Note on local combinatorial formula for Euler class of spherical bundle as an invariant of chains of combinatorial subdivisions

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We sketch universal local combinatorial formulas for Euler class of an oriented PL fiber bundle with fiber S^n. They can be interpreted as certain Euler $n+1$ cocycles on the nerve $\mathcal{N}\mathfrak{S}^n$ of introduced in [Mnë07] category $\mathfrak{S}^n$ of n-dimensional oriented spherical regular cell complexes and combinatorial subdivisions. I.e. the formula is an invariant of $n+1$- chain of combinatorial subdivisions measuring certain twist of the chain together with a way to compute this invariant. All the local formulas for Euler cocycles can be described using N.A. Berikasvili predifferential - a chain-level disassembly of transgression differential in Serre spectral sequence. Fixing freedoms in the predifferential can be done using combinatorial Hodge theory, which produces combinatorially invariant rational formula expressed in terms of iterated Green operators of the cell complexes in the chain of subdivisions.

1. Euler class Consider oriented piecewise-linear oriented S^d spherical bundle $S^d \to E \xrightarrow{p} B$ on a finite PL polyhedron B. It has an integer Euler characteristic class. Topologically Euler class is an invariant of associated fibration, i.e. it survives reduction of bundle structure group to the group-like monoid of orientation preserving homotopy equivalences of d-dimensional sphere and lives on its classifying space [May75]. Euler class of oriented spherical Serre fibration $S^d \to E \xrightarrow{p} B$ is the integer cohomology class $e_p \in H^{d+1}(B)$, the cohomlogy avatar of boundary map ∂_p in exact homotopy sequence of the fibration

$$\cdots \to \pi_{d+1}E \to \pi_{d+1}B \xrightarrow{\partial_p} \pi_d(S^d) \to \cdots$$

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reflecting the twist of the bundle. One may clever apply Hurewicz homomorphism ([McC01, Hat01]) to get the base class out of fiber S^d orientation class.

For the Euler class local spherical triviality is irrelevant, dimensions of fibers can jump in the homotopy class. (The tangent spherical "bundle" or "combinatorial microbundle" of a combinatorial manifold is actually a Dold quasifibration [Mnë14]) The associated constant orientation sheaf is the principal player. The local spherical triviality only provides canonical fundamental classes of fibers which are easy to imagine and useful for local formulas.

2. Simplicial local combinatorial formula Piecewise linear category is defined by triangulations. Triangulation of a PL map is a triangulation of source and target such that the map is simplicial relatively to the triangulations. Combinatorics of any such a triangulation fixes the map up to isomorphism. An oriented spherical bundle on a finite polyhedron has countable number of triangulations related by common subdivisions. In a given triangulation over every simplex of the base sits elementary triangulated oriented spherical bundle over the simplex (this means that projection map sends each simplex of total space to a face of base simplex) definig combinatorial elementary oriented S^d bundle (see Figure 1). Suppose that the base

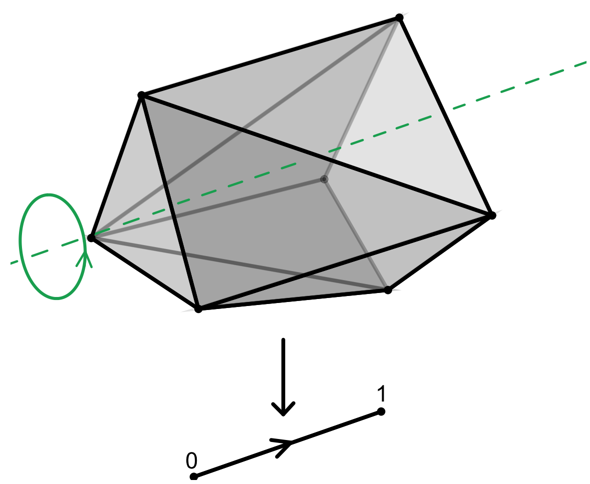

Figure 1: Elementary simplicial circle bundle

simplicial complex is locally ordered. Therefore it has a complex of rational ordered cochains, which algebraic cohomology compute singular cohomology of the base. We wish to find a universal combinatrorial rational function of the combinatorial isomorphism class of elementary triangulated oriented S^d-bundles over $d+1$ simplex such that its value being associated to the base simplex is a cochain reperesnting Euler class of the bundle.

