

ON THE SIMPLEST SPLIT-MERGE OPERATOR ON THE INFINITE-DIMENSIONAL SIMPLEX

Natalia TSILEVICH¹

St. Petersburg State University

E-mail: natalia@pdmi.ras.ru

May 31, 2001

Abstract

We consider the simplest split-merge Markov operator T on the infinite-dimensional simplex Σ_1 of monotone non-negative sequences with unit sum. For a sequence $x \in \Sigma_1$, it picks a size-biased sample (with replacement) of two elements of x ; if these elements are distinct, it merges them and reorders the sequence, and if the same element is picked twice, it splits this element uniformly into two parts and reorders the sequence. We prove that the means along the T -trajectory of the δ -measure at the vector $(1, 0, 0, \dots)$ converge to the Poisson–Dirichlet distribution $PD(1)$.

¹Partially supported by RFBR grant 00–15–96060.

1 Introduction

In this paper we investigate the simplest split-merge Markov operator T originally introduced by A. Vershik. This operator acts on the infinite-dimensional simplex Σ_1 of monotone non-negative sequences with unit sum as follows. For $x = (x_1, x_2, \dots) \in \Sigma_1$, consider a size-biased sample (with replacement) of two elements of x . If two different elements x_i, x_j are picked, then we merge them into $x_i + x_j$ (and reorder the sequence to obtain a point of Σ_1); and if the same element x_i is picked twice, then we split it into two parts $t, x_i - t$ (with t uniformly distributed on $[0, x_i]$) and reorder the sequence.

Conjecture (A. Vershik) *The only T -invariant distribution on Σ_1 is the Poisson–Dirichlet distribution $PD(1)$.*

The Poisson–Dirichlet distribution $PD(1)$ introduced by J. F. C. Kingman [3] is perhaps the most distinguished distribution on the infinite-dimensional simplex. It arises in many various problems from different areas of mathematics and applications and has many remarkable properties and characterizations. The conjecture above may be considered as an attempt to give a new very important characterization of this measure.

In the original setting the operator T arises from the representation theory of the infinite symmetric group \mathfrak{S}_∞ . Namely, the group \mathfrak{S}_∞ acts by shifts R_g on the projective limit \mathfrak{S}^∞ of finite symmetric groups (the so-called space of virtual permutations). On the other hand, there is a projection from \mathfrak{S}^∞ onto the infinite-dimensional simplex which associates with a virtual permutation the sequence of its relative cycle lengths. The simplest split-merge operator T under consideration is the projection on Σ_1 of the shift $R_{(1,2)}$ corresponding to a transposition in \mathfrak{S}_∞ . Following this approach, N. Tsilevich [6, 7] proved the uniqueness of invariant measure for the family of Markov operators $\{T_g\}_{g \in \mathfrak{S}_\infty}$ arising from the action of the whole group \mathfrak{S}_∞ . We would also like to mention the paper [1] proving the uniqueness of invariant measure for a closely related Markov operator.

Recently, the same operator T appeared as a simplified model in a quite different context of triangulations of random Riemannian surfaces (R. Brooks). Inspired by this motivation, E. Mayer-Wolf, O. Zeitouni and M. Zerner have proved (by purely analytic methods) the uniqueness of T -invariant measure under strong smoothness assumptions on the measure (A. Vershik, personal communication). However, the problem for arbitrary Borel distributions is still open.

In this note we prove another partial result. Consider the trajectories of the δ -measures $\delta_x, x \in \Sigma_1$, under T . We show that for almost all points x with finitely many non-zero coordinates (e.g., $x = (1, 0, \dots)$), the binomial means along the trajectories converge to the Poisson–Dirichlet measure $PD(1)$.

We would like to emphasize that we essentially exploit the relation of this problem to symmetric groups and the techniques developed in [6, 7]. We hope that these methods will eventually allow to prove Vershik’s conjecture.

Let us describe the problem more precisely. Let

$$\Sigma = \left\{ x = (x_1, x_2, \dots) : x_1 \geq x_2 \geq \dots \geq 0, \sum_{i=1}^{\infty} x_i \leq 1 \right\}$$

be the simplex of non-increasing non-negative sequences with sum at most one, and $\Sigma_1 = \{x \in \Sigma : \sum_{i=1}^{\infty} x_i = 1\}$ be its subsimplex consisting of sequences with sum exactly one. The most elementary definition of the Poisson–Dirichlet measure $PD(1)$ is as follows.

