APPLICATION OF INTEGRAL GEOMETRY TO LINEAR INEQUALITIES

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A. M. Vershik suggested identifying the space of problems of linear programming with the corresponding Grassman manifold. On this manifold there is defined a probability measure, and the measures of sets of problems with finite and infinite extrema and the mean number of admissible bases in the problems are calculated.

0. Introduction

In the study of various analytic characteristics of problems of linear algebra, in particular, problems of linear programming, one can use methods of integral geometry. For this it is necessary to parametrize the set of corresponding problems (for example, problems of linear programming) and to study the geometric properties of the space of problems. Such a space can be constructed in various ways. A. M. Vershik suggested identifying the space of problems with a Grassman manifold ([2]). The corresponding construction is carried out in Sec. 1. The convenience of using Grassman manifolds for constructing spaces of problems is that on it there exists a unique normalized measure which is invariant with respect to the natural action of the orthogonal group (cf. [3]), and one can speak of the probability that a problem has a specific property as the measure of the corresponding set of problems. In the simplest cases the answer can be obtained not only for the invariant measure, but also for a wider class of measures. In this paper we shall calculate the probability that an extremum in a linear programming problem is finite and the average number of admissible bases. The result is based on the Steiner-Schläfli formula [8] on the number of parts into which hyperplanes in general position partition the space. It has already been used to study characteristics of random cones in [4, 5]. One can find other definitions of the space of problems of linear programming in [6, 7].

1. Construction of the Space of Problems

In what follows we shall denote by the symbol $G(n,k)$ $(n \geqslant k \geqslant 0)$ the Grassman manifold of k-dimensional subspaces of R^n. For each subspace E the dimension and orthogonal complement are denoted respectively by $\dim E$ and $E^\perp$. If $A=\{a_j^i\}$ is a real matrix of size $n\times k$, then we denote by a^i the vector $a^i=(a_1^i, a_2^i,\dots,a_n^i)$ in R^n, and by a_i we denote the vector $a_i=(a_i^1, a_i^2,\dots,a_i^k)$ in R^k. The set of integers $\{k, k+1, \dots, n\}$ is denoted by $k:n$. A possibly empty subset s of $1:n$ will be called a set of indices from $1:n$ and we shall denote by $|s|$ the number of elements of s, and by s † the collection $1:n\setminus s$. Writing a problem in the form $\sup\{f(x) \mid B(x)\}$ means that one must find the supremum of the function f on vectors for which condition B holds.

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Such a vector is said to be admissible. We define on matrices of size $n \times k$ $(n \geq k)$ of rank k a mapping Sub with values in $G(n,k)$. It associates with each matrix $A = \{a^i_j\}$ the subspace of R^n which is the linear span of the vectors $a^1, a^2, \ldots, a^k$.†

Let n, m, k be nonnegative integers, where $n \geq m$, $n \geq k$, $m+k \geq n$. We consider the linear programming problem:

$$\sup\{x_{n+1} \mid xA = 0;\ x_i \geq 0,\ i \in 1:m;\ x_{n+2} = 1\}, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_{n+2})$ is a vector in R^{n+2}, A is a real matrix of size $(n+2) \times (k+1)$ of rank $k+1$.

Let $E = Sub(A)$. Then the condition on admissible vectors in (1) $xA = 0$ is equivalent with x being orthogonal to all vectors generating E, that is, $x \in E^{\perp}$. Using this, we get the following problem:

$$\sup\{x_{n+1} \mid x \in E^{\perp};\ x_i \geq 0,\ i \in 1:m;\ x_{n+2} = 1\}. \tag{2}$$

We note that although to each subspace E of $G(n+2, k+1)$ there corresponds a whole class of matrices, whose column-vectors generate E, the conditions on an admissible vector in (1) for the arbitrary matrix A, such that $Sub(A) = E$, and in (2) are equivalent. Hence in studying the characteristics of problems of linear programming connected with the structure of the set of admissible vectors, it is natural to take as the space of problems the manifold $G(n+2, k+1)$, at each point E of which the problem (2) is considered. The connection between the spaces of problems defined in this way and those defined matrically is discussed in Sec. 5.