Since Euler class in an integer characteristic class the rational formula should have integer periods, now in the combinatorial setting. I.e. its evaluation on integer $d+1$ simplicial cycles in the base are integer numbers depending on isomorphism class of the bundle and homology class of the cycle and independent of triangulation. Particularly, if we triangulate differential S^d bundle on differential closed $d+1$ base we should obtain the same Euler number of the bundle out of combinatorial and out of differential considerations. Therefore the arithmetics of the formula is nontrivial.

There are two situations where a simple local combinatorial formula is known - the tangent microbundle bundle of combinatorial manifold and triangulations of circle bundles. In the case of tangent bundle the answer is simple, can be obtained by resorting terms in standard Euler-Poincare expression for Euler characteristics [Gaĭ05].

In the case of circle bundle the answer for all powers of Euler class is surprisingly simple, has canonical look. It was obtaned in [Igu04] for Allain Connes' cyclic category combinatorics and by ignorance "rediscovered" in [MS17] for simplicial combinatorics. The formula jumps out of the fact that triangulated circle bundle has canonical associated piecewise-differential structure and associated to combinatorics piecewise-differential Kontsevich's cyclic invariant connection form. Then the computation of curvature integral by base simplex collapses to very simple formula using sum of minors - Pfaffian identities ([MS17]) Both tricks has no straightforward extension to the case of general triangulated oriented PL spherical bundles.

3. Category of oriented spherical ball complexes $\mathfrak{S}^n$ The object of our current interest is the categorty of spherical abstract ball complexes endowed by homology orientation with an assembly maps preserving orientation. We denote this category by $\mathfrak{S}^n$. To define it we recall few points from [Mnë07, Sec. 2]

Let X - compact PL manifold. Geometric ball complex B on X is a covering of X by collection of closed embedded PL balls B such that interiors of balls forms a partition of X and boundary of a ball is a union of balls. The notion is not more irrational than the classic notion of PL triangulation of manifold since order complex of the poset of the balls ordered by face inclusions is PL homeomorphic to X and defines a triangulation - "derived subdivision" of B ([Bjö84]). All the PL combinatorics irrationality sits the in fact that generally we don't know what is simplicial sphere. But we know its properties and special interesting classes. Geometric ball complex B_0 is a subdivision of of B_1 (or B_1 is assembly of B_0) if relative interior of any ball from B_0 is contained in relative interior of a ball from B_1. The specific of PL category is that any two geometric ball complexes on X has common subdivision (as triangulations has), therefore all the poset $\mathbf{R}(X)$ of ball complexes on X ordered by subdivisions is contractible. Taking apbstract poset of closed balls and inclusions $\mathbf{P}(B)$ we arrive to notion of "abstract ball complex" where k balls are represented by rank k principal ideals, Face inclusions - by inclusions of the ideals. The nice thing is that we have combinatorial Poincare duality – the maximal ideals of the opposite poset is abstract ball complex of Poincare dual to B. Geometric assemblies goes to maps of ball posets which we call combinatorial assemblies and opposite arrow - combinatorial subdivisions. They can be easily combinatorially characterised, but the point that they are representable by some geometric assembly, which is now not unique, but defined only up to homeomorphism modified by coherent system of isotopies. Thus we got a category $\mathfrak{R}(X)$ of abstract ball complexes of type X and abstract assemblies. The discrete group $\mathbf{PL}^{\delta}(X)$ acts on contratible poset of geometric ball complexes $\mathbf{R}(X)$ and $\mathfrak{R}(X)$ is the category of orbits of this highly non-free action. It was speculated [Mnë07, Theorem A] that this action can be improved up to free action of simplicial group $\mathbf{PL}(X)$ on contractible space and thus the nerve $\mathcal{N}\mathfrak{R}(X)$ of $\mathfrak{R}(X)$ has homotopy of $B\mathbf{PL}(X)$. This strong statement has only philosophical use here.

