The intrinsic hyperplane arrangement in an arbitrary irreducible representation of the symmetric group

N. V. Tsilevich* A. M. Vershik† S. Yuzvinsky‡

Abstract

For every irreducible complex representation π_λ of the symmetric group $\mathfrak{S}_n$, we construct, in a canonical way, a so-called intrinsic hyperplane arrangement $\mathcal{A}_\lambda$ in the space of π_λ. This arrangement is a direct generalization of the classical braid arrangement (which is the special case of our construction corresponding to the natural representation of $\mathfrak{S}_n$), has a natural description in terms of invariant subspaces of Young subgroups, and enjoys a number of remarkable properties.

1 Introduction

In this paper, for an arbitrary irreducible complex representation π_λ of the symmetric group $\mathfrak{S}_n$ we construct an arrangement of hyperplanes in the space of π_λ. This arrangement, which we have called "intrinsic," is defined canonically in representation-theoretic terms. In the case of the natural representation of $\mathfrak{S}_n$, it coincides with the so-called braid arrangement, studied, in particular, by Arnold [1] in connection with the cohomology of the group

*St. Petersburg Department of Steklov Institute of Mathematics and St. Petersburg State University, St. Petersburg, Russia. E-mail: natalia@pdmi.ras.ru. Supported by the RFBR grant 17-01-00433.

†St. Petersburg Department of Steklov Institute of Mathematics and St. Petersburg State University, St. Petersburg, Russia; Institute for Information Transmission Problems, Moscow, Russia.t E-mail: avershik@pdmi.ras.ru. Supported by the RFBR grant 17-01-00433.

‡University of Oregon, Eugene OR, USA. E-mail: yuz@uoregon.edu.

of colored braids. An attempt to generalize Arnold's construction to other irreducible representations of symmetric groups has led us to quite dissimilar arrangements, whose complements, in particular, are not $K(\pi, 1)$ spaces.

Let us recall the main result of [1]. Consider the natural representation of the symmetric group $\mathfrak{S}_n$ by permutations of coordinates in $\mathbb{C}^n$. For any distinct $i, j \in \{1, \dots, n\}$, let $H_{ij} = \{z \in V : z_i = z_j\}$ be the set of fixed points of the transposition $(ij) \in \mathfrak{S}_n$; obviously, H_{ij} is a hyperplane in V and (ij) acts as the reflection with respect to this hyperplane ("mirror"). The collection of the $\binom{n}{2}$ mirrors H_{ij} is called the *braid arrangement* Br_n of hyperplanes. Let M be the complement to all these mirrors. Arnold [1] proved that M is a $K(\pi, 1)$ space with π being the group of colored braids, whence the cohomology ring $H^*(M)$ of M is isomorphic to the cohomology ring of the group of colored braids. He also proved that the Poincaré polynomial of M is equal to $\mathrm{Poin}(M, t) = (1 + t)(1 + 2t) \dots (1 + (n - 1)t)$.

Arnold's results aroused much interest and were generalized in several directions. For instance, Brieskorn [3] proved that the ring $H^*(M)$ for an arbitrary arrangement of hyperplanes is isomorphic to the ring generated by all the logarithmic differential forms $\frac{d\alpha}{\alpha}$ where $\alpha = 0$ is the equation of a hyperplane. Then Orlik and Solomon [8] showed that this ring is determined just by the intersection lattice of the arrangement (see [9]). There are also results for the case where the permutation group $\mathfrak{S}_n$ acts on M, and hence on $H^*(M)$. It was proved in [5] that for the braid arrangement in $\mathbb{C}^n$ and the natural action of $\mathfrak{S}_n$, the module M is isomorphic to $2\,\mathrm{Ind}_{\mathfrak{S}_2}^{\mathfrak{S}_n} \mathrm{id}_2$, where id_2 is the identity representation of $\mathfrak{S}_2$. It is important to note that the braid arrangement and its direct generalizations to other series of Coxeter groups have another description: their hyperplanes are the mirrors (sets of fixed points) of elements of some finite reflection groups. The arrangements introduced in this paper do not, in general, have this property.

We suggest a development of the described framework in quite another direction, replacing the simplest natural representation of the symmetric group by other irreducible representations. Here, the naive approach, obviously, fails, since the set of fixed points of a transposition for a general irreducible representation is no longer a hyperplane. However, an analysis from the point of view of the representation theory of $\mathfrak{S}_n$ suggests the correct approach.

The main result of the paper says that *for every irreducible complex representation π_λ of the symmetric group $\mathfrak{S}_n$ there exists a canonical "intrinsic" hyperplane arrangement $\mathcal{A}_\lambda$ in the space V_λ of this representation.* This arrangement is a direct generalization of the braid arrangement, has a natural description in terms of invariant subspaces of Young subgroups, and enjoys a number of remarkable properties.

There are several constructions of this intrinsic arrangement. For instance, one can consider the space of the representation of $\mathfrak{S}_n$ induced from some "generalized Young subgroup" (a product of wreath products of symmetric groups associated in a natural way with λ) containing π_λ. Then $\mathcal{A}_\lambda$ is the intersection of the coordinate (Boolean) arrangement in the space of this induced representation with the subspace corresponding to π_λ (see Theorem 1).

