Note on electric circuits and local combinatorial formula for Euler class of vector bundles

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0.1 Introduction

Every $\mathbb{R}^n$ linear vector bundle over a compact polyhedra B can be replaced by its fiberwise compactification - $(S^n, *)$- fiber bundle, i.e. S^n-bundle with a fixed section, and this bundle with a marked section a can be triangulated [3]. Thus we obtain a natural locally finite combinatorial structure on vector bundles and one can pose a question on local combinatorial formulas for characteristic classes of vector bundles in terms of such a combinatorial structure. This question means following.

Triangulation of a $(S^n, *)$- fiber bundle $(E, l) \xrightarrow{p} B$ is a simplicial map $(\mathfrak{E}, \mathfrak{l}) \xrightarrow{\mathfrak{p}} \mathfrak{B}$ of a pair of simplicial complexes $(\mathfrak{E}, \mathfrak{l})$ onto simplicial complex $\mathfrak{B}$ together with fiberwise homeomorphism $|\mathfrak{p}| \approx p$, where $|\mathfrak{p}|$ is a geometric realization of $\mathfrak{p}$. Let $\blacktriangle^k$ be the face complex of simplex Δ^k. We suppose that $\blacktriangle^k$ has a fixed orientation. Consider the set of combinatorial objects $\mathcal{C}(n, k)$. Elements of $\mathcal{C}(n, k)$ are all "elementary (n, k) bundles", i.e. simplicial maps $(\mathfrak{K}, \mathfrak{k}) \to \blacktriangle^k$ which geometric realization is a trivial $(S^n, *)$-fiber bundle on the oriented simplex Δ^k (see Fig. 0.2).

Let $\mathbb{A}$ be some ring and let c be some $\mathbb{A}$-valued k-dimensional characteristic class of $\mathbb{R}^n$ vector bundles. Let $\sigma \in \mathfrak{B}_k$ be an oriented k-simplex of $\mathfrak{B}$. Denote by $\mathfrak{p}_\sigma \in \mathcal{C}(n, k)$ the subbundle of $\mathfrak{p}$ over σ. Denote by $\overline{\sigma}$ defined by oriented simplex σ basis element in the cellular cochain complex of $\mathbb{A}$-valued cochains on $\mathfrak{B}$. Local combinatorial formula for c is a way to describe a "universal cocycle" – a function $\mathcal{C}(n, k) \xrightarrow{\xi_c} \mathbb{A}$ which changes sign under reorientation of base and such that the element $c(p) \in H^k(B, \mathbb{A})$ is always represented by cellular cochain $\Sigma_{\sigma \in \mathfrak{B}_k} \xi_c(\mathfrak{p}_\sigma)\overline{\sigma}$. This

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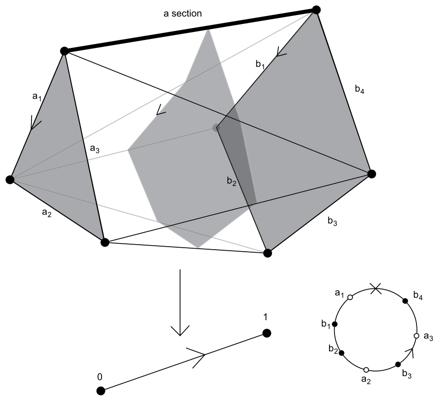

Figure 1: Elenentary bundle over 1-simplex

is a "universal cocycle" because a necessary condition for a function $\mathcal{C}(n,k) \xrightarrow{\varphi} \mathbb{A}$ to represent a k-dimensional characteristic class of vector bundles is a cocycle condition: for any simplicial bundle $(\mathfrak{L}, \mathfrak{l}) \xrightarrow{\mathfrak{t}} \blacktriangle^{k+1} \in \mathcal{C}(n, k+1)$ the function φ have to satisfy the following:

$$\sum_{i=0}^{k+1} \varphi(\mathfrak{t}_{\blacktriangle_i^{k+1}}) = 0 \tag{1}$$

where $\mathfrak{t}_{\blacktriangle_i^{k+1}}$ is a subbundle of $\mathfrak{t}$ over the i-th face of $\blacktriangle^{k+1}$ with induced orientation on the base.

