

Infinite-dimensional Schur–Weyl duality and the Coxeter–Laplace operator

N. V. Tsilevich* A. M. Vershik*

September 21, 2012

Abstract

We extend the classical Schur–Weyl duality between representations of the groups $SL(n, \mathbb{C})$ and \mathfrak{S}_N to the case of $SL(n, \mathbb{C})$ and the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$. Our construction is based on a “dynamic,” or inductive, scheme of Schur–Weyl dualities. It leads to a new class of representations of the infinite symmetric group, which have not appeared earlier. We describe these representations and, in particular, find their spectral types with respect to the Gelfand–Tsetlin algebra. The main example of such a representation acts in an incomplete infinite tensor product. As an important application, we consider the weak limit of the so-called Coxeter–Laplace operator, which is essentially the Hamiltonian of the XXX Heisenberg model, in these representations.

*St. Petersburg Department of Steklov Institute of Mathematics. E-mail: natalia@pdmi.ras.ru, vershik@pdmi.ras.ru. Supported by the grants RFBR 11-01-12092-ofi-m and RFBR 11-01-00677-a.

ГЛАВНЫЙ РЕДАКТОР

С. В. Кисляков

РЕДКОЛЛЕГИЯ

В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов, А.И.Генералов,
И.А.Ибрагимов, Л.Ю.Колотилина, П.П.Кулиш, Б.Б.Лурье, Ю.В.Матиясевич,
Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин, В.Н.Судаков, О.М.Фоменко

1 Introduction

1.1 General setting

We extend the classical Schur–Weyl duality [15, Chap. 4, Sec. 4] between irreducible representations of the general linear group $GL(n, \mathbb{C})$ (or the special linear group $SL(n, \mathbb{C})$) and the (finite) symmetric group \mathfrak{S}_N to the case of $SL(n, \mathbb{C})$ and the infinite symmetric group $\mathfrak{S}_\mathbb{N}$. Usually, one considers only the “static” Schur–Weyl duality, when the parameter N is fixed. Our construction is based on a “dynamic” view of this duality, which allows us to consider an inductive scheme and pass to the limit, obtaining an infinite-dimensional version of the Schur–Weyl duality. This construction leads to a new class of representations of the infinite symmetric group, which have not appeared earlier. The main example is the so-called tensor representation, which is realized in an incomplete infinite tensor product.

In this paper, we consider only the simplest case $N = 2$, since the case of a general N can be handled in exactly the same way.

One of our motivations for considering Schur–Weyl representations was to study the behavior of the so-called Coxeter–Laplace operator L_N , or the Hamiltonian of the XXX Heisenberg model, in these representations. In particular, we show that a generalized Schur–Weyl scheme allows one to construct a representation in which the weak limit of $\frac{1}{N}L_N$ is a scalar operator with the scalar arbitrarily close to the maximum possible value c_{\max} .

Our representations have natural links to representations of the Virasoro algebra, Glimm algebra, and other important representation-theoretic objects. Further analysis should clarify these relations. We would also like to mention the paper [9], where another infinite-dimensional generalization of the Schur–Weyl scheme is developed. The difference is as follows: starting from the classical Schur–Weyl duality between $GL(n, \mathbb{C})$ and \mathfrak{S}_N , we keep n fixed and send N to infinity, obtaining a duality between $GL(n, \mathbb{C})$ and $\mathfrak{S}_\mathbb{N}$; in [9], on the contrary, N is kept fixed and n goes to infinity, resulting in a duality between \mathfrak{gl}_∞ and $\mathfrak{S}_\mathbb{N}$. Another related paper is [1], where an inductive construction of representations of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$ is suggested, which starts from the tensor representations of \mathfrak{sl}_2 and uses the notion of fusion product of representations.

1.2 Main results

Now let us describe our results in more detail.

We consider the representations of $\mathfrak{S}_{\mathbb{N}}$ that are the inductive limits of two-row representations of the finite symmetric groups under so-called *Schur–Weyl embeddings*, which send a representation of \mathfrak{S}_N to a representation of \mathfrak{S}_{N+2} and respect both the actions of $SL(2, \mathbb{C})$ and \mathfrak{S}_N . The structure of a general representation of this kind (which we also call *Schur–Weyl representations* of $\mathfrak{S}_{\mathbb{N}}$) is described in Theorem 1. Namely,

$$\mathcal{H} = \sum_k \Pi_k \otimes M_{k+1}, \quad (1)$$

where Π_k is an irreducible representation of $\mathfrak{S}_{\mathbb{N}}$ (the inductive limit of a sequence of irreducible representations of the symmetric groups), M_k is the irreducible representation of $SL(2, \mathbb{C})$ of dimension k , and the sum is taken over either odd or even k ; moreover, $\Pi_k \otimes M_{k+1}$ is an irreducible representation of $\mathfrak{S}_{\mathbb{N}} \times SL(2, \mathbb{C})$, and the operator algebras generated by the actions of $\mathfrak{S}_{\mathbb{N}}$ and $SL(2, \mathbb{C})$ are mutual commutants.

Thus it suffices to study the irreducible representations Π_k of $\mathfrak{S}_{\mathbb{N}}$ obtained in this way. In particular, we show that the spectral type of such a representation with respect to the Gelfand–Tsetlin algebra is determined by a σ -finite, Bernoulli-like, noncentral measure on the space of infinite Young tableaux (Theorem 2).

The most interesting example of a Schur–Weyl representation is the so-called *tensor* representation, obtained from the unique Schur–Weyl embeddings that preserve the tensor product structure of the space $(\mathbb{C})^{\otimes n}$ to which the Schur–Weyl duality applies. This representation is studied in Section 5. It can be realized in the incomplete tensor product of the spaces \mathbb{C}^4 , the distinguished vector being the unique $SL(2, \mathbb{C})$ -invariant vector in \mathbb{C}^4 .

