_{a<b} \chi_L(a, b),$$

for $L \in \mathrm{UOM}(2, 9)$.

For a rank-three M on ten elements define the forcing term

$$G_3(M) := \frac{1}{1024}\sum_i (-1)^{i-1}\varepsilon_2(M/i).$$

It is the contraction differential applied to the rank-two seed, up to the totalization sign.

24.1 A corrected false start

An initially attractive unmarked formula was

$$\ell_3^{\mathrm{old}}(N) = \frac{1}{1024}\left(1 + \sum_i (-1)^i b_i(N)\right), \qquad b_i(N) = \prod_{X \ni i,\ |X|=3} \chi_N(X).$$

This formula is *not* a universal unmarked cochain on the bare deletion–contraction bicomplex. Exact tests on realizable configurations produced a nonzero defect.

The missing datum is a normal/marking sign. In a flagged presentation the corrected primitive contains factors

$$\nu_i(M, k) = \varepsilon_1(M/\{i, k\}) = \prod_{j \neq i,k} \chi_M(i, k, j).$$

The error is instructive: the bare rank-three Orchard expression arose only after trivializing a normal-sign gauge. The flag formalism explains why that trivialization is not globally canonical.

Status: reported computation. A counterexample was reported in the working computations, but its matrix, face conventions, and nonzero defect are not included here. Until those data are supplied, the failure and the proposed flagged correction are unverified computational claims. The complete-product factorization below proves the sign factorization itself; it does not by itself establish a deletion–contraction primitive.

25 Complete-product factorization

For a rank-s uniform oriented matroid N on an ordered n-element set define

$$\varepsilon_s(N) = \prod_{|B|=s} \chi_N(B).$$

Assume $2 \leq s < n$, fix a representative chirotope, and use the contraction convention

$$\chi_{N/e}(i_1, \dots, i_{s-1}) = \chi_N(e, i_1, \dots, i_{s-1}).$$

Let e have one-based position p in the order.

Proposition 25.1 (Complete-product factorization). *Up to the displayed ordering sign,*

$$\boxed{\varepsilon_s(N) = (-1)^{(p-1)\binom{n-2}{s-2}} \varepsilon_s(N \setminus e)\, \varepsilon_{s-1}(N/e).}$$

Proof. Split the product into bases avoiding e and bases containing e. The first product is $\varepsilon_s(N \setminus e)$. For each of the $p-1$ labels preceding e, exactly $\binom{n-2}{s-2}$ bases contain both that label and e. Moving e to the first position therefore creates a total of $(p-1)\binom{n-2}{s-2}$ inversions, giving the displayed sign. □

This identity is the multiplicative finite-difference rule behind the normal-star signs, Hessian rows, and Orchard transitions appearing below.

26 The rank-two Orchard shadow

26.1 Definition

Let

$$M \in \mathrm{UOM}(r, r+7).$$

For every
$$K \in \binom{E(M)}{r-2}$$
define
$$\boxed{h_M(K) := \varepsilon_2(M/K).}$$
The contraction M/K has rank two on exactly nine labels. Since
$$\binom{9}{2} = 36$$
is even, $h_M(K)$ is independent of the global replacement $\chi \mapsto -\chi$.

The subsets K are the vertices of the hypersimplex
$$\Delta(r-2, r+7),$$
so h_M is a sign function on a hypersimplex vertex set.

26.2 Low ranks

For $r = 3$, K is a singleton and
$$h_M(i) = \varepsilon_2(M/i),$$
the rank-three Orchard vector used in G_3.

For $r = 4$, $K = \{i, j\}$ and
$$h_M(i, j) = \varepsilon_2(M/\{i, j\}),$$
which is the Hessian sign matrix appearing in the two-normal descent calculation.

For higher rank the same object is a signed $(r-2)$-uniform hypergraph.

26.3 Reorientation invariance

Proposition 26.1. *The shadow* h_M *is invariant under reorientation of any single element of* M*.*

Proof. If the reoriented element belongs to K, then all 36 pair signs in the rank-two contraction are changed, an even number. If it does not belong to K, then precisely the eight pair signs containing that element are changed. Again the parity is even. $\square$

Thus the shadow is substantially coarser than the oriented matroid; in particular it factors through reorientation classes.

27 Boolean linearization and modular homology

Write

$$\chi_M(B) = (-1)^{c_B}, \qquad h_M(K) = (-1)^{\eta_K},$$

with $c_B, \eta_K \in F_2$, using increasing tuples for basis signs. The shuffle from (K, a, b) to increasing order contributes no net sign to $h_M(K)$: each inversion between a label in K and a label outside K occurs in eight pair factors. Hence

$$\boxed{\eta_K = \sum_{B \supset K,\ |B|=r} c_B \pmod 2.}$$

Let $\mathcal{M}_j(E) = F_2[\binom{E}{j}]$ and define the two-step inclusion map

$$\partial_{[2]} : \mathcal{M}_r(E) \longrightarrow \mathcal{M}_{r-2}(E), \qquad B \longmapsto \sum_{K \subset B,\ |K|=r-2} K.$$

Then

$$\boxed{\eta = \partial_{[2]} c.}$$

Because an $(r-4)$-subset is contained in six intermediate $(r-2)$-subsets of an r-set,

$$\partial_{[2]}^2 = 0$$

over F_2. Thus the shadow is a boundary, and in particular a cycle, in the two-step inclusion complex studied by Mnukhin–Siemons and successors; its homology class in that complex is zero.

27.1 Universal parity relations

For every

$$A \in \binom{E}{r-4},$$

one obtains

$$\boxed{\prod_{K \supset A,\ |K|=r-2} h_M(K) = 1.}$$

Indeed every rank-r basis containing A appears six times in the product.

In the entire p_2 band $|E| = r + 7$, the complement $E - A$ has size 11. Hence each parity check is uniformly the product over all 55 edges of a K_{11} link.

For $r = 4$ there is only one relation:

$$\prod_{i<j} h_M(i,j) = 1.$$

The associated linear map

$$\mathbb{F}_2^{\binom{11}{4}} \longrightarrow \mathbb{F}_2^{\binom{11}{2}}$$

has rank 54.

Status: proved/formal. For $4 \le r \le 8$, the displayed checks generate all formal linear relations. Wilson's inclusion-matrix formula [14] gives

$$\operatorname{rank}_{\mathbb{F}_2} \partial_{[2]} = \sum_{\substack{0 \le i \le r-2 \\ \binom{r-i}{2} \text{ odd}}} \left(\binom{r+7}{i} - \binom{r+7}{i-1} \right), \qquad \binom{n}{-1} = 0.$$

Applying the same formula to the check map from $(r-2)$-subsets to $(r-4)$-subsets gives exactly the codimension of this image. Independent binary Gaussian elimination verifies all rows of the following table.

27.2 Code dimensions

The formal shadow-code dimensions are:

r	$\binom{r+7}{r-2}$	$\dim \operatorname{im} \partial_{[2]}$	linear relations
2	1	1	0
3	10	10	0
4	55	54	1
5	220	208	12
6	715	638	77
7	2002	1652	350
8	5005	3744	1261

These are constraints before any Grassmann–Plücker axiom is imposed. The genuinely oriented-matroid restrictions, if any, are nonlinear constraints inside this binary code.

28 The rank-three current and Hodge primitive

Assume $r \ge 3$ and put $t = r - 2$. Use the augmented oriented simplex chain complex on the ordered ground set E, including its degree--1

empty-simplex generator. Write $\mathrm{pos}_E(s) \in \{0, \dots, |E|-1\}$ for the zero-based position of s. For a subset S put

$$\lambda_E(S) := (-1)^{\sum_{s\in S} \mathrm{pos}_E(s)}, \qquad \widetilde{h}_M := \sum_{|K|=r-2} \lambda_E(K) h_M(K)[K].$$

Here $[K]$ carries increasing orientation. This complementary-orientation twist is essential for matching the alternating contraction sum. Define

$$\boxed{J_M := \frac{1}{2}\partial \widetilde{h}_M.}$$

For $A \in \binom{E}{r-3}$ there are exactly ten cofaces, and

$$\begin{aligned} J_M(A) &= \frac{1}{2}\sum_{x\notin A} [A+x : A]\lambda_E(A+x) h_M(A+x) \\ &= \frac{\lambda_E(A)}{2} \sum_{x\notin A} (-1)^{\mathrm{pos}_{E-A}(x)} h_M(A+x) \in \{-5, -4, \dots, 5\}. \end{aligned}$$

Since M/A has rank three on ten elements and

$$h_M(A+x) = \varepsilon_2((M/A)/x),$$

the precise comparison with the previously defined forcing term is

$$\boxed{J_M(A) = \lambda_E(A)\, 512\, G_3(M/A).}$$

This expresses these auxiliary currents in terms of rank-three ten-element contractions. Without the twist, the proposed identity already fails for the alternating rank-three chirotope on ten labels: all $h_i = 1$, so the ordinary half-boundary is 5, whereas $512 G_3 = 0$.

28.1 Conservation law

Since $\partial^2 = 0$,

$$\boxed{\partial J_M = 0.}$$

For every $B \in \binom{E}{r-4}$ this gives an eleven-term relation among the rank-three currents on the contractions $M/(B+x)$.

For $r = 4$ this becomes

$$\sum_{i=1}^{11} (-1)^{i-1} G_3(M/i) = 0$$

in the standard ordering convention.

28.2 Canonical Hodge primitive

On the augmented complete simplex with $n = r + 7$ vertices, give the increasing simplex basis the standard orthonormal metric and let ∂^* be the adjoint boundary. The simplicial Cartan identity is

$$\partial\partial^* + \partial^*\partial = nI.$$

Since $\partial J_M = 0$, the chain

$$KJ_M := \frac{1}{n}\partial^* J_M$$

satisfies

$$\partial(KJ_M) = J_M.$$

It is the orthogonal/minimal-norm primitive of the conserved current.

This constructs a canonical primitive for the auxiliary current. Identifying a flagged p_2 primitive with a scalar multiple of KJ_M requires a separate comparison with the deletion–contraction descent equations; no such comparison is proved here.

28.3 Uniform bound

Because $|J_M(A)| \leq 5$ and a $(r-2)$-subset has $r - 2$ facets,

$$|(KJ_M)(S)| \leq \frac{5(r-2)}{r+7} < 5.$$

In the provisional recursive normalization of the working formulas this yields a rank-independent local estimate after the normalization factor is extracted. The existence of this bound at the auxiliary Hodge level should not yet be confused with a proved global bound for the final unflagged p_2 cocycle.

29 Orchard–Hessian matrices and two-normal diamonds

29.1 A minimal flagged presentation

Let

$$P \in \mathrm{UOM}(r, r+7), \qquad S \subset E(P), \quad |S| = r - 2,$$

and put $E = E(P) - S$, so $|E| = 9$. The normal set S is one vertex of $\Delta(r-2, r+7)$. Define

$$c(P, S) := h_P(S)$$

and, for $s \in S$, $x \in E$,

$$H_P(s, x) := h_P(S - s + x).$$

The matrix

$$H_P \in \{\pm 1\}^{S \times E}$$

is the radius-one star of the shadow around the normal vertex S.

For fixed s, contracting all normals except s gives a rank-three oriented matroid

$$Q_s = P/(S - s)$$

on $E \sqcup \{s\}$, and its ten Orchard entries are exactly

$$c(P, S), \qquad H_P(s, x),\ x \in E.$$

Thus the matrix contains all transverse rank-three Orchard vectors simultaneously.

29.2 Pairwise transition

For $s, t \in S$ define

$$z_x^{st} := \frac{H_P(s, x) - H_P(t, x)}{2} \in \{-1, 0, 1\}.$$

The support

$$\{x : H_P(s, x) \neq H_P(t, x)\}$$

is an Orchard disagreement class for the symmetric separation function of the two normals after contracting the remaining normals.

The scalar p_2 transition between the two distinguished-normal primitives is a fixed alternating functional of z^{st}:

$$\mathcal{R}_{st} \propto \sum_{x \in E} \epsilon_x z_x^{st}.$$

Since

$$z^{st} + z^{tu} + z^{us} = 0,$$

the pairwise Bianchi identity inside a common parent is tautological.

29.3 Two-normal diamonds

For a rank-four two-normal parent P with old labels E and normals k, l, write

$$N = P \setminus \{k, l\} \in \mathrm{UOM}(4, 9),$$

and define

$$A_T = \chi_P(T, k), \qquad B_T = \chi_P(T, l), \qquad C_{xy} = \chi_P(x, y, k, l).$$

The triple (A, B, C) is the complete two-normal local data over N.

The three lower-rank objects on the same nine labels are

$$Q_k = (P/k) \setminus l, \qquad Q_l = (P/l) \setminus k, \qquad R_{kl} = P/\{k, l\}.$$

For an explicit convention, place the old labels before $k < l$ and insert contracted labels first. Define

$$a_x := \varepsilon_2(Q_k/x), \qquad b_x := \varepsilon_2(Q_l/x), \qquad t_x := \varepsilon_1(R_{kl}/x).$$

Splitting the pair product into pairs avoiding and containing the remaining normal gives

$$H(k, x) = b_x t_x, \qquad H(l, x) = a_x t_x.$$

The row swap is forced by $H(k, x) = h_P(\{l, x\})$ and $H(l, x) = h_P(\{k, x\})$. Thus the relative Hessian sign is

$$H(k, x)H(l, x) = a_x b_x.$$

This is the Orchard morphism applied to the symmetric separation function

$$s_{kl}(T) = \chi_P(T, k)\chi_P(T, l).$$

30 Integrability of two-normal data

Local two-normal data need not glue to an oriented matroid. The obstruction can be tested in rank two by Las Vergnas' localization theorem.

Let Q be a uniform rank-two oriented matroid with chirotope A_{ab}, and let $C_a \in \{\pm 1\}$ prescribe $\chi_{\widehat{Q}}(a, e)$ for a proposed new label e. Reorient the old labels by C:

$$\widehat{A}_{ab} = C_a C_b A_{ab}.$$

For $a < b < c$ set

$$p = \widehat{A}_{ab}, \qquad q = -\widehat{A}_{ac}, \qquad r = \widehat{A}_{bc},$$

and define

$$\boxed{\Omega_{abc}(Q, C) = \frac{1}{4}(p + q + r + pqr) \in \{-1, 0, 1\}.}$$

Then Ω_{abc} is nonzero exactly when the reoriented rank-two oriented matroid has a positive or negative circuit on $\{a, b, c\}$.

Proposition 30.1. *The proposed signature* C *defines a generic single-element extension of* Q *if and only if*

$$\Omega_{abc}(Q, C) = 0$$

for every triple.

In the rank-four two-normal problem, contractions of one old label and one normal give necessary defects of this form. For a complete sufficiency test, assemble the alternating sign function from N, A, B, C and check every rank-two Grassmann–Plücker relation: for each two-subset I and four distinct labels $a, b, c, d \notin I$, the three signs

$$\chi(I, a, b)\chi(I, c, d), \quad -\chi(I, a, c)\chi(I, b, d), \quad \chi(I, a, d)\chi(I, b, c)$$

must not all agree. This includes contractions with two old labels as well as those containing normals. Together with nonvanishing and alternation, these conditions are necessary and sufficient for the assembled data to be a uniform rank-four chirotope.

30.1 No-collision does not imply amalgamation

A proposed implication was that if two single-element extensions occupy distinct projective sectors in every rank-two contraction, then their pair signs are forced and automatically integrable. The working computations report a counterexample; its full chirotope certificate has not been supplied.

Status: reported computation. The reported computation starts from the alternating rank-three oriented matroid on six labels and finds two generic extensions with the stated property. To verify the claim, one must supply both extension signatures, the precise noncollision criterion, the forced pair signs, and the specific Grassmann–Plücker relation claimed to have three positive terms. Local uniqueness of candidate pair signs alone supplies no proof of their global integrability.

The correct statement is that after the locally forced pair signs are assembled, they must themselves define a valid rank-two localization in every relevant rank-three contraction. In arbitrary rank, compatibility of a transverse pair reduces to compatibility on all rank-three contractions.

31 Exact rank-two cyclic-atlas descent

There is a more intrinsic way to package the previous observations.

Let M be a uniform rank-r oriented matroid on E. For every

$$K \in \binom{E}{r-2}$$

set

$$R_K = M/K,$$

a rank-two oriented matroid on $E - K$.

Theorem 31.1 (Rank-two descent). *Assume $r \geq 3$. A coherently oriented family*

$$\{R_K \in \mathrm{UOM}(2, E-K)\}_{|K|=r-2}$$

comes from a unique rank-r uniform chirotope, up to one global sign, if and only if for every

$$A \in \binom{E}{r-3}, \qquad x, y \notin A,$$

the rank-one contractions agree:

$$\boxed{R_{A+x}/y = R_{A+y}/x}$$

on the common ground set $E - (A \cup \{x,y\})$, with the canonical determinant-line identification.

Proof. Let B be an r-subset. Choose any $(r-2)$-subset $K \subset B$, and write $B - K = \{a, b\}$. Define

$$\chi(B) = \text{shuffle sign} \cdot \chi_{R_K}(a, b).$$

The graph of $(r-2)$-subsets of B is connected. For adjacent choices $K = A + x$ and $K' = A + y$, the overlap condition evaluated on the remaining element of B gives equality of the two proposed values. Hence $\chi(B)$ is well-defined.

Fix I of size $r-2$ and four labels $a, b, c, d \notin I$. The three-term rank-r Grassmann–Plücker condition for χ is exactly the rank-two condition in R_I. Thus χ is a uniform rank-r chirotope. Uniqueness is immediate. $\square$

This theorem is classical in spirit: hyperline sequences encode oriented matroids through rank-two contraction data. The point for the present program is the specialization to a single fixed diagonal.

31.1 The p_2 specialization

When $|E| = r+7$, every R_K has exactly nine labels. Therefore an arbitrary object on the p_2 diagonal is equivalent to

$$\boxed{\text{a compatible atlas of nine-point signed cyclic orders,}}$$

with overlap conditions given by eight-point rank-one sign vectors.

This is arguably the smallest exact finite model of the underlying uniform oriented matroid that remains compatible with the p_2 hypersimplicial organization.

32 Circular orders and parity

Choose a representative chirotope of a nine-point rank-two chart R and take its least label as anchor a. Define

$$\sigma(x) = \chi_R(a, x), \qquad x \neq a.$$

After reorienting the other labels by these signs, all eight lie in the positive semicircle from a to $-a$ and acquire a linear order

$$\pi \in S_8.$$

For a chosen representative chirotope this gives a pair $(\sigma, \pi) \in \{\pm 1\}^8 \times S_8$. Passing to the oriented matroid identifies global sign reversal with

$$(\sigma, \pi) \sim (-\sigma, \operatorname{rev} \pi),$$

where $\operatorname{rev} \pi$ reverses the eight-letter order. Consequently

$$\boxed{R \longleftrightarrow (\{\pm 1\}^8 \times S_8)/\sim .}$$

The complete product is

$$\varepsilon_2(R) = \mathrm{sgn}(\pi),$$

because every lift sign $\sigma(x)$ occurs eight times in the product.

Projectivizing gives a circular order on nine labels. Since a cyclic shift of nine labels and reversal of a nine-cycle are both even permutations, its parity is well-defined. Hence

$$\boxed{h_M(K) = \text{parity of the nine-point projective circular order } R_K.}$$

This is the most direct bridge between the p_2 shadow and the rank-two cyclic-category picture.

32.1 Signed-permutation CSP

In a rank-two atlas choose for every chart K:

$$\sigma_K \in \{\pm 1\}^8, \qquad \pi_K \in S_8.$$

The prescribed p_2 shadow fixes only

$$\mathrm{sgn}(\pi_K) = h(K).$$

All overlap equations are affine-linear in the lift-sign bits and in the pairwise comparison bits of the permutations. The only nonlinear local condition is transitivity of each tournament of pairwise comparisons, equivalently the exclusion of directed 3-cycles.

For $r = 4, n = 11$, this becomes especially concrete: one places a signed eight-letter permutation over each edge of K_{11}, with prescribed parity, and requires the induced eight-point rank-one cuts to agree around every triangle.