An important property of the intrinsic arrangement $\mathcal{A}_\lambda$ is that the collection of unit normal vectors to its hyperplanes is an orbit of an action of the symmetric group; the convex hull of this orbit is an important polytope.

The general theory of hyperplane arrangements is the subject of much research (see [9]). From the point of view of this theory and the theory of lattices, our examples are, as far as we know, new.[1] Their main feature and the key point of our approach is that we consider arrangements and lattices invariant under an action of $\mathfrak{S}_n$ and use this invariance and the representation theory of $\mathfrak{S}_n$ to analyze them. Although there are some papers related to group actions on hyperplane arrangements and lattices, they deal with quite different questions and within a quite different approach. Chapter 6 of [9] is devoted to arrangements consisting of the fixed hyperplanes of reflections of finite reflection groups; while our arrangements are not generated by finite reflection groups, i.e., the groups generated by the reflections at the hyperplanes are infinite (at least for nontrivial hook diagrams, see Sec. 4). This poses an independent problem of studying such groups.

To describe the intersection lattice of $\mathcal{A}_\lambda$ for an arbitrary irreducible representation π_λ of $\mathfrak{S}_n$ is a quite difficult problem. We consider in detail the simplest nontrivial case, corresponding to hook diagrams of the form $\lambda = (n-k, 1^k)$. Here we can already observe important differences with the Arnold case $k = 1$. In particular, the complement to $\mathcal{A}_\lambda$ in the case $k > 1$ is no longer $K(\pi, 1)$, and its fundamental group is Abelian (see Theorem 3).

For a background on hyperplane arrangements and related objects, we

[1] When this article was under preparation, we have come upon the paper [11] in which closely related objects are considered. However, the approach in [11] is completely different, in that, first, no hyperplane arrangements are considered, and, second, the definitions are given just in a purely combinatorial form, while our constructions are systematically defined in invariant representation-theoretic terms.

refer the reader to [9], for that on the classical representation theory of symmetric groups, to [4], and for that on symmetric functions, to [6].

The paper is organized as follows. In Sec. 2, we present our main construction of a hyperplane arrangement $\mathcal{A}_\lambda$ canonically associated with an irreducible representation π_λ of $\mathfrak{S}_n$. Section 3 describes an alternative construction of this arrangement as the intersection of the coordinate arrangement in a larger space with the subspace corresponding to π_λ. In Sec. 4, we study in detail the hyperplane arrangements corresponding to hook diagrams. Finally, in Sec. 5 we study a natural join homomorphism from the partition lattice Π_n to the lattice of invariant subspaces of Young subgroups, which, in particular, provides a natural characterization of the intrinsic arrangement $\mathcal{A}_\lambda$.

2 The intrinsic hyperplane arrangement associated with an irreducible representation of the symmetric group

First, we introduce necessary notation.

For $n \in \mathbb{N}$, let $\mathbb{Y}_n$ be the set of partitions of n (or, which is the same, the set of Young diagrams with n cells) and $\mathbb{P}_n$ be the set of partitions of the set $[n] = \{1, \ldots, n\}$. For $\mu \in \mathbb{Y}_n$, we say that a partition $\alpha \in \mathbb{P}_n$ is of type μ if the sizes of the blocks of α form the partition μ of n; we denote by $\mathbb{P}_n(\mu)$ the set of all partitions $\alpha \in \mathbb{P}_n$ of type μ.

For $\lambda \in \mathbb{Y}_n$, let π_λ be the irreducible representation of $\mathfrak{S}_n$ with diagram λ and V_λ be the space of this representation.

For a set A, denote by $\mathfrak{S}[A]$ the group of permutations of its elements. Given a partition $\alpha \in \mathbb{P}_n$ with blocks $A_1, \ldots, A_k$, let $\mathfrak{S}_\alpha = \mathfrak{S}[A_1] \times \ldots \times \mathfrak{S}[A_k]$ be the corresponding Young subgroup in $\mathfrak{S}_n$. If α is of type μ, we say that $\mathfrak{S}_\alpha$ is of type μ.

Given a set T of transpositions in $\mathfrak{S}_n$, consider the graph $\Gamma(T)$ on the set of vertices $[n]$ in which vertices i and j are connected by an edge if and only if the transposition (ij) lies in T. The partition of $\Gamma(T)$ into connected components determines a partition $\alpha_T \in \mathbb{P}_n$, which in turn determines a Young subgroup $\mathfrak{S}_T := \mathfrak{S}_{\alpha_T}$.

For an arbitrary element $\sigma \in \mathfrak{S}_n$ or a subgroup $G \subset \mathfrak{S}_n$, we denote

$$V_\lambda^\sigma = \{v \in V_\lambda : \pi_\lambda(\sigma)v = v\}, \qquad V_\lambda^G = \{v \in V_\lambda : \pi_\lambda(g)v = v \text{ for all } g \in G\}.$$

The following observation is easy to prove.

