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DISTRIBUTION OF THE MEAN VALUE FOR CERTAIN RANDOM MEASURES

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Let τ be a probability measure on $[0,1]$. We consider a generalization of the classic Dirichlet process – the random probability measure $F = \sum P_i \delta_{X_i}$, where $X = \{X_i\}$ is a sequence of independent random variables with the common distribution τ and $P = \{P_i\}$ is independent of X and has the two-parameter Poisson–Dirichlet distribution $PD(\alpha, \theta)$ on the unit simplex. The main result is the formula connecting the distribution μ of the random mean value $\int x dF(x)$ with the parameter measure τ. Bibliography: 12 titles.

1. Introduction. Let τ be an arbitrary probability distribution on $[0,1]$. We consider a random discrete probability measure

$$F = \sum_{i=1}^{\infty} P_i \delta_{X_i}, \tag{1.1}$$

where $X = (X_1, X_2, \dots)$ is a sequence of independent random variables with common distribution τ, and $P = (P_1, P_2, \dots) \in \Sigma = \{Y = (Y_1, Y_2, \dots) : Y_i \geqslant 0, \sum Y_i \leqslant 1\}$ is a random sequence of masses that is independent of X and has distribution GEM(1). This distribution is generated by the simplest *stick--breaking process*, namely, we choose a number P_1 uniformly distributed in $[0,1]$, then we choose a number P_2 uniformly distributed in the "remaining" part of the interval, i.e., in $[0, 1-P_1]$, and so on; P_n is uniformly distributed in $[0, 1 - P_1 - \dots - P_{n-1}]$. Thus,

$$P_n = W_n(1-W_1)\dots(1-W_{n-1}), \tag{1.2}$$

where $W_1, W_2, \dots$ is a sequence of independent random variables uniformly distributed in $[0,1]$.

The described measure is called a *random Dirichlet measure* with parameter measure τ. We denote it by $\mathcal{D}(\tau)$. It is also convenient to represent this random measure in the form

$$F = \sum_{i=1}^{\infty} \tilde{P}_i \delta_{X_i}, \tag{1.3}$$

where $\tilde{P}_1 \geqslant \tilde{P}_2 \geqslant \dots$ is the permutation of (P_i) in nonincreasing order.

We consider the mean value $\int x dF(x)$ of a random measure $F \in \mathcal{D}(\tau)$, and denote by μ the distribution of this functional. As was shown in [2], the original measure τ and the resulting measure μ are related by the following remarkable identity:

$$\int \frac{d\mu(u)}{z-u} = \exp \int \ln \frac{1}{z-u} d\tau(u), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.4}$$

Now let α, θ be two real parameters. We consider a generalized model where the positions $X_1, X_2, \dots$ are to be chosen as above, and the magnitudes $P_1, P_2, \dots$ are to be constructed by a *nonstationary* stick-breaking process (a *residual allocation model*): in representation (1.2), W_j has a beta distribution $B(1-\alpha, \theta + j\alpha)$ with density

$$\frac{\Gamma(1+\theta+(j-1)\alpha)}{\Gamma(1-\alpha)\Gamma(\theta+j\alpha)} t^{-\alpha}(1-t)^{\theta-1+j\alpha}, \quad t \in [0,1]. \tag{1.5}$$

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We denote the obtained random measure by $\mathcal{D}(\tau, \alpha, \theta)$ and call it a *generalized Dirichlet measure.* The range of admissible parameters is

$$\{(\alpha, \theta) : 0 \leqslant \alpha < 1, \theta > -\alpha\} \quad \bigcup \quad \{(\alpha, -m\alpha),\ \alpha < 0, m \in \mathbb{N}\}. \tag{1.6}$$

Let μ be the distribution of the mean value $\int x dF(x)$ with $F \in \mathcal{D}(\tau, \alpha, \theta)$. In this paper, we prove a formula generalizing (1.4). If $\alpha, \theta \neq 0$, it takes the form

$$\left(\int (z-u)^{-\theta} d\mu(u)\right)^{-\frac{1}{\theta}} = \left(\int (z-u)^{\alpha} d\tau(u)\right)^{\frac{1}{\alpha}}, \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{1.7}$$

The author is grateful to S. V. Kerov for bringing this problem to her attention and for necessary references and numerous fruitful discussions.