The category $\mathfrak{R}(X)$ is a place for parametrised combinatorics of complexes. For example stellar subdivisions one can consider as some (not all perhaps) generators of the category. In the case of sphere S^n the simplicial set $\mathcal{N}\mathfrak{R}(S^n)$ has interesting subsets. For example realisable convex subdivisions of convex polytopes one can not compose, but realisable those chains forms a simplicial subset. Deletion maps of pseudosphere arrangements forms very interesting subcategory of $\mathfrak{R}(S^n)$ Realizable chains of delitions of realizable arrengements forms a simplicial subset of $\mathcal{N}\mathfrak{R}(S^n)$.

Here we are interested in the category of spherical abstract ball complexes endowed by homology orientation with assembly maps preserving orientation. This category we denote by $\mathfrak{S}^n$.

4. Resorting the triangulated bundle combinatorics to combinatorial subdivisions Consider elementary triangulated S^n bundle over simplex $\mathfrak{E} \xrightarrow{p} \Delta^k$ with ordered vertices $v_0, ...v_k$ and the interior point $x \in int\,\Delta^k$. Take 0-face $\Delta^{k-1} \xrightarrow{\delta_0} \Delta^k$ and a point in 0-face $x_0 \in int\,\Delta^{k-1}$. The fiber $p^{-1}(x)$ has induced from triangulation $\mathfrak{E}$ cellular structure of multi-simplicial complex composed from simplicial prisms. This prisms are products of simplices. It comes from the fact that general fiber of simplicial projection of a simplex onto simplex is a product of simplices – the fibers of the projection over the base vertices. When we move the point x in the base to point x_0 in the 0-face all the multipliers of the prisms in fiber coming from $p^{-1}(v_0)$ shrinks to points. This creates multi-simplicial boundary degeneration

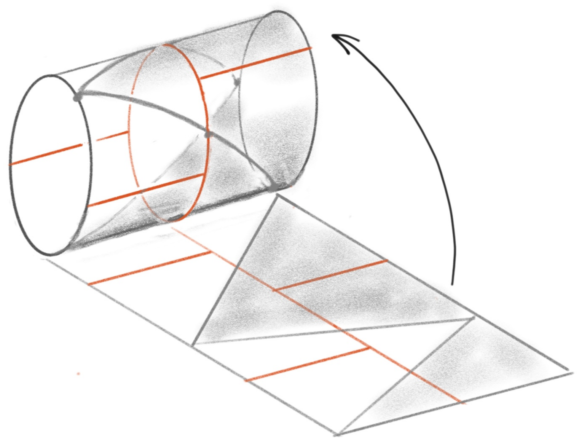

Figure 2: Elementary triangulated circle bundle over interval and dual pattern of circle subdivisions

maps $p^{-1}(x) \xrightarrow{\delta_0^*} p^{-1}(x_0)$. So we see over 1-st barycentric subdivision of Δ^k iterated cylinders of this maps. This boundary degeneration are "simple maps" – the maps having contractible preimages of prisms [WJR13] – the fundamental case of simple homotopy equivalences. Consider the Poincare dual complexes $\widetilde{p^{-1}}(x)$ and $\widetilde{p^{-1}}(x_0)$. On the Poincare duals this degenerations canonically goes to combinatorial subdivision morphism $\widetilde{p^{-1}}(x_0) \xrightarrow{\widetilde{\delta_0^*}} \widetilde{p^{-1}}(x)$. Therefore over 1-st barycentric subdivision we got a combinatorial subdivisions diagram of dual ball complexes (see Fig. 2). Poincare dual of all the total space of elementary bundle is a decomposition of the bundle on prismatic combinatorial [Mnë07] bundles corresponding to the diagram of subdivision morphisms.