33 Computational evidence and cautions

The list distinguishes independently rerun finite checks from reports whose underlying data were not supplied. The accompanying verification script reproduces item 2 and the corrected item 3. Items 1, 4, and 5 remain unverified reports pending matrices or chirotopes, enumeration conventions, and checking code.

1. For a fixed generic realizable rank-four configuration on nine old labels, varying a single generic extension produced all $2^8 = 256$ Orchard two-partitions of the old labels. Thus the Orchard partition alone can be completely unconstrained even with the old oriented matroid fixed.

2. The linear map from rank-four basis-sign bits on 11 labels to the 55 Hessian/shadow bits has rank 54 over $\mathbb{F}_2$; the only universal linear relation is the even total parity

$$\prod_{i<j} h_{ij} = 1.$$

3. Across all labelled $\mathrm{UOM}(3,6)$, the six-bit shadow attains exactly 52 of the 64 possible vectors. Independent exhaustive enumeration of the 2^{19} sign assignments with $\chi(1,2,3) = +1$ finds 11904 chirotopes and confirms this count. After the normalization $g_i = (-1)^{i-1}h_i$, the 12 missing vectors have exactly one or exactly five negative entries, so the missing unoriented partitions have type $1+5$. A single fixed alternating oriented matroid has only one shadow; the enumeration is over all chirotopes.

4. For generic realizable rank-three configurations on ten labels, exact-integer searches found witnesses for all $2^{10} = 1024$ possible rank-two-shadow vectors. Thus an individual transverse $\mathrm{UOM}(3,10)$ slice appears to impose no shadow restriction whatsoever.

5. A no-collision pair of generic extensions of the alternating $\mathrm{UOM}(3,6)$ was found which does not amalgamate, providing the counterexample discussed above.

The contrast between items 3 and 4 is important. Shadow surjectivity is not formal for arbitrary (r,n), but the particular sizes forced by the p_2 diagonal may be unusually flexible.

Part IV

Pontryagin classes as A_2-local sign descent

34 The five-point A_2 atom and the p_1 character

The foundation picture supplies a conceptual explanation for the oldest explicit oriented-matroid Pontryagin cocycle. Let

$$\alpha_5(R) := \varepsilon_2(R) = \prod_{\{i,j\}\subset E(R)} \chi_R(i,j), \qquad R \in \mathrm{UOM}(2,5).$$

By duality the same character on $\mathrm{UOM}(3,5)$ is ε_3. The Gabrielov–Gelfand–Losik normalization is

$$\omega_{p_1} = \frac{1}{48}\alpha_5$$

on the rank-two component, with the dual rank-three component and the usual incidence signs. The point of the present section is that the *shape* of this formula is forced by the A_2 geometry before the factor $1/48$ is inserted.

Proposition 34.1 (Uniqueness of the alternating five-point chamber character). *Modulo reorientation, rank-two uniform oriented matroids on five labelled elements are unoriented circular orders, hence form the homogeneous S_5-set S_5/D_5. The sign representation of S_5 occurs in $\mathbb{Q}[S_5/D_5]$ with multiplicity one. It is generated by α_5.*

Proof. By Frobenius reciprocity,

$$\mathrm{Hom}_{S_5}(\mathrm{sgn}, \mathbb{Q}[S_5/D_5]) \cong \mathrm{Hom}_{D_5}(\mathrm{sgn}\,|_{D_5}, 1).$$

Every element of D_5 is even: a five-cycle is even, and a reflection of a pentagon is a product of two transpositions. Thus the right-hand side is one-dimensional. The complete product α_5 is nonzero and alternating under relabelling, so it spans this line. □

Status: proved/formal.

Hence, once one asks for a relabelling-alternating scalar local function on the five-point A_2 chamber set, there is no freedom except scale. The rational constant $1/48$ belongs to characteristic-class normalization; the pentagon sign itself is representation-theoretically forced.

This suggests a concise slogan:

> p_1 is the normalized alternating orientation character of the real $A_2 = M_{0,5}$ pentagon.

The slogan is rigorous at the level of the rank-two endpoint and heuristic only insofar as one identifies it directly with a motivic "orientation residue"; the latter interpretation is developed below.

35 The A_2 norm and the universal rank-two Pontryagin seed

The most useful bridge between the A_2 note and the p_2 calculations is an elementary multiplicative identity.

Theorem 35.1 (A_2 norm identity). *Let $R \in \mathrm{UOM}(2,n)$ and let F run over the five-element subsets of the ground set. Then*

$$\prod_{F\in\binom{E(R)}{5}} \alpha_5(R|F) = \varepsilon_2(R)^{\binom{n-2}{3}}.$$

In particular, if $n \equiv 1 \pmod 4$, then

$$\boxed{\varepsilon_2(R) = \prod_{F\in\binom{E(R)}{5}} \alpha_5(R|F).}$$

If $n \equiv 0, 2, 3 \pmod 4$, the displayed product is 1 whenever the parity of $\binom{n-2}{3}$ is even.

Proof. A fixed pair $\{i,j\}$ is contained in exactly $\binom{n-2}{3}$ five-subsets. Multiplying all five-point complete products therefore raises every pair-sign $\chi_R(i,j)$ to that power. For $n = 4k+1$, Lucas' theorem gives $\binom{4k-1}{3} \equiv 1 \pmod 2$. $\square$

Status: proved/formal.

There is a stronger pre-sign version. For a generic real rank-two configuration let

$$D_n := \prod_{1\le i<j\le n} p_{ij}$$

be the product of all nonzero Plücker coordinates, and for a five-subset F put $D_F := \prod_{\{i,j\}\subset F} p_{ij}$. Then the monomial identity

$$\boxed{\prod_{F\in\binom{[n]}{5}} D_F = D_n^{\binom{n-2}{3}}}$$

holds before taking signs. Consequently the same identity survives both sign specialization and valuation/tropicalization:

$$\prod_F \mathrm{sgn}(D_F) = \mathrm{sgn}(D_n)^{\binom{n-2}{3}}, \qquad \sum_F \nu(D_F) = \binom{n-2}{3}\nu(D_n).$$

Thus the A_2 norm is naturally compatible with the coefficient changes

$$\mathbb{R} \longrightarrow \mathbb{S}, \qquad \mathbb{R}((t^\Gamma)) \longrightarrow \mathbb{T}_\pm \longrightarrow \mathbb{S}.$$

For the Pontryagin diagonal of degree $4k$, the rank-two endpoint has

$$n = 4k+1.$$

Hence the same five-point character generates every complete-product endpoint seed:

$$\boxed{\alpha_{4k+1} := \varepsilon_2 = \mathrm{Norm}_{5\to 4k+1}(\alpha_5).}$$

This does not prove that every higher p_k cocycle is obtained from p_1 by a norm construction; it says that the particular rank-two complete-product character which occurs in the present p_1, p_2 program is forced by the same A_2 atom on all $4k+1$ endpoints.

35.1 A star norm

For $n = 4k+1$ and a fixed label e, one already has

$$\boxed{\varepsilon_2(R) = \prod_{\substack{F\in\binom{E(R)}{5} \\ e\in F}} \alpha_5(R|F).}$$

Indeed, a pair containing e appears $\binom{n-2}{3}$ times, and a pair avoiding e appears $\binom{n-3}{2}$ times; both numbers are odd for $n \equiv 1 \pmod 4$. This "star norm" is potentially better adapted to deletion/contraction and to pointed Goncharov flags than the fully symmetric product over all five-subsets.

36 The first coherence network: $U_{3,6}$

The foundation of $U_{3,6}$ is assembled from six deletion pentagons and six contraction pentagons glued along thirty A_1 overlaps. This is the first small configuration in which many A_2 charts interact nontrivially.

Let $M \in \mathrm{UOM}(3,6)$ and put

$$a_i := \varepsilon_2(M/i), \qquad b_j := \varepsilon_3(M \setminus j).$$

Each a_i and b_j is a five-point p_1/A_2 character.

Proposition 36.1 (Twelve-pentagon conservation). *For every $M \in \mathrm{UOM}(3,6)$,*

$$\boxed{\prod_{i=1}^{6} a_i = \varepsilon_3(M) = \prod_{j=1}^{6} b_j.}$$

Equivalently,

$$\prod_i a_i \prod_j b_j = 1.$$

Proof. A rank-three basis B occurs in a_i exactly for the three elements $i \in B$, hence contributes with odd multiplicity to $\prod_i a_i$. It occurs in b_j exactly for the three complementary labels $j \notin B$, again with odd multiplicity. □

Status: proved/formal.

This is an abelian conservation law on the twelve-pentagon descent network. It should not be confused with a full nonabelian Čech holonomy. The latter remains conjectural at the foundation/groupoid level.

36.1 Correction concerning the integrability defect Ω

Earlier discussion risked interpreting the cubic defect Ω itself as a new six-point algebraic atom. That is too strong. A nonzero Ω is witnessed by a single three-term Grassmann–Plücker relation, hence by five relevant elements and therefore by an A_2-local relation. The role of $U_{3,6}$ is not to provide a new defining relation but to provide the first nontrivial *network* in which many A_2 relations must glue coherently.

Thus the current working distinction is

$$\boxed{\begin{aligned} \Omega &= A_2\text{-local failure/curvature certificate,} \\ U_{3,6} &= \text{first substantial } A_2 \text{ descent/coherence network.} \end{aligned}}$$

Status: proved/formal.

37 The p_2 current as a secondary A_2 construction

For $Q \in \mathrm{UOM}(3, 10)$ set

$$h_i := \varepsilon_2(Q/i).$$

The rank-three current introduced earlier is, with the determinant-line incidence convention,

$$J(Q) = \frac{1}{2} \sum_{i=1}^{10} (-1)^{i-1} h_i.$$

Every h_i is itself the A_2 norm over the five-element minors of the rank-two nine-element contraction Q/i. Hence

$$\boxed{J(Q) = \frac{1}{2} \sum_i (-1)^{i-1} \prod_{\substack{F \subset E(Q)-i \\ |F|=5}} \alpha_5\big((Q/i)|F\big).}$$

So the p_2 mechanism has the formal shape

$$\boxed{\alpha_5 \xrightarrow{\mathrm{Norm}_{A_2}} h \xrightarrow{\frac{1}{2}\partial} J \xrightarrow{K_{\mathrm{Hodge}}} \text{auxiliary Hodge potential.}}$$

The first arrow is multiplicative. The second uses the complementary-orientation twist of Section 28, and the third is the canonical linear Hodge primitive on the augmented complete simplex. Its identification with a flagged p_2 potential remains a separate problem.

There is also a useful parity refinement. Up to the standard incidence normalization,

$$(-1)^{J(Q)} = \prod_i h_i = \varepsilon_3(Q).$$

Moreover, because each rank-three basis of Q occurs in $\binom{7}{3} = 35$ six-element restrictions,

$$\varepsilon_3(Q) = \prod_{S \in \binom{E(Q)}{6}} \varepsilon_3(Q|S).$$

Applying Proposition 36.1 inside each $Q|S$ expresses the parity of J entirely as a product of five-point A_2 characters over the $U_{3,6}$ networks contained in Q. Thus the integer current J refines an A_2-built multiplicative parity invariant by an actual signed imbalance.

Part V

Motivic, cluster, signed-tropical, and real-orientation shadows

38 The indexing shift: functional equations rather than functions

A correction is necessary before comparing the UOM Pontryagin cocycles with Grassmannian polylogarithms. The Grassmannian n-logarithm is naturally a function on generic configurations of $2n$ points (or hyperplanes) in projective $(n-1)$-space. Its two basic functional equations—forgetting one point and projecting from one point—are alternating relations on configurations of $2n+1$ points. Thus the sign/UOM endpoint with $2n+1$ labels should be compared not with the polylogarithm itself but with the orientation data of its functional equation.

For a Pontryagin class p_k, complexification relates it to the even Chern class c_{2k}, so $n = 2k$. Hence

$$2n + 1 = 4k + 1,$$

exactly the number of labels at the rank-two endpoint of the degree-$4k$

UOM diagonal. Consequently

weight $2k$ Grassmannian polylogarithm	:	$4k$ arguments,
functional equation	:	$4k+1$ arguments,
rank-two UOM endpoint	:	$\mathrm{UOM}(2, 4k+1)$.

For $k=1$ this is the four-point dilogarithm and its five-term relation; for $k=2$ it is the eight-point weight-four Grassmannian object and its nine-term relation.
Status: proved/formal.

39 A parity selection theorem for circular-order chambers

Modulo reorientation, a rank-two UOM on N labelled elements is an unoriented circular order, hence a point of S_N/D_N.

Theorem 39.1 (Alternating chamber line). *For odd* N*,*

$$\dim \mathrm{Hom}_{S_N}(\mathrm{sgn}, \mathbb{Q}[S_N/D_N]) = \begin{cases} 1, & N \equiv 1 \pmod 4, \\ 0, & N \equiv 3 \pmod 4. \end{cases}$$

When $N \equiv 1 \pmod 4$*, the unique alternating scalar character is* ε_2*.*

Proof. Frobenius reciprocity reduces the multiplicity to whether sgn is trivial on D_N. An N-cycle is even for odd N, while a reflection is the product of $(N-1)/2$ transpositions. Hence reflections are even precisely when $N \equiv 1 \pmod 4$. The nonzero complete-product character ε_2 spans the resulting one-dimensional alternating line. □

Status: proved/formal.

Put $N = 2n+1$, where n is the motivic weight. The scalar alternating chamber character exists exactly when n is even. This matches the basic real-bundle dichotomy after complexification: even Chern classes contain the Pontryagin classes, while odd Chern classes of a complexified real bundle are 2-torsion rather than new rational classes.

For a real bundle ξ,

$$p_k(\xi) = (-1)^k c_{2k}(\xi_{\mathbb{C}}), \qquad \rho_2(p_k) = w_{2k}^2,$$

and the odd Chern classes satisfy the classical 2-torsion formula

$$c_{2k+1}(\xi_{\mathbb{C}}) = \beta\big(w_{2k}(\xi)w_{2k+1}(\xi)\big).$$

Thus the vanishing of a *scalar* alternating chamber function for $N \equiv 3 \pmod 4$ should not be interpreted as disappearance of all integral information: the natural remnant is a nontrivial orientation/sign local system, exactly the sort of coefficient system from which a Bockstein class can arise.

40 From the classical bi-Grassmannian to UOMs

On the classical generic Grassmannian/configuration side there are two semisimplicial face families:

$$\alpha_i = \text{forget the } i\text{th vector}, \qquad \beta_i = \text{project from the } i\text{th vector}.$$

Passing to the sign pattern of all nonzero Plücker coordinates sends these operations to

$$\boxed{\alpha_i \rightsquigarrow M \setminus i, \qquad \beta_i \rightsquigarrow M/i.}$$

Thus the semi-hypersimplicial combinatorics of the classical and UOM bi-Grassmannians is literally the same; only the coefficient object has changed.

Matveiakin–Rudenko's cluster-polylogarithmic work puts a Lie-coalgebra/integrable-symbol coefficient complex over this semihypersimplicial skeleton. The foundation construction instead puts a coefficient-independent pasture over it, with realizations in $\mathbb{R}$, $\mathbb{S}$, and signed tropical coefficient objects. This motivates the diagram

$$\begin{array}{ccc} & \text{universal Plücker/foundation geometry} & \\ \swarrow & & \searrow \\ \text{motivic symbols} & & \text{sign/tropical realizations} \\ \downarrow & & \downarrow \\ \text{motivic Chern cocycles} & & \text{proposed UOM cocycles.} \end{array}$$

No functor realizing the diagonal arrow from motivic symbols directly to UOM cochains is presently known; this is a programmatic diagram, not a theorem.

Status: heuristic/programmatic.

41 Goncharov–Kislinskyi and the degree-four anchor

The degree-four case is not merely analogous. Goncharov–Kislinskyi construct the second universal motivic Chern class in degree 4, weight 2, and explicitly describe their cluster construction as a motivic analogue of the Gabrielov–Gelfand–Losik construction. Since for a real bundle

$$p_1 = -c_2(\xi_{\mathbb{C}}),$$

this gives a direct conceptual anchor for the expectation that the real/sign chamber shadow of the motivic c_2 cocycle should reproduce the UOM/GGL p_1 class after normalization.

The correct endpoint comparison is the five-term functional equation: the four-point dilogarithmic/Grassmannian function has an alternating five-point boundary, and the latter has exactly one scalar alternating circular-order character, namely $\alpha_5 = \varepsilon_2$. Therefore the endpoint sign pattern is forced if the real-orientation residue is nonzero.

Status: heuristic/programmatic.

42 Weight four, Rudenko, and the p_2 target

For weight four, the Grassmannian function has eight arguments and an alternating nine-term functional equation. Our p_2 endpoint is $\mathrm{UOM}(2, 9)$, precisely the chamber set appropriate to that nine-term relation. By Theorem 39.1, the space of scalar alternating chamber characters is one-dimensional. Consequently, any nonzero chamber-orientation residue of the nine-term relation must be proportional to

$$\varepsilon_2(R) = \mathrm{Norm}_{5\to 9}(\alpha_5)(R).$$

This is a strong rigidity statement: matching one calibrated chamber would determine the entire rank-two component up to scalar.

The next target is our rank-three current

$$J(Q) = \frac{1}{2}\sum_i (-1)^{i-1}\varepsilon_2(Q/i), \qquad Q \in \mathrm{UOM}(3, 10).$$

The conjectural comparison is that the first piece of internal motivic cobracket/face information lost when one reduces the cluster/motivic coefficient system to pure chamber signs reappears as the Bockstein-like half-boundary J. At present this is not a theorem; it is the most concrete test of the proposed real-orientation shadow.

Conjecture 42.1 (Real-orientation shadow, first two components). *After restricting the weight-four bi-Grassmannian motivic/cluster cocycle to generic real configurations and retaining the orientation data of the nine-term functional equation, the rank-two chamber component is a nonzero scalar multiple of ε_2. Under the next face/cobracket compatibility, the first secondary chamber defect is a scalar multiple of J.*

Status: conjectural.

43 Signed tropicalization as an intermediate coefficient system

A direct "sign of a polylogarithm" map is neither natural nor expected: for example, single-valued regulators can vanish on real loci even when their branch or chamber structure is nontrivial. A more plausible route passes through a signed valuation. Over a real valued field one may retain both

$$\mathrm{sgn}(c_{\mathrm{lead}}) \qquad \text{and} \qquad \mathrm{val}(x),$$

obtaining a signed-tropical coefficient object. Forgetting the valuation gives the UOM sign point.

The A_2 norm is compatible with this refinement because its defining identity is monomial before taking signs. Thus the diagram

$$\text{cluster/motivic} \longrightarrow \text{signed tropical/chirotropical} \longrightarrow \text{UOM}$$

should be investigated before any attempt to construct a direct motivic-to-sign morphism. The scattering-wall structure conjectured for oriented foundations is a natural candidate for retaining the wall-crossing data which disappears in the pure sign quotient.

Status: heuristic/programmatic.

44 A tentative universal real-sign Chern package

The preceding parity theorem suggests that even and odd motivic weights should leave different kinds of real-sign residue.

Conjecture 44.1 (Integral real-sign Chern package). *There is a refinement of the real chamber reduction of the bi-Grassmannian Chern cocycles such that*

$$\mathrm{Res}_{\mathrm{sign}}(c_{2k}) \rightsquigarrow (-1)^k p_k,$$
$$\mathrm{Res}_{\mathrm{sign}}(c_{2k+1}) \rightsquigarrow \beta(w_{2k}w_{2k+1}).$$

At the rank-two endpoint the even-weight residue is scalar and generated by ε_2; the odd-weight residue lives naturally in the nontrivial dihedral orientation line rather than as a scalar function.

Status: conjectural.

This conjecture should be understood first rationally for the Pontryagin part. Its integral/torsion refinement is substantially more delicate because the UOM and PL classifying objects need not remember a canonical smoothing.