Observe an analogy between the decomposition (1) of a Schur–Weyl representation of the infinite symmetric group and the decomposition (a limiting case of the Goddard–Kent–Olive construction)

$$\mathcal{M}_j = \sum_k L(1, k^2) \otimes M_{k+1}, \quad j = 0, 1/2,$$

where \mathcal{M}_j is the level 1 spin j fundamental representation of the affine Lie algebra $\widehat{\mathfrak{sl}}_2$, $L(1, k^2)$ is the irreducible representation of the Virasoro algebra

Vir with central charge 1 and conformal dimension k^2 , the summation is over all positive integers k of the same parity as j , and the algebras **Vir** and $\widehat{\mathfrak{sl}}_2 \subset \widehat{\mathfrak{sl}}_2$ are mutual commutants (see, e.g., [14]). This analogy suggests that one may introduce a natural action of the Virasoro algebra in a Schur–Weyl module, or, equivalently, a natural action of the infinite symmetric group in the fundamental module \mathcal{M}_j .

The last section of the paper concerns the so-called *periodic Coxeter Laplacian*, or the *Coxeter–Laplace operator*. This is the operator $L_N = Ne - (s_1 + \dots + s_N)$ in the group algebra of the symmetric group \mathfrak{S}_N , where s_k is the Coxeter transposition $(k, k+1)$ (with $N+1 \equiv 1$). If π_N is the standard representation of \mathfrak{S}_N in $(\mathbb{C}^2)^{\otimes N}$, then the operator $\pi_N(L_N)$ is related to the Hamiltonian of the XXX Heisenberg model on the periodic one-dimensional lattice with N sites (see, e.g., [4, 10]) by the formula $H = \frac{J}{4}(2L - N)$, where $J > 0$ corresponds to the ferromagnetic case, and $J < 0$ to the antiferromagnetic one. In Section 6, we find the “antiferromagnetic” weak limit of the Coxeter–Laplace operator in the stationary Schur–Weyl representations of the infinite symmetric groups (Proposition 4). It is a scalar operator with constant depending on the parameter of Schur–Weyl embeddings, and the maximum possible value of this constant is greater than for all other natural representations of $\mathfrak{S}_\mathbb{N}$ considered so far. But it is still smaller than the limiting value c_{\max} of the ground energy first computed in [3] and rigorously proved in [16]. However, in Proposition 5 we show that by extending the construction of Schur–Weyl embeddings one can build a representation of $\mathfrak{S}_\mathbb{N}$ with the corresponding constant arbitrarily close to c_{\max} .

1.3 Notation

Here we present necessary notation from the combinatorics of Young diagrams and tableaux. Let \mathbb{Y}_N be the set of Young diagrams with N cells and $\mathbb{Y}_N^l \subset \mathbb{Y}_N$ be the set of Young diagrams with N cells and at most l rows. Let \mathcal{T}_N be the set of Young tableaux with N cells. Given $\lambda \in \mathbb{Y}_N$, let $\mathcal{T}_N(\lambda) \subset \mathcal{T}_N$ be the set of Young tableaux with diagram λ , that is, the set of paths in the Young graph from the unique vertex \emptyset of zero level to λ . Given Young diagrams $\lambda \in \mathbb{Y}_N$, $\mu \in \mathbb{Y}_{N+k}$ such that $\lambda \subset \mu$, we also denote by $\mathcal{T}(\lambda, \mu)$ the set of paths in the Young graph from λ to μ , and by $H(\lambda, \mu)$ the Hilbert space in which the elements of $\mathcal{T}(\lambda, \mu)$ are an orthonormal basis. By $[t]_N$ we denote the initial segment of length N of a Young tableau t . Finally, given $\lambda \in \mathbb{Y}_N$ and $\mu \in \mathbb{Y}_{N+2}$ with at most two rows each, we say that the

pair (λ, μ) is *nice* if $\lambda \subset \mu$ and μ is obtained from λ by adding one cell to each row. Obviously, if a pair (λ, μ) is nice, then $\mathcal{T}(\lambda, \mu)$ contains exactly two elements.

2 An inductive construction of Schur–Weyl embeddings

The classical Schur–Weyl duality (see, e.g., [2]) is a fundamental theorem that relates irreducible representations of the general linear group $GL(l, \mathbb{C})$ and the symmetric group \mathfrak{S}_N in the tensor power $(\mathbb{C}^l)^{\otimes N}$, where \mathfrak{S}_N permutes the factors and $GL(l, \mathbb{C})$ acts by the simultaneous matrix multiplication:

$$(\mathbb{C}^l)^{\otimes N} = \sum_{\lambda \in \mathbb{Y}_N^l} \pi_\lambda \otimes \rho_\lambda,$$

where π_λ is the irreducible representation of \mathfrak{S}_N corresponding to λ and ρ_λ is the irreducible representation of $GL(l)$ with signature λ . The operator algebras generated by the actions of \mathfrak{S}_N and $GL(l)$, respectively, are mutual commutants in the whole operator algebra $\text{End}((\mathbb{C}^l)^{\otimes N})$.

Consider the particular case $l = 2$. For definiteness, let $N = 2n + 1$ be odd. Then

$$(\mathbb{C}^2)^{\otimes N} = \sum_{\lambda \in \mathbb{Y}_N^2} \pi_\lambda \otimes \rho_\lambda = \sum_{k=0}^n \pi_{\lambda^{(k)}} \otimes \rho_{\lambda^{(k)}},$$

where $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}) = (n + k + 1, n - k)$, so that $\lambda_1^{(k)} - \lambda_2^{(k)} = 2k + 1$.