Lemma 1. *Given a collection T of transpositions in $\mathfrak{S}_n$, let $V_\lambda^T = \cap_{\tau\in T} V_\lambda^\tau$. Then $V_\lambda^T = V_\lambda^{\mathfrak{S}_T}$.*

Lemma 2. *Let $\mathfrak{S}_\alpha$ be a Young subgroup in $\mathfrak{S}_n$ of type μ. Then*

$$\dim V_\lambda^{\mathfrak{S}_\alpha} = K_{\lambda\mu},$$

where $K_{\lambda\mu}$ is the Kostka number.

Proof. The dimension $\dim V_\lambda^{\mathfrak{S}_\alpha}$ is the multiplicity of the identity representation $\mathrm{id}_{\mathfrak{S}_\alpha}$ in the restriction of π_λ to $\mathfrak{S}_\alpha$:

$$\dim V_\lambda^{\mathfrak{S}_\alpha} = \langle \mathrm{Res}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} \pi_\lambda, \mathrm{id}_{\mathfrak{S}_\alpha}\rangle = \langle \pi_\lambda, \mathrm{Ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} \mathrm{id}_{\mathfrak{S}_\alpha}\rangle,$$

where the last equality is Frobenius reciprocity. Now, using the theory of symmetric functions and observing that the Frobenius map sends $\mathrm{Ind}_{\mathfrak{S}_\alpha}^{\mathfrak{S}_n} \mathrm{id}_{\mathfrak{S}_\alpha}$ to the complete symmetric function h_μ, we obtain

$$\dim V_\lambda^{\mathfrak{S}_\alpha} = \langle s_\lambda, h_\mu\rangle = K_{\lambda\mu}$$

(where s_λ is a Schur symmetric function). □

Note that in the case of the standard representation, i.e., $\lambda = (n-1, 1)$, for a single transposition $\tau \in \mathfrak{S}_n$ we have $\dim V_\lambda^\tau = n-2 = \dim V_\lambda - 1$, that is, the invariant subspace of τ is a hyperplane, the collection of all such hyperplanes being exactly the braid arrangement Br_n. The correct way of extending this construction to an arbitrary irreducible representation is suggested by the following lemma.

Using the notation and terminology of the theory of lattices, for subspaces V_1 and V_2 of V_λ we denote by $V_1 \vee V_2$ and $V_1 \wedge V_2$ their join and meet in the sense of the lattice of subspaces, i.e., $V_1 + V_2$ and $V_1 \cap V_2$, respectively.

Lemma 3. *Let $\lambda \in \mathbb{Y}_n$ and λ' be the Young diagram conjugate to λ. Given a partition $\alpha \in \mathbb{P}_n(\lambda')$, consider an arbitrary collection T of transpositions in $\mathfrak{S}_n$ generating the Young subgroup $\mathfrak{S}_\alpha$. Let $H_T = \bigvee_{\tau\in T} V_\lambda^\tau$. Then H_T depends only on α, and* $\operatorname{codim} H_T = 1$.

Proof. We have $H_T^\perp = \bigwedge_{\tau\in T}(V_\lambda^\tau)^\perp$. Now, $(V_\lambda^\tau)^\perp$ is the eigenspace of the action of τ in V_λ with eigenvalue -1, hence $\bigwedge_{\tau\in T}(V_\lambda^\tau)^\perp$ is the subspace of the sign representation of the Young group $\mathfrak{S}_T = \mathfrak{S}_\alpha$ generated by T. Therefore, the codimension in question is equal to the multiplicity of the sign representation $\mathrm{sgn}_{\mathfrak{S}_T}$ of $\mathfrak{S}_T$ in the restriction of π_λ to $\mathfrak{S}_T$. Thus, similarly to the proof of Lemma 2, we have

$$\begin{aligned}\operatorname{codim} H_T &= \langle \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_T}\pi_\lambda, \mathrm{sgn}_{\mathfrak{S}_T}\rangle = \langle \pi_\lambda, \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_T}\mathrm{sgn}_{\mathfrak{S}_T}\rangle \\ &= \langle s_\lambda, e_\mu\rangle = \langle s_{\lambda'}, h_\mu\rangle = K_{\lambda',\mu},\end{aligned}$$

where μ is the type of T. By assumption, $\mu = \lambda'$, and $K_{\lambda',\lambda'} = 1$ by the well-known property of Kostka numbers, so the lemma follows. □

Thus, we arrive at the following construction/definition.

Definition 1. Let $\lambda \in \mathbb{Y}_n$. For every partition $\alpha \in \mathbb{P}_n(\lambda')$, consider a collection of transpositions T_α generating the Young subgroup $\mathfrak{S}_\alpha$ and let

$$H_\alpha := \bigvee_{\tau\in T_\alpha} V_\lambda^\tau.$$

By Lemma 3, the subspace H_α does not depend on the choice of T_α and is a hyperplane in V_λ. Thus, the set $\mathcal{A}_\lambda := \{H_\alpha\}_{\alpha\in\mathbb{P}_n(\lambda')}$ is a hyperplane arrangement in V_λ, which will be called the *intrinsic arrangement associated with the irreducible representation* π_λ of the symmetric group $\mathfrak{S}_n$.