Part VI

Arithmetic, integrality, Hodge denominators, and torsion

45 Integral classes versus integral local coefficients

A recurring source of confusion is the difference between integrality of a characteristic class and integrality of a universal local representative. The classical GGL formula already exhibits this distinction: its canonical local coefficients are rational, while the associated characteristic class on a genuine real vector bundle is integral. MacPherson's Bourbaki exposition emphasizes integral homological sections versus rationally averaged local sections. Gaifullin proves a stronger denominator theorem for purely link-local universal formulas: if f is a universal local formula for p_1, then for every positive integer q, the least common multiple of the denominators of f on all PL spheres with at most ℓ vertices is divisible by q for some ℓ [36, Theorems 4.2–4.3]. In particular, in his complex of local formulas,

$$H^4(T^*(G)) = 0$$

for every proper subgroup $G \subsetneq Q$.

Thus

$$\boxed{\text{integral characteristic class} \not\Rightarrow \text{integer-valued canonical local formula.}}$$

This strongly supports the strategy of separating a resolved/geometric integral object from a canonical local/Hodge representative with potentially large denominators.

46 Why Hodge normalization naturally creates denominators

Let $C_{k+1} \xrightarrow{\partial} C_k$ be an integral based chain complex, and suppose an integral cycle y lies in $\operatorname{im} \partial$. There may exist many integral primitives x with $\partial x = y$. The canonical Moore–Penrose/Hodge primitive is

$$x_H = \partial^*(\partial\partial^*)^\dagger y.$$

Even when an integral primitive exists, x_H need not be integral. Its denominators are controlled by determinants and invariant factors of the integral Laplacian. In graph degree these are governed by the critical/sandpile group and spanning-tree numbers; higher cellular Laplacians have analogous critical groups.

This provides a concrete arithmetic mechanism for unbounded local denominators: symmetry and minimal-norm normalization can force division by increasingly complicated Laplacian determinants even though the underlying homology class is integral. It also connects the present UOM program with the separate Hodge/expander/sandpile investigations.

Status: proved/formal.

47 A Bockstein reading of the half-boundary

For an ordinary mod-2 cocycle a, an integral lift $\tilde{a}$ with $d\tilde{a} = 2b$ gives the Bockstein class $[b] = \beta[a]$. The auxiliary current has a related chain-level shape,

$$\boxed{J_M = \frac{1}{2}\partial \tilde{h}_M,}$$

but two distinctions are essential. First, reduction modulo 2 of every coefficient $\lambda_E(K)h_M(K) \in \{\pm 1\}$ is 1; this reduction is independent of the oriented matroid. Second, the augmented complete simplex is acyclic over $\mathbb{Z}$, so the integral cycle J_M is an integral boundary. Thus its ordinary homological Bockstein class is zero in this auxiliary complex, even when the minimal-norm primitive has denominators.

A nontrivial global Bockstein interpretation would require a different specified complex or coefficient system, a specified mod-2 class, and a comparison with these local currents. None has been constructed here.

Status: heuristic/programmatic.

48 Mod-2 compatibility for matroid bundles

Anderson–Davis construct Stiefel–Whitney classes for matroid bundles and establish the corresponding mod-2 characteristic-class comparison [18]. For a proposed integral UOM refinement P_k, a desired compatibility with real-bundle Pontryagin classes is

$$\boxed{\rho_2(P_k) = w_{2k}^2.}$$

In particular,

$$\rho_2(P_1) = w_2^2, \qquad \rho_2(P_2) = w_4^2.$$

This identity is necessary after pullback to genuine real vector bundles. On the entire UOM classifying object it is an additional requirement to prove: rational agreement alone does not impose it, because integral lifts can differ by torsion with nonzero mod-2 reduction. The odd Chern classes of a complexified real bundle motivate the separate search for torsion information in odd-weight sign local systems.

49 A finite arithmetic experiment for p_1

The following computation was carried out in the present project with exact integer arithmetic. Because it has not yet been independently reimplemented, it is recorded as computational evidence rather than as a theorem.

Consider the labelled UOM total chain complex in the small range needed for the p_1 cocycle,

$$C_m^{\mathrm{UOM}} = \bigoplus_{r=1}^{m} \mathbb{Z}[\mathrm{UOM}(r, m+1)], \qquad D = d + \partial,$$

with determinant-line signs chosen so that $D^2 = 0$. Exhaustive generation through six labels gave chain-group sizes

$$\#C_3 = 40, \qquad \#C_4 = 416, \qquad \#C_5 = 15808,$$

with

$$416 = 16 + 192 + 192 + 16$$

in ranks $1, 2, 3, 4$, and

$$15808 = 32 + 1920 + 11904 + 1920 + 32.$$

The number 11904 is independently reproduced by enumerating every uniform rank-three six-element chirotope modulo global sign, represented by $\chi(1, 2, 3) = +1$. This verifies the largest reported generator count, but does not verify the Smith-normal-form assertions below.

Define the integral numerator cochain

$$a(M) = \begin{cases} \varepsilon_2(M), & M \in \mathrm{UOM}(2, 5), \\ \varepsilon_3(M), & M \in \mathrm{UOM}(3, 5), \\ 0, & \text{endpoint ranks.} \end{cases}$$

The reported matrices satisfy

$$D^* a = 0.$$

The cochain identity is independently verified on all five ranks with six labels: exhaustive enumeration covers the 11904 rank-three chirotopes and 1920 rank-two chirotopes, the latter also giving all rank-four chirotopes by explicit duality; the rank-one and rank-five cases vanish by endpoint support. The reported chain-group sizes 40, 416, and 15808 are also reproduced. A reproducible certificate for the remaining integral calculation should include the ordered generator lists, both boundary matrices, the chosen totalization signs, and the integer cycles and change-of-basis data used below. The Smith-normal-form and cycle certificates were not supplied with this note. The reported Smith/modular calculation gives

$$[a] \in 24H^4(C_{\mathrm{UOM}}; Z), \qquad [a] \notin 48H^4(C_{\mathrm{UOM}}; Z).$$

Interpreted in the free quotient of integral cohomology, these assertions would imply that for $P = [a]/48$,

$$2P \in \mathrm{im}\big(H^4(C_{\mathrm{UOM}}; Z) \longrightarrow H^4(C_{\mathrm{UOM}}; Q)\big),$$

whereas P is not in that image. Divisibility in integral cohomology itself must be distinguished from divisibility modulo torsion. An integral 4-cycle supported on 26 generators was reported, but its coefficients are not supplied. The claimed pairing is

$$\langle a, z\rangle = 24, \qquad \langle P, z\rangle = \frac{1}{2}.$$

Status: reported computation.

This should be interpreted cautiously. It is an arithmetic property of the *bare finite UOM total complex* used in the computation, not a contradiction with integrality of the classical p_1 on vector bundles. Rather it suggests that arbitrary cycles of the bare UOM complex include non-geometric combinations on which the rational GGL cocycle is only half-integral.

49.1 Reorientation quotient

A second reported computation passed to reorientation classes, closer to the sign-point/foundation quotient. The corresponding small chain-group counts became

$$5, \qquad 26, \qquad 494,$$

and the interior GGL numerator represented 12 times a primitive free generator in the relevant degree. Hence the normalized class $a/48$ became quarter-integral on that quotient. Again this is computational evidence awaiting an independent implementation.

Status: reported computation.

If reproduced with a precisely defined integral quotient differential, this comparison would show that forgetting reorientation data can change the integral lattice of this bare chain model. It would not by itself identify either lattice with the integral characteristic classes of geometric bundles.

50 A conjectural arithmetic obstruction on the bare UOM bi-Grassmannian

Suppose the rational UOM Pontryagin class is

$$p_k^{\mathrm{UOM}} \in H^{4k}(X_{\mathrm{UOM}}; \mathbb{Q}).$$

Its image under

$$H^{4k}(X;\mathbb{Q}) \longrightarrow H^{4k}(X;\mathbb{Q}/\mathbb{Z})$$

defines an obstruction

$$\vartheta_k \in H^{4k}(X_{\mathrm{UOM}};\mathbb{Q}/\mathbb{Z})$$

to the existence of an integral lift. The p_1 experiment above suggests a nontrivial 2-primary ϑ_1 on the bare complex.

Let

$$\widehat{X}_{\mathrm{flag}} \longrightarrow X_{\mathrm{UOM}}$$

be the resolved OM Goncharov-flag object. A natural conjecture is

$$\boxed{\pi^*\vartheta_k = 0.}$$

In words: the arithmetic obstruction may be created by forgetting coherent flag/lift data and may disappear on a sufficiently resolved geometric object. This is exactly the distinction one would expect between an integral but noncanonical section and a canonical rationally averaged descent.

Status: conjectural.

51 Three arithmetic levels of the program

The evidence suggests separating three questions.

Level I. Rational UOM characteristic classes. Construct and identify

$$p_k^{\mathrm{UOM}} \in H^{4k}(-;\mathbb{Q})$$

by deletion–contraction/Goncharov-flag descent.

Level II. Mod-2 and torsion refinement. For a specified integral candidate P_k, compare with the Anderson–Davis Stiefel–Whitney classes and test

$$\rho_2(P_k) = w_{2k}^2.$$

Investigate whether odd-weight orientation local systems give the Bocksteins $\beta(w_{2k}w_{2k+1})$.

Level III. Integral lift and arithmetic obstruction. Determine the class

$$\vartheta_k \in H^{4k}(-;\mathbb{Q}/\mathbb{Z}),$$

its order and prime decomposition, and whether it vanishes on the realizable or resolved Goncharov-flag locus.

These questions are logically independent of the existence of arbitrarily large denominators in purely link-local formulas: Gaifullin's theorem shows that such denominator growth is compatible with an integral global characteristic class.

52 Revised master picture

The combined program can now be summarized by the chain

$$\boxed{\begin{array}{c} A_2 = M_{0,5} \text{ universal foundation} \\ \downarrow \\ \alpha_5 = \text{five-point alternating chamber character} \\ \downarrow\ \mathrm{Norm}_{5\to 4k+1} \\ \alpha_{4k+1} = \varepsilon_2 \text{ on the Pontryagin endpoint} \\ \downarrow\ \text{hypersimplicial deletion–contraction descent} \\ \text{integer currents / coherence data} \\ \downarrow\ \text{canonical Hodge or Bockstein transfer} \\ p_k \text{ (rationally, and perhaps with integral/torsion refinements).} \end{array}}$$

On the motivic side, the same semihypersimplicial skeleton carries Grassmannian/cluster polylogarithmic coefficient complexes and motivic Chern cocycles. The conjectural real-orientation shadow compares the $(4k+1)$-term functional equation of the weight-$2k$ motivic object with the UOM endpoint character ε_2.

The strongest immediate tests are therefore:

(1) calibrate the real chamber orientation of the weight-four nine-term relation against ε_2 on one explicit chamber;

(2) identify whether the next chamber-level defect is proportional to J;

(3) independently rerun the p_1 Smith-normal-form computation and test whether the half-integral obstruction disappears on the resolved OM Goncharov-flag complex;

(4) compare any integral lift with w_2^2 and, eventually, compare the p_2 lift with w_4^2;

(5) formulate the A_2 norm and the rank-two endpoint character directly in the foundation/signed-tropical category before specialization to $\mathbb{S}$.

Part VII

Hodge transfer, primitive characteristic classes, conormal geometry, and rank–nullity compression

53 Purpose of the present update and correction ledger

The developments in this part reorganize the $\mathrm{p}_1/\mathrm{p}_2$ program around four structures which were not yet incorporated in the preceding parts of the note:

(i) homological perturbation and transferred differentials for the deletion–contraction bicomplex;

(ii) the logarithmic/Newton/Pontryagin-character primitive rather than the individual Pontryagin polynomial as the natural rational output;

(iii) conormal geometry and tautological matroid K-classes as pre-existing algebraic models for primal/dual characteristic data;

(iv) the rank–nullity ring as a small symmetry-invariant algebra indexed by the same rank/nullity grid as the OM bi-Grassmannian.

Several intermediate claims from the exploratory discussion require correction before the new picture is stated.

Warning 53.1 (Extension spaces). *The ordinary one-element extension space of a rank-three oriented matroid should not be described as an* RP^2*-type object. Sturmfels–Ziegler prove sphericity for all oriented matroids of rank at most three (indeed for the strongly Euclidean class), so in rank three the extension space has the homotopy type of* S^2*; Mnëv–Richter-Gebert exhibit nonrealizable rank-four oriented matroids with disconnected extension spaces [42, 43]. An* RP^2*-type quotient can only arise after an additional sign/reorientation quotient and must be named as such.*

Status: proved/formal.

Warning 53.2 (Top reduced Orlik–Solomon dimension for K_4). *For the graphic matroid* $M(K_4)$ *the beta invariant is* $\beta(M) = 2$*, but this is not the dimension of the top reduced Orlik–Solomon group. Since*

$$\chi_M(q) = (q-1)(q-2)(q-3),$$

the Orlik–Solomon Poincaré polynomial is

$$1 + 6t + 11t^2 + 6t^3,$$

and hence

$$\dim \overline{\mathrm{OS}}^2(M(K_4)) = 6.$$

Thus the earlier tentative "$2 \times 2 = 4$ OS states" picture is false. The conormal connection developed below does not use it.

Status: proved/formal.

Warning 53.3 (Pair-row formulas). *A previously proposed explicit pair-row formula* $H_P(i, j)$ *was withdrawn because its sign conventions were not stable under the determinant-line bookkeeping. No result in the present part relies on that formula. The Hodge-transfer constructions below use formal contraction data. Applying them to a marked rank-three formula requires a separately verified descent equation and, for identification with the Hodge solution, the coexact gauge condition.*

Status: proved/formal.

Warning 53.4 (Two logarithms). *The ordinary logarithm of the total Pontryagin series and the convolution logarithm in the matroid-minor Hopf algebra are distinct constructions. The first linearizes Whitney sum and produces Newton/Pontryagin-character primitives. The second removes direct-sum decomposable contributions in the minor Hopf algebra. Their possible compatibility under a characteristic map is a conjecture, not a formal identity.*

Status: proved/formal.

54 Hodge contractions and homological perturbation on the bicomplex

Work over Q with a finite bicomplex and a positive-definite rational metric (or over R). Let C_r denote a rank slice after fixing a totalization in which both d and c have total cohomological degree one. Assume that d preserves the rank filtration and c increases it strictly. Write

$$d : C_r^q \longrightarrow C_r^{q+1}$$

for the differential which is to be resolved first and

$$c : C_r^q \longrightarrow C_{r+1}^{q+\epsilon}$$

for the complementary deletion/contraction differential, with

$$d^2 = c^2 = 0, \qquad dc + cd = 0.$$

Fix a symmetry-invariant counting metric. Let

$$H_r = H(C_r, d), \qquad \iota_r : H_r \hookrightarrow C_r, \qquad \pi_r : C_r \twoheadrightarrow H_r$$

be the harmonic inclusion and projection and put

$$h_r = d_r^\dagger = d_r^*(d_r d_r^* + d_r^* d_r)^\dagger.$$

With the standard Hodge side conditions,

$$h_r^2 = h_r \iota_r = \pi_r h_r = 0,$$

and

$$dh + hd = 1 - \iota\pi. \tag{19}$$

Because hc increases the bounded rank filtration, it is nilpotent; hence the inverse below is a finite geometric series. Treating c as a perturbation, the basic perturbation lemma transfers the total differential $d + c$ to the harmonic space. With the present sign convention one obtains

$$D_H = \pi c(1 + hc)^{-1}\iota = \sum_{j \geq 1} D_j, \qquad D_j = (-1)^{j-1}\pi c(hc)^{j-1}\iota. \tag{20}$$

See Crainic [38] for the perturbation lemma in this form and Johnson–Freyd [39] for the diagrammatic/Feynman interpretation of homological perturbation.

Status: proved/formal.

The first operations are

$$D_1 = \pi c\iota, \qquad D_2 = -\pi c h c\iota, \qquad D_3 = \pi c h c h c\iota.$$

Since $D_H^2 = 0$, its homogeneous pieces satisfy

$$D_1^2 = 0,$$

$$D_1 D_2 + D_2 D_1 = 0,$$

and

$$D_1 D_3 + D_2^2 + D_3 D_1 = 0. \tag{21}$$

The identity (21) is the formal reason that $D_3 a$ defines a higher-transgression class once $D_1 a = D_2 a = 0$.

54.1 The p_2 seed and the third transferred differential

For the historical compressed logarithmic gauge, take the rank-two endpoint

$$a_2 = \ell_2 = -\frac{1}{1024}\,\varepsilon_2. \tag{22}$$

If the first harmonic obstruction vanishes, the Moore–Penrose primitive is

$$\ell_3^H = -h_3 c_2 a_2.$$

A marked rank-three formula solving the same descent equation differs from this primitive by a d-closed cochain. Equality with ℓ_3^H requires a separate verification of the coexact gauge $\ell_3 \in \operatorname{im} d_3^*$. The obstruction condition is

$$D_1 a_2 = \pi_3 c_2 a_2 = 0. \tag{23}$$

The next obstruction is

$$D_2 a_2 = -\pi_4 c_3 h_3 c_2 a_2. \tag{24}$$

If it vanishes, the coexact Moore–Penrose gauge gives

$$\ell_4 = h_4 c_3 h_3 c_2 a_2.$$

The first central obstruction is then

$$\boxed{\mathcal{O}_5 = D_3 a_2 = \pi_5 c_4 h_4 c_3 h_3 c_2 a_2.} \tag{25}$$

Status: proved/formal.

Equation (25) is the preferred definition of the central residual. It involves only the trusted endpoint ε_2, the two incidence differentials and the Moore–Penrose contractions. In particular, it does not require a guessed Orchard-type sign polynomial in rank four or five.

Conjecture 54.1 (First two vanishings for the p_2 tower). *For the characteristic sector generated by* (22),

$$D_1 a_2 = 0, \qquad D_2 a_2 = 0,$$

whereas the first potentially nonzero higher transgression is $D_3 a_2$ *on the ten-element central stage.*

Status: conjectural.

A verified marked rank-three primitive would establish the first equality; identifying that primitive with the Moore–Penrose representative requires the additional gauge check above. The second equality remains to be checked in the final determinant-line convention.

54.2 Gauge dependence versus intrinsic higher differential

The vectors $h_3 c_2 a_2$ and $h_4 c_3 h_3 c_2 a_2$ depend on the chosen contraction. The third spectral-sequence differential is intrinsic once its source has survived to the third page. Comparisons between contractions use the induced filtered equivalence and may change the lower correction terms; one cannot hold the seed and all its intermediate representatives fixed without checking this equivalence. The Moore–Penrose metric chooses a preferred finite representative relative to that metric and those identifications.

Status: proved/formal.

This distinction is parallel to the one already familiar from the local Euler cocycle: Hodge normalization chooses a preferred representative, while the characteristic class itself is independent of that gauge.

54.3 Adjoint-state reduction

Suppose the relevant harmonic obstruction space is one-dimensional and generated by a unit vector ψ_5. Its coefficient is

$$\langle \psi_5, c_4 h_4 c_3 h_3 c_2 a_2 \rangle.$$

Move the operators to the other side:

$$\langle \psi_5, c_4 h_4 c_3 h_3 c_2 a_2 \rangle = \langle c_2^* h_3^* c_3^* h_4^* c_4^* \psi_5, a_2 \rangle . \tag{26}$$

Hence, defining

$$\Phi_2 := c_2^* h_3^* c_3^* h_4^* c_4^* \psi_5,$$

one obtains

$$\boxed{\langle \psi_5, \mathcal{O}_5 \rangle = -\frac{1}{1024} \langle \Phi_2, \varepsilon_2 \rangle.} \tag{27}$$

Status: proved/formal.

Thus one can propagate a single sparse harmonic test vector backwards and take one final pairing with the rank-two seed. There is no need to construct the entire intermediate vectors or a global pseudoinverse matrix.

If the complex is compressed to symmetry orbits, the adjoint must be taken with the orbit-size Gram matrix. If

$$G_r = \operatorname{diag}(|\mathcal{O}_1|, \dots, |\mathcal{O}_s|)$$

and D_r is the orbit incidence matrix, then

$$D_r^* = G_r^{-1} D_r^T G_{r+1}. \tag{28}$$

Using the ordinary transpose on orbit representatives changes the Hodge metric and may alter the finite representative.