So, the general problem is to describe combinatorial universal cocycles ξ_c.

There is a classical subject called "local formulas for characteristic classes of *combinatorial manifolds*" (see [1] for a survey). Here formulas are expressed in terms of combinatorics of stars. The structure of stars on a combinatorial manifold represents its tangent bundle. Both combinatorial structures - triangulation of a $(S^n, *)$-bundle and combinatorics of stars of a combinatorial manifold has canonically associated *combinatorial vector bundle* structures in a sense of [4, 5]. Local formulas for combinatorial vector bundles produces local formulas in both classical situations.

Our main observation is that the total space of cellular "prismatic fiber bundle" associated to oriented combinatorial vector bundle in the sense of [4, 5] miraculously come with granted canonical cellular Thom class. The Euler class of a vector bundle is an image of Thom class in the base under the canonical isomorphism induced by bundle section and projection. We apply a trick from elementary electric circuit physics to construct a universal cellular homology section and thus got a formula for the rational local combinatorial value for the universal Euler cochain. The result is easily computable in the terms of codifferentials and combinatorial Laplace operators of total complex of elementary bundle, but the combinatorial meaning of the formula is yet to be investigated. There are miracles. There is a natural local "Euler's formula" for the Euler class of tangent bundle of a combinatorial manifold. This formula results from trivial reassembling of the standard formula for Euler characteristic of the manifold.

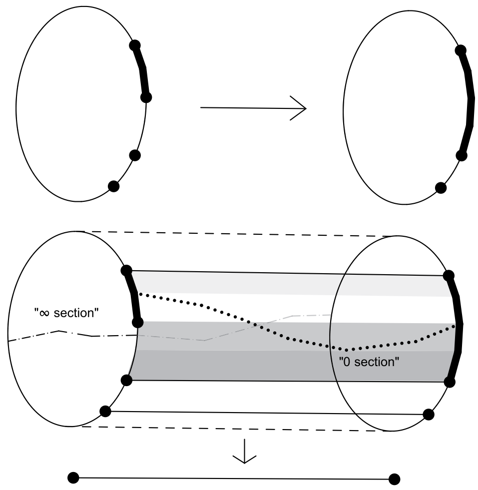

Figure 2: Assembly of ball complexes with marked balls and corresponding prizmatic bundle with marked prizms and virtual sections

Computational experiments shows that our formula in the case of tangent bundles produces just this "Euler's formula", which is not an obvious fact. Generally there is plenty different "local formulas" for one characteristic class. Yet we have no proof for this. Generally we are suspecting that our formula can be expressed as some function of "f-vector" of total complex.

0.2 Combinatorics of vector bundles

We will reproduce basic definitions and constructions related to "combinatorial fiber bundles" from [4]

0.2.1 Combinatorial assemblies of ball complexes

A PL *geometrical ball complex* structure Q on a compact Euclidean polyhedron X is a finite cover Q of X by closed PL balls such that the relative interiors of the balls from Q form a partition of X and the boundary of every ball from Q is the union of balls of smaller dimension. The main example of a PL ball complex is a finite geometrical simplicial complex.

Combinatorial assembly of PL geometrical ball complex structures on X is a map of finite sets balls $Q_1 \xrightarrow{\xi} Q_2$ such that there exist a PL homeomorphism $X \xrightarrow{F} X$ sending each ball from $q \in Q_1$ into the ball $\xi(q) \in Q_2$, i.e. such that $F(q) \subseteq \xi(q)$. Composition of combinatorial assemblies is a combinatorial assembly. With a polyhedron X is associated a small category $\mathfrak{R}(X)$ of combinatorial assemblies of PL ball complex structures on X. "Combinatorial assembly" is a combinatorial abstraction of "geometric subdivision" (the direction is inverted to obtain maps on sets of balls). We should mention that in $\mathfrak{R}(X)$ one can have nontrivial isomorphisms and there can be several different morphisms between two objects - it is possible that one can assemble one complex from another in combinatorially different ways. We should mention that as we will see, the natural constructions associates with simplicial manifolds and triangulated bundles essentially nonsimplicial diagrams

0.2.2 Oriented combinatorial vector bundle

Consider the category $\mathfrak{R}_n^+$ which objects are pairs (Q, q) of geometrical ball complexes Q on the oriented sphere S^n with a marked n-dimensional ball $q \in Q$. Morphisms of $\mathfrak{R}^+{}_n$ are those combinatorial assemblies from $\mathfrak{R}(S^n)$ which preserves orientations and sends the marked ball to the marked ball.