Consider the restriction of $\rho_{\lambda^{(k)}}$ to the subgroup $SL(2, \mathbb{C}) \subset GL(2, \mathbb{C})$. This is an irreducible representation of $SL(2, \mathbb{C})$ that depends only on the difference $\lambda_1^{(k)} - \lambda_2^{(k)} = 2k + 1$, i.e., on k . Let M_m be the irreducible representation of $SL(2, \mathbb{C})$ of dimension m . Then

$$(\mathbb{C}^2)^{\otimes(2n+1)} = \sum_{j=0}^n \pi_{(n+j+1, n-j)} \otimes M_{2j+2}. \quad (2)$$

In a similar way, for even $N = 2n$ we have

$$(\mathbb{C}^2)^{\otimes(2n)} = \sum_{j=0}^n \pi_{(n+j, n-j)} \otimes M_{2j+1}. \quad (3)$$

Now consider embeddings $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ that preserve this Schur–Weyl structure. We endow the tensor spaces under consideration with the standard inner product and regard all representations as unitary representations. Observe that both in $(\mathbb{C}^2)^{\otimes N}$ and $(\mathbb{C}^2)^{\otimes(N+2)}$ we have actions of $SL(2, \mathbb{C})$ and \mathfrak{S}_N (with the standard embedding $\mathfrak{S}_N \subset \mathfrak{S}_{N+2}$).

Definition 1. *Isometric embeddings $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ that are equivariant with respect to both these actions (in other words, equivariant with respect to the action of $\mathfrak{S}_N \times SL(2, \mathbb{C})$) will be called Schur–Weyl embeddings.*

Our first purpose is to describe all Schur–Weyl embeddings $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$. Proposition 1 below says that such an embedding is determined by a sequence of vectors from the one-dimensional complex circle \mathbb{T}^1 .

Proposition 1. *The Schur–Weyl embeddings $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ with $N = 2n - 1$ or $N = 2n$ are indexed by the elements of $(\mathbb{T}^1)^n$, where \mathbb{T}^1 is the one-dimensional complex circle: $\mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\}$.*

Proof. Obviously, under a Schur–Weyl embedding, for each k we have

$$\pi_{(n+k-1, n-k)} \otimes M_{2k} \hookrightarrow \pi_{(n+k, n-k+1)} \otimes M_{2k},$$

where $\pi_{(n+k-1, n-k)} \hookrightarrow \pi_{(n+k, n-k+1)}$ is an embedding of the irreducible representation $\pi_{(n+k-1, n-k)}$ of \mathfrak{S}_{2n-1} into the irreducible representation $\pi_{(n+k, n-k+1)}$ of \mathfrak{S}_{2n+1} . A similar fact holds in the case of an even N .

Obviously, $H(\lambda, \mu)$ is the multiplicity space of π_λ in π_μ , so that the isometric embeddings $\pi_\lambda \hookrightarrow \pi_\mu$ commuting with the action of \mathfrak{S}_N are indexed by the unit vectors $h \in H(\lambda, \mu)$. For $\lambda = (n+k-1, n-k)$ and $\mu = (n+k, n-k+1)$, the pair (λ, μ) is nice, so that the set $\mathcal{T}(\lambda, \mu)$ consists of two tableaux: the tableau t_{21} is obtained by putting the element $2n$ into the second row and the element $2n+1$ into the first row, while the tableau t_{12} is obtained by putting $2n$ into the first row and $2n+1$ into the second row. Thus we can identify the spaces $H((n+k-1, n-k), (n+k, n-k+1))$ for all k . Denote the obtained space by $H^{1,1}$, and fix the standard basis $\{t_{21}, t_{12}\}$ of $H^{1,1}$. Then the isometric embeddings $\pi_\lambda \hookrightarrow \pi_\mu$ commuting with the action of \mathfrak{S}_N are indexed by the elements of the one-dimensional complex circle $\mathbb{T}^1 \subset H^{1,1}$.

In the case of even N , the situation is exactly the same, with the only exception: for the diagrams $\lambda = (n, n)$ and $\mu = (n+1, n+1)$, the multiplicity space $H(\lambda, \mu)$ is one-dimensional. \square

Remark. Note that a Schur–Weyl embedding $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2)}$ does not necessarily preserve the tensor product structure. The important class of Schur–Weyl embeddings that do have this property is considered in Section 5.

The scheme described above has a natural generalization. Namely, we can consider embeddings that preserve the Schur–Weyl structure but “jump” over an arbitrary (even) number of levels instead of two ones.

Definition 2. A generalized Schur–Weyl embedding is an isometric embedding $(\mathbb{C}^2)^{\otimes N} \hookrightarrow (\mathbb{C}^2)^{\otimes(N+2k)}$, $k \geq 1$, that is equivariant with respect to the actions of $SL(2, \mathbb{C})$ and \mathfrak{S}_N (with the standard embedding $\mathfrak{S}_N \subset \mathfrak{S}_{N+2k}$).

The theory of Schur–Weyl representations of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ developed below can easily be extended to the case of generalized Schur–Weyl embeddings. In particular, generalized Schur–Weyl embeddings are used in Section 6 for constructing a representation of $\mathfrak{S}_{\mathbb{N}}$ in which the weak limit of the so-called Coxeter–Laplace operators has an eigenvalue that is arbitrarily close to the maximal possible value.

3 Infinite-dimensional Schur–Weyl duality

Consider an *infinite* chain of Schur–Weyl embeddings

$$(\mathbb{C}^2)^{\otimes 0} \hookrightarrow (\mathbb{C}^2)^{\otimes 2} \hookrightarrow (\mathbb{C}^2)^{\otimes 4} \hookrightarrow \dots \quad \text{or} \quad (\mathbb{C}^2)^{\otimes 1} \hookrightarrow (\mathbb{C}^2)^{\otimes 3} \hookrightarrow (\mathbb{C}^2)^{\otimes 5} \hookrightarrow \dots \quad (4)$$

(in what follows, these two cases will be referred to as “even” and “odd,” respectively) and the corresponding inductive limit Π of representations of the symmetric groups. In the space \mathcal{H} of this representation we have commuting actions of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ and the special linear group $SL(2, \mathbb{C})$. By above, the representation Π is determined by a collection of vectors $h_j^{(k)} \in \mathbb{T}^1$, $k = 0, 1, \dots$, $j = 0, 1, \dots$, where $h_j^{(k)}$ determines the embedding $\pi_{(k+j, j)} \hookrightarrow \pi_{(k+j+1, j+1)}$.