2. The classical Dirichlet process. The Dirichlet process (corresponding to the parameter $\alpha = 0$) was introduced by Ferguson ([3]). He used the following axiomatic definition. First, let β be a positive discrete measure with masses $\beta_1, \dots, \beta_n$ at points $z_1, \dots, z_n$. The set of all discrete probability measures concentrated on $\{z_1, \dots, z_n\}$ may be identified with the simplex $\Delta_n = \{x = (x_1, \dots, x_n) : x_i \geqslant 0, \sum x_i = 1\}$. the Dirichlet distribution $D(\beta_1, \dots, \beta_n)$ on Δ_n has density

$$\frac{\Gamma(\beta_1 + \ldots + \beta_n)}{\Gamma(\beta_1) \ldots \Gamma(\beta_n)} x_1^{\beta_1 - 1} \ldots x_n^{\beta_n - 1}$$

with respect to Lebesgue measure $dx_1 \ldots dx_{n-1}$. A random measure $F = \sum_{i=1}^n P_i \delta_{z_i}$, where $P = (P_1, ..., P_n) \in \Delta_n$ has a Dirichlet distribution $D(\beta_1, \dots, \beta_n)$, is called a Dirichlet process with parameter β. Given an arbitrary finite positive measure β on $[0,1]$, a random measure F is said to be a Dirichlet process with parameter β if, for any finite partition $[0,1] = A_1 \cup \ldots \cup A_n$, the vector $(F(A_1), \dots, F(A_n))$ has a Dirichlet distribution with parameters $(\beta(A_1), \dots, \beta(A_n))$. (See also [8, Chap. 9].) Ferguson showed that a random Dirichlet measure can be represented in the form (1.3) for the described construction with parameters $(\nu, 0, \theta)$, where θ is the total mass of β and $\nu = \beta/\theta$. In this case, the stick-breaking process is stationary, namely, all W_i have the same distribution with density $\theta(1-x)^{\theta-1}$. The measure generated by this process on the simplex Σ is called a *GEM distribution* with parameter θ, and the distribution of the ordered permutation $(\tilde{P}_i)$ on the simplex of monotone sequences is the famous *Poisson–Dirichlet distribution* $PD(\theta)$.

It is convenient to deal with random measures in terms of their samples. A sample from a random distribution is a sequence of random variables obtained by the following two-stage procedure. First, we choose a realization of random measure, and then we construct a sequence of independent variables obeying the distribution chosen at the first step. Such sequences share a special property of *exchangeability.* By definition, a sequence is exchangeable if its joint distributions are invariant under all finite permutations of its elements. In accordance with the classical de Finetti theorem, every exchangeable sequence is a sample from some random measure, and this measure is uniquely determined by the sequence.

The following method of constructing a sample from $\mathcal{D}(\beta)$, called the *Blackwell–MacQueen urn scheme,* is suggested in [1].

(1) X_1 has distribution $\frac{\beta}{\|\beta\|}$;

(2) given $X_1, \dots, X_n$, the conditional distribution of X_{n+1} is $\frac{\beta_n}{\|\beta_n\|}$ with $\beta_n = \beta + \sum_{i=1}^n \delta_{X_i}$.

We denote by $\|\beta\|$ the total mass of β.

There exists a version of this urn scheme called the *Chinese restaurant construction* (see [2]). The following description corresponds to sampling from a Dirichlet process with parameters $\beta = \theta\nu$, $\theta = \|\beta\|$. Assume that we have infinitely many circular tables, and each table can seat infinitely many persons. We associate with these tables random labels $x_1, x_2, \dots$ which are taken independently from distribution ν. The first guest sits at the first table. The nth person sits at the first empty table with probability $\frac{\theta}{n+\theta}$, or sits to the immediate right of the ith already seated person with probability $\frac{1}{n+\theta}$ $(i = 1, \dots, n)$. Let X_i be the label of the table where the ith guest sits. We obtain a sequence of random variables $X_1, X_2, \dots$. This sequence is a sample from $\mathcal{D}(\beta)$.