2. Formulation of the Theorem

Definition. Let L be an n-dimensional linear space, $b = \{b_1, b_2, \ldots, b_n\}$ be a basis in L, E be a k-dimensional subspace $(n \geq k \geq 0)$. We shall say that E is a subspace in general position in L with respect to the basis b, if for any collection of indices s from $1:n$ such that $|s| = k$, $E \cap \{x \mid x_i = 0,\ i \in s\} = 0$, where $x_1, x_2, \ldots, x_n$ are the coordinates of the vector x in the basis b.

In the space R^n we fix the standard basis $e = \{e_1, e_2, \ldots, e_n\}$, where e_i denotes the vector whose i-th coordinate is equal to 1, and the rest are zero. We shall say that E is a subspace in general position in R^n or simply a subspace in general position, when it is clear which R^n is involved, if E is a subspace in general position in R^n with respect to the basis e. Properties of subspaces in general position will be considered in Sec. 3.

Definition. Suppose there is a given a measure μ on the Borel sets of the manifold $G(n,k)$. We shall say that μ is invariant with respect to change of signs of coordinates if for any transformation of the manifold φ_ε, corresponding to the change of coordinate vectors in R^n from $\{x_i\}$ to $\{\varepsilon_i x_i\}$, where $\varepsilon_i = \pm 1$, and any measurable set C $\mu(C) = \mu(\varphi_\varepsilon(C))$.

Let n, m, k be nonnegative integers, where $n \geq m$, $n \geq k$, $m+k \geq n$. At each point E of the manifold $G(n+2, k+1)$ we consider the problem (2). We denote by val the function

†Another notation is $Span\ A$.

on $G(n+2, k+1)$ with values in the extended real line, equal at each point to the supremum of the problem (2), considered at it.

Definition. Let s be a collection of indices from $1:(n+1)$ where $(m+1):(n+1) \subset s$ and $|s| = k+1$, E be a $(k+1)$-dimensional subspace of R^{n+2}. We shall call the collection s an admissible basis of (2), considered in E, if the following conditions hold:

$$E^{\perp} \cap \{x \mid x_i = 0,\ i \in \bar{s} \cup \{n+2\}\} = 0, \tag{3}$$

$$E^{\perp} \cap \{x \mid x_i = 0,\ i \in \bar{s};\ x_i \geq 0,\ i \in s \cap 1:m;\ x_{n+2} = 1\} \neq \emptyset. \tag{4}$$

The naturality of this definition will be discussed in Sec. 4.

We denote by bas the function on $G(n+2, k+1)$ equal at each point to the number of admissible bases of (2) corresponding to this point.

THEOREM 1. Suppose there is given on the Borel sets of the manifold $G(n+2, k+1)$ a probability measure P such that on a set of full measure one can find a subspace in general position, and invariant with respect to change of signs of coordinates. Then

$$P\{val > -\infty\} = 2^{-m} \sum_{i=0}^{n-k} \binom{m}{i},$$

$$P\{val = +\infty\} = 2^{-m} \sum_{i=0}^{n-k-1} \binom{m}{i},$$

$$P\{-\infty < val < +\infty\} = 2^{-m} \binom{m}{n-k},$$

$$M_P\, bas = 2^{n-k-m} \binom{m}{k+m-n}.^*$$

3. Properties of Subspaces in General Position

To prove the theorem we need several properties of subspaces in general position, but before formulating them it is convenient to introduce some notation.

Let s and t be two disjoint collections of indices from $1:n$.

We denote by $\varepsilon(s)$ the set of numbers $\varepsilon(s) = \{\varepsilon_i \mid \varepsilon_i = \pm 1,\ i \in s\}$, which we shall call a collection of signs in what follows. By $R_+^n(s,t)$, $R_+^n(s,\varepsilon(s),t)$, $R^n(t)$ we denote the cone in R^n $R_+^n(s,t) = \{x \mid x_i > 0,\ i \in s;\ x_i = 0,\ i \in t\}$, $R_-^n(s,\varepsilon(s),t) = \{x \mid \varepsilon_i x_i > 0, i \in s;\ x_i = 0,\ i \in t\}$ and the subspace in R^n $R^n(t) = \{x \mid x_i = 0,\ i \in t\}$. In each subspace $R^n(t)$ there is a fixed basis consisting of vectors of the standard basis e which belong to this subspace. We shall speak of subspaces in general position in $R^n(t)$, having this basis in mind. χ is the set function such that $\chi(C) = 0$ if C is empty, and $\chi(C) = 1$ otherwise.