Definition 1 *Let U_1, U_2, \dots be a sequence of i.i.d. random variables uniformly distributed on the interval $[0, 1]$. Let*

$$V_n = U_n \prod_{i=1}^{n-1} (1 - U_i).$$

It is easy to see that $\sum_{i=1}^{\infty} V_i = 1$. The Poisson–Dirichlet measure $PD(1)$ on the simplex Σ_1 is the distribution of the order statistics $V_{(1)} \geq V_{(2)} \geq \dots$ of the sequence V_1, V_2, \dots

Now consider the following Markov operator on Σ_1 :

$$Tx = \begin{cases} V(x_i + x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots), & i < j, \\ \text{with probability } 2x_i x_j; \\ V(t, x_i - t, x_1, \dots, x_{i-1}, x_{i+1}, \dots), & i = 1, 2, \dots, \\ \text{with probability } x_i dt, t \in [0, x_i], \end{cases} \quad (1)$$

where the operator V arranges the elements of a sequence in non-increasing order. Denote by E the identity operator on Σ_1 . We may consider these operators also on the space $\mathcal{M}(\Sigma_1)$ of probability Borel measures on Σ_1 .

Let ν_0 denote the δ -measure at the point $(1, 0, 0, \dots) \in \Sigma_1$. In this note we prove the following theorem.

Theorem 1 *The sequence*

$$\left(\frac{E + T}{2} \right)^m \nu_0$$

converges (weakly) as $m \rightarrow \infty$ to the Poisson–Dirichlet measure $PD(1)$.

Corollary *The sequence*

$$\left(\frac{E + T}{2} \right)^m \delta_x$$

converges (weakly) as $m \rightarrow \infty$ to the Poisson–Dirichlet measure $PD(1)$ for almost all (with respect to the Lebesgue measure) points $x \in \Sigma_1$ with finitely many non-zero coordinates.

The author is grateful to A. Vershik for suggesting the problem and for many fruitful discussions.

2 The space of virtual permutations

In this Section we give a necessary background concerning the space of virtual permutations and central measures on this space.

Definition 2 *Given a subset $J \subset [n]$ and a permutation $w \in \mathfrak{S}_n$, denote by $\pi_{n,J}w$ a permutation of the set J obtained by removing from cycles of w all elements that do not belong to J . The permutation $\pi_{n,J}w$ is called the induced permutation for w on J .*

The induced permutation on the subset $J = [m]$ is denoted by $\pi_{n,m}w$. We shall usually omit the index n if it is clear from the context. For example, if $w = (6\ 3\ 5\ 1)(4\ 2)(7)$, then $\pi_4 w = (3\ 1)(4\ 2)$.

Definition 3 ([2]) *The space of virtual permutations \mathfrak{S}^∞ is the projective limit $\mathfrak{S}^\infty = \varprojlim \mathfrak{S}_n$ of finite symmetric groups \mathfrak{S}_n with respect to canonical projections $\pi_{n+1,n} : \mathfrak{S}_{n+1} \rightarrow \mathfrak{S}_n$.*

Thus a virtual permutation is a sequence $(w_1, w_2, \dots) \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \dots$, such that $\pi_n w_{n+1} = w_n$ for all $n \in \mathbb{N}$.

Note that π_n commutes with shifts on elements of the group \mathfrak{S}_n , i.e. for all $N > n$, the equality $\pi_n(g_1^{-1}h g_2) = g_1^{-1}\pi_n(h)g_2$ holds for all $h \in \mathfrak{S}_N$ and $g_1, g_2 \in \mathfrak{S}_n$. Let $\mathfrak{S}_\infty = \bigcup_{n \geq 1} \mathfrak{S}_n$ be the infinite symmetric group. Then the group $G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty$ acts on the space of virtual permutations \mathfrak{S}^∞ as

$$((g_1, g_2)\omega)_i = \begin{cases} g_1^{-1}w_i g_2, & \text{if } i \geq n; \\ \pi_i(g_1^{-1}w_n g_2), & \text{if } i < n. \end{cases} \quad (2)$$