We note that we can regard the numbers of persons sitting at the same table at the nth step of this procedure as a cycle of some permutation $\pi \in S_n$. Thus, we have obtained measures M_θ^n on symmetric groups S_n, and

$$M_\theta^n(\pi) = \frac{\theta^{c(\pi)}}{\theta(\theta+1)\dots(\theta+n-1)}, \tag{2.1}$$

where $c(\pi)$ is the number of cycles of $\pi \in S_n$. These measures are called *Ewens measures* with parameter $\theta > 0$. They are of great importance in combinatorics and in many applications. We note that M_1^n is the Haar measure on S_n.

3. The generalized Dirichlet process. S. V. Kerov ([5]) and J. Pitman ([11]) proposed independently a generalization of constructions leading to Dirichlet measures. In terms of representation (1.4), this generalization means that we consider random measures

$$F = \sum \tilde{P}_i \delta_{X_i} + (1 - \sum \tilde{P}_i)\tau, \tag{3.1}$$

where $\{X_i\}$ is a sequence of independent random variables with distribution τ, and the vector $\tilde{P} = (\tilde{P}_1, \tilde{P}_2, \dots)$ has an *arbitrary* distribution on the simplex

$$\tilde{\Sigma} = \{Y = (Y_1, Y_2, \dots) : Y_1 \geqslant Y_2 \geqslant \dots \geqslant 0, \quad \sum Y_i \leqslant 1\}.$$

J. Pitman introduced corresponding urn schemes. Now we describe a generalized Chinese restaurant model due to S. V. Kerov ([5]). As above, one parameter of this model is a probability distribution τ. But now there is a second parameter that is a family of measures M^n on symmetric groups S_n satisfying the following two conditions. The first condition is the invariance of M^n under inner automorphisms (i.e., the mass of a permutation depends only on the lengths of its cycles). The second condition is a coherence in the following sense. For every permutation $\pi \in S_n$,

$$M^n(\pi) = \sum_{\sigma' = \pi} M^{n+1}(\sigma), \tag{3.2}$$

where the sum is taken over all permutations $\sigma \in S_{n+1}$ such that π is obtained from σ by removing the element $n+1$ from its cycle. As in the original model, we associate with the tables random labels x_i taken independently with the distribution τ. A generalized rule for guests takes the following form. The first person sits at the first table. Let the first n persons form a permutation $\pi_n \in S_n$. When the $(n+1)$th guest takes his place, we obtain a permutation $\pi_{n+1} \in S_{n+1}$ such that $\pi'_{n+1} = \pi_n$. We assume that the probability of such a permutation equals $M^{n+1}(\pi_{n+1})/M^n(\pi_n)$. Let X_i be the label of the table where the ith person sits. The conditions imposed on $\{M^n\}$ guarantee the exchangeability of the sequence $\{X_i\}$. Therefore, it determines some random measure F. We denote it by $\mathcal{D}(\tau, \{M^n\})$.

The relation of urn schemes and Chinese restaurant models with representations (3.1) is provided by Kingman's theory of random partitions (see [6, 7]).

Let μ be the distribution of the mean value $\int x dF(x)$ with $F \in \mathcal{D}(\tau, \{M^n\})$. A formula from [5] relates the moments of μ and τ.

Theorem ([5], Theorem 4.2.2). *Let $h_n = \int u^n d\mu(u)$ and $p_n = \int u^n d\tau(u)$ be the moments of μ and τ. Then*

$$h_n = \sum_{\pi \in S_n} M^n(\pi) \prod_{j \geqslant 1} p_j^{r_j(\pi)}, \quad n = 1, 2, \dots, \tag{3.3}$$

where $r_j(\pi)$ is the number of cycles of length j in permutation π.