Theorem 1. Let Π be the representation of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ that is the inductive limit of the standard representations of \mathfrak{S}_N in $(\mathbb{C}^2)^{\otimes N}$ with respect to an infinite chain (4) of Schur–Weyl embeddings. Then Π decomposes into a countable direct sum of primary representations

$$\Pi = \sum_{k=0}^{\infty} \Pi_k(h^{(k)}) \otimes M_{k+1}, \quad (5)$$

where $\Pi_k(h^{(k)})$ is the inductive limit of the irreducible representations of the symmetric groups $\mathfrak{S}_k, \mathfrak{S}_{k+2}, \dots$ corresponding to the Young diagrams

$$(k), \quad (k+1, 1), \quad (k+2, 2), \quad \dots, \quad (k+n, n), \quad \dots$$

determined by the sequence $h^{(k)} = (h_0^{(k)}, h_1^{(k)}, h_2^{(k)}, \dots) \in (\mathbb{T}^1)^\infty$, and the sum is taken over even k in the even case and over odd k in the odd case.

The representation $\Pi_k(h^{(k)})$ of $\mathfrak{S}_\mathbb{N}$ is irreducible.

Proof. Follows from the previous considerations. \square

Definition 3. The representation Π of the infinite symmetric group will be called the Schur–Weyl representation of $\mathfrak{S}_\mathbb{N}$ determined by the sequence of Schur–Weyl embeddings with parameters $h_j^{(k)} \in \mathbb{T}^1$, $k = 0, 1, \dots$, $j = 0, 1, \dots$. Representations of the form $\Pi_k(h^{(k)})$ will be called irreducible Schur–Weyl representations.

Thus, if we are interested in the representation theory of the infinite symmetric group, it suffices to study the irreducible Schur–Weyl representations $\Pi_k(h)$ for $h = (h_0, h_1, h_2, \dots) \in (\mathbb{T}^1)^\infty$.

An important class of irreducible Schur–Weyl representations consists of representations determined by sequences of Schur–Weyl embeddings that are homogeneous in N , as defined below.

Definition 4. An irreducible Schur–Weyl representation determined by a sequence $h = (h_0, h_1, h_2, \dots) \in (\mathbb{T}^1)^\infty$ is called a stationary Schur–Weyl representation if all h_k coincide with the same vector of \mathbb{T}^1 .

Denote

$$\mathcal{T}^{\text{odd}} = \{\tau = (\tau^{(1)}, \tau^{(3)}, \tau^{(5)}, \dots) : \tau_n \in \mathbb{Y}_n^2, \tau^{(n)} \subset \tau^{(n+2)} \text{ for all } n, \\ (\tau^{(n)}, \tau^{(n+2)}) \text{ is nice for sufficiently large } n\}.$$

Thus \mathcal{T}^{odd} is the set of restrictions of two-row Young tableaux (regarded as sequences of Young diagrams) to the *odd* levels such that for sufficiently large n , the $(n+2)$ th diagram is obtained from the n th one by adding one cell to each row.

By definition, for $\tau \in \mathcal{T}^{\text{odd}}$ with $\tau^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)})$, the sequence $\tau_1^{(n)} - \tau_2^{(n)}$ stabilizes to some value $k = 1, 3, 5, \dots$: $\tau_1^{(n)} - \tau_2^{(n)} = k$ for sufficiently large n . Denoting the corresponding subset of \mathcal{T}^{odd} by $\mathcal{T}_k^{\text{odd}}$, we have

$$\mathcal{T}^{\text{odd}} = \bigcup_{j=1}^{\infty} \mathcal{T}_{2j-1}^{\text{odd}}.$$

The sets $\mathcal{T}_k^{\text{odd}}$ and \mathcal{T}^{odd} are, obviously, countable. The set $\mathcal{T}_k^{\text{odd}}$ consists of tail-equivalent sequences.

Let $h = (h_0, h_1, \dots)$ be the sequence that determines the irreducible Schur–Weyl representation $\Pi_k(h)$ under study; recall that h_j determines the embedding $\pi_{(k+j,j)} \hookrightarrow \pi_{(k+j+1,j+1)}$. Consider an arbitrary $\tau \in \mathcal{T}^{\text{odd}}$. By the definition of \mathcal{T}^{odd} , for sufficiently large j , the pair $(\tau_{k+2j}, \tau_{k+2j+2})$ is nice, so that we may identify $H(\tau_{k+2j}, \tau_{k+2j+2})$ with $H^{1,1}$ and assume that $h_j \in H(\tau_{k+2j}, \tau_{k+2j+2})$. Let H_τ^h be the *incomplete tensor product* [7] of the spaces $H(\tau_n, \tau_{n+2})$ determined by h , that is, the completion of the set of all finite linear combinations of simple tensor vectors $\bigotimes_{j=0}^{\infty} v_{2j+1}$ with $v_n \in H(\tau_n, \tau_{n+2})$ such that all but finitely many of v_n coincide with h_n .

In the even case, the argument is the same, with the space $\mathcal{T}^{\text{even}}$, defined in a similar way, in place of \mathcal{T}^{odd} . However, in this case we have an exceptional representation Π_0 (which does not depend on h), which is the inductive limit of the representations with Young diagrams (n, n) . This is the so-called “discrete” elementary representation D_{t_0} , which is realized in the l^2 space spanned by all infinite Young tableaux tail-equivalent to the “principal” tableau t_0 with 1, 3, 5, \dots in the first row and 2, 4, 6, \dots in the second row; that is, the representation whose spectral measure with respect to the Gelfand–Tsetlin subalgebra is δ_{t_0} .

Summarizing the above discussion, we obtain the following proposition.

Proposition 2. *Let $k \geq 1$. Then, denoting by $H_k(h)$ the space of the representation $\Pi_k(h)$, we have*

$$H_k(h) = \bigoplus_{\tau \in \mathcal{T}_k^{\text{odd}}} H_\tau^h$$

(however, the subspaces H_τ^h are not invariant under the action of $\mathfrak{S}_{\mathbb{N}}$).