A sequence $\{\mu_n\}$ of distributions on finite symmetric groups \mathfrak{S}_n is called *coherent*, if it accords with projections $\pi_{n+1,n}$, i.e. $\pi_{n+1,n}\mu_{n+1} = \mu_n$ for all $n \in \mathbb{N}$. Each coherent sequence of distributions defines a Borel measure $\mu = \varprojlim \mu_n$ on the space of virtual permutations \mathfrak{S}^∞ , and each Borel measure on \mathfrak{S}^∞ can be presented in this form. *In what follows all measures are assumed to be Borel and normalized.* If all measures μ_n are central, i.e. invariant under inner automorphisms of \mathfrak{S}_n , then the limit distribution μ is invariant under the diagonal subgroup $K = \{(g_1, g_2) \in G : g_1 = g_2\}$.

Definition 4 *K -invariant measures on the space of virtual permutations \mathfrak{S}^∞ are called central. The set of all central measures is denoted by $\mathcal{M}^K(\mathfrak{S}^\infty)$.*

Example 1. Let m_n be the Haar measure on the symmetric group \mathfrak{S}_n . The sequence $\{m_n\}$ is coherent, and the measure $m = \varprojlim m_n$ is called the *Haar measure on the space of virtual permutations*. It is clear that this measure is central. Moreover, it is invariant under the whole group G .

The problem of describing all central measures on the space of virtual permutations is parallel to Kingman's theory of *partition structures* (see, e.g., [3, 4]). Let us reformulate his results in our terms.

Each point $x \in \Sigma$ defines a central measure P^x on the space of virtual permutations. For $x \in \Sigma_1$ they are described as follows. Let us put elements $1, 2, \dots$ at random into cycles and label at random these cycles according to the following rule:

- at the first step the element 1 forms a 1-cycle, and we label it by j with probability x_j ($j = 1, 2, \dots$);
- if the elements $1, \dots, m$ are already placed, and they form k cycles with labels i_1, \dots, i_k , then the element $m + 1$ is inserted in one of possible positions in the j th cycle with probability x_{ij} ($j = 1, \dots, k$), or forms a new 1-cycle labelled by i with probability x_i ($i \neq i_1, \dots, i_k$).

After the n th step of this procedure we obtain a random permutation $w_n \in \mathfrak{S}_n$. It follows from construction that the sequence $\{w_n\}$ is coherent, thus it defines a random element of \mathfrak{S}^∞ , and we denote its distribution by P^x .

Example 2. If $x_0 = (1, 0, \dots)$, then the projection of the measure $\tau = P^{x_0}$ onto \mathfrak{S}_n is the measure τ_n uniformly distributed on one-cycle permutations.

Example 3. The projection of the measure P^x , $x \in \Sigma_1$, on \mathfrak{S}_2 is given by

$$\begin{aligned} P_2^x((1)(2)) &= 2 \sum_{1 \leq i < j < \infty} x_i x_j, \\ P_2^x((12)) &= \sum_{i=1}^{\infty} x_i^2. \end{aligned}$$

Theorem 2 ([4]) *Let μ be a central measure on the space of virtual permutations. Denote by $l_1(w_n) \geq l_2(w_n) \geq \dots$ the cycle lengths of a permutation $w_n \in \mathfrak{S}_n$ in non-increasing order. Then the limits*

$$X_i(\omega) = \lim_{n \rightarrow \infty} \frac{l_i(w_n)}{n}, \quad i = 1, 2, \dots, \quad (3)$$

called the relative cycle lengths of the virtual permutation ω , exist for almost all with respect to μ virtual permutations $\omega = (w_1, w_2, \dots)$. For every $x \in \Sigma$, the conditional distribution of ω , given $(X_1, X_2, \dots) = x$, equals P^x . Thus

$$\mu = \int_{\Sigma} P^x d\nu(x), \quad (4)$$

where ν is the distribution of the vector $X = (X_1, X_2, \dots)$ on Σ .

In particular, if the measure μ is *saturated*, i.e. the sum of relative cycle lengths is equal to 1 a.s. with respect to μ , then the corresponding measure ν is supported by Σ_1 .

Example 4. The distribution of the relative cycle lengths of virtual permutations with respect to the Haar measure is the Poisson–Dirichlet distribution $PD(1)$.