From main results of [4] it follows that the classifying space $B\mathfrak{R}_n^+$ of the category $\mathfrak{R}_n^+$ classifies oriented $\mathbb{R}^n$ vector bundles with a structure group - simplicial group $\mathrm{PL}^+(\mathbb{R}^n)$ of orientation-preserving PL-homeomorphisms of $\mathbb{R}^n$. Here we will not use directly this result, but it explains the naturality of the following definition.

Oriented combinatorial vector bundle on a simplicial complex $\mathfrak{K}$ is a decoration of vertices of $\mathfrak{K}$ by objects of $\mathfrak{R}_n^+$ and 1-dimensional simplices of $\mathfrak{K}$ by morphisms of $\mathfrak{R}_n^+$ – combinatorial assemblies of the complexes at the ends in such a way that all the resulting from 2-simplices of $\mathfrak{K}$ triangles are commutative. If 1-skeleton of $\mathfrak{K}$ we consider as a graph, then our bundle is a map of graphs $\mathfrak{K}_1 \xrightarrow{\mathfrak{Q}} \Gamma\mathfrak{R}_n^+$ (where $\Gamma\mathfrak{R}_n^+$ is a graph of the category) such that for any 2-simplex $\sigma \in \mathfrak{R}_2$ the restriction $\mathfrak{Q}|_{\partial\sigma\subset\mathfrak{K}_1}$ is a commutative triangle in $\mathfrak{R}_n^+$.

0.2.3 Prismatic fiber bundle associated with oriented combinatorial vector bundle

With n-dimensional combinatorial bundle $\mathfrak{K}_1 \xrightarrow{\mathfrak{Q}} \Gamma\mathfrak{R}_n^+$ on a simplicial complex $\mathfrak{K}$ we can associate in a canonical way *"Prismatic fiber bundle"*. Prismatic bundle is a pair $(e(\mathfrak{Q}), t(\mathfrak{Q}))$ where $e(\mathfrak{Q})$ is a cellular S^n fiber bundle $T(\mathfrak{Q}) \xrightarrow{e(\mathfrak{Q})} \mathfrak{K}$ and $t(Q) : \mathfrak{K} \to (T(Q))$ a map assigning to every simplex of $\mathfrak{K}$ a ball of maximal dimension in the total ball complex of the subbundle over σ.

Assume for simplicity that the diagram $\mathfrak{Q}$ is such that commutative triangles $\mathfrak{Q}|_{\partial\sigma\subset\mathfrak{K}_1}$ are acyclic for all $\sigma \in \mathfrak{R}_2$. We can always achieve it by inverting direction of some isomorphisms and the results of our constructions are invariant under this operation.

We will first build prismatic bundles for combinatorial vector bundles over simplices, and then we'll past from such a trivializations a construction of prismatic bundle for general combinatorial vector bundle.

Take the boundary complex of k-simplex $\blacktriangle^k$, take some acyclic decoration $\blacktriangle_1^k \xrightarrow{\mathfrak{P}} \Gamma\mathfrak{R}_n^+$. We'll describe elementary cellular bundle $T(\mathfrak{P}) \xrightarrow{e(\mathfrak{P})} \blacktriangle^k$. The total space $T(\mathfrak{P})$ is ball complex on the polyhedron $\Delta^k \times S^n$, $e(\mathfrak{P})$ is a coordinate projection $\Delta^k \times S^n \xrightarrow{\pi_1} \Delta^k$ which induces cellular morphism.