There is a natural transitive action of the symmetric group $\mathfrak{S}_n$ on the set $\mathbb{P}_n(\lambda')$. For $\alpha \in \mathbb{P}_n(\lambda')$, denote by n_α a unit normal vector to the hyperplane H_α (defined up to a multiplicative constant). Clearly, one can choose the multiplicative constants in such a way that $n_{\sigma(\alpha)} = \sigma(n_\alpha)$ for every $\alpha \in \mathbb{P}_n(\lambda')$ and every $\sigma \in \mathfrak{S}_n$. Thus, *the set of normals* $\{n_\alpha\}_{\alpha\in\mathbb{P}_n(\lambda')}$ *to the hyperplanes of* $\mathcal{A}_\lambda$ *is an orbit of* $\mathfrak{S}_n$.

The convex hull of the set of normals $\{n_\alpha\}_{\alpha\in\mathbb{P}_n(\lambda')}$ is an important polytope (in another form it appeared in [11] under the name of "Specht polytope"). For instance, in the case $\lambda = (n-1,1)$ of the braid arrangement, it is the root polytope of type A_n.

Among the normals n_α there is a distinguished one in the following sense. Consider the Gelfand–Tsetlin basis $\{e_t\}$ in V_λ, indexed by the standard Young tableaux t of shape λ (see, e.g., [7]). Then there is a unique normal n_α that coincides (up to a multiplicative constant) with an element of the Gelfand–Tsetlin basis. Namely, let $\lambda' = (l_1, l_2, \ldots, l_k)$, and let

$$\alpha_0 = \big\{\{1,\ldots,l_1\},\{l_1{+}1,\ldots,l_1{+}l_2\},\ldots,\{l_1{+}\ldots{+}l_{k-1}{+}1,\ldots,n\}\big\} \in \mathbb{P}_n(\lambda').$$

Also, consider the "minimal" Young tableau $\tau_\lambda^{\min}$ of shape λ obtained by successively inserting the numbers $1, 2, \ldots, n$ into the first column of λ from top to bottom, then into the second column of λ from top to bottom, etc. Then n_{α_0} coincides with $e_{\tau_\lambda^{\min}}$ (up to a multiplicative constant), and all the other normals are the elements of the orbit of $e_{\tau_\lambda^{\min}}$ under the action of $\mathfrak{S}_n$. The tableau $\tau_\lambda^{\min}$ and all the related objects (the partition α_0, the normal n_{α_0}, the hyperplane H_{α_0}, the Young subgroup $\mathfrak{S}_{\alpha_0}$) will be called *distinguished*.

The hyperplanes of our arrangement can also be described as follows. For the symmetric group $\mathfrak{S}[A]$ on a set A of cardinality N, denote by Q_A the corresponding antisymmetrizer:

$$Q_A = \frac{1}{N!}\sum_{\sigma\in\mathfrak{S}[A]}(-1)^{\operatorname{sgn}\sigma}\sigma \in \mathbb{C}[\mathfrak{S}_n].$$

Proposition 1. *For a partition $\alpha \in \mathbb{P}_n(\lambda')$, denote by R_α the reflection with respect to the hyperplane H_α. Then*

$$R_\alpha := \pi_\lambda(1 - 2Q_{A_1}\ldots Q_{A_k}) \in \mathbb{C}[\mathfrak{S}_n] \tag{1}$$

where $A_1, \ldots, A_k$ are the blocks of α and 1 is the identity element of $\mathfrak{S}_n$.

Proof. Consider the distinguished hyperplane H_{α_0}. Denoting by P_{α_0} the orthogonal projection to the corresponding normal n_{α_0}, we obviously have $R_{\alpha_0} = 1 - 2P_{\alpha_0}$. But n_{α_0} coincides with $e_{\tau_\lambda^{\min}}$, and it is not difficult to show that the orthogonal projection to $e_{\tau_\lambda^{\min}}$ is exactly

$$\pi_\lambda(Q_{\{1,\ldots,l_1\}}Q_{\{l_1+1,\ldots,l_1+l_2\}}\ldots Q_{\{l_1+\ldots+l_{k-1}+1,\ldots,n\}}),$$

so (1) is proved for H_{α_0}. Now, since any other hyperplane of $\mathcal{A}_\lambda$ is obtained from H_{α_0} by the action of an element $g \in \mathfrak{S}_n$, the corresponding reflection is obtained from R_{α_0} by the conjugation by $\pi_\lambda(g)$, and the proposition follows. □

The number $\#\mathcal{A}_\lambda$ of hyperplanes in the arrangement $\mathcal{A}_\lambda$ is equal to the number $\#\mathbb{P}_n(\lambda')$ of different partitions of $[n]$ of type λ'. Since $\mathfrak{S}_n$ acts transitively on $\mathbb{P}_n(\lambda')$, this number is equal to $\frac{n!}{\#\,\mathrm{Stab}(\alpha)}$ where $\mathrm{Stab}(\alpha)$ is the cardinality of the stabilizer of an arbitrary partition $\alpha \in \mathbb{P}_n(\lambda')$. Let $\lambda' = (1^{m_1}2^{m_2}\dots n^{m_n})$. It is easy to see that this stabilizer is isomorphic to the following subgroup of $\mathfrak{S}_n$:

$$W_{\lambda'} = (\mathfrak{S}_1 \wr \mathfrak{S}_{m_1}) \times (\mathfrak{S}_2 \wr \mathfrak{S}_{m_2}) \times \ldots \times (\mathfrak{S}_n \wr \mathfrak{S}_{m_n}), \tag{2}$$

where $\wr$ stands for wreath product. Then

$$\#\mathcal{A}_\lambda = \frac{n!}{\#W_{\lambda'}} = \frac{n!}{\prod_k (k!)^{m_k} m_k!}.$$