Thus there is a one-to-one correspondence $\rho : \nu \mapsto \mu$ between Borel distributions ν on Σ and central distributions μ on \mathfrak{S}^∞ . Given ν , the corresponding central measure $\mu = \rho(\nu)$ is recovered via (4). And given μ , the corresponding measure $\nu = \rho^{-1}(\mu)$ on Σ is the distribution of the relative cycle lengths with respect to μ .

Remark. The space of virtual permutations \mathfrak{S}^∞ was introduced by S. Kerov, G. Olshanski and A. Vershik [2]. They considered a family of quasi-invariant distributions on \mathfrak{S}^∞ with respect to the action of the group $G = \mathfrak{S}_\infty \times \mathfrak{S}_\infty$ and studied the associated family of unitary representations of the infinite symmetric group.

3 Reduction lemmas

The infinite symmetric group \mathfrak{S}_∞ acts on the space of virtual permutations by two-sided shifts (2). Denote by $R_g : \mathfrak{S}^\infty \rightarrow \mathfrak{S}^\infty$ the right shift on $g \in \mathfrak{S}_\infty$.

One can easily check that the image $\mu^g = R_g\mu$ of a central measure μ enjoys the property that the normalized cycle lengths exist for almost all with respect to μ^g virtual permutations. Hence we may define the projection of R_g on Σ , which is already a Markov operator. It sends a point $x \in \Sigma$ to $\rho^{-1}(\omega g)$, where ω is a random virtual permutation with distribution P^x . Thus the corresponding operator T_g on the space $M(\Sigma)$ of probability measures on Σ completes the commutative diagram

$$\begin{array}{ccc} M^K(\mathfrak{S}^\infty) & \xrightarrow{R_g} & M(\mathfrak{S}^\infty) \\ \rho \uparrow & & \rho^{-1} \downarrow \\ M(\Sigma) & \xrightarrow{T_g} & M(\Sigma). \end{array}$$

One can easily see that the operator T_g depends in fact only on the conjugacy class of g , i.e. on its cycle structure.

The subsimplex $\Sigma_a = \{x \in \Sigma : \sum x_i = a\}$ is T_g -invariant for all $a \in [0, 1]$ and all $g \in \mathfrak{S}_\infty$ (since the shift by a finite permutation does not change the sum of the relative cycle lengths of a virtual permutation). It is proved in [7] that these sets are ergodic components, and the ergodic measure concentrated on Σ_a is the image of $PD(1)$ under the homothetic transformation $\Gamma_a : \Sigma_1 \rightarrow \Sigma_a$. In particular,

Theorem 3 ([7]) *The only measure on Σ_1 which is invariant under the family $\{T_g\}_{g \in \mathfrak{S}_\infty}$ is the Poisson–Dirichlet distribution $PD(1)$.*

The relation of this theorem to our problem is given by the following lemma (note that T_e is obviously the identity operator, and we may consider the identity operator E on Σ_1 as the restriction of T_e on Σ_1).

Lemma 1 *The Markov operator T on Σ_1 defined by (1) is the restriction on Σ_1 of the operator $T_{(1,2)}$ ($= T_g$, where g is an arbitrary transposition in \mathfrak{S}_∞).*

Proof. Follows from the definition of T_g and Example 3.

Now we cite some preliminary lemmas from [7] which we shall use in the sequel. First, we reformulate the problem in terms of the space of virtual permutations. Since there is a one-to-one correspondence between Borel measures on Σ and central measures on \mathfrak{S}^∞ , one may think of the operator T_g as acting on the space $\mathcal{M}^K(\mathfrak{S}^\infty)$. More exactly, denote by \tilde{T}_g the operator on $\mathcal{M}^K(\mathfrak{S}^\infty)$, corresponding to T_g , i.e., the image of T_g under the map ρ .

Lemma 2 ([7]) *Let $\mu \in \mathcal{M}(\mathfrak{S}^\infty)$. Then $\tilde{T}_g\mu = PR_g\mu$, where $P : M(\mathfrak{S}^\infty) \rightarrow M^K(\mathfrak{S}^\infty)$ is the projection (conditional expectation) onto the space of central measures.*

In particular, let $\tilde{T} = \tilde{T}_{(1,2)}$ and $\tilde{E} = \tilde{T}_e$ (the identity operator on $\mathcal{M}(\mathfrak{S}^\infty)$). Then our Theorem 1 is equivalent to the following proposition. Let $\tau = \varprojlim \tau_n = P^{x_0}$ be the central measure on \mathfrak{S}^∞ corresponding to the vector $x_0 = (1, 0, 0, \dots) \in \varprojlim \Sigma_1$ (see Example 3).