This duality has remarkable (and guiding) geometric avatar in the theory of fibered convex polytopes [GKZ08] (and therefore toric varieties). If the total space of elementary bundle is a simplicial convex polytope and projection is linear projection onto base simplex then generic fiber looks like Minkowski sum of polytopes over the vertices (Fig. 3) and the dual picture is common geometric subdivision of those polar dual simple spherical fans in generic position.

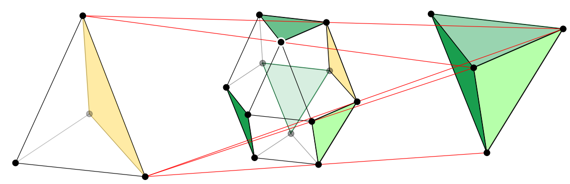

Figure 3: Cuboctahedron is a Minkovski sum $\Delta^3 \oplus_M (\Delta^3)^*$ of simplex and its polar dual, represents triangulation of S^2-bundle over interval

This construction identifies elmenentary triangulated bundles with a subset in Kan simplicial set $\mathbf{Ex}\mathcal{N}\mathfrak{S}^n$, serving as classifying object for triangulated oriented S^n bundles. Therefor a local formula for Euler cocycle on $\mathcal{N}\mathfrak{S}^n$ canonically induces local formula on elementary triangulated bundles.

5. Canonical local system on $\mathcal{N}\mathfrak{S}^n$ After fixing orientations on balls a ball complex became regular CW complex with ± 1 incidence numbers and thus obtains complex of cellular chains $C_*(B)$ with fixed basis marked by balls. Now consider subdivision of ball complexes. It is representable by some homeomorphism. Therefore if cells are oriented they got relative orientation ± 1 which are invariant of the choice of homeomerhism. We can correctly form a chain map $C(B_0) \xrightarrow{\varepsilon(f)} C(B_1)$ sending a k-ball from B_0 to the sum of k-balls in the image with relative orientation. By acyclic carriers argument this maps are quasi-isomorphisms and they are obviously commute with compositions. Therefore we got a functor $\mathfrak{S}^n \xrightarrow{C} \mathbf{Ch}(\mathbb{Z})$ to the category of based chain complexes and quasi-isomorphisms. This functor induces canonical local system (or "constructible sheaf") on $\mathcal{N}\mathfrak{S}^n$.

6. Lere-Serre spectral sequence Euler class of oriented spherical fibration $S^n \to E \xrightarrow{\pi} B$ has a canonical representation using transgression differential in Serre spectral sequence of the fibration.

The second page of Serre spectral sequence looks following

$$E_2^{p,q} = H^p(B; H^q(S^n)) = \begin{cases} H^p(B;\mathbb{Z}) & \text{if } q = n \text{ or } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Particulary $E_2^{0,n} = H^n(B;\mathbb{Z}), E_2^{p,0} = H^p(B;\mathbb{Z})$. The nonzero elements are concentrated on two lines $q = 0, n$, Therefore all pages of the sequence are the same up to page $n+1$ where stays nontrivial transgression differentials $H^n(B) = E_{n+1}^{p,n} \xrightarrow{d_{n+1}} E_{n+1}^{p+n+1,0} = H^{p+n+1}(B)$. Particularly, we got transgression differential $H^0(B) \xrightarrow{d_{n+1}} H^{n+1}(B)$. We can put $e(p) = d_{n+1}(1)$, and this is the Euler class. By multiplicative property of Serre spectral sequance all other transgerssion differentials are cup products with $* \cup e(\pi)$. They form Gysin homomorphisms in Gysin exact sequence of the bundle π.

$$... \to H^p(E) \xrightarrow{\pi_*} H^{p-n}(B) \xrightarrow{*\cup e(\pi)} H^{p+1}(B) \xrightarrow{\pi^*} H^{p+1}(B) \to ...$$

where π_* is integration by fiber homomorphism.