Definition. Let E be a k-dimensional linear space $(k \geq 1)$. A family of hyperplanes $E_1, E_2, \ldots, E_n$ in E is called a family of hyperplanes in general position, if the intersection of each k of these hyperplanes is equal to zero.

Now we can formulate the basic properties of subspaces in general position.

Property 1. Let E be a subspace of R^n. Then the following assertions are equivalent:

1. E is a k-dimensional subspace in general position in R^n,

*The symbol M_P is for the expectation with respect to the measure P.

2. $E^\perp$ is an $(n-k)$ -dimensional subspace in general position in R^n,

3. for any collection of indices s from $1:n$ such that $|s|=m$ and $k\geq m\geq 0$, the subspace $E\cap R^n(s)$ is a $(k-m)$ -dimensional subspace in general position in $R^n(s)$, and if $k-m\geq 1$, then the family of subspaces $E\cap R^n(s)\cap R^n(\{i\})$ for $i\in\bar{s}$ is a family of hyperplanes in general position in $E\cap R^n(s)$,

4. $\dim E=k$ and for any two disjoint collections of indices s and t from $1:n$ such that $|s|\geq 1$, and any collection of signs $\varepsilon(s)$, the following assertions are equivalent:

a) $E\cap\{x\,|\,\varepsilon_i x_i\geq 0,\ i\in s;\ x_i=0,\ i\in t\}\neq \mathbb{0}$,

b) $E\cap R^n_+(s,\varepsilon(s),t)\neq\emptyset$.

Proof. Equivalence of 1 and 2. Let 1 hold, and fix an arbitrary collection of indices s from $1:k, |s|=k$. It follows from the definition of general position that $E\cap R^n(s)=\mathbb{0}$, so

$$R^n=(E\cap R^n(s))^\perp=E^\perp+R^n(s)^\perp=E^\perp+R^n(\bar{s}).$$

Using this equation, we have $\dim(E^\perp\cap R^n(\bar{s}))=\dim E^\perp+\dim R^n(\bar{s})-\dim(E^\perp+R^n(\bar{s}))=(n-k)+k-n=0.$ $\dim E^\perp=n-k$ and $\bar{s}$ is an arbitrary collection of indices from $1:n$ with $n-k$ elements, so $E^\perp$ is an $(n-k)$-dimensional subspace in general position in R^n. Using the equation $(E^\perp)^\perp=E$, we get the converse.

Equivalence of 1 and 3. 1 follows from 3 since it suffices to take the empty set as s. Conversely, let 1 hold. We fix an arbitrary collection of indices s from $1:n, |s|=m$, $k\geq m\geq 0$. It suffices to show that $\dim(E\cap R^n(s))=k-m$, and the rest of assertion 3 is an obvious consequence of the definition of general position,

$$\dim(E\cap R^n(s))=\dim E+\dim R^n(s)-\dim(E+R^n(s))\geq k-m.$$

Let $L=E\cap R^n(s)$; if $\dim L>k-m$, then for any collection of indices t from $1:n$ such that s and t are disjoint and $|t\cup s|=k$, we have

$$0=\dim(L\cap R^n(t))=\dim L+\dim(R^n(t))-\dim(L+R^n(t))>$$

$$>(k-m)+(n-k+m)-\dim(L+R^n(t)).$$

From this we get that $\dim(L+R^n(t))>n$, which is impossible.