Acyclic decoration $\mathfrak{P}$ creates a total order $v_0, ..., v_k$ on the vertices of $\blacktriangle^k$: $i < j$ if the morphism $\mathfrak{P}([v_i, v_j])$ has $\mathfrak{P}(v_i)$ as a source $\mathfrak{P}(v_j)$ and as a target. Let a be a subset of $\{1, ..., k\}$. Denote by $\max(a)$ the maximal element of a. The polyhedron of the ball complex $T(\mathfrak{P})$ is $\Delta^k \times S^n$, $T(\mathfrak{P}) = \{b_{(a,s)}\}$ is a collection of embedded closed balls in $\Delta^k \times S^n$ which are numbered by pairs $(a \subseteq \{1, ..., k\}, s \in \mathfrak{P}(\max(a))$. The dimension of the ball $b_{(a,s)}$ is $\#a - 1 + \dim(s)$ (here "#" stands for "cardinality"); $b_{(a_1,s_1)} \subseteq b_{(a_2,s_2)}$ iff $a_1 \subseteq a_2$ and $\mathfrak{P}_{[\max a_1, \max a_2]}(s_1) = s_2$.

Now we will describe the embeddings. Choose some homeomorphisms $F_{i,i+1}$ representing $\mathfrak{P}([v_i, v_{i+1}])$. If $i \in \{1, ...k\}$ Then put $\mathfrak{P}(v_i) = (P_i, p_i), p_i \in P_i$. So, by definition for 1-simplex $[v_i, v_j]$ morphism $\mathfrak{P}_{[v_i,v_j]}$ is a combinatorial assembly morphism $P_i \to P_j$ such that $\mathfrak{P}_{[v_i,v_j]}(p_i) = p_j$ Define $F_{i,j}$ as a composit

$$S^n \xrightarrow{F_{i,i+1}} S^n \xrightarrow{F_{i+1,i+2}} \dots \xrightarrow{F_{j-1,j}} S^n$$

then $F_{[i,j]}$ represents $\mathfrak{P}([v_i, v_j])$ Denote by $\partial_a \Delta^k \subseteq \Delta^k$ the face of Δ^k with vertices $\{v_i\}_{i \in a}$. Now define an imbedded $(\#a - 1 + \dim(s))$-dimensional ball

$$b_{(a,s)} = \partial_a \Delta^k \times F_{[\max a,k]}(s) \subset \Delta^k \times S^n$$

Projection $\Delta^k \times S^n \xrightarrow{\pi_1} \Delta^k$ projects the ball $b_{(a,s)}$ onto the face $\partial_a \Delta^k$ of the base simplex, so the cellular bundle $e(\mathfrak{P})$ is defined. The remaining part of structure - the map $t(\mathfrak{P})$ assigns to the face $\partial_a \blacktriangle^k$ the ball $b_{\max(a),p_{\max(a)}}$.

Now for arbitrary acyclic decoration $\mathfrak{K}_1 \xrightarrow{\mathfrak{Q}} \Gamma\mathfrak{R}_n^+$ of a simplicial complex $\mathfrak{K}$ we can build as was described the individual trivial bundles ("trivializations") over the simplices and past them together into the cellular fiber bundle $T(\mathfrak{Q}) \xrightarrow{e(\mathfrak{Q})} \mathfrak{K}$ using fibred Alexander trick. The resulting bundle is defined by $\mathfrak{Q}$ uniquely up to isomorphism of the bundles. The "section" $t(\mathfrak{Q})$ is composed from local sections over simplices automatically since this sections by construction commutes with face embeddings in $\mathfrak{K}$. So prismatic bundle – the pair $(e(\mathfrak{Q}), t((Q))$ is defined. The balls $b_{a,s}$ looks like *"prisms"* and structure homomorphisms of prismatic bundle respects this prismatic cellular structure, this was a starting point for [4].