7. Berikashvili predifferential N.A. Berikashvili ([Ber76], see [Kad76], [Roh74]) developed local disassembly of Serre spectral sequence, producing constructive presentation of transgression differentials using local "predifferentials" out of cellular structure on the bundle. Applied for our situation this gives exactly all possible local formulas for Euler cocycles on $\mathcal{N}\mathfrak{S}^n$ in the terms of canonical local system. We will formulate the output of the machinery in the simplest possible form.

Let us introduce notations. Simplices of dimension k in $\mathcal{N}\mathfrak{S}^n$ are all k-chains of abstract subdivisions

$$\mathcal{N}_k\mathfrak{S}^n = \{A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} ... \xrightarrow{f_{k-1}} A_k\}$$

Simplicial face maps we denote as follows

$$\partial_i(A_0 \xrightarrow{f_0} ... \xrightarrow{f_{k-1}} A_k) = \begin{cases} A_1 \xrightarrow{f_1} ... \xrightarrow{f_{k-1}} A_k & \text{if } i = 0 \\ A_0 \xrightarrow{f_0} ... \xrightarrow{f_{k-2}} A_{k-1} & \text{if } i = k \\ A_0 ... \xrightarrow{f_{i-2}} A_{i-1} \xrightarrow{f_i f_{i-1}} A_{i+1} \xrightarrow{f_{i+1}} ... A_k & \text{if } i \neq 0, k \end{cases}$$

Predifferential $t = (t_0, ..., t_n)$ on $\mathfrak{S}^n$ is a sequence of functions on $\mathcal{N}_0\mathfrak{S}^n ... \mathcal{N}_n\mathfrak{S}^n$, satisfying two conditions.

To every k-simplex x_k in $\mathcal{N}_k\mathfrak{S}^n$,

$$x_k = (A_0 \xrightarrow{f_0} ... \xrightarrow{f_{k-1}} A_k)$$

it asignes k-chain $t_k(x_k) \in C_k(A_k)$ such holds condition (A):

$$dt_k(x_k) = \sum_{i=0}^{k-1} (-1)^i t_{k-1}(\partial_i x_k) + (-1)^k \varepsilon_{k-1} t_{k-1}(\partial_k x_k) \qquad (1)$$

where ε_i is $\varepsilon(f_i)$.

Denote by $c_{k-1}(x_k, t_{k-1}) \in C_{k-1}(A_k)$ the chain in left part of (1). It is obvoiusly a cycle. The conition (A) (1) is a recursive skeletal definition of predifferential

$$dt_k(x_k) = c_{k-1}(x_k, t_{k-1}) \tag{2}$$

On 1-simplices we have equations

$$c_0(x_1, t_0) = t_0(A_1) - \varepsilon_0 t_0(A_0) \in C_0(A_1)$$
$$dt_1(x_1) = c_0(x_1, t_0)$$

I.e. c_0 is 1 - cycle. ε_0 is a quasi-isomorphism, $\mathcal{N}_1\mathfrak{S}^n$ is connected, therefore $t_0(A_1), t_0(A_0)$ should represent the same 0-homology class of S^n which is number $\alpha \in \mathbb{R} = H_0(S^n; \mathbb{R}) = H^n(S^n; \mathbb{R})$ and $t_0(A) \in C_0(A)$ is a 0 - chain having α as the sum of its coordinates. Simultaniously, assigning the same number α to all A is a 0-cocycle in $C^0(\mathcal{N}\mathfrak{S}^n)$ representing $\alpha \in \mathbb{R} = H^0(\mathcal{N}\mathfrak{S}^n; H^n(S^n; R))$.

We require (B): $\alpha(t) = 1$.