Proposition 1 *Let $\tau^q = \left(\frac{\tilde{E} + \tilde{T}}{2}\right)^q \tau$. Then*

$$\lim_{q \rightarrow \infty} \tau^q = m,$$

where m is the Haar measure on the space of virtual permutations.

The ergodic method allows one to express the operator \tilde{T}_g in the following form.

Lemma 3 ([7]) *Let μ be a central measure on the space of virtual permutations. Then*

$$\tilde{T}_g \mu = \lim_{N \rightarrow \infty} \frac{1}{N!} \sum_{u \in \mathfrak{S}_N} u^{-1} (R_g \mu) u. \quad (5)$$

Finally, the formula (5) may be restated in terms of distributions on *finite* symmetric groups.

Lemma 4 ([7]) *Let μ be a central measure on the space of virtual permutations, and μ_n be the corresponding coherent family of measures on finite symmetric groups. Then the finite-dimensional projections of the measure $\tilde{T}_g \mu$ are given by the following formula,*

$$(\tilde{T}_g \mu)_n(u) = \sum_{\substack{w \in \mathfrak{S}_{n+k} \\ \pi_n(w) = u}} \mu_{n+k}(wg^n) \quad \forall n, k \in \mathbb{N}, \forall g \in \mathfrak{S}^k, \forall u \in \mathfrak{S}_n, \quad (6)$$

where $\mathfrak{S}^k = \mathfrak{S}[n+1, \dots, n+k] \subset \mathfrak{S}_{n+k}$.

4 Shifted projection of irreducible characters of symmetric groups

In this section we present a formula obtained in [6] which describes the action of the operator \tilde{T}_g on irreducible characters χ_λ of the symmetric group \mathfrak{S}_{n+1} . It generalizes a known formula for the action on χ_λ of the canonical projection π_n .

Fix $n, k \in \mathbb{N}$, $N = n + k$ and $g \in \mathfrak{S}^k = \mathfrak{S}[n+1, \dots, n+k]$. Define a projection $\pi_{N,n}^g : \mathfrak{S}_N \rightarrow \mathfrak{S}_n$ as

$$\pi_{N,n}^g(h) = \pi_n(hg).$$

Then the formula (6) from Lemma 4 may be formulated as follows:

$$(\tilde{T}_g \mu)_n = \pi_{N,n}^g \mu_N$$

for all $n, k \in \mathbb{N}$, $N = n + k$, $g \in \mathfrak{S}^k$. We may consider the central measure μ_n as a central function on the symmetric group \mathfrak{S}_n . Since the characters of irreducible representations

form a basis in the space of central functions, it is interesting to consider the action of $\pi_{N,n}^g$ on the irreducible characters. Recall that the irreducible representations of the symmetric group \mathfrak{S}_n are indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ of the number n (which we identify with the corresponding Young diagrams). Denote by Π_n the set of all partitions of n . Thus we are interested in a formula for

$$\pi_{N,n}^g \chi_\lambda(u) = \sum_{\substack{w \in \mathfrak{S}_N \\ \pi_n(w)=u}} \chi_\lambda(wg), \quad u \in \mathfrak{S}_n.$$

Remark. If $g = e$ this is the action of the canonical projection π_n on the irreducible characters. As follows from the arguments of [5, Sect. 4.6],

$$\pi_n \chi_\lambda = \sum_{\mu \nearrow \lambda} (c(\lambda \setminus \mu) + 1) \chi_\mu, \quad \lambda \in \Pi_{n+1}, \quad (7)$$

where $\mu \nearrow \lambda$ means that the diagram $\mu \in \Pi_n$ is obtained from λ by removing one cell.