As was shown in [10, 12], we may naturally include Poisson–Dirichlet measures $PD(\theta)$ in a two-parameter family $PD(\alpha, \theta)$ of distributions generated by the residual allocation models described above in

Sec. 1. Namely, let $\{W_i\}$ be independent random variables on $[0,1]$ such that W_i has a beta distribution $B(1-\alpha, \theta+i\alpha)$. We denote $P_i = W_i \prod_{j=1}^{i-1}(1-W_j)$. Then $PD(\alpha,\theta)$ is the distribution of $\{\tilde{P}_i\}$, where $\tilde{P}_1 \geqslant \tilde{P}_2 \geqslant \dots$ is the permutation of $\{P_i\}$ in nonincreasing order. The range of admissible parameters is (1.6).

Let τ be an arbitrary probability measure on $[0,1]$, and let α, θ be admissible parameters. We consider a random measure $F = \sum_{i=1}^{\infty} \tilde{P}_j \delta_{X_j}$, where X is a sequence of independent variables with common distribution τ, and the vector $\tilde{P}$ is independent of X and has distribution $PD(\alpha,\theta)$. We call this measure a generalized Dirichlet measure $\mathcal{D}(\tau,\alpha,\theta)$.

The corresponding family of distributions on symmetric groups $M^n_{\alpha,\theta}$ is given by the formula

$$M^n_{\alpha,\theta}(\pi) = \frac{(\theta+\alpha)(\theta+2\alpha)\dots(\theta+(k-1)\alpha)}{[\theta+1]_{n-1}} \prod_{j\geqslant 1} \left(\frac{[1-\alpha]_{j-1}}{(j-1)!}\right)^{r_j}, \tag{3.4}$$

where $k = c(\pi)$ is the total number of cycles of π, r_j is the number of cycles of length j, and $[x]_m = x(x+1)\dots(x+m-1)$ is the Pochhammer symbol.

Thus, we have the following rule for sampling from $\mathcal{D}(\tau,\alpha,\theta)$. We choose a sequence $\{x_i\}$ of independent variables with distribution τ. The first element X_1 equals x_1. If we have already constructed n elements $X_1, \dots, X_n$, and n_j of them equal x_j $(j = 1, \dots, k)$, then the conditional distribution of X_{n+1}, given $X_1, \dots, X_n$, is

$$\sum_{j=1}^{k} \frac{n_j-\alpha}{\theta+n}\delta_{x_j} + \frac{\theta+k\alpha}{\theta+n}\delta_{x_{k+1}}. \tag{3.5}$$

The following schemes correspond to limiting values of parameters.

1. If $\alpha = 1$, $\theta > -1$, then $M^n_{1,\theta}$ is a δ-measure at the identity permutation; therefore, $X_i = x_i$ are independent variables with distribution τ, i.e., $\mathcal{D}(\tau,1,\theta)$ coincide with τ almost everywhere (in representation (3.1), all $\tilde{P}_i = 0$ a.e.).

2. If $\theta = -\alpha$, $0 < \alpha < 1$, then $M^n_{\alpha,-\alpha}$ is concentrated on permutations with exactly one cycle. Thus, for all i, $X_i = x_1$, i.e., $\mathcal{D}(\tau,\alpha,-\alpha)$ is δ_x, where x obeys τ ($\tilde{P}_1 = 1$ a.e.). The same random measure corresponds to the case $\theta \to \infty$, $\alpha \geqslant 0$.

3. If $m = -\frac{\theta}{\alpha} \in \mathbb{N}$ is fixed, $\alpha \to -\infty$, $\theta \to \infty$, then

$$M^n_{-\infty,m\infty} = \frac{1}{\prod_{i\geqslant 1}((i-1)!)^{r_i}} \cdot \frac{m(m-1)\dots(m-k+1)}{m^n}. \tag{3.6}$$

Thus, in (3.5), the conditional distribution of X_{n+1}, given $X_1, \dots, X_n$, takes the form

$$\sum_{j=1}^{k} \frac{1}{m}\delta_{x_j} + \frac{m-k}{m}\delta_{x_{k+1}}. \tag{3.7}$$

In this case, in (3.1), we have $\tilde{P}_i = 1/m$, $i \leqslant m$, i.e.,

$$F = \sum_{i=1}^{m} \frac{1}{m}\delta_{X_i} = \frac{1}{m}\sum_{i=1}^{m}\delta_{X_i}. \tag{3.8}$$