Consequently, $\dim(E\cap R^n(s))=k-m.$

Equivalence of 1 and 4. We shall show that 4 follows from 3. Let 3 hold; we fix two disjoint collections of indices s and t from $1:n$ such that $|s|\geq 1$, and a collection of signs $\varepsilon(s)$. It is obvious that a) follows from b). We shall prove the converse. We prove this by induction on the dimension of the space R^n. It is clear that for $n=1$ the assertion is true. Suppose the assertion is true for spaces of dimension less than n; we shall prove it for dimension n. By hypothesis there exists a vector y such that $y\in E\cap\{x|\varepsilon_i x_i\geq 0,\ i\in s;\ x_i=0,\ i\in t\}$ and $y\neq\mathbb{0}$. If $\varepsilon_i y_i>0$ for all $i\in s$, then the assertion is true. If not, then one can find an index $j\in s$ such that $y\in E\cap R^n(t\cup\{j\})$.

$$\dim(E\cap R^n(t\cup\{j\}))=\dim(E\cap R^n(t))-1,$$

so there exists a vector z in $E \cap R^n(t)$ such that $\varepsilon_j z_j = 1$. If $\varepsilon_i y_i > 0$ for all $i \in s \setminus \{j\}$, then the line segment joining y and z intersects the cone $R^n_+(s, \varepsilon(s), t)$, and hence, $E \cap R^n_+(s, \varepsilon(s), t) \neq \emptyset$. It remains to show that one can choose the vector y so that this condition holds. If $|s| = 1$, then there is such a vector already; if not, the possibility of finding it follows from the application of the inductive hypothesis to the $(n-1)$-dimensional space $R^n(\{j\})$ and the subspace $E \cap R^n(\{j\})$ of it.

We shall show that 1 follows from 4. Let 4 hold; we fix an arbitrary collection of indices s from $1:n$, $|s| = k$. If $E \cap R^n(s) \neq \emptyset$, then we consider a system of vectors in E y, y^i for $i \in s$ such that $y \in E \cap R^n(s)$, $y^i \in E \cap R^n_+(\{i\}, s \setminus \{i\})$ for $i \in s$. By hypothesis such vectors exist, but they are linearly independent, so $\dim E \geqslant k+1$. We have obtained a contradiction with the fact that $\dim E = k$. Thus, $E \cap R^n(s) = \emptyset$.

Property 2. Let E be a k-dimensional subspace in general position in $R^n (n > k \geqslant 1)$; we fix two disjoint collections of indices s and t from $1:n$, where $|s| = p$, $|t| = m$, $p \geqslant 1$, $m < k$. Then

$$\sum_{\varepsilon(s)} \chi(E \cap R^n_+(s, \varepsilon(s), t)) = 2 \sum_{i=0}^{k-m-1} \binom{p-1}{i},$$

where the summation is over all distinct collections of signs $\varepsilon(s)$.

Remark. To prove this property we shall use the Steiner—Schläfli formula which says that a family of $n\ (n \geqslant 1)$ hyperplanes in general position in a k-dimensional linear space $(k \geqslant 1)$ partitions the space into $2 \sum_{i=0}^{k-1} \binom{n-1}{i}$ open parts. It is easy to get the proof of this formula by induction on the dimension of the space, and besides, it is in [4, 8].

Proof. Let $L = E \cap R^n(t)$. By property 1, L is a $(k-m)$-dimensional space and the family of subspaces $L \cap R^n(\{i\})$ for $i \in s$ is a family of hyperplanes in general position in L. Consequently, these hyperplanes divide L into $2 \sum_{i=0}^{k-m-1} \binom{p-1}{i}$ open parts. It is obvious that each of these parts is the intersection of E with one of the cones $R^n_+(s, \varepsilon(s), t)$, and E does not intersect the rest of the cones.

4. Proof of the Theorem

LEMMA 1. Suppose there is given a normalized measure μ on the Borel sets of the manifold $G(n,k)\ (n > k \geqslant 1)$, such that on a set of full measure one can find a subspace in general position, and invariant with respect to change of signs of coordinates. We fix two disjoint collections of indices s and t from $1:n$, where $|s| = p$, $|t| = m$, $p \geqslant 1$, $m < k$. Then for any fixed collection of signs $\varepsilon^1(s)$

$$\int_{G(n,k)} \chi(E \cap R^n_+(s, \varepsilon^1(s), t))\, d\mu = 2^{1-p} \sum_{i=0}^{k-m-1} \binom{p-1}{i}.$$