We recall some definitions related to Young diagrams. The *Young diagram* D_λ of the partition $\lambda \in \Pi_n$ is the set $\{(i, j) : i, j \in \mathbb{Z}, i \geq 1, \lambda_i \geq j \geq 1\}$. If $x = (i, j) \in D_\lambda$, then x is called a *cell* of the diagram D_λ . The *content* of a cell $x = (i, j)$ is the number $c(x) = j - i$. Let $m < n$, $\lambda \in \Pi_n$, $\mu \in \Pi_m$. We write $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for all $i = 1, 2, \dots$, that is, the Young diagram of λ contains the Young diagram of μ . The set-theoretic difference of these diagrams is called a *skew diagram*. A skew diagram is called a *skew hook*, if it is connected and does not contain two cells in one diagonal. If a skew hook contains k cells, it is called a *skew k -hook*. Denote by Θ_k the set of skew k -hooks. The *height* of a skew hook $\lambda \setminus \mu$ is the number $l(\lambda \setminus \mu)$ equal to the number of rows it intersects minus one. A skew diagram is called a *horizontal m -strip*, if it contains m cells and has at most one cell in each row.

Proposition 2 ([6]) *Let χ_λ be the character of the irreducible representation of the symmetric group \mathfrak{S}_N corresponding to the partition $\lambda \in \Pi_N$, and let $g \in \mathfrak{S}^k$ be a permutation consisting of one cycle of length k . Then*

$$\pi_{N,n}^g \chi_\lambda = \sum_{\lambda \setminus \mu \in \Theta_k} (-1)^{l(\lambda \setminus \mu)} \prod_{x \in \lambda \setminus \mu} (c(x) + 1) \chi_\mu. \quad (8)$$

Corollary *Let χ_λ be the character of the irreducible representation of the symmetric group \mathfrak{S}_n corresponding to the partition $\lambda \in \Pi_{n+k}$, and let $g \in \mathfrak{S}^k$ be a permutation with cycle structure $\nu = (\nu_1, \dots, \nu_s) \in \Pi_k$. Then*

$$\pi_{N,n}^g \chi_\lambda = \sum_{\mathbf{M}} (-1)^{l(\mathbf{M})} \prod_{x \in \lambda \setminus \mu} (c(x) + 1) \chi_\mu, \quad (9)$$

where the sum is taken over all s -tuples $\mathbf{M} = \{\mu_1, \dots, \mu_s\}$ such that $\lambda = \mu_0 \supset \mu_1 \supset \dots \supset \mu_s \in \Pi_n$, $\mu_{i-1} \setminus \mu_i$ is a skew ν_i -hook for all $i = 1, \dots, s$, and $l(\mathbf{M}) = \sum_{i=1}^s l(\mu_{i-1} \setminus \mu_i)$.

5 Proof of Proposition 1

Let $N = n + 2m$, and $\lambda \in \Pi_N$.

Lemma 5

$$\frac{1}{2^m}(\tilde{E} + \tilde{T})^m \chi_\lambda = \sum_{\mu \in \Pi_n} \chi_\mu \cdot \left(\prod_{x \in \lambda \setminus \mu} (c(x) + 1) \right) \cdot p(\lambda, \mu), \quad (10)$$

where $p(\lambda, \mu)$ is the number of all sequences $M = \{\mu_0 = \mu, \mu_1, \dots, \mu_m = \lambda\}$ such that $\mu_k \in \Pi_{n+2k}$, and $\mu_{k+1} \setminus \mu_k$ is a horizontal 2-strip for all $k = 0, \dots, m-1$.

Proof. Note that

$$\begin{aligned} \frac{1}{2^m}(\tilde{E} + \tilde{T})^m \chi_\lambda = \\ \left(\frac{\pi_{n+2,n} + \pi_{n+2,n}^{\text{tr}}}{2} \right) \left(\frac{\pi_{n+4,n+2} + \pi_{n+4,n+2}^{\text{tr}}}{2} \right) \dots \left(\frac{\pi_{N,N-2} + \pi_{N,N-2}^{\text{tr}}}{2} \right) \chi_\lambda, \end{aligned}$$

where $\pi_{l+2,l}^{\text{tr}} = \pi_{l+2,l}^{(l+1,l+2)}$. Then it follows from Proposition 2 that the projection of $\frac{1}{2^m}(E + T)^m \chi_\lambda$ onto \mathfrak{S}_n equals

$$\sum_{\mu \in \Pi_n} \chi_\mu \cdot \left(\prod_{x \in \lambda \setminus \mu} (c(x) + 1) \right) \cdot \sum_M \alpha(M),$$