4. The distribution of the mean value for $\mathcal{D}(\tau,\alpha,\theta)$.

Theorem. *Let τ be a probability measure on $[0,1]$, and let (α,θ) be admissible parameters. If μ is the distribution of the mean value $\int x dF(x)$ of a random measure $F \in \mathcal{D}(\tau,\alpha,\theta)$, then μ and τ are related by the formula*

(1) *if $\alpha, \theta \neq 0$,*

$$\left(\int (z-u)^{-\theta} d\mu(u)\right)^{-\frac{1}{\theta}} = \left(\int (z-u)^{\alpha} d\tau(u)\right)^{\frac{1}{\alpha}}, \quad z \in \mathbb{C} \setminus \mathbb{R}; \tag{4.1}$$

(2) *if* $\theta = 0$,

$$\exp \int \ln(z-u)^{\alpha} d\mu(u) = \int (z-u)^{\alpha} d\tau(u), \quad z \in \mathbb{C} \setminus \mathbb{R}; \tag{4.2}$$

(3) *if* $\alpha = 0$,

$$\int (z-u)^{-\theta} d\mu(u) = \exp \int \ln(z-u)^{-\theta} d\tau(u), \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{4.3}$$

In our proof, we use the following statement.

Lemma ([9], Example I.2.11). *Let*

$$f(t) = \sum_{n=1}^{\infty} \frac{f_n t^n}{n!}, \quad g(t) = \sum_{n=1}^{\infty} \frac{g_n t^n}{n!} \tag{4.4}$$

be formal power series. Consider the composition $H(t) = f(g(t))$ *and its expansion in a power series* $H(t) = \sum_{n=1}^{\infty} H_n t^n/(n!)$. *Then the coefficients* H_n *take the form*

$$H_n = \sum_{k=1}^{n} f_k B_{n,k}(g), \tag{4.5}$$

where $B_{n,k}$ *is a polynomial of the coefficients of* g. *This polynomial is given by the formula*

$$B_{n,k} = \sum_{\lambda} \frac{n!}{\prod_{i \geqslant 1} r_i!(i!)^{r_i}} \prod_{i \geqslant 1} g_i^{r_i}, \tag{4.6}$$

where the sum is taken over all partitions λ *of* n *that have exactly* k *summands, and* r_j *is the number of summands of* λ *equal to* j.

For our purposes, it is more convenient to use a version of (4.6) in which the sum is taken over elements of a symmetric group S_n. Since for an arbitrary partition λ of n, the number of permutations of cycle structure λ equals

$$z_{\lambda} = \frac{n!}{\prod_{i \geqslant 1} r_i! i^{r_i}}, \tag{4.7}$$

we can represent (4.5) in the form

$$H_n = \sum_{\pi \in S_n} f_k \prod_{i \geqslant 1} \left(\frac{g_i}{(i-1)!} \right)^{r_i}, \tag{4.8}$$

where k is the total number of cycles of π, and r_j is the number of cycles of length j.

Proof of the theorem. Let $h_n = \int u^n d\mu(u)$, $p_n = \int u^n d\tau(u)$ be the moments of μ and τ. According to (3.3),

$$h_n = \sum_{\pi \in S_n} M^n_{\alpha,\theta}(\pi) \prod_{j \geqslant 1} p_j^{r_j} = \sum_{\pi \in S_n} \frac{(\theta+\alpha)\dots(\theta+(k-1)\alpha)}{[\theta+1]_{n-1}} \cdot \prod \left(\frac{[1-\alpha]_{j-1}}{(j-1)!} \right)^{r_j} \prod p_j^{r_j}. \tag{4.9}$$

Let $\alpha, \theta \neq 0$. Then (4.9) takes the form

$$h_n = \sum_{\pi \in S_n} \frac{[\frac{\theta}{\alpha}]_k}{[\theta]_n} \prod_{j \geqslant 1} \left(\frac{-[-\alpha]_j p_j}{(j-1)!} \right)^{r_j}. \tag{4.10}$$

We denote $f_k = [\frac{\theta}{\alpha}]_k$, $g_j = -[-\alpha]_j p_j$, $H_n = [\theta]_n h_n$ $(k, j, n \geqslant 1)$. Then (4.10) takes the form

$$H_n = \sum_{\pi \in S_n} f_k \prod \left(\frac{g_j}{(j-1)!} \right)^{r_j}. \tag{4.11}$$