Example 1. The standard representation (braid arrangement). Let $\lambda = (n-1,1)$. Then $\lambda' = (21^{n-2})$, so a collection of transpositions of type λ' is just a transposition τ, the corresponding Young group $\mathfrak{S}_\tau$ is the group (isomorphic to $\mathfrak{S}_2$) generated by this transposition, and the corresponding hyperplane H_τ, which is the orthogonal complement to the sign representation of $\mathfrak{S}_\tau$, is just the invariant subspace $H_\tau = \{v : \tau v = v\}$ of τ. Thus, we obtain the braid arrangement. More exactly, the classical braid arrangement Br_n is defined by the same construction in the space $V_{\mathrm{nat}} = \mathbb{C}^n$ of the *natural* representation $\pi_{\mathrm{nat}} = \pi_{(n-1,1)} + \pi_{(n)}$ of $\mathfrak{S}_n$, where the one-dimensional subspace corresponding to the identity representation $\pi_{(n)}$ belongs to all hyperplanes. Thus, $\mathrm{Br}_n = \widetilde{\mathrm{Br}}_n \times \Phi_1$ where Φ_1 is the empty 1-arrangement and $\widetilde{\mathrm{Br}}_n$ is an irreducible arrangement which exactly coincides with $\mathcal{A}_{(n-1,1)}$.

Example 2. Let $\lambda = (2, 1^{n-2})$. Denote by χ_n the characteristic polynomial[2] of the arrangement $\mathcal{A}_\lambda = \mathcal{A}_{(2,1^{n-2})}$ (that is, the characteristic polynomial of the intersection lattice of $\mathcal{A}_\lambda$). In this case, we can calculate χ_n explicitly.

Proposition 2. *The characteristic polynomial of $\mathcal{A}_\lambda$ is given by the formula*

$$\chi_n(t) = (t-1)^{n-1} - (t-1)^{n-2} + \ldots + (-1)^n(t-1) = \frac{t-1}{t}\big((t-1)^n - (-1)^{n-1}\big).$$

Proof. Since $\lambda' = (n-1,1)$, the hyperplanes of $\mathcal{A}_\lambda$ are indexed by $(n-1)$-subsets of $[n]$, or, passing to complements, by $j \in [n]$. Observe that $\pi_{(2,1^{n-2})}|_{\mathfrak{S}_{n-1}} =$

[2]Recall that the characteristic polynomial $\chi(t)$ of a hyperplane arrangement and its Poincaré polynomial $\mathrm{Poin}(t)$ are related by the formula $\chi(t) = t^\ell\,\mathrm{Poin}(-t^{-1})$ where ℓ is the dimension of the ambient space.

$\pi_{(1^{n-1})} + \pi_{(2,1^{n-3})}$, so the hyperplane H_j is exactly the space of the representation $\pi_{(2,1^{n-3})}$ of the Young subgroup $\mathfrak{S}_{\widehat{j}}$, where we denote $\widehat{j} = [n] \setminus \{j\}$. It follows that for any $A \subset [n]$ with $|A| = k \le n-2$, the intersection $\cap_{j\in A} H_j$ is the $(n-k-1)$-dimensional space of the representation $\pi_{(2,1^{n-k-2})}$ of the Young subgroup $\mathfrak{S}_{[n]\setminus A}$.

Consider the distinguished hyperplane H_n. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_n\}$ and $\mathcal{A}'' = \mathcal{A}^{H_n}$ be the restriction of $\mathcal{A}$ to H_n. Then it is not difficult to see from above that $\mathcal{A}'' = \mathcal{A}_{(2,1^{n-3})}$, while $\mathcal{A}'$ is the Boolean arrangement in the $(n-1)$-dimensional space. Since the characteristic polynomial $\chi_{\mathcal{A}'}$ of the Boolean arrangement is equal to $(t-1)^{n-1}$, from the deletion-restriction theorem $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}'}(t) - \chi_{\mathcal{A}''}(t)$ (see [9, Corollary 2.57]) we obtain a recurrence relation $\chi_n(t) = (t-1)^n - \chi_{n-1}(t)$ for χ_n. Since, obviously, $\chi_2(t) = t-1$, the desired formula follows. □

Example 3. The table below shows the basic characteristics of the intrinsic arrangements $\mathcal{A}_\lambda$ for all diagrams λ with 5 cells (except the trivial cases of the identity and sign representations, where V_λ is one-dimensional): the dimension $\dim V_\lambda$ of the ambient space, the number $\#\mathcal{A}_\lambda$ of hyperplanes, and the characteristic polynomial $\chi_{\mathcal{A}_\lambda}(t)$.