0.2.4 Vector bundle associated to combinatorial vector bundle

The cellular fiber bundle $e(\mathfrak{Q})$ has a canonical triangulation as $(S^n, *)$ fiber bundle, where the marked section is contained in the interior of the marked prism $t(\mathfrak{Q})(\sigma)$ for any simplex of $\mathfrak{K}$. This section is not included in $T(\mathfrak{Q})$ as a subcomplex, but it virtually exist in some canonical subdivision. This section we will consider as a zero section. Consider subcomplex $T_\infty(\mathfrak{Q}) \subset T(\mathfrak{Q})$ which is formed by all but all but marked prisms in $T(\mathfrak{Q})$, i.e. $T_\infty(\mathfrak{Q}) = T(\mathfrak{Q}) \setminus \cup_{\sigma \in \mathfrak{K}} t(\mathfrak{Q})(\sigma)$. The total polyhedron of $|T_\infty(\mathfrak{Q})|$ is a subpolyhedron of $|T(\mathfrak{Q})|$ which is a complementary to the union of the interiors of all marked prisms.

According to theory of Quiper and Lashof [3] the subcomplex $T_\infty(\mathfrak{Q})$ contains in its interior as a retract yet one more section of the bundle $e(\mathfrak{Q})$ (we will consider it as a virtual "∞"-section) of $e(\mathfrak{Q})$. So our bundle we can consider as fiberwise compactification by virtual ∞ section of an $\mathbb{R}^n$-bundle $e'(\mathfrak{Q})$ with a virtual zero section. The freedom in the choices of these two sections preserves isomorphism class of $e'(\mathfrak{Q})$.

0.2.5 Combinatorial vector bundle associated with triangulated $(S^n, *)$-bundle

here we present a canonical construction of cvb on first barycentric subdivision of the base of triangulated $(S^n, *)$-bundle. We use [2] for this. The construction is illustrated by Figure 3.

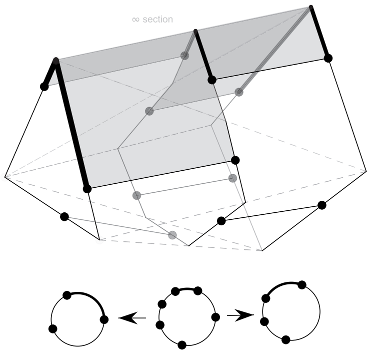

Figure 3: Transforming triangulated $(S^1, *)$-bundle from Fig. 1 into cvb

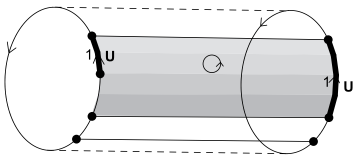

Figure 4: Thom class of prismatic bundle

0.2.6 Tangent combinatorial vector bundle of a combinatorial manifold

Here we reproduce [5]

0.3 Thom space and Thom class of combinatorial vector bundle

What follows from the discussion in §0.2.4: The CW complex $T(\mathfrak{Q})/T_\infty(\mathfrak{Q})$ is a cellular Thom space of the vector bundle $e'(\mathfrak{Q})$.

Let $\mathsf{T}_\bullet(\mathfrak{Q}) \xrightarrow{\mathsf{e}(\mathfrak{Q})} \mathsf{K}_\bullet$ and $\mathsf{T}^\bullet(\mathfrak{Q}) \xleftarrow{\mathsf{e}^*(\mathfrak{Q})} \mathsf{K}^\bullet$ be the morphisms of cellular chain and cochain complexes associated to $e(\mathfrak{Q})$. Choose a special n-dimensional cochain $U(\mathfrak{Q}) \in \mathsf{T}^n(\mathfrak{Q})$. Cochain $U(\mathfrak{Q})$ assigns value 1 to every marked n-ball $t(\mathfrak{Q})(v), v \in \mathfrak{K}_0$ with the orientation induced from the orientation of the bundle (see Fig. 4). Cochain $U(\mathfrak{Q})$ is a cellular n-cocycle in $\mathsf{T}^\bullet(\mathfrak{Q})$. One can see this because the only $n+1$-prisms of $T(\mathfrak{Q})$ which contains a marked ball $t(\mathfrak{Q})(v)$ are the $n+1$-dimensional marked prisms $t(\mathfrak{Q})([v,w])$ for all 1-dimensional simplices $[v,w] \in \mathfrak{K}_1$, containing vertex v. For any $[v,w] \in \mathfrak{K}_1$ the orientations of $t(\mathfrak{Q})(v)$ and $t(\mathfrak{Q})(w)$ induced from the orientation of the bundle induces opposite orientations of $t(\mathfrak{Q})([v,u])$. Hence $d^*(U) = 0$ in the cellular cochain complex $\mathsf{T}^\bullet(\mathfrak{Q})$. The map $c \mapsto \mathsf{e}^*(\mathfrak{Q})(c) \smile U$ induces Thom isomorphism