Proof. It is obvious that the integrand is measurable, so the integration is possible. Using the fact that on a set of full measure one can find a subspace in general position, and property 2, we have

$$\int_{G(n,k)} \sum_{\varepsilon(s)} \chi(E \cap R^n_+(s, \varepsilon(s), t))\, d\mu = 2 \sum_{i=0}^{k-m-1} \binom{p-1}{i}, \tag{5}$$

where the summation is taken over all distinct collections of signs $\varepsilon(s)$. We consider 2^p cones $R_+^n(s,\varepsilon(s),t)$, all of them obtained from $R_+^n(s,t)$ by changes of signs of coordinates, and a measure invariant with respect to such changes. Consequently, the measures of subspaces intersecting these cones are equal. We get the answer from this and (5).

LEMMA 2. Let $A=\{a_j^i\}$ be a real matrix of size $n\times k$ of rank $k\,(n\geq k\geq 1)$, let the subspace $E=Sub(A)$, let s be a collection of indices from $1:n$, $|s|=k$. Then the k vectors a_i for $i\in s$ are linearly independent in R^k, if and only if $E^\perp\cap R^n(\bar{s})=\mathbb{0}$.

Proof. This assertion is an obvious consequence of the fact that the homogeneous system of linear equations

$$\sum_{i\in s} x_i a_i^j = 0 \qquad \text{for } j\in 1:k$$

has only the zero solution if and only if

$$E^\perp\cap R^n(\bar{s})=\mathbb{0}.$$

Proof of the Theorem. We fix an arbitrary $(k+1)$-dimensional subspace E in R^{n+2} and a real matrix A of size $(n+2)\times(k+1)$ of rank $k+1$, such that $Sub(A)=E$. We consider the problems (1) and (2) and we shall show that the definition of an admissible basis in (2) is natural in the sense that each admissible basis of (1) is an admissible basis in (2) and conversely. In fact, the concept of admissible basis in (1) is well known. A collection of indices s from $1:(n+1)$ such that $(m+1):(n+1)\subset s$ and $|s|=k+1$ is called an admissible basis in (1), if the vectors a_i for $i\in s$ are linearly independent in R^k and the corresponding basis solution is admissible. The first of these conditions is equivalent with (3) by Lemma 2, and the second coincides with (4). If E is a subspace in general position, then by Property 1 condition (3) always holds, and condition (4) is equivalent with the following:

$$E^\perp\cap R_+^{n+2}((s\cap 1:m)\cup\{n+2\},\bar{s})\neq\emptyset. \tag{6}$$

The value of the function val at E is greater than $-\infty$ if and only if there exists an admissible vector in (2), that is, if $E^\perp\cap\{x\,|\,x_i\geq 0,\ i\in 1:m;\ x_{n+2}=1\}\neq\emptyset$, which for subspaces in general position by Property 1 is equivalent with the condition:

$$E^\perp\cap R_+^{n+2}(1:m\cup\{n+2\},\emptyset)\neq\emptyset. \tag{7}$$

It is known (cf. [1]), that the supremum in (1) is equal to $+\infty$ if and only if there exists a vector y such that $yA=\mathbb{0},\ y_i\geq 0$ for $i\in 1:m,\ y_{n+2}=0,\ y_{n+1}>0$, and there is an admissible vector in the problem. We reformulate these conditions for (2). The existence of a vector y with the properties needed is equivalent with the condition: $E^\perp\cap\{x\,|\,x_i\geq 0,\ i\in 1:m;\ x_{n+2}=0;\ x_{n+1}>0\}\neq\emptyset$.

If E is a subspace in general position, then by Property 1, this condition is equivalent with the following:

$$E^\perp\cap R_+^{n+2}(1:m\cup\{n+1\},\{n+2\})=\emptyset. \tag{8}$$

Again by Property 1 for subspaces in general position it follows that if (8) holds then (7) holds, and hence $val(E)=+\infty$ if and only if (8) holds. Now we can proceed to the calculation of the probabilities of interest to us. We note that passage to orthogonal complements of subspaces carries the measure from the Borel sets of the manifold $G(n+2,k+1)$ to the Borel sets of $G(n+2,n-k+1)$ in the standard way (the measure of a set is equal to the measure of its preimage), the measure μ obtained is invariant with respect to changes of signs of coordinates, and on a set of full measure one can find a subspace in general position. We calculate the expectation of bas. Since on a set of full measure one can find a subspace in general position, one can use (6) and Lemma 1. We have