55 Deletion-first versus contraction-first transfer

Matroid duality exchanges deletion and contraction:

$$(M \setminus e)^* = M^*/e, \qquad (M/e)^* = M^* \setminus e.$$

Consequently a Hodge transfer which resolves the deletion direction first is not expected to commute strictly with duality. Duality converts it into the opposite filtration in which contraction is resolved first.

Let

$$\mathcal{O}^{D\to C}, \qquad \mathcal{O}^{C\to D}$$

denote the resulting central representatives. They initially belong to different transferred models. After constructing comparison maps to a common target, and verifying that an involution there exchanges their images, the symmetric and antisymmetric combinations are

$$\mathcal{O}^{+} = \frac{1}{2}\left(\mathcal{O}^{D\to C} + \mathcal{O}^{C\to D}\right), \qquad \mathcal{O}^{-} = \frac{1}{2}\left(\mathcal{O}^{D\to C} - \mathcal{O}^{C\to D}\right). \tag{29}$$

Status: proved/formal.

The formal construction (29) should be distinguished from the conjectural identification of these pieces with classical characteristic classes. On the realizable tautological/complementary bundle pair one has, rationally,

$$p(E)p(E^{\perp}) = 1.$$

Hence

$$p_1(E^{\perp}) = -p_1(E),$$

and

$$p_2(E^{\perp}) = p_1(E)^2 - p_2(E). \tag{30}$$

Therefore the degree-eight even and odd combinations are

$$p_1^2, \qquad \Theta_2 := 2p_2 - p_1^2.$$

Conjecture 55.1 (Two-filtration characteristic interpretation). *After comparison with the realizable/Gelfand–MacPherson normalization, the symmetric part of the degree-eight transfer represents the decomposable p_1^2 direction and the antisymmetric connected part represents the primitive direction $2p_2 - p_1^2$.*

Status: conjectural.

The word "connected" is important: at the ten-element central stage there are also odd wheel/Borel classes discussed below, so an arbitrary antisymmetric harmonic vector should not be identified with p_2 without first separating the adjacent odd regulator sector.

56 Logarithmic characteristic classes: Newton classes and the Pontryagin character

The complement formula becomes linear after taking a logarithm. Write

$$P(t) = 1 + p_1 t + p_2 t^2 + p_3 t^3 + \cdots .$$

In the torsion-free/rational Pontryagin ring let

$$\log P(t) = \sum_{k\geq 1} (-1)^{k-1} \frac{q_k}{k} t^k,$$

where q_k is the Newton power sum in the squared formal roots. Thus

$$q_1 = p_1,$$

$$q_2 = p_1^2 - 2p_2,$$
$$q_3 = p_1^3 - 3p_1p_2 + 3p_3,$$

and so on. Complement duality sends

$$q_k \longmapsto -q_k.$$

Status: proved/formal.

For a real bundle E, define the Pontryagin character by

$$\mathrm{ph}_k(E) := \mathrm{ch}_{2k}(E_{\mathbb{C}}).$$

If $y_i = x_i^2$ are the squared formal roots, then

$$\mathrm{ph}_k(E) = \frac{2}{(2k)!} \sum_i y_i^k,$$

so

$$q_k = \frac{(2k)!}{2} \mathrm{ph}_k\,. \tag{31}$$

Consequently

$$\boxed{\log P(t) = \sum_{k\geq 1} (-1)^{k-1} (2k-1)!\ \mathrm{ph}_k\ t^k.} \tag{32}$$

Status: proved/formal.

In particular,

$$q_2 = p_1^2 - 2p_2 = 12\,\mathrm{ph}_2,$$

so

$$\boxed{2p_2 - p_1^2 = -12\,\mathrm{ph}_2\,.} \tag{33}$$

This is the natural rational primitive direction in degree eight.

56.1 Why p_2 is misleadingly simple

If $P^C = (P^D)^{-1}$ at the cohomology level and $L = \log P^D$, then

$$P^D - P^C = e^L - e^{-L} = 2\sinh L.$$

In weight two,

$$[t^2](P^D - P^C) = 2L_2,$$

so the raw duality difference is already primitive. Beginning in weight three this fails:

$$[t^3](P^D - P^C) = 2L_3 + \frac{1}{3}L_1^3. \tag{34}$$

Since $L_3 = 120\,\mathrm{ph}_3$, the raw antisymmetrization contains the decomposable term $p_1^3/3$.

Thus the correct higher object is the logarithmic/cumulant part, not the raw deletion/contraction difference.

56.2 Newton reconstruction of the integral Pontryagin polynomials

Once the primitive Newton classes q_k are known, all p_n are recovered from Newton's recursion

$$np_n = \sum_{k=1}^{n} (-1)^{k-1} q_k p_{n-k}, \qquad p_0 = 1. \tag{35}$$

Equivalently, writing $\Phi_k = \mathrm{ph}_k$,

$$np_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{(2k)!}{2} \Phi_k p_{n-k}.$$

The first formulas are

$$p_1 = \Phi_1,$$

$$p_2 = \frac{1}{2}\Phi_1^2 - 6\Phi_2, \tag{36}$$

$$p_3 = \frac{1}{6}\Phi_1^3 - 6\Phi_1\Phi_2 + 120\Phi_3, \tag{37}$$

$$p_4 = \frac{1}{24}\Phi_1^4 - 3\Phi_1^2\Phi_2 + 18\Phi_2^2 + 120\Phi_1\Phi_3 - 5040\Phi_4. \tag{38}$$

Status: proved/formal.

This suggests that the rational UOM theory should aim first at one primitive cocycle Φ_k^{UOM} in each Pontryagin degree, with p_k reconstructed afterward by universal algebra.

Conjecture 56.1 (Primitive UOM characteristic classes). *There are canonical rational cocycles Φ_k^{UOM} whose restriction to the realizable Grassmannian characteristic sector equals* ph_k.

Status: conjectural.

57 The cochain logarithm and the first $\smile_i$ corrections

Choose a rational differential graded cochain algebra with an associative product, for example normalized simplicial cochains with the Alexander–Whitney product. An arbitrary cellular cup product requires additional choices and need not be strictly associative. Let

$$\mathcal{P}(t) = 1 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots$$

be a total series of even cocycles. Its formal noncommutative logarithm has coefficients

$$\Lambda_1 = P_1,$$

$$\Lambda_2 = P_2 - \frac{1}{2} P_1 \smile P_1, \tag{39}$$

and

$$\Lambda_3 = P_3 - \frac{1}{2}(P_1 \smile P_2 + P_2 \smile P_1) + \frac{1}{3} P_1^{\smile 3}. \tag{40}$$

Status: proved/formal.

For even cocycles and the convention

$$d(P_1 \smile_1 P_2) = P_1 \smile P_2 - P_2 \smile P_1,$$

formula (40) becomes

$$\Lambda_3 = P_3 - P_1 P_2 + \frac{1}{3} P_1^3 + \frac{1}{2} d(P_1 \smile_1 P_2). \tag{41}$$

This exhibits the exact difference between the symmetrized logarithm coefficient and the chosen ordered product. The formal logarithm itself needs no additional $\smile_1$ correction. Higher cup-i operations can organize coherent comparisons of representatives, but the appearance of a specific $\smile_2$ term at weight four is not implied by this identity.

Status: proved/formal. The algebraic identities above are formal once the chosen cup-i convention is fixed. Their identification with the specific UOM flag operations is still programmatic.

Conjecture 57.1 (Transferred E_∞ logarithm). *The flag hierarchy in the UOM bi-Grassmannian realizes the homotopy-transferred logarithm of the total characteristic series. Minor flags account for the cumulant/decomposability corrections, while the cup-*i *hierarchy accounts independently for homotopy commutativity of cellular cochains.*

Status: conjectural.

58 Minor Hopf algebra, convolution logarithm, and flags

The restriction–contraction Hopf algebra of matroids has coproduct

$$\Delta M = \sum_{A \subseteq E} M|A \otimes M/A. \tag{42}$$

Crapo–Schmitt study its primitive elements [40]; Billera–Jia–Reiner construct a Hopf morphism from the matroid Hopf algebra to quasisymmetric functions [41]. The same restriction/contraction formalism is compatible with the modern hyperfield viewpoint in which oriented matroids are matroids over the sign hyperfield [20].

For a unital character φ with values in a commutative $\mathbb{Q}$-algebra, define the convolution logarithm

$$\log_\star \varphi = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} (\varphi - \epsilon)^{\star m}. \tag{43}$$

The m-fold reduced coproduct is indexed by strict flags

$$\varnothing = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_m = E,$$

so

$$(\log_\star \varphi)(M) = \sum_{m \geq 1} \frac{(-1)^{m-1}}{m} \sum_{A_\bullet} \prod_{j=1}^{m} \varphi\big((M|A_j)/A_{j-1}\big). \tag{44}$$

Status: proved/formal.

The universal coefficients

$$1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \ldots$$

are intrinsic to this minor-flag logarithm; the sum is finite on each matroid because $m \le |E|$. Moreover $\log_\star \varphi$ is an infinitesimal character, so it vanishes on direct sums of two nonempty matroids. This supplies a precise flag mechanism that could be compared with UOM flags once a characteristic map is constructed.

One must not, however, identify $\log_\star$ with the ordinary logarithm of $P(t)$. The ordinary logarithm extracts classes additive under Whitney sum; the convolution logarithm extracts the part of a matroid invariant which is indecomposable under direct sum. A characteristic map intertwining these two logarithms would be additional structure.

Conjecture 58.1 (Cumulant compatibility). *There is a cochain-valued characteristic morphism from an oriented/minor enhancement of the matroid Hopf algebra to the UOM characteristic complex such that its convolution cumulants map to the Newton/Pontryagin-character primitives.*

Status: conjectural.

This conjecture should be formulated only after the determinant-line coefficient system has been incorporated: the ordinary matroid Hopf algebra forgets precisely the real orientation information which is essential to the integral UOM formulas.

59 Odd matroid complexes, wheel classes, and the adjacent Borel diagonal

Bruce–Bucciarelli–Zacovic construct deletion and contraction bicomplexes on matroid classes equipped with ground-set orientations and show that direct sum and restriction–contraction give a connected graded Hopf-algebra framework. They compute several variants through ground-set size 9, and the connected quotient of the simple loopless regular complex through size 15; the resulting nontrivial homology suggests a conjectural pattern generated by odd-wheel matroids [44].

For a wheel graph W_m,

$$|E(W_m)| = 2m, \qquad \operatorname{rk} M(W_m) = m, \qquad \operatorname{null} M(W_m) = m.$$

Hence

$$M(W_{2k+1}) \in C_{2k+1,2k+1} \qquad (|E| = 4k + 2). \tag{45}$$

Thus

$$W_3 \in C_{3,3}, \qquad W_5 \in C_{5,5}, \qquad W_7 \in C_{7,7}.$$

This is exactly the same central sequence which arises one step after the even Pontryagin diagonals.

59.1 Brown canonical forms

Brown constructs invariant differential forms on graph complexes whose primitive generators represent the stable Borel classes of GL; the associated canonical periods detect graph-homology classes [45]. Brown–Schnetz compute the canonical periods of odd wheels and prove nonvanishing proportional to odd zeta values [46].

This leads to an important grading correction. The 4k-dimensional Pontryagin/Pontryagin-character class belongs to the even diagonal, whereas the primitive Brown/Borel form has degree $4k+1$ and naturally lives on the next, $4k + 2$-element central stage. Thus a genuinely harmonic class on $C_{2k+1,2k+1}$ should not automatically be called another representative of ph_k; on the regular/graphic connected sector it may instead be an odd Borel/regulator class.

The current working picture is therefore

$$\boxed{\begin{array}{ll} \text{even diagonal} & ph_k,\ p_k \\ \qquad\downarrow & \\ C_{2k+1,2k+1} & \text{odd wheel/Borel sector } \beta_{4k+1}. \end{array}} \tag{46}$$

Status: heuristic/programmatic.

For the p_2 program, a useful proposed diagnostic is evaluation of the corrected ten-element residual on a wheel cycle, after constructing a comparison with the relevant regular graphic complex and checking the coefficient system. If that cycle represents a valid obstruction functional, a nonzero evaluation obstructs closure of the full descent. Calling it a regulator component does not remove that obstruction; the Brown interpretation is an additional comparison to be proved.

60 The exact $W_3 = K_4$ local test

The smallest wheel can be analysed completely. Consider the regular oriented realization

$$B = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \tag{47}$$

with labels

$$1 = 01, \quad 2 = 02, \quad 3 = 03, \quad 4 = 12, \quad 5 = 13, \quad 6 = 23.$$

Exactly four maximal minors vanish:

$$p_{124} = p_{135} = p_{236} = p_{456} = 0. \tag{48}$$

They are the four triangular circuits of K_4.

Writing $B = [I_3 \mid A]$, the functions

$$z_1 = p_{124} = a_{31}, \qquad z_2 = p_{135} = -a_{22}, \qquad z_3 = p_{236} = a_{13}, \qquad z_4 = p_{456} = \det A$$

have independent differentials at the matrix (47); for instance the first three use three distinct entries and

$$\frac{\partial \det A}{\partial a_{11}} = 1$$

at the base point. Hence the wheel stratum is locally transverse of codimension four, and all 16 sign resolutions

$$(s_1, s_2, s_3, s_4) \in \{\pm 1\}^4$$

are realized in sufficiently small generic perturbations.

Status: proved/formal.

Let

$$A_i := \Pi_3(M_s \setminus i), \qquad B_i := \Pi_2(M_s/i),$$

where the complete products use the contraction convention in which the contracted label is inserted first. A direct determinant calculation gives

$$\boxed{(A_1, \dots, A_6) = (s_3s_4, -s_2s_4, s_1s_4, -s_2s_3, s_1s_3, -s_1s_2),} \tag{49}$$

and

$$\boxed{(B_1, \dots, B_6) = (-s_1s_2, s_1s_3, -s_2s_3, s_1s_4, -s_2s_4, s_3s_4).} \tag{50}$$

Status: reported computation.

These formulas have been rechecked directly from (47) for all 16 local sign chambers. If

$$C = -s_1 s_2 s_3 s_4,$$

then componentwise

$$B_i = CA_i. \tag{51}$$

Multiplication by C sends each pair monomial in four variables to minus its complementary pair monomial. Thus the six ground-set edges of K_4 are naturally identified with the six two-subsets of the four circuit normals, and deletion/contraction exchanges complementary pairs. After orientation signs this is the same $3 + 3$ pattern as the self-dual/anti-self-dual decomposition of $\Lambda^2 R^4$.

The alternating sums are

$$\sum_{i=1}^{6} (-1)^{i-1} A_i = e_2(s_1, s_2, s_3, s_4), \tag{52}$$

and

$$\sum_{i=1}^{6} (-1)^{i-1} B_i = -e_2(s_1, s_2, s_3, s_4), \tag{53}$$

where

$$e_2(s) = \sum_{a<b} s_a s_b.$$

Equivalently,

$$(1 + C)e_2(s) = 0. \tag{54}$$

Indeed, if $s_1 s_2 s_3 s_4 = +1$ then $C = -1$, while if the product is -1 an odd number of signs are negative and $e_2(s) = 0$.

Status: proved/formal.

Equation (54) explains the local closure of the p_1 tower around the first nonuniform wheel boundary. It also clarifies the significance of the previously observed p_2 central defect of the schematic form

$$(1 + C) \sum_i t_i :$$

the elementary four-sign identity which closes the p_1 model has no automatic higher-rank analogue.

61 Conormal geometry as a primal–dual local model

Ardila–Denham–Huh associate to a loopless and coloopless matroid M a conormal fan Σ_{M,M^*} supported on the product of the primal and dual Bergman fans, and prove Poincaré duality, Hard Lefschetz and Hodge–Riemann relations for its Chow ring [48, 49]. The maximal cones are indexed by biflags. This is structurally close to the UOM bi-Grassmannian because primal and dual minor directions are simultaneously present.

Zharkov identifies the projective Orlik–Solomon algebra with the tropical cohomology of the Bergman fan [50]; Eur–Lam construct canonical Orlik–Solomon forms for topes of arbitrary oriented matroids [51]. Thus primal and dual oriented-matroid canonical forms naturally occupy the two factors of conormal geometry, although a direct Brown–Eur–Lam pairing is degree-incompatible and should not be asserted.

61.1 The universal middle hyperbolic lattice

Let $k \geq 1$ and let M be loopless and coloopless of rank $2k+1$ on $4k+2$ elements. Its conormal fan has dimension $4k$. Write

$$\gamma = \pi^* \alpha_M, \qquad \bar{\gamma} = \bar{\pi}^* \alpha_{M^*}$$

for the pullbacks of the standard Bergman hyperplane classes, and put

$$u_k = \gamma^{2k}, \qquad v_k = \bar{\gamma}^{2k}. \tag{55}$$

These are integral classes in middle Chow degree $2k$. Since the two Bergman factors have dimension $2k$, their pullbacks satisfy

$$u_k^2 = v_k^2 = 0.$$

The subdivision map from the conormal fan to the product Bergman fan preserves the fundamental weight 1 [48, Section 3.4, Proposition 3.7 and its proof]. The projection formula [48, Section 3.1], together with $\deg(\alpha_M^{2k}) = \deg(\alpha_{M^*}^{2k}) = 1$ [47, Propositions 5.8 and 5.10], therefore gives

$$\deg(u_k v_k) = 1.$$

Here the degree is normalized to 1 on maximal biflag monomials, as in [48, Section 3]. Thus the pairing on $L_k^{\mathrm{con}} = \mathbb{Z}u_k \oplus \mathbb{Z}v_k$ is

$$\boxed{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}. \tag{56}$$

In particular this is a primitive integral summand: the map $x \mapsto \deg(xv_k)u_k + \deg(xu_k)v_k$ retracts the middle Chow group onto it.
Status: proved/formal.

Set $g_k = u_k + v_k$ and $e_k = u_k - v_k$. Then

$$u_k = \frac{g_k + e_k}{2}, \qquad v_k = \frac{g_k - e_k}{2},$$

and the determinant of this change of generators has absolute value 2. Hence

$$\boxed{\frac{\mathbb{Z}u_k \oplus \mathbb{Z}v_k}{\mathbb{Z}g_k \oplus \mathbb{Z}e_k} \cong \mathbb{Z}/2.} \tag{57}$$

Status: proved/formal.

Factor exchange identifies the conormal ring of M with that of M^*; it is not in general an involution of the ring of a fixed M. The terms *even* and *odd* apply internally only after choosing a self-duality identification that induces an involution, or after working on the direct sum of the two dual rings. Under such an identification, g_k and e_k are its respective eigenvectors.

Thus the conormal middle lattice provides a precise candidate geometric model for the index-two gluing which appeared computationally in the p_1 lattice. The grading correction is important: in the $k = 1$ case the plausible comparison is

$$h \longleftrightarrow \gamma^2, \qquad \tau h \longleftrightarrow \bar{\gamma}^2, \tag{58}$$

not $h \leftrightarrow \gamma$.
Status: conjectural.

Under (58), the reported finite identities

$$[a] = 24(h + \tau h), \qquad [b_1] = 2(h - \tau h)$$

become the conormal pattern

$$[a] \sim 24(\gamma^2 + \bar{\gamma}^2), \qquad [b_1] \sim 2(\gamma^2 - \bar{\gamma}^2),$$

and

$$h = \frac{[a] + 12[b_1]}{48}$$

is simply the reconstruction of one integral factor from its duality-even and duality-odd coordinates. The coefficients 24 and 2 remain specific to the UOM normalization; conormal geometry explains the index-two lattice mechanism, not those numerical transfer factors.
Status: reported computation.

61.2 Lefschetz primitive projection rather than an algebraic indecomposable quotient

The conormal Chow ring is generated in degree one, so the naive quotient

$$A^2/(A^1A^1)$$

does not provide the desired notion of a new primitive characteristic direction. The correct Hodge-theoretic analogue is Lefschetz primitiveness.