$H^\bullet(\mathsf{K}) \approx H^{\bullet+n}(\mathsf{T}(\mathfrak{Q}), \mathsf{T}_\infty(\mathfrak{Q}))$. This means that cochain U represents Thom class of the vector bundle $e'(\mathfrak{Q})$.

0.4 Electric homotopy operator

Here we consider classical electrostatic Kirchhoff problem of computation currents in a "linear" (i.e. only with resistors) electric circuit by a distribution of charges at the nodes together with its generalization to higher dimensional complexes of "conducting meshes". This is a subject of elementary combinatorial Hodge theory and it exists in literature in zillions reincarnations.

Consider finite dimensional acyclic chain complex Y over $\mathbb{Q}$, Y_k is a vector space k-chains, $Y_k \xrightarrow{d} Y_{k-1}$. Fix a nondegenerate scalar product on each Y_k and consider dual acyclic cochain complex $Y_{k-1} \xrightarrow{d^*} Y_k$. where the codifferential operator d_i^* is adjoint to d_i. We are in situation of combinatorial Hodge theory. Consider Laplace operator $\mathcal{L} = d^*d + dd^* : Y \to Y$. According to Hodge theory the kernel of $\mathcal{L}$ is equal to homology of Y. The complex Y is acyclic, hence the operator $\mathcal{L}$ is invertible. Consider Green operator $\mathcal{L}^{-1}$ and an operator $Y_k \xrightarrow{\mathcal{T}} Y_{k+1}$, $\mathcal{T} = d^*\mathcal{L}^{-1}$. The operator $\mathcal{T}$ satisfies identity for a homotopy operator

$$\mathrm{id} = d\mathcal{T} + \mathcal{T}d \tag{2}$$

an is a homotopy operator for the retraction $Y \to 0$ of Y on its homology - the zero subcomplex. Let $BC = \ker d = \operatorname{im} d$. The operator $\mathcal{T}$ has an extremal property: it sends $x \in BC$ to the unique $\tilde{y} \in Y$ such that $\|\tilde{y}\| = \min\{\|y\| \,|y : dy = x\}$, where $\|\ \|$ is a norm in the chosen scalar product.

For us the most useful property of the operator $\mathcal{T}$ is that it provides a canonical solution of the following extension problem. Suppose that Z is enother finite chain complex and we have a chain map $f_{<k} : Z_{<k} \to Y_{<k},\ k > 0$ of chain complexes

$$Z_{<k} = \bigoplus_{i<k} Z_i \to Y_{<k} = \bigoplus_{i<k} Y_i.$$

Our purpose is to extend this map to a chain map $f = f_{<k+1} : Z_{<k+1} \to Y_{<k+1}$, in a canonical way. Such extensions are always exist since Y is acyclic, and they form some linear space. But we want to make a definite canonical choice. This choice is: put

$$f_{k+1} = \mathcal{T} f_k d \tag{3}$$

The electric meaning of the operator $\mathcal{T}$ is following. One can consider a directed graph Γ as a linear electric circuit with resistance equal 1 on each arc. Consider a distribution of positive and negative charges on the nodes of graph such that the sum of charges is zero. Then the current flow on the arcs of graph induced by charges is $\mathcal{T}_\Gamma$(charges), where operator $\mathcal{T}_\Gamma$ is built by reduced cellular chain complex of the graph .