λ	$\dim V_\lambda$	$\#\mathcal{A}_\lambda$	$\chi_{\mathcal{A}_\lambda}(t)$
(41)	4	10	$(t-4)(t-3)(t-2)(t-1)$
(32)	5	15	$t^5 - 15t^4 + 90t^3 - 260t^2 + 350t - 166$
(31^2)	6	10	$t^6 - 10t^5 + 45t^4 - 115t^3 + 175t^2 - 147t + 51$
(21^3)	4	5	$t^4 - 5t^3 + 10t^2 - 10t + 4$

3 The "coordinate" model of the intrinsic arrangement

The intrinsic arrangement $\mathcal{A}_\lambda$ is defined for the irreducible representation π_λ of $\mathfrak{S}_n$. However, this representation has different realizations. By a slight abuse of language, *we will denote by the symbol $\mathcal{A}_\lambda$ the corresponding arrangement in whatever space the representation π_λ is realized*; this should not cause any ambiguity.

Let $\lambda' = (1^{m_1}2^{m_2}\dots n^{m_n})$, and let $W_{\lambda'}$ be the group given by (2). For each subgroup of the form $\mathfrak{S}_k \wr \mathfrak{S}_{m_k}$, consider the representation $\mathrm{sgn}_k \wr \mathrm{id}_{m_k}$ obtained by taking the sign representation in each factor $\mathfrak{S}_k$ and the identity representation of $\mathfrak{S}_{m_k}$. Consider the corresponding induced representation of $\mathfrak{S}_n$:

$$\xi_{\lambda'} = \operatorname{Ind}_{W_{\lambda'}}^{\mathfrak{S}_n} \big((\mathrm{sgn}_1 \wr \mathrm{id}_{m_1}) \times (\mathrm{sgn}_2 \wr \mathrm{id}_{m_2}) \times \ldots \times (\mathrm{sgn}_n \wr \mathrm{id}_{m_n})\big). \tag{3}$$

Observe that $\dim \xi_{\lambda'} = \#\mathcal{A}_\lambda$.

Lemma 4. *The multiplicity of π_λ in the decomposition of $\xi_{\lambda'}$ into irreducible representations is equal to* 1.

Proof. Consider the Young subgroup $\mathfrak{S}_{\lambda'} \subset \mathfrak{S}_n$ corresponding to the diagram λ', and let $\eta_{\lambda'} = \operatorname{Ind}_{\mathfrak{S}_{\lambda'}}^{\mathfrak{S}_n} \mathrm{sgn}_{\lambda'}$ be the representation of $\mathfrak{S}_n$ induced from the sign representation of $\mathfrak{S}_{\lambda'}$. It is well known from the classical representation theory of symmetric groups that the multiplicity of π_λ in $\eta_{\lambda'}$ is equal to 1.

Now, observe that we have a natural embedding $\mathfrak{S}_{\lambda'} \subset W_{\lambda'}$ and denote $\Xi_{\lambda'} = \operatorname{Ind}_{\mathfrak{S}_{\lambda'}}^{W_{\lambda'}} \mathrm{sgn}_{\lambda'}$. By the transitivity of induction, we have $\eta_{\lambda'} = \operatorname{Ind}_{W_{\lambda'}}^{\mathfrak{S}_n} \Xi_{\lambda'}$. It is not difficult to see that

$$\Xi_{\lambda'} = (\mathrm{sgn}_1 \wr \mathrm{Reg}_{m_1}) \times (\mathrm{sgn}_2 \wr \mathrm{Reg}_{m_2}) \times \ldots \times (\mathrm{sgn}_n \wr \mathrm{Reg}_{m_n}),$$

where Reg_k is the regular representation of $\mathfrak{S}_k$. Comparing with (3), it follows that $\xi_{\lambda'} \subset \eta_{\lambda'}$, and hence the multiplicity of π_λ in the decomposition of $\xi_{\lambda'}$ is at most 1.

It remains to prove that π_λ is indeed contained in $\xi_{\lambda'}$. To this end, recall that the representation $\eta_{\lambda'}$ can be realized in the space $M^{\lambda'}$ spanned by the elements $e_t = \sum_{\sigma \in R_t} \mathrm{sgn}(\sigma)\sigma(t)$ where t is a Young tableau of shape λ' and R_t is the row stabilizer of t (and we mean the permutation action of $\mathfrak{S}_n$ on Young tableaux). The subrepresentation π_λ is spanned by the elements of the form

$$\sum_{\sigma \in C_t} \sigma(e_t) \tag{4}$$

where C_t is the column stabilizer of t. Now, observe that the subspace of $M^{\lambda'}$ corresponding to $\xi_{\lambda'}$ consists of the linear combinations of e_t such that if tableaux t_1 and t_2 differ only by the order of rows of equal length, then e_t and $e_{t'}$ have equal coefficients. But it is easy to see that the elements (4) satisfy this property, hence $\pi_\lambda \subset \xi_{\lambda'}$. ☐

Remark. If λ' has no rows of equal length greater than 1, then the group $W_{\lambda'}$ coincides with the Young subgroup $\mathfrak{S}_{\lambda'}$ and the representation $\xi_{\lambda'}$ coincides with $\eta_{\lambda'}$. In this case, Lemma 4 reduces to a key fact of the classical approach to the representation theory of symmetric group (the Frobenius–Young correspondence).