We see that this formula coincides with (4.8). Thus, for $f(z) = \sum_{k=1}^{\infty} \frac{f_k}{k!} z^k$, $g(z) = \sum_{k=1}^{\infty} \frac{g_k}{k!} z^k$, and $H(z) = \sum_{k=1}^{\infty} \frac{H_k}{k!} z^k$, we have

$$H(z) = f(g(z)). \tag{4.12}$$

It remains to compute the functions f, g, and H. One can easily check that

$$\begin{aligned} f(z) &= (1-z)^{-\frac{\theta}{\alpha}} - 1; \\ g(z) &= -\sum_{k=1}^{\infty} \frac{[-\alpha]_k}{k!} p_k z^k = -\sum_{k=1}^{\infty} \frac{[-\alpha]_k}{k!} z^k \int u^k d\tau(u) \\ &= 1 - \int (1-zu)^{\alpha} d\tau(u); \\ H(z) &= \int (1-zu)^{-\theta} d\mu(u) - 1. \end{aligned} \tag{4.13}$$

Substituting these expressions in (4.12), and replacing z by $\frac{1}{z}$, we obtain (4.1).

In the case $\theta = 0$, we have

$$h_n = \sum_{\pi \in S_n} \frac{(k-1)!}{\alpha(n-1)!} \prod \left(\frac{-[-\alpha]_j p_j}{(j-1)!} \right)^{r_j}. \tag{4.14}$$

Thus, (4.11) holds with $H_n = \alpha(n-1)! h_n$, $f_k = (k-1)!$, $g_j = -[-\alpha]_j p_j$. The corresponding functions are $H(z) = -\int \ln(1-uz)^{\alpha} d\mu(u)$, $f(z) = \ln(1-z)$, $g(z) = 1 - \int (1-zu)^{\alpha} d\tau(u)$. Substituting these expressions in (4.12), we obtain (4.2).

The relation with $\alpha = 0$ is given in [5]. To make the picture complete, we present the corresponding formulae. In this case,

$$h_n = \sum_{\pi \in S_n} \frac{1}{[\theta]_n} \prod (\theta p_j)^{r_j}; \tag{4.15}$$

thus, $H_n = [\theta]_n h_n$, $f_k = 1$, $g_j = \theta p_j (j-1)!$, i.e., $H(z) = \int \frac{d\mu(u)}{(1-zu)^{\theta}}$, $f(z) = e^z$, $g(z) = \int \ln \frac{1}{(z-u)^{\theta}} d\tau(u)$, and (4.3) follows.

Remarks. 1. At first glance, it is not clear that the right side of (4.1) tends to the right-hand side of (4.3) as $\alpha \to 0$. However, this fact is easy to check, since

$$\begin{aligned} \frac{1}{\alpha} \ln \int \left(1 - \frac{u}{z}\right)^{\alpha} d\tau(u) &= \frac{1}{\alpha} \ln \int \left(1 + \alpha \ln\left(1 - \frac{u}{z}\right) + o(\alpha^2)\right) d\tau(u) \\ &= \frac{1}{\alpha} \ln \left(1 + \alpha \int \ln\left(1 - \frac{u}{z}\right) d\tau(u) + o(\alpha^2)\right) \to \int \ln\left(1 - \frac{u}{z}\right) d\tau(u). \end{aligned}$$

2. For limiting values of parameters, the formula takes the following form.

(1) If $\alpha = 1$, $\theta > -1$, then (4.1) implies

$$\left(\int (z-u)^{-\theta} d\mu(u)\right)^{-\frac{1}{\theta}} = \int (z-u) d\tau(u) = z - p_1. \tag{4.16}$$

Thus, μ is a δ-measure at point p_1. This fact is in accord with our observation that $\mathcal{D}(\tau, 1, \theta)$ equals τ a.e.