0.5 Local formula for Euler class of combinatorial vector bundles

The Euler class of a vector bundle $\mathbb{R}^n \to E \xrightarrow{\pi} B$ is a value of Thom class $s^*(U) \in H^n(B)$, where $U \in H^n(E)$ is a Thom class and $B \xrightarrow{s} E$ is any section of π. To obtain a cochain $\mathcal{E}(\mathfrak{Q}) \in \mathsf{K}^n$ representing the Euler class for our vector bundle $e'(\mathfrak{Q})$ form the combinatorial data of $e(\mathfrak{Q})$ it is sufficient to build a homology section $\mathsf{K}_\bullet \xrightarrow{\mathsf{s}} \mathsf{T}_\bullet(\mathfrak{Q})$ of the morphism of complexes $\mathsf{T}_\bullet(\mathfrak{Q}) \xrightarrow{\mathsf{e}(\mathfrak{Q})} \mathsf{K}$, $\mathsf{e}(\mathfrak{Q})\mathsf{s} = \mathrm{id}$, which is zero on the cells which has interiour in the interiour of $T_\infty(\mathfrak{Q})$. Then we can put $\mathcal{E}(\mathfrak{Q}) = \mathsf{s}^*(U(\mathfrak{Q})) \in \mathsf{K}^n$. By additivity of all construction

$$\mathcal{E}(\mathfrak{Q}) = \mathsf{s}^*(U(\mathfrak{Q})) = \sum_{\sigma \in \mathfrak{K}_n} \mathsf{s}^*|_\sigma(U(\mathfrak{Q}|_\sigma))\overline{\sigma} \tag{4}$$

So, to get a local formula for Euler class in the form (1) it is sufficient to get a universal local homology section $\mathsf{s}(\mathfrak{P})$ of the morphism $\mathsf{T}_\bullet(\mathfrak{P}) \xrightarrow{\mathsf{e}(\mathfrak{P})} \blacktriangle^k$ for a combinatorial vector bundle $\blacktriangle_1^k \xrightarrow{\mathfrak{P}} \Gamma\mathfrak{R}_n^+$ on a simplex. Without loss of generality we will suppose that $\mathfrak{P}$ is acyclic on $\blacktriangle^k$ and creates an order $v_0, ..., v_k$ on the vertices. This order creates distinguished orientation of $\blacktriangle^k$. Let $\delta_i(\mathfrak{P})$ denote restriction of $\mathfrak{P}$ on the i-th face (formed by vertices with numbers $1, .., i-1, i+1, ..., k$) of $\blacktriangle^k$. We have prismatic bundle $T(\mathfrak{P}) \xrightarrow{e(\mathfrak{P})} \blacktriangle^k$ and the section $\blacktriangle^k \xrightarrow{t(\mathfrak{P})} T(\mathfrak{P})$ sending the face of simplex to the marked prism. We have induced morphism of cellular chain complexes $\mathsf{T}_\bullet(\mathfrak{P}) \xrightarrow{\mathsf{e}(\mathfrak{P})} (\blacktriangle^k)_\bullet$ We have distingushed acyclic chain subcomplex $\mathsf{h}_\bullet(\mathfrak{P}) \hookrightarrow \mathsf{T}_\bullet(\mathfrak{P})$. The complex $\mathsf{h}_\bullet(\mathfrak{P})$ corresponds to the maximal marked prism $t(\mathfrak{Q})(\Delta^k)$. We denote by $\overline{\delta}_i$ natural embedding $\mathsf{T}_\bullet(\delta_i\mathfrak{P}) \xrightarrow{\overline{\delta}_i} \mathsf{T}_\bullet(\delta_i\mathfrak{P})$.

The last and the most important element for our canonical section is the electric homotopy operator $\mathsf{h}_\bullet(\mathfrak{P}) \xrightarrow{\mathcal{T}_\bullet(\mathfrak{P})} \mathsf{h}_{\bullet+1}(\mathfrak{P})$. In our situation complexes $\mathsf{h}_\bullet(\mathfrak{P})$ are acyclic and hence has invertible combinatorial Laplace operators. And so the Green operator is just the inverse Laplace operator. So, according to [Gosha] $\mathcal{T}_k(\mathfrak{P}) = d_{k-1}^*\mathcal{L}_{k-1}^{-1}$, where $\mathcal{L}_\bullet$ is combinatorial Laplace operator of $\mathsf{h}_\bullet(\mathfrak{P})$.