Theorem 1. *Let $\mathcal{B}_\lambda$ be the Boolean (coordinate) arrangement of hyperplanes in the space $M^{\lambda'}$ of $\xi_{\lambda'}$. Then the restriction of $\mathcal{B}_\lambda$ to the subspace $V_\lambda \subset M^{\lambda'}$ of the irreducible representation π_λ coincides with the intrinsic arrangement $\mathcal{A}_\lambda$.*

Proof. By Lemma 4, there is a unique subspace $V_\lambda \subset M^{\lambda'}$ such that the restriction of $\xi_{\lambda'}$ to V_λ is isomorphic to π_λ. Denote by $P : M^{\lambda'} \to V_\lambda$ the orthogonal projection to V_λ; it is an operator commuting with the action of $\mathfrak{S}_n$. Now, consider the distinguished tableau $\tau_\lambda^{\min}$ and the conjugate tableau $\tau_{\lambda'}^{\max} = (\tau_\lambda^{\min})'$ of shape λ'. Let $w_0 \in M^{\lambda'}$ be the tabloid corresponding to $\tau_{\lambda'}^{\max}$. Note that w_0 is a cyclic vector of $M^{\lambda'}$ and, consequently, Pw_0 is a cyclic vector of V_λ, so $Pw_0 \neq 0$. Further, w_0 is the normal to one of the hyperplanes B_0 of the coordinate arrangement $\mathcal{B}_\lambda$. Hence Pw_0 is the normal to $B_0 \cap V_\lambda$. By construction, the distinguished Young subgroup $\mathfrak{S}_{\alpha_0}$ acts on the vector Pw_0 as the sign representation. Since, by Lemma 3, the multiplicity of the sign representation of $\mathfrak{S}_{\alpha_0}$ in π_λ is equal to 1, it follows that Pw_0 coincides (up to a multiplicative constant) with n_{α_0} and, therefore, $B_0 \cap V_\lambda = H_{\alpha_0}$, i.e., $B_0 \cap V_\lambda \in \mathcal{A}_\lambda$. The rest follows from the fact that all the other hyperplanes both from $\mathcal{B}_\lambda$ and $\mathcal{A}_\lambda$ are obtained from B_0 and H_{α_0}, respectively, by the action of $\mathfrak{S}_n$ and the projection P commutes with this action. □

4 Representations corresponding to hook diagrams

Let $\lambda = (n-k, 1^k)$ be an arbitrary hook diagram. Then $M^{\lambda'} = \Lambda^{k+1}(\mathbb{C}^n)$ is the $(k+1)$th exterior power of $\mathbb{C}^n$ (the space of totally antisymmetric tensors T of valence $k+1$), the coordinate arrangement $\mathcal{B}_{(n-k,1^k)}$ consists of the $\binom{n}{k+1}$ "coordinate" hyperplanes $H_I := \{T_I = 0\}$, where by $I = \{i_1, \ldots, i_{k+1}\}$ we denote a $(k+1)$-subset of $[n]$, assuming that $1 \le i_1 < i_2 < \ldots < i_{k+1} \le n$, and the restriction of this arrangement to $V_{(n-k,1^k)}$ is exactly the arrangement $\mathcal{A}_{(n-k,1^k)}$.

The representation of $\mathfrak{S}_n$ in $\Lambda^{k+1}(\mathbb{C}^n)$ decomposes into the sum of two irreducible representations isomorphic to $\pi_{(n-k,1^k)}$ and $\pi_{(n-k-1,1^{k+1})}$. Denote the corresponding subspaces by $V_k^{(k+1)}$ and $V_{k+1}^{(k+1)}$, respectively, so $\Lambda^{k+1}(\mathbb{C}^n) = V_k^{(k+1)} \oplus V_{k+1}^{(k+1)}$. Then the equivariant embedding $V_k^{(k)} \hookrightarrow V_k^{(k+1)}$ is given by the formula $V_k^{(k)} \ni \alpha \mapsto \partial\alpha \in V_k^{(k+1)}$ where $\partial : \Lambda^k(\mathbb{C}^n) \to \Lambda^{k+1}(\mathbb{C}^n)$ is the usual differential in the exterior algebra:

$$(\partial T)_{i_1 \dots i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} T_{i_1 \dots \widehat{i_j} \dots i_{k+1}}$$

(as usual, a hat over an index means that the index is omitted). The following theorem shows that the arrangement $\mathcal{A}_{(n-k,1^k)}$ can be essentially given by the set of equations $\partial T = 0$.

Theorem 2. *Consider the hyperplane arrangement* $\mathcal{C}_{(n-k,1^k)} = \{H_I\}$ *in* $\Lambda^k(\mathbb{C}^n)$ *where*

$$H_I = \{T \in \Lambda^k(\mathbb{C}^n) : (\partial T)_I = 0\}$$

and I *ranges over all* $(k+1)$*-subsets of* $[n]$*. Then*

$$\mathcal{C}_{(n-k,1^k)} = \mathcal{A}_{(n-k,1^k)} \times \Phi_\ell$$

where Φ_ℓ *is the empty arrangement in the* ℓ*-dimensional space with* $\ell = \dim \pi_{(n-k+1,1^{k-1})} = \binom{n-1}{k-1}$.