(2) If $\theta = -\alpha$, $0 < \alpha < 1$, formula (4.1) turns into identity, hence, $\mu = \tau$. Indeed, in this case, $\mathcal{D}(\tau, \alpha, -\alpha)$ is δ_x, where x has distribution τ, and the distribution of the mean value of this random measure coincides with τ.

(3) If $m \equiv -\frac{\theta}{\alpha} \in \mathbb{N}$, $\alpha \to -\infty$, $\theta \to \infty$, we may use (3.6) and an argument similar to the proof of the theorem to obtain the relation

$$\int e^{umz} d\mu(u) = \left(\int e^{uz} d\tau(u)\right)^m, \tag{4.17}$$

i.e., in terms of characteristic functions, $f_\mu(t) = f_\tau(t/m)^m$. This fact is in accord with (3.8), since this formula shows that, in this case, μ is the distribution of the normalized sum $\frac{1}{m}\sum_{i=1}^m x_i$ of independent random variables with distribution τ.

3. The symmetry of (4.1) with respect to the change $(\tau, \alpha, \theta) \leftrightarrow (\mu, -\theta, -\alpha)$ shows that (4.9) remains valid if we substitute $h_i \leftrightarrow p_i$, $\alpha \to -\theta$, $\theta \to -\alpha$. Thus, we have the following "inversion formula" for restoring the moments of τ, given the moments of μ:

$$p_n = \sum_{\pi \in S_n} (-1)^{k-1} \frac{(\alpha+\theta)\dots(\alpha+(k-1)\theta)}{[1-\alpha]_{n-1}} \cdot \prod \left(\frac{[1+\theta]_{j-1}}{(j-1)!}\right)^{r_j} \prod h_j^{r_j}.$$

4. Our proof is based on the fact that Ewens–Pitman measures $M^n_{\alpha,\theta}$ can be represented in the form

$$M^n(\pi) = \frac{a_k}{b_n} \cdot \prod_{i \geqslant 1} c_i^{r_i}. \tag{4.18}$$

Distributions of this form were considered in [4], where they were generated by the so-called *Kolchin model*. This model can be described as follows. Let $\varkappa = \{\varkappa_i\}$ and $\sigma = \{\sigma_i\}$ be two probability distributions on the set of natural numbers. We pick a random number K distributed according to $\varkappa$. Then we consider a random K-vector $(S_1, \dots, S_K)$ consisting of independent integer random variables with common distribution σ. This vector defines a random partition $\Lambda = (1^{R_1} 2^{R_2} \dots)$ of $N = S_1 + \dots + S_K$ ($R_j = \#\{i : S_i = j\}$). For a fixed $n \in \mathbb{N}$, we consider the conditional distribution of Λ, given $N = n$, and we denote it by P_n. The obtained measure on the set of partitions of n corresponds, in a natural way, to some measure on the symmetric group S_n which is invariant under inner automorphisms. Namely, $M^n(g) = P_n(\lambda)/z_\lambda$, where the partition λ describes the cycle structure of $g \in S_n$, and z_λ is the number (4.7) of permutations of cycle structure λ. One can easily check that

$$M^n(g) = \frac{k! \varkappa_k}{n! C_n} \cdot \prod_{i \geqslant 1} (i\sigma_i)^{r_i}. \tag{4.19}$$

Now the proof of the theorem shows that we may reformulate its statement in terms of the Kolchin model. Namely,

$$\int \varphi_N(uz) d\mu(u) = \varphi_K\left(\int \varphi_S(uz) d\tau(u)\right), \tag{4.20}$$

where φ_K and φ_S are the generating functions of the distributions $\varkappa$ and σ, and $\varphi_N(u) = \varphi_K(\varphi_S(u))$ is the generating function of the sum $N = S_1 + \dots + S_K$. It should be mentioned that for fixed values of α and θ, the Ewens–Pitman measure $M^n_{\alpha,\theta}$ can be obtained in the Kolchin model with different parameters $\varkappa$ and σ (see [4]), but the substitution of the corresponding generating functions in (4.20) yields formulae which can be reduced to (4.1)–(4.3) by trivial transformations. We should also note that, as was shown in [4], among all the distributions that can be obtained by the Kolchin model, only the Ewens–Pitman measures and their limiting variants satisfy the coherence condition (3.2).

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