where the sum is taken over all sequences $M = \{\mu_0 = \mu, \mu_1, \dots, \mu_m = \lambda\}$ such that $\mu_k \subset \mu_{k+1}$, $\mu_k \in \Pi_{n+2k}$, and $\alpha(M) = \prod_{k=1}^m \alpha(\mu_k \setminus \mu_{k-1})$, where $\alpha(\cdot) = \frac{\alpha_1(\cdot) + \alpha_2(\cdot)}{2}$ with

$$\begin{aligned} \alpha_1(\lambda \setminus \nu) &= \begin{cases} 1, & \text{if } \lambda \setminus \nu \text{ is a 2-row or a 2-column,} \\ 2, & \text{otherwise.} \end{cases} \\ \alpha_2(\lambda \setminus \nu) &= \begin{cases} 1, & \text{if } \lambda \setminus \nu \text{ is a 2-row,} \\ -1, & \text{if } \lambda \setminus \nu \text{ is a 2-column,} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

that is,

$$\alpha(\lambda \setminus \nu) = \begin{cases} 1, & \text{if } \lambda \setminus \nu \text{ is a horizontal 2-strip,} \\ 0, & \text{otherwise (that is, if } \lambda \setminus \nu \text{ is a 2-column).} \end{cases}$$

The Lemma follows.

Now each central measure on \mathfrak{S}_N can be decomposed into a combination of irreducible characters. In particular, let

$$\tau_N(g) = \sum_{\lambda \in \Pi_N} a_\lambda \chi_\lambda(g).$$

It is easy to see that the coefficients of this decomposition equal

$$a_\lambda = \frac{1}{N!} \chi_\lambda((12 \dots N)).$$

It follows from the Murnaghan–Nakayama rule that these coefficients are not zero only for hook partitions $\lambda_k^N = (k1^{N-k})$, for $k = 1, \dots, N$, and $a_{(k1^{N-k})} = \frac{(-1)^{N-k}}{N!}$. Thus

$$\tau_N = \frac{1}{N!} \sum_{k=1}^N (-1)^{N-k} \chi_{(k1^{N-k})}.$$

Now we are ready to prove Proposition 1.

First let $n = 2$. Denote $c(\lambda, \mu) = \prod_{x \in \lambda \setminus \mu} (c(x) + 1)$. Then the projection τ_2^q of τ^q on \mathfrak{S}_2 can be written as $a_q \chi_{(2)} + b_q \chi_{(1^2)}$, where

$$\begin{aligned} a_q &= \sum_{k=1}^N \frac{(-1)^k}{N!} \cdot c(\lambda_k^N, (2)) \cdot p(\lambda_k^N, (2)), \\ b_q &= \sum_{k=1}^N \frac{(-1)^k}{N!} \cdot c(\lambda_k^N, (1^2)) \cdot p(\lambda_k^N, (1^2)). \end{aligned}$$

It is clear that the coefficient $p(\lambda_k^N, (2))$ is not zero only for $k = N$, and $c((N), (2)) = 3 \cdot 4 \cdot \dots \cdot N = N!/2$. Thus we obtain that $a_q = 1/2$. The corresponding term $\frac{1}{2} \chi_{(2)} = m_2$ is just the Haar measure on \mathfrak{S}_2 . Thus it suffices to show that $b_q \rightarrow 0$ as $q \rightarrow \infty$.

Let us calculate the coefficient $p(\lambda_k^N, (1^2))$. It follows from Lemma 5 that we must calculate the number of paths from (1^2) to $(k1^{N-k})$ such that at each step we add either two cells in one row or two cells in different rows and different columns. It is easy to see that this number equals $\binom{q}{N-k-1}$. The content coefficient equals $c((k1^{N-k}), (1^2)) = (-1)^{k+1} k!(N-k-1)!$, thus the summand in b_q corresponding to λ_k^N equals

$$-\frac{q!k!(N-k-1)!}{N!(N-k-1)!(k-q-1)!} = -\frac{k!q!}{N!(k-q-1)!}.$$

Hence,

$$\begin{aligned} b_q &= \frac{q!}{N!} \sum_{k=q+1}^{2q+1} \frac{k!}{(k-q-1)!} = \frac{q!(q+1)!}{N!} \sum_{k=q+1}^{2q+1} \frac{k!}{(q+1)!(k-q-1)!} = \\ &= \frac{q!(q+1)!}{N!} \sum_{k=q+1}^{2q+1} \binom{k}{q+1} = \frac{q!(q+1)!}{N!} \binom{2q+2}{q+2} = \frac{1}{q+2} \rightarrow 0 \end{aligned}$$

as $q \rightarrow \infty$, as desired.