Now are able to write down a recursive expression for our section. If $k = 0$ then $\mathfrak{P}$ is just some object of $\mathfrak{R}_n^+$ so it is a ball on oriented S^n complex Q with marked ball q. $\mathsf{T}(\mathfrak{P})$ is a cellular chain complex of Q. Let V be the set of vertices of the marked ball q. Define

$$\mathsf{s}_0(\mathfrak{P}) = \sum_{v \in V} \frac{1}{\#V} v \in \mathsf{h}_0(\mathfrak{P}) \subset \mathsf{T}_0(\mathfrak{P})$$

Now define recursion by k

$$\mathsf{s}_k(\mathfrak{P}) = \mathcal{T}_{k-1}(\mathfrak{P})(\sum_{i=0}^{k}(-1)^k\overline{\delta}_i\mathsf{s}_{k-1}(\delta_i\mathfrak{P}))$$

If $k = n$ and $\blacktriangle^n \xrightarrow{\mathfrak{P}} \mathfrak{R}_n^+$ the local universal value for Euler class $\mathcal{E}(\mathfrak{P})$ on oriented by $\mathfrak{P}$ simplex $\blacktriangle^n$ is

$$\mathcal{E}(\mathfrak{P}) = \mathsf{s}_n^*(\mathfrak{P})(U(\mathfrak{P}))$$

which is equal to the sum of coefficients of $\mathsf{s}_n(\mathfrak{P})$ numbered by marked n-balls in fibers of $T(\mathfrak{P}) \xrightarrow{e(\mathfrak{P})} \blacktriangle^n$ over the vertices of $\blacktriangle^n$.

Some first properties of our formula for $\mathcal{E}(\mathfrak{P})$:

- $\mathcal{E}(\mathfrak{P}) = 0$ if the diagram $\mathfrak{R}$ contains combinatorial isomorphism;
- $\mathcal{E}(\mathfrak{P}) = -\mathcal{E}(-P)$ where $-P$ is combinatorially "mirror symmetric" to $\mathfrak{P}$ relatively to orientation;
- $\mathcal{E}(\mathfrak{P}) = 0$ if $\mathfrak{P}$ has a combinatorial mirror symmetry.

0.5.1 Example: an electrician way to compute Euler class of an oriented 1-bundle

Here just from symmetry of one dimensional electric circuit which show up we conclude that we will got always 0 in our formula and hence that Euler class of 1-dimensional oriented bundle is always 0, which is true.

0.5.2 Euler's cochain for Euler class of tangent bundle of a combinatorial manifold

Consider n-dimensional combinatorial manifold M^n and a simplex in its first barycentric subdivision. This simplex can be identified with a complete flag $s_0 \subset s_1 \subset ...s_n$ of simplices of manifold. Let $f(s_i)$ be the number of n-dimensional simplices of M^n containing s_i, Then put

$$\mathrm{E}(s_0, ..., s_n) = \sum_{k=0}^{n} (-1)^k \frac{1}{f(s_k)(k+1)!(n-k)!}$$

For combinatorial tangent bundle of combinatorial manifold it seems that we will get faimous "Euler's" local formula, which in dimension 2 looks like

$$\frac{1}{2f(s_0)} - \frac{1}{12}$$

Experiments with concrete small examples shows that our formula applied for the bundle from pp... produces numerically the same results.

0.6 Simplifications for spherical fiber bundles

Here we mention that for spherical bundles everything looks the same but there are simplifications. At the same time in PL category "vector bundles" and spherical bundles are not the same thing, and tangent bundle of combinatorial manifold is a "vector bundle" and generally may be not related to any spherical bundle. But rational classes on these different types of the input data are the same. So at this time we forced to treat them separately. But generally it looks that theory works for some Serre bundles, and this will unify the two kinds of bundles.

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