Let $d = 4k$, $m = 2k$, and work in the real Chow ring. Choose a Lefschetz class $\lambda \in A^1(\Sigma_{M,M^*})_{\mathbb{R}}$. Hard Lefschetz gives

$$\lambda^2 : A^{m-1}_{\mathbb{R}} \xrightarrow{\sim} A^{m+1}_{\mathbb{R}}.$$

For $x \in A^m_{\mathbb{R}}$, define

$$\mathrm{pr}_{\mathrm{prim},\lambda}(x) = x - \lambda(\lambda^2)^{-1}(\lambda x). \tag{59}$$

Then $\lambda\,\mathrm{pr}_{\mathrm{prim},\lambda}(x) = 0$. For $e_k = \gamma^{2k} - \bar{\gamma}^{2k}$, put

$$\Theta^{\mathrm{con}}_{k,\lambda} = e_k - \lambda(\lambda^2)^{-1}(\lambda e_k) \in P^{2k}_{\lambda}. \tag{60}$$

If a chosen self-duality involution exchanges γ and $\bar{\gamma}$ and fixes λ, this class lies in the odd primitive subspace $P^{2k,-}_{\lambda}$. A rational Lefschetz class makes the formula rational; Hard Lefschetz does not assert that its inverse is integral.

Status: proved/formal.

The projection depends on λ and may vanish. For $k = 2$ it supplies a finite candidate, relative to this choice, for comparison with a UOM transfer. Such a comparison must also account for the distinct meanings of Lefschetz primitive and Whitney-sum primitive.

62 Tautological matroid K-classes and the rational characteristic sector

Berget–Eur–Spink–Tseng construct tautological equivariant K-classes

$$[\mathcal{S}_M], \qquad [\mathcal{Q}_M]$$

on the permutohedral variety for every matroid M, extending the tautological subbundle and quotient bundle of a realizable linear subspace

[52]. The construction depends only on the matroid, including in the nonrealizable case.

Use the inverse torus-action convention of [52, Definition 3.9]. At the fixed point indexed by σ,

$$[\mathcal{S}_M]_\sigma = \sum_{i \in B_\sigma(M)} T_i^{-1}, \qquad [\mathcal{Q}_M]_\sigma = \sum_{i \notin B_\sigma(M)} T_i^{-1},$$

where $B_\sigma(M)$ is the lexicographically first basis. Hence $[\mathcal{S}_M] + [\mathcal{Q}_M] = [\underline{C}^E_{\mathrm{inv}}]$, and after forgetting equivariance,

$$\mathrm{ch}_j(\mathcal{Q}_M) = -\,\mathrm{ch}_j(\mathcal{S}_M), \qquad j > 0. \tag{61}$$

This is the subbundle/quotient identity on one permutohedral variety; identifying it with matroid duality requires the appropriate Cremona and bundle-dual operations.

Status: proved/formal.

For a real realization $L \subseteq \mathbb{R}^E$, let $i : X_E(\mathbb{R}) \hookrightarrow X_E(\mathbb{C})$ and let $\mathcal{S}_L^{\mathbb{R}}$ be the real tautological bundle there. Over the dense real torus its fiber is $t^{-1}L$. The base-correct comparison is

$$\boxed{i^*\,\mathrm{ch}_{2k}(\mathcal{S}_M) = \mathrm{ph}_k(\mathcal{S}_L^{\mathbb{R}}) \quad \text{in } H^{4k}(X_E(\mathbb{R}); \mathbb{Q}).} \tag{62}$$

Here the Chow class on the left is first sent to singular cohomology.

Warning 62.1 (A fixed real toric locus does not recover universal Pontryagin classes). *In fact both sides of* (62) *vanish for* $k > 0$*. The Chow ring of* X_E *is generated by torus-invariant divisor classes. Each associated line bundle is defined over* $\mathbb{R}$*, so its restriction to* $X_E(\mathbb{R})$ *is the complexification of a real line bundle. Its first Chern class is* 2*-torsion and vanishes rationally. Thus every positive-degree Chow class restricts to zero rationally. Any proposed nonzero UOM characteristic class must use additional gluing or descent over a varying family; it cannot be obtained by this fixed-locus restriction alone.*

Status: proved/formal.

The BEST classes remain natural algebraic input for a comparison program, but no universal UOM characteristic map is supplied by the restriction identity.

Conjecture 62.2 (BEST descent comparison). *The rational primitive UOM class is the deletion–contraction/Hodge descent of the BEST tautological Chern character:*

$$\Phi_k^{\mathrm{UOM}} = \mathrm{Desc}_{\mathrm{UOM}}\big(\mathrm{ch}_{2k}(\mathcal{S}_M)\big). \tag{63}$$

Status: conjectural.

Equation (63) remains a conjectural construction, including at the rational level. Its first task is to specify a common base or a family-level descent complex and explain how a nonzero class survives the fixed-real-locus vanishing above. Only after that step can determinant lines, integral gluing, and possible KO refinements be compared.

62.1 Adjacent permutation chambers as a discrete connection

Suppose adjacent permutations σ, σ' differ by one braid-wall crossing. Either their greedy bases agree or

$$B_{\sigma'} = B_\sigma - \{a\} + \{b\}.$$

At the level of Chern characters the fixed-point jump is then

$$\Delta_{\sigma,\sigma'} \operatorname{ch}_m(\mathcal{S}_M) = \frac{(-t_b)^m - (-t_a)^m}{m!}. \tag{64}$$

For $m = 2k$ this is

$$\frac{t_b^{2k} - t_a^{2k}}{(2k)!},$$

which is divisible by the braid-wall weight $t_b - t_a$. This is the expected GKM gluing condition.

Status: proved/formal.

For an oriented matroid the elementary basis exchange additionally has a chirotope orientation. This strongly suggests that the normal determinant-line factors appearing in the corrected marked UOM formulas are the real-oriented refinement of the ordinary BEST wall transport.

Status: heuristic/programmatic.

63 The rank–nullity ring and the same bidegree grid

Fife–Mannino–Rincón introduce the rank–nullity ring $R^*(M)$, a subring of the permutohedral Chow ring generated by sums of its subset classes with fixed rank and nullity. Their presentation includes $x_E = -\alpha$, a redundant full-set class rather than a boundary divisor. They prove that $R^*(M)$ contains all tautological Chern classes and, for a uniform matroid, coincides with the full S_n-invariant subring of the permutohedral Chow ring; they also give an integral basis and a Gröbner presentation in the uniform case [53].

For $S \subseteq E$ put

$$i = r_M(S), \qquad j = |S| - r_M(S).$$

If M has rank r and nullity c, then

$$M|S \text{ has bidegree } (i, j),$$

while

$$M/S \text{ has bidegree } (r - i, c - j).$$

Thus the restriction–contraction coproduct refines as

$$\Delta M = \sum_{i=0}^{r} \sum_{j=0}^{c} \Delta_{i,j} M, \tag{65}$$

where each term pairs opposite points

$$(i, j) \longleftrightarrow (r - i, c - j) \tag{66}$$

in exactly the same rank/nullity rectangle as the deletion–contraction bi-Grassmannian.

Status: proved/formal.

Schematically, a degree-one rank–nullity generator is

$$y_{i,j}(M) = \sum_{\substack{\varnothing \neq S \subseteq E \\ r_M(S) = i \\ |S| - r_M(S) = j}} x_S. \tag{67}$$

Hence $R^*(M)$ forgets the individual subset while retaining precisely its position in the rank/nullity grid.

For a uniform matroid $U_{r,n}$, the nonzero types trace the boundary staircase

$$(1, 0), (2, 0), \ldots, (r, 0), (r, 1), \ldots, (r, n - r).$$

Fife–Mannino–Rincón prove

$$\boxed{R^*(U_{r,n}) = A^*(U_{n,n})^{S_n}.} \tag{68}$$

Status: proved/formal.

Thus, on the uniform ordinary-matroid locus, rank–nullity compression loses no permutation-invariant Chow information. This is a precise algebraic analogue of the symmetry reduction repeatedly used in the finite UOM calculations.

Conjecture 63.1 (Rank–nullity shadow of the UOM bi-Grassmannian). *After forgetting chirotope signs and the chain-level coherence data, the rank–nullity ring is the natural commutative/invariant coefficient shadow of the UOM deletion–contraction bi-Grassmannian. The UOM refinement restores the orientation local system and the higher homotopies which are invisible in* $R^*(M)$.

Status: conjectural.

The recent rank–nullity formula for tautological Chern classes is organized by chains of subsets. This provides an independent algebraic reason for the appearance of flags in the characteristic formulas: flags are intrinsic to tautological matroid characteristic classes, not merely an artifact of Moore–Penrose inversion.

64 A reorganized four-layer picture

The evidence now suggests separating four layers which had previously been mixed.

Layer A: ordinary-matroid rational characteristic coefficients. BEST tautological K-classes and the rank–nullity ring supply $\mathrm{ch}_{2k}(\mathcal{S}_M)$ and its permutation-invariant rank/nullity compression for every underlying matroid.

Layer B: deletion–contraction descent and coherence. The UOM bicomplex, resolved by Hodge/HPL, supplies coherent representatives across minor maps. The transferred differentials D_j measure the higher obstruction to strict descent.

Layer C: real orientation refinement. Chirotope ratios and determinant lines refine ordinary basis exchange to oriented basis exchange. The corrected marked factors such as $\nu_i(M, k)$ belong here.

Layer D: integral KO and 2-primary gluing. The index-two conormal lattice and the reported p_1 arithmetic indicate that recovering an integral real characteristic class from the duality eigensectors involves genuine parity/gluing data. This is the natural place to compare Stiefel–Whitney and Bockstein information.

Thus the working dictionary is

$$\boxed{\begin{aligned} \text{BEST/rank–nullity} &= \text{ordinary-matroid rational coefficient,}\\ \text{UOM Hodge flags} &= \text{minor descent/coherence,}\\ \text{chirotope determinant lines} &= \text{real orientation refinement,}\\ \text{index-two/torsion data} &= \text{integral KO refinement.} \end{aligned}} \tag{69}$$

Status: heuristic/programmatic.

Conjecture 64.1 (Oriented tautological KO refinement). *There exists an oriented-matroid refinement of the BEST tautological class whose complexification is the ordinary matroid class $\mathcal{S}_M$, whose rational Pontryagin character is represented by the Hodge-transferred UOM primitives, and whose 2-primary descent obstruction is encoded by determinant-line/flag data.*

Status: conjectural.

This is stronger than the rational comparison conjecture. It also requires a precise base space and a specified complexification map: an ordinary algebraic K-class on $X_E(\mathbb{C})$ is not automatically the complexification of a KO-class on a UOM classifying space.

65 Relation with Brown and Eur–Lam: what is established and what is not

Brown's canonical odd forms and Eur–Lam's oriented-matroid canonical forms should be kept distinct.

Eur–Lam attach canonical classes in the reduced Orlik–Solomon algebra to topes of an oriented matroid and characterize them by residue recursion [51]. These classes are defined for arbitrary oriented matroids.

Brown's invariant forms are odd-degree Borel-type forms on metric graph/matroid realizations; their odd-wheel integrals are nonzero odd-zeta multiples [45, 46].

Three gradings must be distinguished. For rank $2k+1$ on $4k+2$ elements, each reduced Orlik–Solomon algebra has top exterior degree $2k$, so the primal/dual tensor product has top exterior degree $4k$. Brown's form has differential degree $4k+1$. The conormal Chow ring, by contrast, has top *Chow* degree $4k$, a codimension grading that corresponds to degree $8k$ under the usual cycle-class convention. These are not interchangeable degree conventions.

Status: proved/formal.

Consequently a pairing such as

$$\langle \overline{\Omega}_{\mathcal{M}}, \omega_M^{4k+1} \rangle$$

has no specified common domain or degree-compatible pairing map. The adjacent exterior degrees of the two-factor Orlik–Solomon data and Brown forms motivate a possible secondary construction, but they do not establish a one-step operation from the conormal Chow ring.

Conjecture 65.1 (Conormal–Borel secondary transgression). *On the regular realizable locus, suitably paired primal/dual Orlik–Solomon data may admit a secondary comparison with Brown's form* ω_M^{4k+1}. *A formulation must specify its complexes, degree shift and relation to the conormal Chow ring.*

Status: conjectural.

This conjecture is motivated by degree, duality and deletion/contraction compatibility; no construction is presently claimed.

66 Immediate computational tests

The new picture suggests a sequence of small tests, ordered so that each can falsify a structural conjecture before a large computation is attempted.

Problem 66.1 ($k = 1$: BEST versus the known UOM p_1 cocycle). *Compute* $\mathrm{ch}_2(\mathcal{S}_M)$ *in the rank–nullity/invariant algebra for the five- and six-element models, attach the oriented determinant-line signs along basis exchanges, and compare its Hodge descent with the already computed UOM* p_1 *cocycle. The target identity is*

$$\mathrm{Desc}_{\mathrm{UOM}}\big(\mathrm{ch}_2(\mathcal{S})\big) = p_1^{\mathrm{UOM}}. \tag{70}$$

Problem 66.2 ($K_4 = W_3$: conormal lattice comparison). *For* $M(K_4)$ *use the proved primitive hyperbolic summand generated by* $\gamma^2, \bar{\gamma}^2$ *and choose a self-duality involution. Construct an explicit comparison with the two integral directions* $h, \tau h$ *in the finite* p_1 *computation, and test whether the relations*

$$[a] = 24(h + \tau h), \qquad [b_1] = 2(h - \tau h)$$

match the conormal hyperbolic-plane dictionary (58).

Problem 66.3 ($k = 2$: rational primitive). *Compute*

$$\mathrm{ch}_4(\mathcal{S}_M)$$

in the rank–nullity ring and compare its UOM descent with the HPL third transfer

$$D_3 a_2 = \pi_5 c_4 h_4 c_3 h_3 c_2 a_2.$$

The expected realizable characteristic class is ph_2*.*

Problem 66.4 ($C_{5,5}$: wheel/Borel filter). *Construct a coefficient-compatible comparison to the regular connected matroid complex and then evaluate the ten-element residual on* W_5*. If this evaluation is a valid obstruction pairing, a nonzero result obstructs the full descent. Determine separately whether Brown's canonical form identifies that functional; removing it would require a newly specified quotient problem.*

Problem 66.5 (Odd primitive conormal comparison). *For loopless and coloopless rank-five ten-element matroids choose a rational Lefschetz class and compute*

$$\Theta^{\mathrm{con}}_{2,\lambda} = \mathrm{pr}_{\mathrm{prim},\lambda}(\gamma^4 - \bar{\gamma}^4).$$

For a self-dual example also specify the involution and an invariant λ*. Test nonvanishing and dependence on* λ *before proposing a comparison with the UOM residual; retain every obstruction component until a quotient construction is justified.*

Problem 66.6 (Integral refinement). *After the rational comparison is established, compute the Smith normal form of the transition from primal/dual integral generators to the* $\pm$ *duality eigenlattices. Compare the resulting 2-primary discriminant with the determinant-line flag class and with the expected* KO*/Stiefel–Whitney data.*

67 Pre-v5 master picture: Hodge, conormal, and rank–nullity

Warning 67.1. *This section records the organization reached on 21 September 2026 and is retained because the Hodge/conormal/rank–nullity comparisons remain useful. Its bottom-up logic from a* *rank-two sign seed to a putative Pontryagin class is superseded by Part VIII: the hypersimplicial PL bundle already defines the geometric rational classes, and the current problem is to identify explicit Thom–Bier and compressed representatives with them.*

The pre-v5 program was summarized as follows:

$$
\begin{array}{c}
A_2 \text{ / chirotope endpoint signs} \\
\downarrow \\
\text{rank-two seed } \varepsilon_2 \\
\downarrow \text{ Hodge/HPL minor descent} \\
\text{primitive UOM cocycles } \Phi_k^{\mathrm{UOM}} \\
\updownarrow \text{ comparison conjecture} \\
\mathrm{ch}_{2k}(\mathcal{S}_M) \quad \text{(family-level comparison still required)} \\
\downarrow \text{ Newton identities} \\
p_k
\end{array}
$$

rank–nullity ring: symmetry-invariant ordinary-matroid compression
conormal fan: primal/dual Hodge lattice and index-two gluing
chirotope flags: real orientation and determinant-line transport
odd central wheel/Borel sector: adjacent regulator obstruction.

The established ingredients still provide useful comparison targets: HPL obstruction formulas, tautological matroid classes, the conormal hyperbolic summand, and rank–nullity compression. What has changed in v5 is that the rational family-level characteristic map is now supplied geometrically by the hypersimplicial PL-bundle theorem. The unresolved task is the comparison of explicit Hodge/sign representatives with that already existing class, together with the integral refinement.

Part VIII

Geometric revision: the hypersimplicial PL bundle, Thom classes, Bier residues, and the Pontryagin program

68 Revision v5: what has changed

This part records a substantial reorganization of the characteristic-class program which became possible after the construction of the hypersimplicial PL bundle in the companion note [54]. The earlier parts of this working note are retained because they contain useful sign identities, finite computations, and unsuccessful ansätze, but several of their characteristic-class interpretations are now superseded by the geometry below.

The main changes are the following.

(i) The deletion–contraction bicomplex is no longer treated merely as a formal incidence gadget. The companion paper constructs a hypersimplicial CW realization and a canonical stable PL sphere bundle over it. Consequently the rational UOM Pontryagin classes already exist geometrically, independently of any sign or Hodge formula.

(ii) The fixed-rank input is reorganized around a relative Thom class. The cone pair of a cellular sphere is the suspension of its augmented cellular chain complex. Thus the Hodge construction underlying the local Euler cocycle is naturally reinterpreted as Thom descent.

(iii) For an even rank $2k$ bundle the critical Pontryagin datum is born at rank $2k$ from the Euler square $e_{2k}^2 = p_k$. On the Thom space, however, the correctly graded representative of p_k is not U^2 but

$$U^3 = p_k U.$$

The square $U^2 = eU$ is the Euler/Thom relation; the cube is the Thom representative of the degree-$4k$ Pontryagin class.

(iv) Contraction is geometrically a residue along a coordinate pseudosphere. For a uniform oriented matroid, iterated contractions meet with the expected codimension. The scalar deletion–contraction incidence complex is therefore a normal-crossing-type residue complex.

(v) The universal minor-incidence complex has a closed combinatorial model: it is the Bier sphere of a uniform simplex skeleton, equivalently the facet nerve of the simple generic cube slice

$$P_{\lambda,n} = [0,1]^n \cap \left\{ \sum_i x_i = \lambda \right\}, \qquad r < \lambda < r+1.$$

The two colors are deletion ($x_i = 0$) and contraction ($x_i = 1$), while the unique unused label at a top facet is the occurrence mark.

(vi) The universal contraction direction is a truncated Koszul complex and can be integrated out explicitly. After this reduction the remaining universal deletion complex is an S_n-equivariant Bier–Pieri resolution of the sign representation. Thus the large universal matrices in the old p_2 program can be removed before any oriented-matroid-dependent calculation begins.

(vii) The old constants $1/9$, 24, 512, 1024, and the raw central residuals have distinct origins and must not be conflated. The factor $1/9$ in the marked rank-three primitive has a universal Koszul/Bier explanation. The factors 24 and 512 concern compressed sign representatives and lattice saturation. They are not the defining normalization of the geometric Pontryagin classes.

(viii) The foundational construction of p_2 should start with the rank-four Thom/Euler data, not with the rank-two compressed Newton seed. The rank-two cocycle $\varepsilon_2/512$ remains useful for the primitive/logarithmic gauge, but it is secondary to the geometric p_2 construction.

Status: proved/formal. Items (iii)–(vi) below are proved/formal under the stated orientation and metric conventions. The comparison of the explicit Thom–Bier cocycle with the geometrically defined PL Pontryagin class on the full UOM space remains a comparison theorem; it is not silently assumed.

69 Geometric anchor from the hypersimplicial PL-bundle theorem

We quote the principal result of the companion paper in the form needed here. Put
$$\mathfrak{G}^{\mathrm{UOM}}_{p,q} = \mathrm{UOM}(q+1, p+q+2),$$
with deletion decreasing p and contraction decreasing q.

Theorem 69.1 (Hypersimplicial realization and universal PL bundle; companion paper). *There is a hypersimplicial CW space $\mathscr{B}^{\mathrm{UOM}}$ with the following properties.*

(i) After orienting a rank-r cell by $(-1)^{r-1}$ times the standard hypersimplex orientation,

$$C^{\mathrm{cell}}_m\big(\mathscr{B}^{\mathrm{UOM}}, \mathscr{B}^{\mathrm{UOM}}_{(0)}; Z\big) \cong \bigoplus_{p+q=m-1} Z[\mathfrak{G}^{\mathrm{UOM}}_{p,q}],$$

and the cellular boundary is the deletion–contraction differential $d + c$.