$$M_p\, bas = \int_{G(n+2,k+1)} bas(E)\, dP = \int_{G(n+2,k+1)} \sum_{s} \chi(E^{\perp}\cap R_{+}^{n+2}((s\cap 1:m)\cup$$

$$\{n+2\},\bar{s}))\, dP = \int_{G(n+2,n-k+1)} \sum_{s} \chi(E\cap R_{+}^{n+2}((s\cap 1:m)\cup\{n+2\},\bar{s}))\, d\mu =$$
$$= 2^{n-k-m}\binom{m}{k+m-n},$$

where the summation is over all distinct collections of indices s from $1:(n+1)$ such that $(m+1):(n+1)\subset s$ and $|s|=k+1$. Analogously, using (7), (8), and Lemma 1, we get

$$P\{val>-\infty\} = 2^{-m}\sum_{i=0}^{n-k}\binom{m}{i},$$
$$P\{val=+\infty\} = 2^{-m}\sum_{i=0}^{n-k-1}\binom{m}{i},$$
$$P\{-\infty<val<+\infty\} = P\{val>-\infty\} - P\{val=+\infty\} = 2^{-m}\binom{m}{n-k}.$$

Remark. It is evident from the proof of the theorem that it suffices to require of the measure invariance with respect to changes of signs of the coordinates only for coordinates with indices from the collection $1:m\cup\{n+1\}\cup\{n+2\}$.

5. Basic Example

We shall consider real matrices $(k\geq 1,\ n\geq 1)$ of the form $\mathcal{U}=\{u_i^j \mid i\in 0:(n-1);\ j\in 0:(k-1);\ u_0^0=0\}$.

We denote by $M(n,k)$ the space of such matrices with the natural topology.

We fix nonnegative integers n, m, k, p such that $n\geq m\geq p,\ n\geq k\geq p,\ m+k\geq n$ and we consider a linear programming problem of general form:

$$\text{find the supremum of the function } \sum_{i=1}^{n-p} x_i u_i^0 \tag{9a}$$

on vectors of R^{n-p}, satisfying the conditions

$$\sum_{i=1}^{n-p} x_i u_i^j \leq u_0^j \quad \text{for} \quad j\in 1:p, \tag{9b}$$

$$\sum_{i=1}^{n-p} x_i u_i^j = u_0^j \quad \text{for} \quad j\in (p+1):k, \tag{9c}$$

$$x_i\geq 0 \quad \text{for} \quad i\in 1:(m-p), \tag{9d}$$

where $\mathcal{U}=\{u_i^j\}$ is a matrix from $M(n-p+1, k+1)$. We reduce this problem to the form (1). For this we introduce new variables $x_{n-p+1},\dots, x_{n+2}$ and we consider a problem equivalent with the original one: find the supremum of the function x_{n+1} on vectors from R^{n+2}, satisfying the conditions

$$\sum_{i=1}^{n-p} x_i u_i^0 - x_{n+1} = 0, \quad \sum_{i=1}^{n-p} x_i u_i^j + x_{n-j+1} - u_0^j x_{n+2} = 0$$

for $j \in 1:p$,

$$\sum_{i=1}^{n-p} x_i u_i^j - u_0^j x_{n+2} = 0 \qquad \text{for } j \in (p+1):k,$$

$$x_i \geq 0 \qquad \text{for } i \in 1:(m-p) \cup (n-p+1):n, \; x_{n+2} = 1.$$

This problem now has the form (1) up to the indexing of the coordinates.

We reindex the coordinates so that the coordinates with indices $m-p+1,\dots,n-p$ respectively have indices $m+1,\dots, n$, and the coordinates with indices $n-p+1,\dots,n$, respectively, have indices $m-p+1,\dots,m$, while the indexing of the remaining coordinates is unchanged. We denote the matrix of the problem obtained by $M_3(\mathcal{U})$. Thus, we have the problem equivalent with the original one:

$$\sup\{ x_{n+1} \mid x \cdot M_3(\mathcal{U}) = \mathbb{O};\ x_i \geq 0, i \in 1:m;\ x_{n+2} = 1\}. \tag{10}$$

There exist different definitions of admissible bases in (9).