4 Spectral measures of Schur–Weyl representations

Now we want to construct a realization of the representation $\Pi_k(h)$, $k \geq 1$, in the space $L^2(\mathcal{T}, \nu)$ for some measure ν on the space of all infinite Young tableaux \mathcal{T} .

We have $h_j \in H(\lambda^{(j)}, \lambda^{(j+1)})$, where $\lambda^{(j)} = (k+j, j)$. Let $h_j = p_j t_{21} + q_j t_{12}$, where $\{t_{21}, t_{12}\}$ is the standard tableaux basis of $H(\lambda^{(j)}, \lambda^{(j+1)}) \simeq H^{(1,1)}$. By unitarity, $p_j^2 + q_j^2 = 1$.

Let $\mathcal{T}^{\text{proper}}$ be the set of finite Young tableaux $t \in \mathcal{T}_N$, $N \in \mathbb{N}$, such that the pair $([t]_{N-2}, t)$ is not nice (that is, $N-1$ and N lie in the same row of t), and let $\mathcal{T}_k^{\text{proper}} = \{t \in \mathcal{T}^{\text{proper}} : t \in \mathcal{T}_N(\lambda) \text{ with } \lambda_1 - \lambda_2 = k\}$. Given $t \in \mathcal{T}_k^{\text{proper}}$, consider the measure μ_t^h on \mathcal{T} that is the distribution of the following random walk on \mathcal{T} : we start from $t \in \mathcal{T}_N$, and at each step, passing from the n th level to the $(n+2)$ th level, we choose the path corresponding to t_{21} with probability $\alpha_n = p_j^2$ and the path corresponding to t_{12} with probability $\beta_n = q_j^2$, where $j = [(n-k)/2]$.

Let

$$\mu^{(k)} = \sum_{t \in \mathcal{T}_k^{\text{proper}}} \mu_t.$$

Theorem 2. *The representation $\Pi_k(h)$ has a simple spectrum with respect to the Gelfand–Zetlin algebra, and*

$$\Pi_k(h) \simeq L^2(\mathcal{T}, \mu^{(k)}). \quad (6)$$

Proof. The norm in $L^2(\mathcal{T}, \mu^{(k)})$ will be denoted by $\|\cdot\|$. By definition, $\Pi_k(h)$ is an inductive limit of irreducible representations $\pi_{(k)}, \pi_{(k+1,1)}, \pi_{(k+2,2)}, \dots$. The representation $\pi_{(k)}$ is one-dimensional; to the only tableau t with diagram (k) we associate the function $\phi_t = \delta_t$, where δ_t is the cylinder function in $L^2(\mathcal{T}, \mu^{(k)})$ such that $\delta_t(s) = 1$ if $[s]_k = t$ and $\delta_t(s) = 0$ otherwise. Obviously, $\|\delta_t\| = 1$. Now assume that we constructed functions $\phi_t \in L^2(\mathcal{T}, \mu^{(i)})$ for all t with diagram $(k+l, l)$. Denoting $N = k+2l$ and considering the restriction of the irreducible representation $\pi = \pi_{(k+l+1, l+1)}$ of \mathfrak{S}_{N+2} to \mathfrak{S}_N , we have $\pi = \pi_{(k+l, l)} \oplus \pi'$, where $\pi' = \pi_{(k+l+1, l-1)} + \pi_{(k+l-1, l+1)}$, the latter term missing if $k < 2$. Now let $t \in \mathcal{T}_{N+2}((k+l+1, l+1))$. If $[t]_N \in \mathcal{T}_N(\lambda)$ with $\lambda = (k+l+1, l-1)$ or $\lambda = (k+l-1, l+1)$, then $t \in \mathcal{T}_k^{\text{proper}}$ and we put $\phi_t = \delta_t$. If $[t]_N \in \mathcal{T}_N((k+l, l))$, then $\phi_{[t]_N}$ is already constructed and we put

$$\phi_t = \phi_{[t]_N} \cdot \begin{cases} \frac{1}{p} \delta_t & \text{if } N+1 \text{ lies in the second row in } t, \\ \frac{1}{q} \delta_t & \text{if } N+1 \text{ lies in the first row in } t. \end{cases}$$

It is not difficult to verify that the map $t \mapsto \phi_t$ thus defined is an isometry $\pi_{(k+l, l)} \rightarrow L^2(\mathcal{T}, \mu^{(k)})$ and these maps agree with the embeddings $\pi_{(k+l, l)} \hookrightarrow \pi_{(k+l+1, l+1)}$. \square

In particular, for a stationary Schur–Weyl representation (see Definition 4), we have $\alpha_j \equiv \alpha$, $\beta_j \equiv \beta$ for all j . Thus a stationary Schur–Weyl representation is determined by a number $p \in [-1, 1]$, where $h_j = pt_{21} + qt_{12}$ and $q = \sqrt{1 - p^2}$ (it suffices to consider only positive q , because the embeddings determined by parameters (p, q) and $(-p, -q)$ are obviously equivalent). The corresponding measure $\mu^{(k)}$ on the space of infinite Young tableaux is the distribution of a stationary (“Bernoulli”) random walk.

Example 1. If each h_j coincides with t_{21} (or with t_{12}), i.e., $p \in \{0, 1\}$, then the measure $\mu^{(k)}$ is the δ -measure at an infinite Young tableau, and the corresponding representation $\Pi_k(h)$ is the discrete elementary representation of $\mathfrak{S}_{\mathbb{N}}$ associated with this tableau.

Remark 1. The measure $\mu^{(k)}$ on the set of infinite Young tableaux is not central. It is σ -finite, continuous, and ergodic with respect to the tail equivalence relation on partitions (since the corresponding representation is irreducible); the representation under study acts in the scalar L^2 space over this measure.

Remark 2. The action of the infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ in the space $L^2(\mathcal{T}, \mu^{(k)})$ providing the isomorphism (6) does not coincide with the standard action of permutations on Young tableaux determined by Young’s orthogonal form.