(ii) For every $N \geq 1$, the rank-$\leq N$ subspace carries a canonically determined concordance class of PL bundles

$$S^{N-1} \longrightarrow \mathcal{E}^{\mathrm{UOM}}_N \longrightarrow \mathscr{B}^{\mathrm{UOM}}_{\leq N}.$$

Over the cell labelled by a rank-r oriented matroid M, a fiberwise ball-complex model is

$$\mathscr{S}_N(M) = \mathrm{sd}\, P(M) * S^0_1 * \cdots * S^0_{N-r},$$

where $P(M)$ is the poset of nonzero covectors.

(iii) The construction is natural for labelled deletion and contraction and is stable under suspension:

$$\mathcal{E}^{\mathrm{UOM}}_{N+1}\big|_{\mathscr{B}^{\mathrm{UOM}}_{\leq N}} \simeq \Sigma_{\mathrm{fib}} \mathcal{E}^{\mathrm{UOM}}_N.$$

After fiberwise coning the associated PL microbundles satisfy

$$\xi^{\mathrm{UOM}}_{N+1} \simeq \xi^{\mathrm{UOM}}_N \oplus \varepsilon^1.$$

Status: proved/formal. This theorem is quoted from [54]. Its essential technical point is that repeated contraction orders are related by canonical permutohedral diagrams of simultaneous orthant completions; these higher coherences are retained by the labelled face-occurrence category.

The theorem supplies a stable classifying map

$$\bar{\kappa} : \mathscr{B}^{\mathrm{UOM}} \longrightarrow \mathrm{BPL}$$

and hence geometrically defined rational classes

$$p_j^{\mathrm{UOM}} := \bar{\kappa}^* p_j^{\mathrm{PL}} \in H^{4j}(\mathscr{B}^{\mathrm{UOM}}; \mathbb{Q}). \tag{71}$$

Since $\mathrm{BO} \to \mathrm{BPL}$ is a rational equivalence, these restrict on the realizable locus to the usual rational Pontryagin classes. In cellular degree $4j$, the relevant UOM cells have

$$|E| = 4j + 1.$$

Thus the five-label and nine-label calculations are not arbitrary finite truncations: they are the first cellular degrees in which p_1 and p_2 can be evaluated.

Warning 69.2 (Existence versus explicit formula). *Theorem 69.1 establishes the target classes* (71)*. It does not identify any previously guessed sign, Orchard, Hodge, or logarithmic cocycle with them. Every such identification is a separate comparison theorem.*

70 Fixed-rank Thom theory and the cone–suspension identification

Let S be an oriented regular cellular $(r-1)$-sphere and CS its cone, with apex a. There is a canonical suspension identification of relative chains

$$C_j(CS, S) \cong \widetilde{C}_{j-1}(S). \tag{72}$$

Indeed, for a cell $\sigma \subset S$,

$$\partial(c\sigma) = \sigma - c(\partial\sigma),$$

so modulo S the relative boundary is the suspended augmented boundary. With the standard suspension sign, (72) is an isomorphism of chain complexes.

Equip corresponding cells with compatible inner products. Then the cellular Laplacians, Green operators, and Hodge contractions correspond under suspension. Consequently any Hodge twisting-cochain construction on the augmented sphere complex can be transported verbatim to the relative cone pair.

Status: proved/formal.

Let

$$U \in H^r(CS, S; \mathcal{O})$$

be the relative Thom class, with coefficients in the orientation local system when necessary. If s denotes the zero section, the Euler class is

$$e = s^* U.$$

At cohomology level the Thom identity is

$$U^2 = \pi^* e\, U. \tag{73}$$

Thus $\Phi^{-1}(U^2) = e$ under the Thom isomorphism.

70.1 The total descended Thom cocycle

For a chain of aggregations, write the Hodge-descended Thom cocycle as

$$\mathbb{U} = U_0 + U_1 + \cdots + U_r, \qquad \deg_h U_j = j, \quad \deg_v U_j = r - j.$$

The descent equations are

$$dU_0 = 0,$$

$$\delta U_j + dU_{j+1} = 0 \qquad (0 \le j < r),$$

and

$$\delta U_r = 0,$$

with the expected totalization signs. In the minimal Hodge gauge one has schematically

$$U_{j+1} = -h\delta U_j$$

whenever the relevant harmonic obstruction vanishes.

The top horizontal component U_r is a scalar on the cone apex. Under the suspension identification (72), the assertion that U_r is the local Euler cocycle e_{CH} is a comparison of two descriptions of the same Hodge twisting cochain. It is exact in the rank-two calculation below and is the natural general fixed-rank identification to verify for arbitrary spheres.

Status: heuristic/programmatic. The cone–suspension equivalence is formal. The equality with every sign convention in the existing e_{CH} manuscript should be checked once in the final merged notation before being promoted to a quoted theorem.

71 Critical rank: Euler square, Thom cube, and the correct Pontryagin seed

Let $r = 2k$. For an oriented real 2k-plane bundle,

$$p_k = e_{2k}^2. \tag{74}$$

For a possibly nonorientable bundle let X_{2k} be the twisted Euler class. Massey proves in the universal case that

$$X_{2k}^2 = p_k \in H^{4k}(BO(2k); \mathbb{Z}). \tag{75}$$

This is an integral statement, not merely a rational identity [56].

There is, however, an important degree distinction on the Thom space. Since

$$\Phi(x) = \pi^* x \, U,$$

the Thom representative of p_k is

$$\Phi(p_k) = p_k U = e^2 U = U^3. \tag{76}$$

Hence

$$\boxed{\Phi^{-1}(U^3) = p_k.} \tag{77}$$

The older language in which U^2 itself was called the Pontryagin seed was therefore one Thom degree too low. The square produces the Euler class; the cube produces the critical Pontryagin class.

Status: proved/formal.

71.1 A chain-level Thom transgression

Let

$$\mathbb{E} = \pi^* s^* \mathbb{U}$$

be the total Euler cocycle pulled back to the Thom complex. Choose the canonical Hodge primitive

$$\mathsf{T} = D^{*}G\big(\mathbb{U}^2 - \mathbb{E}\mathbb{U}\big), \qquad D\mathsf{T} = \mathbb{U}^2 - \mathbb{E}\mathbb{U}, \tag{78}$$

after removing the zero harmonic component. Since r is even,

$$\boxed{\mathbb{U}^3 - \mathbb{E}^2\mathbb{U} = D\big((\mathbb{U} + \mathbb{E})\mathsf{T}\big).} \tag{79}$$

More generally,

$$\mathbb{U}^{m+1} - \mathbb{E}^m\mathbb{U} = D\left[\mathsf{T}\sum_{j=0}^{m-1}\mathbb{U}^{m-1-j}\mathbb{E}^j\right].$$

Equation (79) is the finite combinatorial analogue of a Chern–Simons transgression between two Thom-space representatives of the same class. **Status: proved/formal.**

72 Contraction as a Thom residue

Let M be a rank-r oriented matroid. In the Folkman–Lawrence pseudosphere model its covector sphere is

$$S(M) = |\mathcal{L}(M) \setminus \{0\}| \simeq S^{r-1}.$$

For a nonloop e,

$$\mathcal{L}(M/e) = \{X|_{E-e} : X \in \mathcal{L}(M),\ X_e = 0\}.$$

Thus $S(M/e)$ is the coordinate pseudosphere

$$S_e = \{X : X_e = 0\} \subset S(M).$$

For a subset I of an independent set,

$$S(M/I) = \bigcap_{i \in I} S_i. \tag{80}$$

If M is uniform and $|I| < r$, this intersection has the expected codimension and is a sphere $S^{r-|I|-1}$.

Coorient S_e by declaring $X_e = +$ to be the positive side. Normal-first orientation gives a relative Gysin/residue map

$$\rho_e : H^r(D(M), S(M); \mathcal{O}_M) \longrightarrow H^{r-1}(D(M/e), S(M/e); \mathcal{O}_{M/e})$$

with

$$\rho_e(U_M) = U_{M/e}. \tag{81}$$

For $e \neq f$, swapping the two ordered normals changes orientation, hence

$$\rho_f \rho_e = -\rho_e \rho_f. \tag{82}$$

Deletion of e and residue along f commute geometrically when $e \neq f$; after incidence signs this becomes the bicomplex relation

$$d\rho + \rho d = 0.$$

Status: proved/formal. The geometric statements follow directly from the pseudosphere realization and the standard normal-first orientation convention. The exact sign comparison with every earlier computer convention is a separate bookkeeping audit.

73 The universal minor complex is a Bier sphere

Fix an ordered n-element ground set E and a rank bound r. Let

$$K_{r,n} = \{C \subset E : |C| \leq r\}.$$

Its Alexander dual is

$$K^{\circ}_{r,n} = \{D \subset E : |D| \leq n - r - 1\}.$$

The Bier sphere is the deleted join

$$\mathrm{Bier}(K_{r,n}) = K_{r,n} *_{\Delta} K^{\circ}_{r,n}.$$

Its faces are exactly pairs of disjoint subsets

$$(C, D), \qquad C \cap D = \varnothing, \qquad |C| \leq r, \quad |D| \leq n - r - 1. \tag{83}$$

Interpreting C as contracted labels and D as deleted labels identifies the scalar Thom/residue incidence complex with the cellular complex of this Bier sphere. Bier's theorem gives

$$\mathrm{Bier}(K_{r,n}) \simeq_{PL} S^{n-2}$$

[57].

Status: proved/formal.

A top facet has

$$|C| = r, \qquad |D| = n - r - 1,$$

and one unused label m:

$$E = C \sqcup D \sqcup \{m\}. \tag{84}$$

Thus the occurrence mark is not auxiliary decoration. It is the unique unused vertex of a top Bier simplex.

73.1 Generic hypersimplex realization of the Bier sphere

Let

$$P_{\lambda,n} = \left\{ x \in [0,1]^n : \sum_i x_i = \lambda \right\}, \qquad r < \lambda < r+1.$$

Because λ is nonintegral, $P_{\lambda,n}$ is a simple $(n-1)$-polytope. Its facets are $x_i = 0$ and $x_i = 1$. An intersection

$$x_i = 1 \ (i \in C), \qquad x_j = 0 \ (j \in D)$$

is nonempty exactly under the conditions (83). Hence the facet nerve is the Bier sphere and

$$\boxed{\partial(P^*_{\lambda,n}) \cong \mathrm{Bier}(K_{r,n}).} \tag{85}$$

At a vertex, exactly one coordinate is fractional:

$$x_C = 1, \qquad x_D = 0, \qquad x_m = \lambda - r.$$

As $\lambda \downarrow r$, these vertices collapse to the rank-r hypersimplex vertex 1_C; as $\lambda \uparrow r+1$, they collapse to $1_{C \cup \{m\}}$. Thus the occurrence mark m is literally the coordinate which changes rank across the integer hypersimplex wall.

Status: proved/formal.

74 Koszul reduction and the universal 1/N propagator

Fix a deletion set D and let $A = E \setminus D$ be the active labels, $|A| = N$. On the one-dimensional Thom-line sector identify a q-fold contraction

$I = \{i_1 < \cdots < i_q\}$ with

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_q} \in \Lambda^q Q^A.$$

After compatible orientation trivializations the contraction differential is

$$d_c = \nu \wedge -, \qquad \nu = \sum_{i \in A} e_i. \tag{86}$$

For the standard metric,

$$d_c^* = \iota_\nu$$

and the Clifford identity yields

$$d_c d_c^* + d_c^* d_c = N\,\mathrm{id} \tag{87}$$

away from the terminal truncation. Hence the canonical Hodge contraction is

$$\boxed{h_c = \frac{1}{N}\iota_\nu.} \tag{88}$$

Status: proved/formal.

This explains the $1/9$ factor in the marked rank-three formula from the old p_2 investigation. After marking k in a ten-label object, there are exactly nine active labels $i \neq k$, and the canonical Koszul Hodge inverse is $\frac{1}{9}\iota_\nu$.

74.1 Exact sign identity for the marked rank-three primitive

For a rank-three chirotope on $E = [10]$ define

$$\nu_i(M, k) = \prod_{j \neq i,k} \chi_M(i, k, j).$$

Let $p_k(i)$ be the one-based position of i in $E \setminus \{k\}$. Then a direct split of the product defining $\varepsilon_2(M/i)$ gives

$$\boxed{(-1)^{p_k(i)}\nu_i(M,k)\varepsilon_2((M \setminus k)/i) = (-1)^{k-1}(-1)^{i-1}\varepsilon_2(M/i).} \tag{89}$$

Consequently the marked primitive

$$\widetilde{Q}_3(M; k) = -\frac{1}{9 \cdot 512}\sum_{i \neq k}(-1)^{p_k(i)}\nu_i(M,k)\varepsilon_2((M \setminus k)/i)$$

may be rewritten as

$$\widetilde{Q}_3(M;k) = -\frac{(-1)^{k-1}}{9\cdot 512}\sum_{i\neq k}(-1)^{i-1}\varepsilon_2(M/i).$$

Summing the deletion coboundary over k counts every contraction face exactly nine times and cancels the $1/9$. Thus the marked descent equation is a universal incidence identity. The $1/9$ is simultaneously the multiplicity correction and the Koszul Green factor (88).

Status: proved/formal. The identity (89) is a direct parity calculation once the contraction convention is fixed. It does not establish that $\varepsilon_2/512$ is the geometric p_2 seed; that normalization belongs to the compressed logarithmic gauge discussed below.

75 The Bier–Pieri resolution after contraction

The contraction complex is truncated at degree r. Its terminal cohomology at a deletion set D is

$$H^r_c(D) \cong \Lambda^r\left(Q^{E\setminus D}/\langle 1\rangle\right). \tag{90}$$

Therefore the contraction-first reduced complex is

$$R^p_{r,n} = \bigoplus_{|D|=p}\Lambda^r\left(Q^{E\setminus D}/\langle 1\rangle\right), \qquad 0\le p\le n-r-1, \tag{91}$$

with deletion induced by coordinate-forgetting maps.

Let $W_m = Q^m/\langle 1\rangle$, the standard representation of S_m. Since

$$\Lambda^r W_m \cong S^{(m-r,1^r)},$$

one has

$$R^p_{r,n} \cong \mathrm{Ind}^{S_n}_{S_{n-p}\times S_p}\left(S^{(n-p-r,1^r)}\boxtimes \mathrm{sgn}_p\right). \tag{92}$$

The vertical-strip Pieri rule is multiplicity-free. Every non-top irreducible is of the form

$$\lambda_{L,b} = (L,2^b,1^{n-L-2b}), \tag{93}$$

with

$$2\le L\le n-r, \qquad 0\le b\le \min(r,n-r-L).$$

It occurs in exactly two consecutive deletion degrees

$$p_0 = n - r - L, \qquad p_0 + 1.$$

The only unpaired representation is $S^{(1^n)} = \mathrm{sgn}_n$ in the final degree. Since the total Bier complex is a sphere, the repeated blocks are isomorphically paired. Hence, over $\mathbb{Q}$,

$$\boxed{R^{\bullet}_{r,n} \simeq \mathrm{sgn}_n[-(n-r-1)] \oplus \bigoplus_{L,b} \left[S^{\lambda_{L,b}} \xrightarrow{\sim} S^{\lambda_{L,b}} \right].} \tag{94}$$

Status: proved/formal. The decomposition follows from (92), the Pieri rule, and the known sphere homology of the Bier complex. It is independent of oriented-matroid topology.

75.1 Closed Hodge eigenvalues

With the quotient Euclidean metric on W_m, d^*d is scalar on every multiplicity-one Pieri block. A cross-transposition Casimir computation gives

$$\boxed{\alpha_{L,b} = \frac{(L+b-1)(n-b)}{r+L-1}.} \tag{95}$$

Thus the Green operator on that block is

$$\boxed{G_{L,b} = \frac{r+L-1}{(L+b-1)(n-b)}\,\mathrm{id}.} \tag{96}$$

It is zero on the surviving sign line.

Status: proved/formal. Formula (95) is formal for the stated invariant metric; it reduces to the content eigenvalue of the cross-transposition Casimir. A machine-readable independent verification should be included with the next computational supplement.

For the critical p_2 band $(r,n) = (4,9)$ the reduced dimensions are

$$70 \longrightarrow 315 \longrightarrow 540 \longrightarrow 420 \longrightarrow 126,$$

and the ten Green factors are

$$\frac{2}{9}, \frac{7}{27}, \frac{7}{32}, \frac{1}{3}, \frac{1}{4}, \frac{3}{14}, \frac{5}{9}, \frac{5}{16}, \frac{5}{21}, \frac{5}{24}.$$

Thus the universal incidence calculation is ten scalar propagators plus one final sign representation. No large universal matrix inversion is necessary.

76 Balanced critical bands and the geometry of duality

For p_k the critical rank and cellular size are

$$r = 2k, \qquad n = 4k + 1 = 2r + 1. \tag{97}$$

A top Bier facet is therefore

$$E = C \sqcup D \sqcup \{m\}, \qquad |C| = |D| = 2k.$$

This is the geometric origin of the old middle bidegree $C_{2k,2k+1}$.

On the generic slice

$$P_\theta = \left\{ x \in [0,1]^{2r+1} : \sum_i x_i = r + \theta \right\}, \qquad 0 < \theta < 1,$$

matroid duality is represented by

$$x \longmapsto 1 - x.$$

It sends

$$(C, D; m, \theta) \longmapsto (D, C; m, 1 - \theta).$$

At $\theta = 1/2$ the generic hypersimplex is centrally symmetric and the occurrence mark is fixed. Thus the central deletion/contraction equation is geometrically the compatibility of the two sides of the rank-changing hypersimplex cobordism.

Status: proved/formal.

This reflection controls duality parity but not, by itself, Hopf primitivity. Beginning in degree twelve, duality-odd decomposable classes such as ℓ_1^3 coexist with the primitive logarithmic class ℓ_3. Primitivity must therefore be extracted by the Hopf Eulerian idempotent, not by midpoint antisymmetrization alone.

77 Bier residues as a combinatorial inverse Thom map

Let $\tau = (C, D; m)$ be a top Bier facet with ordered contraction set C. Write

$$\rho_C = \rho_{c_r} \cdots \rho_{c_1}$$

for the iterated residue and let d_D denote restriction along the deletion face. Define

$$\mathrm{Res}_\tau(\alpha) = \epsilon(\tau)\, \rho_C(d_D \alpha),$$

where $\epsilon(\tau)$ is the Bier orientation sign. Pairing with the oriented Bier fundamental cycle gives a normalized operator

$$\mathcal{I}_{r,n}(\alpha) = \frac{1}{N_{r,n}} \sum_{\tau \text{ top facet}} \mathrm{Res}_\tau(\alpha), \qquad N_{r,n} = n\binom{n-1}{r}. \tag{98}$$

The boundary of the Bier fundamental cycle is zero. Internal differentials commute with residues with the usual degree sign. Therefore the codimension-one incidence terms cancel in pairs and one obtains the combinatorial Stokes formula

$$\boxed{D\mathcal{I}_{r,n} = \mathcal{I}_{r,n} D.} \tag{99}$$

Moreover, on a Thom-form cochain,

$$\boxed{\mathcal{I}_{r,n}(\mathbb{U}\,\pi^* x) = x.} \tag{100}$$

Thus the Bier residue operator realizes the inverse Thom isomorphism on the Thom sector.

Status: heuristic/programmatic. Equation (99) follows formally once the residue maps are installed with the precise occurrence-category orientation signs. A complete sign-level proof should be imported into the companion PL-bundle paper or supplied as a dedicated lemma before (100) is quoted without qualification.

Combining (79) and (100) yields the canonical comparison

$$\boxed{P_k^{\mathrm{TB}} := \mathcal{I}_{2k,4k+1}(\mathbb{U}_{2k}^3) = \mathbb{E}_{2k}^2 + DC_k,} \tag{101}$$

where

$$C_k = \mathcal{I}_{2k,4k+1}\big((\mathbb{U}_{2k} + \mathbb{E}_{2k})T_{2k}\big). \tag{102}$$

This is the chain-level form of the Thom–Bier program: the direct inverse-Thom representative and the Euler-square representative differ by a canonical Hodge Chern–Simons coboundary.