We consider one of them (cf. [1]).

<u>Definition.</u> By an admissible basic pair in (9) we mean two collections of indices s from $1:(n-p)$ and t from $1:k$, such that one has:

$$\bar{s} \subset 1:(m-p), \ \bar{t} \subset 1:p, \ |s| = |t|,$$

the system of equations

$$\sum_{i \in s} x_i u_i^j = u_0^j \qquad \text{for } j \in t,$$

$x_i = 0$ for $i \in \bar{s}$ has a unique solution y, and y is an admissible vector.

It is easy to see that the admissible pairs in (9) are in one-to-one correspondence with admissible bases in (10). The correspondence associates with the pair s, t the collection of indices

$$\{m-i+1 \mid i \in \bar{t}\} \cup (s \cap 1:(m-p)) \cup (m+1):n \cup \{n+1\}.$$

Suppose at each point $\mathcal{U}$ of the space $M(n-p+1, k+1)$ we consider the problem (9). We denote by $value$ the function on $M(n-p+1, k+1)$ with values in the extended real line, which is equal at each point to the supremum of the problem considered at it.

We denote by $basis$ the function $M(n-p+1, k+1)$, which is equal at each point to the number of admissible basic pairs of the problem considered at it.

<u>Definition.</u> Suppose there is given a measure μ on the Borel set of the space $M(n+1, k+1)$ $(n \geq 0, k \geq 0)$. We shall say that μ is invariant with respect to changes of signs of coordinates by columns, if for any transformation of the space φ_ε, corresponding to the change of columns of matrices from $\{u^i\}$ to $\{\varepsilon_i u^i\}$, where $\varepsilon_i = \pm 1$,

$u^i = (u_0^i, u_1^i, \ldots, u_n^i)$ for $i = 0, 1, \ldots, k$, and any measurable set c, $\mu(c) = \mu(\varphi_\varepsilon(c))$. Analogously one defines invariance with respect to changes of signs of coordinates by rows.

COROLLARY. Suppose given a probability measure P on Borel sets of the space $M(n-p+1, k+1)$, which is invariant with respect to change of signs of coordinates by columns and rows, and the measure of the set of matrices u, such that each $k+1$ rows of the matrix $M_s(u)$ are linearly independent, is equal to one. Then

$$P\{\text{value} > -\infty\} = 2^{-m} \sum_{i=0}^{n-k} \binom{m}{i},$$

$$P\{\text{value} = +\infty\} = 2^{-m} \sum_{i=0}^{n-k-1} \binom{m}{i},$$

$$P\{-\infty < \text{value} < +\infty\} = 2^{-m} \binom{m}{n-k},$$

$$M_p\, \text{basis} = 2^{n-k-m} \binom{m}{k+m-n}$$

Proof. Let M° be the set of matrices u in $M(n-p+1, k+1)$ such that each $k+1$ rows of the matrix $M_s(u)$ are linearly independent. M° is open and has full measure. Hence it suffices to consider the problems on the set M°. For any matrix $u \in M^\circ$ the subspace of R^{n+2} $E = \mathrm{Sub}(M_s(u))$ is a subspace in general position by Lemma 2 and Property 1. The number of admissible basic pairs in (9) coincides with the number of admissible bases in (2), and the values of the suprema of these problems are equal. In the standard way, we carry the measure from the Borel sets of M° to the Borel sets of $G(n+2, k+1)$. It is easy to see that the invariance of the measure on M° with respect to change of signs of coordinates by columns and rows guarantees the invariance of the measure obtained on $G(n+2, k+1)$ with respect to change of signs of coordinates in R^{n+2}. Applying the theorem we get the answer.

Remark. It suffices to require the invariance of the measure on $M(n-p+1, k+1)$ with respect to change of signs of coordinates only by rows with indices from the collection $0:(m-p)$ and by columns with indices from the collection $0:p$, so that for problems with $p=0$, $m=n$, just invariance with respect to change of signs of coordinates by rows suffices, and for problems with $p=k$, $m=p$, by columns.

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