5 The main example: tensor Schur–Weyl representations

There is a distinguished Schur–Weyl embedding that agrees with the tensor product structure of the space $(\mathbb{C}^2)^{\otimes N}$. Namely, observe that $\mathbb{C} \otimes \mathbb{C} = M_1 \oplus M_3$ as $SL(2, \mathbb{C})$ -modules, where M_3 is the three-dimensional irreducible $SL(2, \mathbb{C})$ -module and M_1 is the trivial one-dimensional $SL(2, \mathbb{C})$ -module. Thus we may embed $(\mathbb{C}^2)^{\otimes N}$ into

$$(\mathbb{C}^2)^{\otimes(N+2)} = (\mathbb{C}^2)^{\otimes N} \otimes (\mathbb{C}^2)^{\otimes 2} = (\mathbb{C}^2)^{\otimes N} \otimes (M_1 \oplus M_3)$$

along this one-dimensional representation.

In other words, this is the unique Schur–Weyl embedding that has the form

$$(\mathbb{C}^2)^{\otimes N} \ni v \mapsto v \otimes v_0 \in (\mathbb{C}^2)^{\otimes(N+2)}$$

for some $v_0 \in \mathbb{C}^2$. It is easy to see that, denoting the standard basis of \mathbb{C}^2 by $\{e_1, e_2\}$, we have $v_0 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$; this is the unique (up to a constant) $SL(2, \mathbb{C})$ -invariant vector in \mathbb{C}^4 .

Definition 5. *The embedding described above will be called the tensor Schur–Weyl embedding.*

Just another way to describe the tensor Schur–Weyl embedding is to say that the image of $(\mathbb{C}^2)^{\otimes N}$ in $(\mathbb{C}^2)^{\otimes(N+2)}$ should lie in the eigenspace of the last Coxeter transposition $\sigma_{N+1} = (N+1, N+2)$ with eigenvalue -1 .

Proposition 3. *For the tensor Schur–Weyl embedding described above, we have*

$$p = -\sqrt{\frac{r-1}{2r}}, \quad q = \sqrt{\frac{r+1}{2r}},$$

so that the corresponding Bernoulli measure μ_t has the weights

$$\alpha = \frac{r-1}{2r}, \quad \beta = \frac{r+1}{2r};$$

here $r = k+1$ for $t \in \mathcal{T}_k^{\text{proper}}$.

Proof. Let $\lambda = (k+n, n)$, $\mu = (k+n+1, n+1)$, and $h \in H(\lambda, \mu)$ be the vector determining the tensor Schur–Weyl embedding in these components, so that $h = pt_{21} + qt_{12}$. Recall that Young’s orthogonal form (see, e.g., [5]) says that the Coxeter transposition σ_{N+1} , where $N = 2n+k$, acts in $H(\lambda, \mu)$ and has the following matrix in the basis $\{t_{21}, t_{12}\}$:

$$\begin{pmatrix} r^{-1} & \sqrt{1-r^{-2}} \\ \sqrt{1-r^{-2}} & -r^{-1} \end{pmatrix}$$

where the axial distance r (defined as $c_{N+2} - c_{N+1}$, where c_j is the content of the cell containing j) in our case is equal to $k+1$. The proposition now follows from the fact that the image of $(\mathbb{C}^2)^{\otimes N}$ in $(\mathbb{C}^2)^{\otimes(N+2)}$ should lie in the eigenspace of $\sigma_{N+1} = (N+1, N+2)$ with eigenvalue -1 . \square

Obviously, the tensor Schur–Weyl representation can be realized in the incomplete tensor product

$$(\mathbb{C}^4)_{v_0}^{\otimes \infty}, \quad v_0 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1) \in \mathbb{C}^4.$$

It follows that in the space of the tensor Schur–Weyl representation we have also an action of the UHF algebra $\mathcal{G} = \varinjlim \text{Mat}_{4^n}(\mathbb{C})$ (the Glimm algebra of type 2^∞ ; see, e.g., [8]), thus obtaining a new representation of this algebra. One can show that this is the only Schur–Weyl representation with this property.

There is a natural question: given a vector $w \in (\mathbb{C}^4)_{v_0}^{\otimes \infty}$, determine in what primary component $\mathcal{H}_k = \Pi_k \otimes M_{k+1}$ of the corresponding “spin” decomposition (5) it lies. The answer is as follows. By the definition of an incomplete tensor product, we have $w = u \otimes v_0 \otimes v_0 \otimes \dots$, where $u \in (\mathbb{C}^4)^{\otimes N}$ for some finite N . Then w has the same spin k as the finite vector u does according to decomposition (2) or (3). To find this k , one may write $(\mathbb{C}^4)^{\otimes N}$ as $(\mathbb{C}^2)^{\otimes 2N}$ and then use the results on tensor representations of the finite symmetric groups from [13, Section 5].

In the class of generalized Schur–Weyl representations (see Definition 2) there are many other tensor representations. Namely, the “jump factor” $(\mathbb{C}^2)^{\otimes k}$ contains the one-dimensional representation M_1 of $SL(2, \mathbb{C})$ with multiplicity equal to the Catalan number C_k . Choosing any vector from this C_k -dimensional $SL(2, \mathbb{C})$ -invariant subspace as v_0 (in general, we may choose different vectors at different steps) and constructing the corresponding incomplete tensor product, we obtain a tensor generalized Schur–Weyl representation.

6 The Coxeter–Laplace operator in Schur–Weyl representations

In this section we study the so-called Coxeter–Laplace operator in Schur–Weyl representations. But first let us briefly describe the general setting.