The proof for $n > 2$ goes in the same way with a bit complicated calculations. It is clear that the decomposition of τ_n^q contains only hook characters, that is,

$$\tau_n^q = \sum_{l=1}^n a_q(l) \chi_{(l1^{n-l})},$$

and

$$a_q(l) = \sum_{k=1}^N \frac{(-1)^{N-k}}{N!} \cdot c(\lambda_k^N, \lambda_l^n) \cdot p(\lambda_k^N, \lambda_l^n).$$

If $l = n$, there is again only one non-zero summand corresponding to $k = N$,

$$a_q(n) = \frac{1}{N!} \cdot \frac{N!}{n!} \cdot 1 = \frac{1}{n!},$$

and $a_q(n)\chi_{(n)} = m_n$, that is, the Haar measure on \mathfrak{S}_n . Thus we only have to prove that $a_q(l) \rightarrow 0$ as $q \rightarrow \infty$ for $l < n$. Easy calculations show that $p(\lambda_k^N, \lambda_l^n) = \binom{q}{2q-k+l}$, and $c(\lambda_k^N, \lambda_l^n) = (-1)^{l-k} \frac{k!(N-k-1)!}{l!(n-l-1)!}$. Thus

$$a_q(l) = \frac{(-1)^{l+N}}{l!(n-l-1)!} \cdot \frac{q!}{N!} \cdot \sum_{k=q+l}^{2q+l} \frac{k!(N-k-1)!}{(2q-k+l)!(k-l-q)!}.$$

But $\frac{(N-k-1)!}{(N-k-n+l)!} \leq (N-k-1)^{n-l-1} \leq (q+n-l-1)^{n-l-1}$. Then

$$\begin{aligned} |a_q(l)| &\leq \text{const} \cdot \frac{(q+n-l-1)^{n-l-1} q! (q+l)!}{N!} \sum_{k=q+l}^{2q+l} \binom{k}{q+l} \leq \\ &\leq \text{const} \cdot \frac{(q+n-l-1)^{n-l-1} q! (q+l)!}{N!} \cdot \binom{2q+l+1}{q+l+1} \leq \\ &\leq \text{const} \cdot \frac{(q+n-l-1)^{n-l-1}}{q+l+1} \cdot \frac{(2q+l+1)!}{(2q+n)!} \leq \\ &\leq \text{const} \cdot \frac{1}{q+l+1} \left(\frac{q+n-l-1}{2q+l+2} \right)^{n-l-1} \rightarrow 0 \end{aligned}$$

as $q \rightarrow \infty$. Proposition follows.

References

- [1] A. Gnedin, S. Kerov. A characterization of GEM distributions via a split-and-merge transformation, *PDMI preprint* 13/1999 (1999), available at <ftp://ftp.pdmi.ras.ru/pub/publicat/preprint/1999/13-99.ps.gz>
- [2] S. V. Kerov, G. I. Olshanski, A. M. Vershik, Harmonic Analysis on the Infinite Symmetric Group, *Comptes Rend. Acad. Sci. Paris* **316** (1993), 773–778.
- [3] J. F. C. Kingman, Random discrete distributions, *J. R. Stat. Soc.* **37** (1975), 1–22.
- [4] J. F. C. Kingman, The representation of partition structures, *J. London Math. Soc.* (2) **18** (1978), 374–380.

- [5] G. I. Olshanski, Unitary representations of (G, K) -pairs related to the infinite symmetric group, *Algebra and Analysis* **1** (1989), 178–209.
- [6] N. V. Tsilevich, Poisson–Dirichlet measures and virtual permutations, Ph.D thesis, Steklov Institute of Mathematics at St.Petersburg, 1998.
- [7] N. V. Tsilevich, Stationary random partitions of positive integers, *Theory of Prob. and Appl.* **44** (1999), 60–74.