Having predifferential t we can assign a n-cycle $c_{n+1}(x_{n+1}, t) \in C_n(A_{n+1})$ to every $n+1$-simplex in $\mathcal{N}\mathfrak{S}^n$. The cycle $c_{n+1}(x_{n+1}, t)$ is proportional to fundamental cycle $c_{n+1} = e(x_n, t)[A_{n+1}] \in H_n(A_{n+1})$. The correspondence $x_{n+1} \mapsto e(x_{n+1}, t)$ is a real cocycle on $\mathcal{N}\mathfrak{S}^n$.

The predifferential t determines local formula $e(t) = e(*, t)$ for the Euler class. All the local formulas for Euler class come from a predifferential since one can choose chains t_n up to arbitary constants, shifting cocycle on arbitrary coboundary.

8. Procedure of construction a predifferential So, to build an Euler cocycle on $\mathcal{N}\boldsymbol{\mathfrak{S}}^n$ one should first choose for any ball complex A a 0-chain $t_0(A) \in C_0(A)$ representing 1, i.e having 1 as sum of its coordinates – a "probability measure" on vertices on A. In the case of integer coefficients this is just choosing a vertex. Then for any simplex $x_1 = (A_0 \xrightarrow{f} A_1)$ the chain $c_0(x_1, t_0)$ is a boundary and one can can choose $t_1(t_0)$ in coordinate affine subspace of $C_1(A_1)$ – shifted kernel of differential d_1. We know that $1, ..., n-1$ homology of all the complexes $C(A)$ are zero and c_i are cycles. Therefore they are boundaries. So we are choosing t_i in affine subspace – shifted kernel of d_i in $C_i(A_i)$. Finally we arrive to Euler cochain defined as liner functon on big affine space of all these freedoms. Making integer choices we will obtain integer local formulas. Over a field of characteristic 0 one can make average of a set of choices, etc. The problem of local formula is therefore how to make this choices.

9. Fixing choices by Hodge strong homology retractions Over rationals $\mathbb{Q}$ we can fix all the choices canonically in predifferentials by combinatorial Hodge theory. It will produce a rational formula for Euler cocycle which is invariant under combinatorial symmetries of the chain of subdivisions. We will write down this formula. For abstract oriented ball complex A on sphere S^n Let $\Delta_i(A), i = 0, ..., n-1$ the i-th combinatorial Laplace operator of the augmented complex $C_*(A) \to \mathbb{Q}$. Let

$$R^i(A) = C_i(A) \xrightarrow{d^* \Delta_i^{-1}} C_{i+1}(A), i = 0, ..., n-1$$

– Hodge theory strong retraction on homology operator, where d^* is metric adjoint codifferential. For every n simplex

$$x_n = (A_0 \xrightarrow{f_0} ... \xrightarrow{f_{n-1}} A_n)$$

in $\mathcal{N}\mathfrak{S}^n$ we have n-chain in $C_n(A_n)$:

$$T_n(x_n) = (-1)^n R_n^{n-1} \varepsilon_{n-1} R_{n-1}^{n-2} ... \varepsilon_1 R_1^0((1_*)_1 - \varepsilon_0(1_*)_0)$$

where R_i^{i-1} is operator $R^{i-1}(A_i)$, $(1_*)_i$ – harmonic unit 0-chain on A_i , assigbing to every vertex the number $1/\#(A_i)_0$.

Then the expression for rational combinatorial Hodge Euler cocycle on $\mathcal{N}_{n+1}\mathfrak{S}^n$ is following. To a $n+1$ simplex x_{n+1} it assignes rational number

$$e(x_{n+1}) = [\sum_{i=0}^{n+1} (-1)^i T_n(\partial_i x_{n+1}) + (-1)^{n+1} \varepsilon_n T_n(\partial_{n+1} x_{n+1}), 1^*] \tag{3}$$

where [– ,1*] is paring of n-cycle and harmonic unit n-cocycle 1^* in $C_n(A_{n+1})$.

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