In the general theory of random walks, the Laplacian of the random walk on a finite group G with probability measure μ is the operator $E - \sum_g \mu(g)L_g$ in the group algebra of G (or in the space where a representation of G is defined), where the sum is over all $g \in G$ with $\mu(g) > 0$, L_g is the operator of left multiplication by g , and E is the identity operator. Let G be the symmetric group \mathfrak{S}_N and μ be the uniform measure on the set of Coxeter generators $\sigma_k = (k, k+1)$, $k = 1, \dots, N$, where we set $\sigma_N = (N, 1)$. The

corresponding Laplacian has the form

$$L_N = e - \frac{1}{N} \sum_{k=1}^N \sigma_k.$$

We call it the *periodic Coxeter Laplacian*, or the *Coxeter–Laplace operator*. It follows from the Schur–Weyl duality that if π_N is the representation of \mathfrak{S}_N in the tensor product $(\mathbb{C}^2)^{\otimes N}$ by permutations of factors, then the operator $\pi_N(L_N)$ is related to the Hamiltonian of the XXX Heisenberg model on the periodic one-dimensional lattice with N sites (see, e.g., [4, 10]) by the formula $H = \frac{J}{4}(2L - N)$, where $J > 0$ corresponds to the ferromagnetic case, and $J < 0$ corresponds to the antiferromagnetic case (see also [11, 6]).

Thus we have the following natural problem. For a given (irreducible or not) representation of the symmetric group, find the eigenvalues and eigenfunctions of the Coxeter Laplacian. Usually, one considers the spectrum of operators in $L^2(G)$. In our case, it is natural, both from the point of view of representation theory and applications to physics, to consider the asymptotics of the Coxeter Laplacian L_N and its spectrum as $N \rightarrow \infty$. The case most important for applications (in particular, for the Heisenberg model) is that of representations of \mathfrak{S}_N corresponding to Young diagrams with finitely many rows; e.g., at most two rows. Besides, there are two different asymptotic modes, “ferromagnetic” mode and “antiferromagnetic” mode. The first case is easier: it means considering representations corresponding to Young diagrams with fixed second row and growing first row; for asymptotic results in this case, see [11]. But the most interesting case is when both rows grow; namely, we are interested in the asymptotics of the largest eigenvalue of L_N , which corresponds to the ground energy of the Heisenberg antiferromagnet. Some related results are obtained in [12] and cited below.

Here we study the asymptotic behavior of the Coxeter–Laplace operator in the limit of the representations π_N corresponding to a stationary chain of Schur–Weyl embeddings. We are especially interested in the “leading” components Π_0 and Π_1 for the even and odd case, respectively, because they correspond to the ground state of the antiferromagnetic Heisenberg chain (see [12]). But Π_0 is just the discrete representation D_{t_0} corresponding to the principal tableau t_0 (see Example 1), and it is shown in [12] that the weak limit of the operators $\frac{1}{N}\pi_N(L_N)$ in this representation is the scalar operator with constant $5/4$.

Proposition 4. *The weak limit of the operators $\frac{1}{N}L_N$ in the leading component Π_1 of the stationary Schur–Weyl representation of \mathfrak{S}_N with parameters $p, q = \sqrt{1-p^2}$ is the identity operator $\phi(p)E$, where*

$$\phi(p) = \frac{13}{12} + \frac{8}{6}p^4 - \frac{7}{6}p^2 - \frac{\sqrt{3}}{2}p\sqrt{1-p^2}.$$

Proof. The representation Π_1 is the inductive limit of the irreducible representations $\pi_{(1)}, \pi_{(2,1)}, \pi_{(3,2)}, \dots$ of the symmetric groups $\mathfrak{S}_1, \mathfrak{S}_3, \mathfrak{S}_5, \dots$ under the stationary sequence $i_{2k+1} : \mathfrak{S}_{2k+1} \hookrightarrow \mathfrak{S}_{2k+3}$ of Schur–Weyl embeddings determined by p . Thus a basis of Π_1 consists of the images in Π_1 of the finite Young tableaux $t \in \mathcal{T}((k+1, k))$, $k = 0, 1, \dots$. Let t, s be such tableaux and $n = 2k + 1$. Obviously,

$$\lim_{N \rightarrow \infty} \frac{1}{N}(L_N t, s) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=n+1}^{N-1} (\sigma_j t, s).$$

Further, since the Schur–Weyl embeddings under consideration are stationary, we have $((\sigma_{j+2} + \sigma_{j+3})t, s) = ((\sigma_j + \sigma_{j+1})t, s)$ for all $j > n$, whence

$$\lim_{N \rightarrow \infty} \frac{1}{N}(L_N t, s) = ((\sigma_{n+1} + \sigma_{n+2})t, s),$$

the latter expression depending only on the image of t and s in $\pi_{(k+3, k+2)}$ under the embedding $i = i_{n+2} \circ i_n$. Now we have

$$i(t) = p^2 t_{pp} + q^2 t_{qq} + pq(t_{pq} + t_{qp}),$$

where t_{pp} is the tableau obtained from t by putting $n+1$ into the second row and $n+2$ into the first row, and then $n+3$ into the second row and $n+4$ into the first row; t_{pq} is the tableau obtained from t by putting $n+1$ into the second row and $n+2$ into the first row, and then $n+3$ into the first row and $n+4$ into the second row; etc. A similar formula holds for s , and the rest follows from straightforward calculations based on Young’s orthogonal form. \square

In particular, for the tensor Schur–Weyl embedding we have $\phi(-1/2) = 5/4$ (in fact, one can prove that $\frac{1}{N}L_N$ has the same limit in all components Π_k , $k = 0, 1, 2, \dots$). The maximum possible value $\phi(p)$ corresponds to $p = -0.95543\dots$ and is equal to $c_{SW} = 1.3736684\dots$. The limiting operators for other natural representations of \mathfrak{S}_N are found in [12]; they are also scalar, and the corresponding constants are smaller:

- 1.3736684... for the stationary Schur–Weyl representation with $p = -0.95543\dots$;
- 1.25 for the discrete representation D_{t_0} (see Example 1);
- 1 for the representation induced from the identity representation of the Young subgroup $\mathfrak{S}_{\{1,3,5,\dots\}} \times \mathfrak{S}_{\{2,4,5,\dots\}}$;
- 0.5 for the factor representation with Thoma parameters $\alpha = (1/2, 1/2)$, $\beta = 0$.