Conjecture 77.1 (Thom–Bier Pontryagin comparison). *The cocycle P_k^{TB} of* (101)*, constructed on the UOM occurrence model with the exact PL-bundle residue maps, represents the geometrically defined class p_k^{UOM} of* (71)*.*

Status: conjectural. On the realizable vector-bundle locus the cohomological identity is ordinary Thom theory. The new content is the equality with the canonical PL class on the full UOM space and the global occurrence-category sign/coherence statement.

78 Only one diagonal of the Thom cube contributes

The vertical degree of a monomial $U_i U_j U_\ell$ is

$$3r - (i + j + \ell).$$

Inverse Thom removes exactly r units of vertical degree. Hence only

$$\boxed{i + j + \ell = 2r} \tag{103}$$

can contribute to the degree-$2r = 4k$ base class. Therefore

$$P_k^{TB} = \mathcal{I}_r\left(\sum_{i+j+\ell=2r} U_i U_j U_\ell\right). \tag{104}$$

Equivalently, if $a = r - i$, $b = r - j$, $c = r - \ell$ are vertical degrees, then $a + b + c = r$. The Pontryagin formula is a sum over ways of distributing the r normal directions among three Thom legs.

Status: proved/formal.

For p_2, $r = 4$, and only four unordered monomial types occur:

$$(0, 4, 4), \quad (1, 3, 4), \quad (2, 2, 4), \quad (2, 3, 3). \tag{105}$$

In a strictly graded-commutative transferred model this reads

$$\Theta_{p_2} = 3U_0 U_4^2 + 6U_1 U_3 U_4 + 3U_2^2 U_4 + 3U_2 U_3^2, \qquad P_2^{TB} = \mathcal{I}_4(\Theta_{p_2}). \tag{106}$$

For an Alexander–Whitney product the ordered terms should be retained until the relevant $\smile_i$ homotopies are inserted.

79 Rank two: the circle, sawtooth propagator, and the degree-four dictionary

Let a cellular circle have vertices $v_0, \ldots, v_{m-1}$ and oriented edges $e_i = [v_i, v_{i+1}]$. The relative cone complex has

$$C_2(CS, S) = Q^m, \qquad C_1(CS, S) = Q^m, \qquad C_0(CS, S) = Q,$$

with
$$\partial(ce_i) = cv_i - cv_{i+1}, \qquad \partial(cv_i) = -a.$$
The normalized harmonic Thom cochain is
$$U_0(ce_i) = \frac{1}{m}. \tag{107}$$
For a mean-zero top cochain $z = (z_i)$ put
$$S_j = \sum_{a=0}^{j-1} z_a, \qquad \bar{S} = \frac{1}{m}\sum_j S_j.$$
Then the minimal Hodge primitive is
$$(h_2 z)_j = \bar{S} - S_j. \tag{108}$$
This is the centered discrete sawtooth kernel.
Status: proved/formal.

The Thom descendants are
$$U_1 = -h_2 \delta U_0, \qquad U_2 = E_H = e_{CH}$$
with the final equality subject only to the global totalization convention. The rank-two Thom relation simplifies strongly: if
$$R = \mathbb{U}^2 - E_H \mathbb{U},$$
then the only nonzero Hodge secondary term is
$$T = h_2(U_1^2). \tag{109}$$
Consequently the degree-three comparison cochain is
$$\boxed{CS_3^{CH} = \mathcal{I}_2\big(U_1 h_2(U_1^2)\big),} \tag{110}$$
and
$$\boxed{P_1^{TB} = E_H^2 + DCS_3^{CH}.} \tag{111}$$
The correction is a cubic expression in the centered-sawtooth connection. Status: proved/formal. within the transferred commutative product model. For a strict Alexander–Whitney model the equality contains the standard product-transfer homotopies.

79.1 Five-label normalization and the factor 48

The classical five-label sign formula on the critical rank-two/rank-three cells has coefficient 1/48:

$$\omega_{p_1} = \frac{1}{48}\varepsilon_r.$$

This agrees with the MacPherson/Gabrielov–Gelfand–Losik normalization already recorded in the earlier p_1 section. Hence the correct critical rank-two UOM seed for p_1 is

$$\boxed{\varepsilon_2/48.} \tag{112}$$

The earlier temptation to identify $a/24$ directly with the geometric p_1 should be discarded.

On the cyclic side, Igusa constructs explicit cocycles for powers of the Euler class on the category of finite cyclically ordered sets [55]. The precise numerical comparison depends on whether one uses the oriented circle Euler class or the projectivized/double-cover normalization. In the normalization used in the current rank-two sign computation, the projective Euler class is twice the signed Euler class, and its square is therefore four times p_1. Thus any comparison with an Igusa degree-four cocycle must include this factor of four before identifying its five-point sign normalization.

Status: proved/formal. for the double-cover scaling; the exact pointwise normalization of the chosen Igusa cocycle should be quoted only after matching his conventions to those used here.

The conceptual degree-four dictionary is therefore

$$\frac{\varepsilon_2}{48} \quad \leftrightsquigarrow \quad \text{cyclic Euler-square representative} \quad \leftrightsquigarrow \quad P_1^{\mathrm{TB}},$$

with differences given by explicit degree-three product/Hodge homotopies such as (110).

80 The new p_2 program

The geometric class p_2^{UOM} already exists by Theorem 69.1. The explicit problem is to compare it with the Thom–Bier cocycle. The foundational steps are now:

(1) construct the rank-four total Thom cocycle

$$\mathbb{U}_4 = U_0 + U_1 + U_2 + U_3 + U_4$$

for the combinatorial S^3 aggregation system;

(2) identify U_4 with the rank-four local Euler cocycle $e^{(3)}_{\mathrm{CH}}$ in the exact cone/suspension convention;

(3) form the four-term selected Thom cube (106);

(4) apply four contraction residues and the Bier inverse-Thom projection on the balanced $(4, 9)$ occurrence sphere;

(5) compare the resulting cocycle with p_2^{UOM} and with E_4^2 via the explicit transgression (101).

The canonical balanced top facet may be taken to be

$$C_0 = \{1, 2, 3, 4\}, \qquad D_0 = \{5, 6, 7, 8\}, \qquad m = 9.$$

Its stabilizer is $S_4 \times S_4$. Frobenius reciprocity identifies the global S_9 sign channel with the local

$$\mathrm{sgn}_4 \boxtimes \mathrm{sgn}_4$$

channel at this one facet. Thus a global alternating projection can be replaced by six adjacent-transposition conditions locally. Together with the ten universal Green factors of Section 75, this removes almost all universal linear algebra from the p_2 calculation.

Warning 80.1 (Role of the old 512/1024 formulas). *The rank-two sign class $\varepsilon_2/512$, the marked rank-three primitive with coefficient $1/(9 \cdot 512)$, and the corresponding $1/1024$ logarithmic normalization belong to a compressed representative of the primitive/Newton degree-eight class. They are no longer taken as the definition or initial geometric seed of p_2. Their proper place is a later gauge comparison between*

$$q_2 = p_1^2 - 2p_2$$

and a compact sign representative.

The old raw central residual, including the reported value 16 in one realizable rank-five example, therefore has no direct obstruction meaning until it has been projected to the tautological alternating coefficient channel and all universal Bier-propagator corrections have been included.

81 Primitive classes, duality, and Brown's 2-primary corrections

Rationally write
$$P(t) = 1 + p_1 t + p_2 t^2 + \cdots$$
and
$$\log P(t) = \sum_{k\geq 1} (-1)^{k-1} \frac{q_k}{k} t^k.$$
The Newton classes satisfy
$$q_1 = p_1, \qquad q_2 = p_1^2 - 2p_2, \qquad q_3 = p_1^3 - 3p_1p_2 + 3p_3,$$
and are primitive for Whitney sum. Complement duality sends $q_k \mapsto -q_k$ rationally. The Hopf Eulerian idempotent, equivalently the convolution logarithm of the total Pontryagin series, is the correct general primitive projector.

Status: proved/formal.

At the integral level strict primitivity acquires 2-primary corrections. Brown's integral Whitney-sum formula contains odd Bockstein classes. In low degree, writing $b = \beta w_1$,
$$\Delta p_1 = p_1 \otimes 1 + 1 \otimes p_1 + b \otimes b,$$
so
$$\Delta q_2 = q_2 \otimes 1 + 1 \otimes q_2 + b^2 \otimes b^2.$$
On the oriented cover $b = 0$, and the rational/odd-primary primitive picture is recovered. Thus an integral UOM theory should separate:

(i) the free/rational Thom–Bier Pontryagin cocycle;

(ii) mod-2 Steenrod data from $Sq^i U = w_i U$;

(iii) Bockstein/Brown corrections measuring failure of strict integral primitivity.

This fits naturally with the existing cellular $\smile_i$ program: modulo two,
$$Sq^i[U] = [U \smile_{r-i} U],$$
so the same combinatorial Thom object controls both the free Pontryagin seed and the Stiefel–Whitney/Bockstein sector.

82 Status hierarchy after the geometric revision

It is useful to separate the current program into four levels.

Level 1. **Geometric existence.** The hypersimplicial PL bundle and the stable rational classes $\mathrm{p}_j^{\mathrm{UOM}}$ exist by Theorem 69.1.

Level 2. **Universal incidence algebra.** The residue anticommutation, Bier-sphere model, generic-hypersimplex interpretation, truncated Koszul reduction, and Bier–Pieri representation theory are independent of the detailed oriented-matroid coefficient system.

Level 3. **Fixed-rank Thom comparison.** The cone/suspension model and Thom-cube identity are formal; the exact identification of the total combinatorial Thom descendant with the existing e_{CH} formulas, and the occurrence-level Stokes signs for $\mathcal{I}$, must be audited in the final common convention.

Level 4. **Pontryagin identification and compression.** Conjecture 77.1 identifies the explicit Thom–Bier cocycle with the already existing PL Pontryagin class. Only after this should one compare with compressed sign representatives such as $\varepsilon_2/48$ for p_1 and the 512/1024 primitive degree-eight gauges.

This hierarchy should replace the older logic in which a low-rank sign seed was first guessed and a global characteristic interpretation inferred from successful descent equations.

83 Immediate computations suggested by the revision

The following tests now have a clear order of priority.

1. **Rank-two Thom/cyclic normalization.** Implement the circle Thom descendants using the centered sawtooth kernel, evaluate $\mathrm{P}_1^{\mathrm{TB}}$ on the five-label occurrence complex, and compare directly with the established $\varepsilon_2/48$ MacPherson/GGL cocycle.

2. **Cone–Euler convention audit.** Check, for the exact total differential used in the local Euler paper, that the top Thom descendant

equals e_{CH} in ranks two and four without an unnoticed global sign.

3. **Residue Stokes certificate.** Generate the oriented top Bier facets and verify (99) over Z for small (r,n), including $(2,5)$ and $(4,9)$. This is a finite convention check, not evidence for a conjectural identity.

4. **Bier–Pieri spectrum certificate.** Construct $R^{\bullet}_{r,n}$ for small (r,n) and verify the decomposition (94) and eigenvalues (95) exactly over Q.

5. **Rank-four Thom square/cube.** Compute $U_0,\dots,U_4$, form the four monomial types (105), and project only to the local $\mathrm{sgn}_4 \boxtimes \mathrm{sgn}_4$ channel at the canonical $(4,4;1)$ facet.

6. **Only afterwards: compressed primitive comparison.** Form

$$q_2 = (P_1^{\mathrm{TB}})^2 - 2P_2^{\mathrm{TB}}$$

and compare its rank-two restriction with the compact $\varepsilon_2/512$ representative. The marked $1/9$ primitive then becomes a gauge-transformation calculation rather than the foundation of p_2.

84 Revised master picture

The current architecture can be summarized as

$$\begin{gathered}
\text{UOM deletion–contraction cells } \downarrow \\
\text{hypersimplicial CW realization and stable PL bundle } \downarrow \\
\text{relative combinatorial Thom cocycle } \mathbb{U}_{2k} \ \downarrow \\
\text{Euler descendant } \mathbb{E}_{2k} = s^*\mathbb{U}_{2k} \ \downarrow \\
\mathbb{U}_{2k}^3 \simeq \mathbb{E}_{2k}^2\mathbb{U}_{2k} \ \downarrow \text{ Bier residues / inverse Thom} \\
P_k^{\mathrm{TB}} \simeq \mathbb{E}_{2k}^2 \ \downarrow \text{ comparison} \\
p_k^{\mathrm{UOM}} \in H^{4k}(\mathscr{B}^{\mathrm{UOM}};Q) \ \downarrow \text{ log /Eulerian idempotent} \\
q_k,\ ph_k \text{ and primitive gauges } \downarrow \\
\text{integral/mod-2/Bockstein refinements.}
\end{gathered}$$

The universal deletion–contraction incidence is no longer the difficult part: it is the Bier sphere and its explicit Koszul–Pieri Hodge contraction. The genuinely subtle information lies in the variation of the cellular Thom complex and its off-harmonic modes over the UOM coefficient system. In HPL language, after the Thom cohomology line is trivialized, all nontrivial effective curvature comes from excursions

$$H \longrightarrow A \xrightarrow{h} A \longrightarrow H$$

through the acyclic complement. This is the discrete superconnection/Chern–Weil mechanism behind the local Euler cocycle and, through the Thom cube, behind the proposed Pontryagin formulas.

Part IX

Status, conjectures, and research agenda

85 Partial theorems summarized

The following statements are established with the conventions and qualifications specified in their respective sections. Unlike earlier revisions, the existence of the stable rational UOM Pontryagin classes is now geometric: it follows from the hypersimplicial PL-bundle theorem. What is not yet established is the equality of the proposed explicit Thom–Bier/sign formulas with those classes.

1. The hypersimplicial UOM incidence object has a CW realization carrying a canonical stable PL sphere/microbundle, natural for labelled deletion and contraction and compatible with stabilization. Consequently there are geometrically defined stable rational classes p_j^{UOM}.

2. The universal scalar deletion–contraction minor complex at fixed (r, n) is the Bier sphere $\mathrm{Bier}(K_{r,n})$, equivalently the facet nerve of the generic nonintegral hypersimplex slice. Contraction on the Thom line is a truncated Koszul differential with canonical propagator $|A|^{-1}\iota_1$.

3. After contraction-first Hodge reduction, the universal deletion complex is the Bier–Pieri resolution (94): all irreducible blocks are paired in two consecutive degrees except the terminal sign representation. With the invariant quotient metric the Green factors are given by (96).

4. The uniform OM bi-Grassmannian admits the exact hypersimplex dictionary
$$\Delta^{p,q} \leftrightarrow \mathrm{UOM}(q+1, p+q+2),$$
with $x_i = 0$ as deletion and $x_i = 1$ as contraction for the represented positive-dimensional faces.

5. Realizable generic Grassmannian configurations map functorially to the OM bi-Grassmannian. The map intertwines deletion and contraction with the two classical face families.

6. Compatible OM labels on an oriented hypersimplicial decomposition define a chain map to the positive-dimensional OM bi-Grassmannian; internal faces cancel with the rank-dependent orientation convention.

7. The complete-product factorization relates deletion and contraction signs and explains all normal-star terms.

8. The rank-two shadow
$$h_M(K) = \varepsilon_2(M/K)$$
is reorientation-invariant and has a Boolean two-step-boundary linearization.

9. The universal parity checks on h are the $(r-4)$-link relations; in the p_2 band they are K_{11} edge-parity equations.

10. With the complementary-orientation twist $\widetilde{h}_M$ defined in Section 28,
$$J_M = \frac{1}{2}\partial\widetilde{h}_M$$
is an integer-valued conserved auxiliary current. Its canonical Hodge primitive on the augmented complete simplex is $(r+7)^{-1}\partial^* J_M$. Comparing that primitive with a p_2 descent cochain remains open.

11. A uniform oriented matroid is exactly recoverable from its coherently oriented rank-two contraction atlas with rank-one overlap conditions.

12. In each nine-point rank-two chart, the p_2 shadow bit is the parity of the projective circular order.

86 Conjectures and hopes

86.1 Shadow-surjectivity in rank four

Conjecture 86.1 (Rank-four p_2 shadow surjectivity). *Every sign function*

$$h : \binom{[11]}{2} \to \{\pm 1\}$$

satisfying

$$\prod_{i<j} h_{ij} = 1$$

is the rank-two shadow of some $M \in \mathrm{UOM}(4, 11)$.

Equivalently: every even edge-coloring of K_{11} admits a compatible atlas of nine-point signed cyclic orders with the prescribed local permutation parities.

Status: conjectural. The $\mathrm{UOM}(3, 6)$ computation shows that such a statement is false in general sizes. The conjecture is specific to the p_2 diagonal.

86.2 A characteristic-class quotient of the OM bi-Grassmannian

The full rank-two atlas contains much more information than p_2 appears to use. One hopes to construct a canonical quotient

$$\mathfrak{B}^{\mathrm{OM}} \longrightarrow \mathfrak{B}^{\mathrm{shadow}}_{p_2}$$

whose objects are shadows/currents and whose face maps are induced by the complete-product factorization. The local p_2 cocycle should factor through this quotient.

A successful quotient would separate two problems:

1. oriented-matroid integrability of the sign atlas;

2. characteristic-class descent on the resulting shadow complex.

86.3 Harmonic normalization and uniqueness

The current J_M has a canonical Hodge-minimal primitive on the complete simplex. This suggests a normalization principle for symmetry-invariant local characteristic cocycles: among cohomologous flagged primitives, choose the one orthogonal to cycles/minimal in the natural ℓ^2 norm.

Conjecture 86.2 (Harmonic normalization principle). *After passing to the occurrence/groupoid model carrying the hypersimplicial PL bundle, the preferred explicit representative of a rational characteristic class is characterized by a discrete Hodge condition together with its prescribed critical-rank Thom datum. For p_k the datum is the Thom cube $\mathbb{U}_{2k}^3$, equivalently the Euler square after inverse Thom.*

This formulation replaces the older requirement of a prescribed rank-two sign seed. Rank-two compressed representatives remain useful gauges for primitive classes, but they are secondary to the geometric Thom datum.

86.4 Higher Pontryagin classes

The v5 pattern is uniform in k. At critical rank $2k$ construct the total Thom cocycle

$$\mathbb{U}_{2k} = \sum_{j=0}^{2k} U_j,$$

form the selected vertical-degree-$2k$ component of $\mathbb{U}_{2k}^3$, and apply the balanced Bier inverse-Thom operator on $4k+1$ labels. The resulting candidate

$$P_k^{\mathrm{TB}} = \mathcal{I}_{2k,4k+1}(\mathbb{U}_{2k}^3)$$

should be compared with the already existing geometric class p_k^{UOM}. Only afterwards should one pass to the Hopf logarithm/Newton classes q_k and search for compressed sign formulas.

The combinatorics of the universal incidence part is already uniform: the critical Bier sphere has top facets

$$E = C \sqcup D \sqcup \{m\}, \qquad |C| = |D| = 2k,$$

and duality exchanges C and D. The new difficulty in higher k is therefore the fixed-rank Thom/Euler Hodge system and its matroid-dependent off-harmonic coefficient variation, not the universal deletion–contraction incidence.

86.5 Modular homology and Steenrod-type operations

The appearance of

$$\partial_{[2]}^2 = 0 \quad \text{over } \mathbb{F}_2$$

and the mixed relation between the integer current and higher Boolean inclusion transforms suggest a connection with divided-power or Steenrod-type structures. One should determine whether the relevant modular homology operations can be interpreted directly on the OM bi-Grassmannian and whether they interact with characteristic classes.

Status: heuristic/programmatic. At present no Steenrod operation on the OM bi-Grassmannian has been constructed.

86.6 Relation with Λ_{inj} and higher cyclic structures

Rank-two charts are nine-point circular orders; deleting labels stays in the circular category. Higher-rank objects are glued from such charts through contraction overlap data. This suggests that the OM bi-Grassmannian may be viewed as a "higher cyclic" object whose local one-dimensional pieces lie in Λ_{inj} but whose gluing records higher oriented-matroid incidence.