As proved in [16], the maximum eigenvalue λ_N of $\pi_N(L_N)$ satisfies $\frac{\lambda_N}{N} \rightarrow c_{\max} = 2 \log 2 = 1.38629436\dots$. Thus the “deficiency” of c_{SW} as compared with c_{\max} is just about 0.0126. It would be very interesting to find a representation of the infinite symmetric group in which the limit of the Coxeter–Laplace operator has the eigenvalue c_{\max} . A step in this direction is Proposition 5 below, which shows that we can construct a representation with an eigenvalue arbitrarily close to c_{\max} .

Proposition 5. *For every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ and a stationary sequence of generalized Schur–Weyl embeddings*

$$\mathfrak{S}_1 \subset \mathfrak{S}_{1+2k} \subset \mathfrak{S}_{1+4k} \subset \dots$$

such that the weak limit of the operators $\frac{1}{N}L_N$ in the leading component of the corresponding generalized Schur–Weyl representation of $\mathfrak{S}_{\mathbb{N}}$ is a scalar operator cE with $c > c_{\max} - \varepsilon$.

Proof. Let $t, s \in \mathcal{T}((mk + 1, mk))$, and let $N = 2nk + 1$ for $n > m$. Arguing as in the proof of Proposition 4, we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} (L_N t, s) &= \frac{1}{2k} ((\sigma_{mk+2} + \dots + \sigma_{(m+2)k+1})t, s) \\ &= \frac{1}{2k} ((T^{mk} L_{2k+1})t, s) + \frac{1}{2k} (R_k t, s), \end{aligned}$$

where $R_k = \sigma_{(m+2)k+1} - \sigma_{mk+1} - (mk + 1, (m + 2)k + 1)$ and T is the endomorphism of $\mathfrak{S}_{\mathbb{N}}$ defined by the formulas $(Tg)(1) = 1$, $(Tg)(i) = g(i - 1) + 1$ for $i > 1$ (the infinite shift). Now let v_{\max} be the eigenvector of L_{2k+1} corresponding to the largest eigenvalue $\lambda_{\max}^{(k)}$. As proved in [12], v_{\max} lies in the irreducible representation $\pi_{(k+1,k)}$. Choose an embedding $\pi_{(1)} \hookrightarrow \pi_{(k+1,k)}$

such that the only tableau in $\mathcal{T}((1))$ goes to v_{\max} . Then, since the generalized embedding is stationary, we have

$$\frac{1}{2k}((T^{mk}L_{2k+1})t, s) = \frac{1}{2k}(L_{2k+1}[t]_{2k+2}, [s]_{2k+2}) = \frac{\lambda_{\max}^{(k)}}{2k},$$

and the proposition follows. \square

The calculation in [16] of the limit value c_{\max} , based on the Bethe ansatz, is very indirect and, in particular, relies on considering a more general model that has no interpretation in terms of the symmetric groups. However, studying the asymptotics of the spectrum of the Coxeter–Laplace operator is a natural problem for the theory of symmetric groups. Thus a challenge is to obtain a representation-theoretic proof of this result.

References

- [1] B. Feigin and E. Feigin, Integrable $\widehat{\mathfrak{sl}}_2$ -modules as infinite tensor products, in: *Fundamental Mathematics Today*, Independent University of Moscow, Moscow (2003), pp. 304–334; [arXiv:math/0205281v1](#).
- [2] W. Fulton and J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991.
- [3] L. Hultén, *Arkiv Mat. Astron. Fys.*, **25A**, 1 (1938).
- [4] Yu. A. Izyumov and Yu. N. Skryabin, *Statistical Mechanics of Magnetically Ordered Systems*, Consultants Bureau, New York, 1988.
- [5] G. D. James, *The Representation Theory of the Symmetric Groups*, Springer-Verlag, Berlin–Heidelberg–New York (1978).
- [6] E. Mukhin, V. Tarasov, and A. Varchenko, Bethe algebra of homogeneous XXX Heisenberg model has simple spectrum, *Comm. Math. Phys.*, **288**, No. 1, 1–42 (2009).
- [7] J. von Neumann, On infinite direct products, *Compos. Math.*, **6**, 1–77 (1938).
- [8] G. K. Pedersen, *C*-Algebras and Their Automorphism Groups*, Academic Press, London–New York (1979).

- [9] I. Penkov and K. Styrkas, Tensor representations of classical locally finite Lie algebras, in: *Developments and Trends in Infinite-Dimensional Lie Theory*, Progr. Math., **288**, Birkhäuser Boston, Inc., Boston, (2011), pp. 127–150.
- [10] L. A. Takhtadzhyan and L. D. Faddeev, The spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model, *Zap. Nauchn. Semin. LOMI*, **109**, 134–178 (1981).
- [11] N. V. Tsilevich, Spectral properties of the periodic Coxeter Laplacian in the two-row ferromagnetic case, *J. Math. Sci. (N. Y.)*, **174**, No. 1, 58–70 (2011).
- [12] N. V. Tsilevich, On the behavior of the periodic Coxeter Laplacian in some representations related to the antiferromagnetic asymptotic mode and continual limits, *J. Math. Sci. (N. Y.)*, **181**, No. 6, 914–920 (2012).
- [13] N. V. Tsilevich and A. M. Vershik, Markov measures on Young tableaux and induced representations of the infinite symmetric group, *Prob. Theory Appl.*, **51**, No. 1, 211–223 (2006).
- [14] A. Wasserman, Kac–Moody, Virasoro and vertex algebras in infinite dimensions, Lecture Notes, <http://www.dpmms.cam.ac.uk/~ajw/course11-2.pdf>.
- [15] H. Weyl, *The Classical Groups. Their Invariants and Representations*, Princeton Univ. Press, Princeton, N.J., 1997.
- [16] C. N. Yang and C. P. Yang, One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system, *Phys. Rev.* **150**, 327–339 (1966).