A concrete goal is to define a category or semi-Segal object whose objects are these cyclic atlases and whose nerve maps naturally to the OM bi-Grassmannian. It should avoid weak maps at the first stage; degenerations can be added later as a compactification if needed.

86.7 Motivic shadow

The classical pipeline is schematically

$$\text{decorated flags} \longrightarrow \text{bi-Grassmannian} \longrightarrow \text{Bloch/motivic complex} \\ \longrightarrow \text{regulator/characteristic class.}$$

The OM pipeline presently reads

$$\text{OM Goncharov flags} \longrightarrow \text{OM bi-Grassmannian} \longrightarrow \text{Orchard/shadow currents} \\ \longrightarrow \text{combinatorial characteristic cocycle.}$$

It is tempting to call the middle-right object a *motivic shadow*, but this terminology should remain heuristic until an actual universal property or comparison with motivic complexes is found.

86.8 A characteristic-one metaphor

Oriented matroids have long invited the metaphor of "linear geometry over a field of characteristic one". The OM bi-Grassmannian gives this metaphor a precise testing ground: it is obtained from the algebraic bi-Grassmannian by retaining the sign combinatorics of generic real Plücker coordinates and the exact deletion/projection incidence pattern.

However, the object is not currently a Grassmannian over an actual $\mathbb{F}_1$ theory. The useful mathematical question is not whether the slogan is literally true, but which algebraic structures on the classical bi-Grassmannian survive at the oriented-matroid level and which require more than sign data.

87 Corrected statements and pitfalls

For later reference we collect points where the present program corrected earlier tempting simplifications.

1. The unmarked rank-three Orchard formula

$$1 + \sum (-1)^i b_i$$

   was reported to fail the bare deletion–contraction equation. The counterexample and its incidence conventions must be supplied before this reported failure can be used as a proved result.

2. The horizontal deletion complex in the resolved flag model is a truncated Koszul complex; its homology is not generally one-dimensional.

3. Failure of two-element amalgamation despite local no-collision was reported, but the witness is not supplied. In all cases the complete Grassmann–Plücker integrability conditions must be checked.

4. The ternary transition vector z is sufficient for the scalar pairwise p_2 transition but is not stable under arbitrary parent changes. The full shadow or the pair of Hessian rows is the stable coefficient object.

5. The OM bi-Grassmannian should not be identified with a MacPhersonian, a flag matroid space, or a weak-map compactification.

6. The motivic comparison is structural. No claim is made that the OM complex computes motivic cohomology or algebraic K-theory.

7. The extension-space theorem must be read with the hypotheses and the extension poset of Sturmfels–Ziegler. In the rank-three setting covered there, the homotopy type is S^2; this is not intrinsically an RP^2-object. An RP^2 description requires an additional sign/reorientation quotient. Rank four is the first place where disconnected nonrealizable extension spaces occur.

8. For $M(K_4)$ one has $\beta(M) = 2$, but $\dim \overline{OS}^2(M) = 6$. The beta invariant must not be confused with the top reduced Orlik–Solomon dimension.

9. A previously used pair-row expression $H_P(i, j)$ had unstable sign conventions and has been withdrawn. The revised p_2 discussion uses formal Hodge-transfer operators; any proposed marked primitive still requires its descent and gauge checks.

10. The ordinary logarithm of the total Pontryagin series and the convolution logarithm in the matroid-minor Hopf algebra are different constructions. The first extracts Whitney-additive Newton/Pontryagin-character primitives; the second extracts an indecomposable minor cumulant. Their compatibility is a conjectural comparison, not a formal identity.

11. A harmonic class at the self-dual central stage $C_{2k+1,2k+1}$ is not automatically the even class ph_k. A comparison with a regular graphic complex may detect an adjacent wheel class. If its pairing is an obstruction to descent, it must vanish in the full problem; omitting it requires a justified quotient construction.

12. The primitive hyperbolic summand in §61 follows from the projection formula with the standard integral degree normalization. Its comparison with the reported UOM lattice remains conjectural; an internal duality eigenspace also requires a specified self-duality involution.

13. The hypersimplicial PL-bundle theorem removes the former ambiguity about whether a rational family-level Pontryagin class exists on the UOM incidence space. It does. The open problem is to identify concrete local representatives with its pullback.

14. The Thom square and Thom cube have different roles. $\Phi^{-1}(U^2) = e$, whereas $\Phi^{-1}(U^3) = e^2 = p_k$ at the critical rank $2k$. Any formulation calling U^2 itself the Thom-space p_k seed has a degree error.

15. Midpoint deletion/contraction duality detects duality parity but not Hopf primitivity in degrees ≥ 12. The logarithmic/Newton primitive must be extracted by the Hopf Eulerian idempotent (or an equivalent logarithm), not merely by antisymmetrization.

16. The factors $1/9$, 24, 512, and 1024 have different meanings. In particular $1/9$ is a universal Koszul/Bier Hodge factor in the marked ten-label calculation, while $512/1024$ belongs to a compressed primitive degree-eight gauge. None of these constants defines the geometric p_2 class.

88 Research agenda

The most concrete next problems appear to be the following.

1. **Complete the occurrence-level Thom/residue sign theorem.** Import the permutohedral coherence conventions from the hypersimplicial PL-bundle paper and prove the chain-level Stokes identity $D\mathcal{I} = \mathcal{I}D$ with exactly the determinant-line signs used by the Hodge complexes.

2. **Finish the five-label Thom–Bier regression test.** Compute the circle total Thom cocycle with the centered sawtooth kernel and verify explicitly that P_1^{TB} is cohomologous to the classical $\varepsilon_2/48$ GGL/MacPherson cocycle, including the product-transfer Chern–Simons term.

3. **Complete the Thom–Bier p_2 comparison.** Construct the rank-four total Thom cocycle, evaluate the selected component of $\mathbb{U}_4^3$ by the balanced $(4,9)$ Bier residue, and compare it with the geometrically defined PL class. Only afterwards compare the resulting primitive $q_2 = p_1^2 - 2p_2$ with the compressed $512/1024$ sign gauge.

4. **Test shadow surjectivity for $\mathrm{UOM}(4,11)$.** Use the signed-permutation/cyclic-atlas CSP rather than mutation graphs.

The first target is either one infeasible even edge-coloring or a structural proof that all even colorings lift.

5. **Characterize fibers of the shadow map.** Determine how much oriented-matroid information can vary while h, J, and the resulting local p_2 value remain fixed.

6. **Build the OM Goncharov flag complex computationally.** Since every fixed diagonal is finite, one can compute low-degree homology, compare the realizable and full subcomplexes, and test whether nonrealizable cells change characteristic cohomology.

7. **Compare to the classical real bi-Grassmannian.** The sign map from generic real configurations should induce a chain map. It is natural to ask what homology is lost by collapsing each realization chamber to its oriented matroid.

8. **Explore higher cyclic structure.** Start from the rank-two circular atlas and formulate the higher-rank compatibility as a Segal-type gluing condition. The goal is a geometric category before any weak-map completion.

9. **Investigate higher classes by Thom cubes.** For each k, construct the rank-$2k$ Thom descendants, keep only the diagonal $i+j+\ell = 4k$ in $\mathbb{U}^3_{2k}$, and compare the balanced Bier inverse-Thom cocycle on $4k+1$ labels with the geometric p_k^{UOM}. Gabrielov's hypersimplicial chains should then be used as a comparison, not as a guessed normalization.

10. **Boundedness.** Study whether the Hodge-current formulation produces uniform estimates for flagged representatives, and identify precisely where unboundedness can enter when descending to an unflagged local cocycle.

11. $k = 1$ **BEST–UOM comparison.** Compute the descent of $\mathrm{ch}_2(\mathcal{S}_M)$ through the five-/six-element rank–nullity coefficient system and compare it cell by cell with the established p_1^{UOM} cocycle. This is the smallest decisive test of

$$\mathrm{Desc}_{\mathrm{UOM}}(\mathrm{ch}_2\, \mathcal{S}) = p_1^{\mathrm{UOM}}.$$

12. $K_4 = W_3$ **conormal normalization.** Use the primitive hyperbolic plane spanned by $\gamma^2, \bar{\gamma}^2$, choose the K_4 self-duality explicitly, and

compare its eigensublattices with a reproducibly computed $(h, \tau h)$ lattice.

13. **Rational p_2 primitive after the geometric comparison.** Once P_2^{TB} has been identified with p_2^{UOM}, form $q_2 = (P_1^{TB})^2 - 2P_2^{TB}$ and then compare its compressed rank-two/three HPL gauge with $\varepsilon_2/512$, the marked $1/9$ primitive, and $ch_4(\mathcal{S}_M) = ph_2$.

14. **W_5/Borel filter.** Evaluate the fully corrected ten-element residual on the regular connected W_5 direction. A nonzero value is an obstruction when the cycle pairing is valid. Identifying a Brown/Borel component does not remove it; that interpretation requires its own comparison map.

15. **Oriented KO refinement.** Search for a real/oriented refinement of the BEST tautological K-class whose complexification recovers $\mathcal{S}_M$. Compare the resulting 2-primary obstruction with the determinant-line flags, the index-two lattice, Stiefel–Whitney classes, and Bockstein corrections.

16. **Higher logarithmic test.** At the prospective p_3 stage verify that raw deletion/contraction antisymmetrization contains decomposable lower terms. Then check that the cochain logarithm, including the required $\smile_1$ and eventually $\smile_2$ homotopies, isolates the primitive ph_3 component.

89 A compact dictionary

Classical/hypersimplicial	Oriented-matroid counterpart	Comment
$G_{p+q+2}(q+1)$	$\mathrm{UOM}(q+1, p+q+2)$	generic Grassmannian point $\mapsto$ chirotope
$\Delta^{p,q}$	same bidegree (p, q)	common incidence polytope
facet $x_i = 0$	deletion $M \setminus i$	forget vector
facet $x_i = 1$	contraction M/i	quotient/project
generic decorated flags	OM Goncharov flags	compatible cell labeling
internal-face cancellation	minor cancellation	produces chain map

Classical/hypersimplicial	Oriented-matroid counterpart	Comment
classical bi-Grassmannian	OM bi-Grassmannian	varieties versus finite sign objects
MacPhersonian	not the same object	weak maps versus minors
rank-two configuration	signed cyclic order	hyperline sequence
rank-two projectivization	circular order	Λ_{inj}-type datum
$\varepsilon_2(M/K)$	shadow bit $h_M(K)$	parity of 9-point circular order in p_2 band
$\partial\widetilde{h}/2$	rank-three current J	orientation-corrected current
$\partial^* J/n$	Hodge primitive	canonical minimal-norm primitive
HPL transfer $\pi c(hc)^{j-1}$	higher deletion–contraction transgression	intrinsic spectral-sequence class, Hodge representative
Newton power sum q_k	logarithmic primitive	$q_k = (2k)!\ \mathrm{ph}_k/2$
BEST $\mathcal{S}_M, \mathcal{Q}_M$	tautological matroid coefficient system	ordinary-matroid rational characteristic data
rank–nullity ring $R^*(M)$	symmetry-invariant (r, ν) compression	uniform case equals full S_n-invariant permutohedral Chow ring
conormal primal/dual factors	deletion/contraction local model	candidate source of the index-two duality lattice
odd wheel W_{2k+1}	central $C_{2k+1,2k+1}$ regulator direction	Brown/Borel sector, one degree above the even Pontryagin class
hypersimplicial PL bundle	stable UOM PL microbundle	geometric source of p_j^{UOM}
relative Thom class U	cone/suspended covector complex	$U^2 = eU$, $U^3 = p_k U$ at rank $2k$
Bier sphere $K *_\Delta K^\circ$	universal minor-incidence sphere	deletion/contraction colors; mark is the unused label

Classical/hypersimplicial	Oriented-matroid counterpart	Comment
Bier–Pieri resolution	universal incidence Hodge reduction	two-term Specht blocks plus terminal sgn_n
Thom–Bier cocycle	$\mathcal{I}(\mathbb{U}^3)$	candidate explicit representative of geometric p_k^{UOM}
Bloch/motivic target	no established analogue	open problem

90 Concluding perspective

The geometric status of the UOM bi-Grassmannian has changed substantially in this revision. It is no longer only a finite combinatorial analogue of the classical bi-Grassmannian: the companion hypersimplicial theorem realizes its deletion–contraction incidence as a CW space carrying a canonical stable PL sphere bundle. Stable rational UOM Pontryagin classes therefore exist before any explicit sign formula is chosen.

The central explicit problem is now one of comparison. At fixed even rank, the covector sphere and its cone provide a combinatorial Thom pair, and the Hodge construction of the local Euler cocycle becomes Thom descent. The correct Thom-space representative of the critical class is $U^3 = p_k U$. Contraction acts as residue along coordinate pseudospheres, while the complete universal minor incidence is the Bier sphere of a uniform simplex skeleton. Its generic-hypersimplex model explains simultaneously deletion, contraction, the occurrence mark, and the central duality reflection.

The universal incidence algebra is unexpectedly rigid: contraction is a truncated Koszul complex, and the subsequent deletion complex is a Bier–Pieri resolution of the sign representation with explicit Green factors. Consequently the universal part of the former large p_2 matrices can be integrated out analytically. What remains is the genuinely oriented-matroid-dependent variation of the Thom/Hodge fiber complex, especially its excursions through off-harmonic modes.

This produces a new hierarchy for the characteristic-class program. First construct the Thom–Bier representative $P_k^{\mathrm{TB}} = \mathcal{I}(\mathbb{U}^3_{2k})$ and compare it with the geometrically defined p_k^{UOM}. Next pass to the Hopf logarithm to obtain primitive Newton/Pontryagin-character classes. Only at that stage should compact sign representatives, the 512/1024 gauges, conormal lattices, BEST classes, and possible motivic or Borel shadows be compared. The older sign computations remain valuable, but they are now compression and comparison data rather than the source of existence of the characteristic classes.

The smallest decisive checks are correspondingly concrete: the five-label circle Thom–Bier calculation should reproduce the classical $\varepsilon_2/48$ class with an explicit sawtooth Chern–Simons coboundary; the nine-label rank-four calculation should evaluate only the four degree-allowed monomial types in $\mathbb{U}_4^3$ and only in the local $\mathrm{sgn}_4 \boxtimes \mathrm{sgn}_4$ channel. Success in these two cases would turn a large part of the previous experimental $\mathrm{p}_1/\mathrm{p}_2$ program into a geometric Hodge–Thom theory with a clear stable target.

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A Reproducible checks supplied with this revision

The accompanying verification scripts use exact integer and rational arithmetic. The supplied output records:

- all 2^{19} normalized candidate rank-three chirotopes on six labels tested against the uniform three-term axioms, yielding 11904 oriented matroids and 52 distinct six-bit shadows;
- exhaustive rank-two counts 24, 192, and 1920 on four, five, and six labels, and total chain-group sizes 40, 416, and 15808;
- zero defects in the numerator cocycle equation $D^*a = 0$ on every rank on six labels (rank-three enumeration, rank-two enumeration and its dual, and the vanishing endpoint cases);
- exact binary elimination confirming every entry of the displayed shadow-code table and the ranks of its parity-check maps;
- the alternating rank-three, ten-label counterexample to the untwisted current comparison, together with its corrected value;
- the formal paired-removal identities in 45 rank/size cases and the geometric incidence signs in 42 exact determinant checks.

These checks do not verify the reported Smith normal forms, the asserted 26-generator integral cycle, the two-extension nonamalgamation witness, or the proposed global p_2 normalization. Those claims still require their own matrices, witnesses, or derivations. Finite checks of a formula supplement, and do not replace, the general proofs given in the text.

B Conventions fixed here and remaining normalization checks

The incidence conventions used in this revision are:

1. $[n] = \{1, \ldots, n\}$, with zero-based positions in alternating sums and inherited increasing orders after removal.
2. Contracted labels are placed first, in increasing order, in chirotope representatives.

3. The two raw alternating minor operators anticommute; $D = d + c$ has no additional row sign.

4. Hypersimplex orientations are $(-1)^{r-1}$ times the positive-normal-first orientation. The bi-Grassmannian chain complex omits vertices.

5. The ordinary augmented-simplex boundary acts on the orientation-twisted shadow $\widetilde{h}$, as specified in the current section.

The coefficient $-1/1024$ is retained only for the historical compressed logarithmic/primitive gauge; it is not the geometric definition of p_2. The current normalization audit starts from the rank-four Thom cube and the geometrically defined PL class. The factors $1/9$, 512, and 1024 must be tracked separately as, respectively, universal Hodge incidence and compressed-lattice normalizations.

C Suggested computational representation

For explicit computation on the p_2 diagonal, the following representation appears efficient.

For each rank-two chart R_K on nine labels:

1. choose an anchor;

2. store the eight lift-sign bits σ_K;

3. store the eight-letter permutation π_K;

4. constrain $\mathrm{sgn}(\pi_K)$ to equal the desired shadow bit $h(K)$;

5. express rank-one overlap equations linearly in lift-sign and pair-comparison bits;

6. enforce that the comparison bits of each chart form a transitive tournament.

For $r = 4, n = 11$ the charts are indexed by the 55 edges of K_{11} and the overlap constraints live on its 165 triangles. This formulation avoids mutation graphs and extension-space connectivity entirely.

D Working glossary

OM bi-Grassmannian The bigraded family $\mathrm{UOM}(q+1, p+q+2)$ with deletion and contraction arrows.

Fixed diagonal $\mathfrak{B}_m^{\mathrm{OM}} = \bigsqcup_r \mathrm{UOM}(r, m+1)$.

OM Goncharov flag A compatible uniform oriented-matroid labeling of all hypersimplices in a hypersimplicial simplex decomposition.

Resolved flag A parent OM together with active, contraction-tail, and deletion-tail labels.

Rank-two shadow $h_M(K) = \varepsilon_2(M/K)$ for $|K| = r-2$.

Rank-three current $J_M = \frac{1}{2}\partial\widetilde{h}_M$, with the complementary-orientation twist of Section 28.

Orchard–Hessian matrix The radius-one star $H_P(s, x) = h_P(S-s+x)$ around a normal vertex S.

Two-normal diamond The compatible triple of data (A, B, C) describing two simultaneous normal extensions over a rank-four old minor.

Integrability defect The cubic sign function Ω detecting failure of the proposed localization on rank-two triples.

Cyclic atlas The complete family of nine-point rank-two contractions R_K with rank-one overlap gluing.

Transferred differential $D_j = (-1)^{j-1}\pi c(hc)^{j-1}\iota$, the j-th HPL operation on the d-harmonic space.

Pontryagin-character primitive $\Phi_k = \mathrm{ph}_k = 2q_k/(2k)!$, with q_k the Newton power sum.

Conormal lattice The primal/dual middle lattice generated in the central case by γ^{2k} and $\bar{\gamma}^{2k}$.

Rank–nullity ring The Fife–Mannino–Rincón subring $R^*(M)$ generated by divisor sums of fixed subset rank and nullity.

BEST tautological classes The Berget–Eur–Spink–Tseng matroid K-classes $[\mathcal{S}_M], [\mathcal{Q}_M]$ and their Chern classes.

Odd wheel sector The adjacent central classes represented in regular matroid complexes by W_{2k+1} and related to Brown/Borel canonical forms.

Hypersimplicial PL bundle The stable PL sphere/microbundle over the hypersimplicial UOM realization constructed in the companion paper; it defines p_j^{UOM} geometrically.

Total Thom cocycle $\mathbb{U} = \sum_j U_j$, the Hodge-descended relative Thom class with horizontal degree j and vertical degree $r - j$.

Thom–Bier cocycle $P_k^{\mathrm{TB}} = \mathcal{I}_{2k,4k+1}(\mathbb{U}_{2k}^3)$, the proposed explicit inverse-Thom representative of p_k^{UOM}.

Bier minor sphere $\mathrm{Bier}(K_{r,n})$, whose two vertex colors encode contraction and deletion; equivalently the facet nerve of a generic nonintegral hypersimplex slice.

Bier–Pieri resolution The S_n-equivariant reduction of the universal minor complex to two-term Specht blocks plus one terminal sign representation.