Proof. Follows from the above considerations and the identity $\partial^2 = 0$. Namely, each hyperplane from $\mathcal{C}_{(n-k,1^k)}$ is the direct sum of a hyperplane from $\mathcal{A}_{(n-k,1^k)}$ and the space $V_{k-1}^{(k)}$. □

This construction is a direct generalization of the case of the braid arrangement Br_n corresponding to the standard representation of $\mathfrak{S}_n$. Namely, let $\lambda = (n-1,1)$, that is, $k = 1$. Then $\Lambda^2(\mathbb{C}^n)$ can be identified with the space $M_n^{\text{skew}} = \{(a_{ij})_{i,j}^n : a_{ij} = -a_{ji}\}$ of skew-symmetric $n \times n$ matrices. The subspace $V_1^{(2)}$ of the representation $\pi_{(n-1,1)}$ is given by

$$V_1^{(2)} = \{(a_{ij})_{i,j}^n : a_{ij} = \alpha_i - \alpha_j \text{ where } (\alpha_j)_{j=1}^n \in \mathbb{C}^n,\ \sum_{j=1}^n \alpha_j = 0\}.$$

The coordinate arrangement $\mathcal{B}_{(n-1,1)}$ in M_n^{skew} consists of the $\frac{n(n-1)}{2}$ hyperplanes $H_{ij} := \{M \in M_n^{\text{skew}} : a_{ij} = 0\}$ for $1 \le i < j \le n$, and the restriction of this arrangement to $V_1^{(2)}$ is exactly the irreducible braid arrangement $\widetilde{\mathrm{Br}}_n$. While the arrangement $\mathcal{C}_{(n-1,1)}$ in $\mathbb{C}^n$ consists of the hyperplanes of the form $\{\alpha_i - \alpha_j = 0\}$, which is the standard definition of the braid arrangement Br_n. Thus, $\mathcal{C}_{(n-1,1)} = \mathrm{Br}_n$ and $\mathcal{B}_{(n-1,1)} = \widetilde{\mathrm{Br}}_n$.

Arnold's result [1] that the complement of the braid arrangement is a $K(\pi, 1)$ space ignited attempts to generalize it to a wider class of reflection arrangements. This generalization was proved in full by Bessis [2]. In the remaining part of this section, we will prove that the complements of our arrangements corresponding to hook diagrams with $k > 1$ are never $K(\pi, 1)$. In particular, in this case the group generated by all reflections at the hyperplanes is infinite.

Suppose there exists a linear dependence between the left-hand sides of several hyperplane equations, i.e., there exist nonzero $a_1, \ldots, a_m \in \mathbb{C}$ and $(k+1)$-subsets $I_1, \ldots, I_m \subset [n]$ such that $\sum_{r=1}^m a_r (\partial T)_{I_r} = 0$. Here the vanishing of the left-hand side means that the coefficients of the elements with the same index sum to 0. Thus we can focus our attention on the indices I_r only, which brings an abstract simplicial complex into the picture.

The abstract complex we need is just the simplex S on the set of vertices $[n]$ (i.e., each subset of $[n]$ is a simplex of S). Consider the chain complex $C = C(S)$ on S with coefficients from $\mathbb{C}$ and denote its boundary map by d. Then the above equality is equivalent to $\sum_{r=1}^m a_r dI_r = d(\sum_{r=1}^m a_r I_r) = 0$, where I_r is viewed as a simplex in the abstract complex and as a generator of the respective chain complex. Thus, $z = \sum_{r=1}^m a_r I_r$ is a cycle in the chain complex. The following observation is obvious.

Lemma 5. *If a chain in a simplicial complex is a cycle of dimension greater than one, then it cannot be a linear combination of less than* 4 *natural generators (simplices).*

Lemma 5 immediately implies the following.

Lemma 6. *The intersection of any two hyperplanes in $\mathcal{A}_{(n-1,1^k)}$ for $k > 1$ is double (i.e., there is no other hyperplane containing this intersection).*

Theorem 3. *Let $\lambda = (n-k, 1^k)$ with $k > 1$. Then the fundamental group π_1 of the complement to the arrangement $\mathcal{A}_\lambda$ is free Abelian of rank equal to the size of the arrangement. Besides, the complement is not $K(\pi, 1)$.*

Proof. By factoring out we can reduce the problem to an essential arrangement.

Recall from [9] that the generators of π_1 are one for each hyperplane, and relations correspond to every intersection of hyperplanes of codimension 2. More precisely, assume that m hyperplanes have a common subspace J of codimension 2 and there are no other hyperplanes containing J. Denote the generators corresponding to these m hyperplanes by $a_1, a_2, \ldots, a_m$. Then the relations corresponding to J are

$$a_1 a_2 \cdots a_m = a_2 a_3 \cdots a_m a_1 = \cdots = a_m a_1 \cdots a_{m-1}.$$

As Lemma 6 implies, in our case for each codimension 2 subspace J we have $m \leq 2$, and $m = 2$ only for J that lies in two hyperplanes. Thus, we have the relations $ab = ba$ for any two generators a and b. The group given by these generators and relations is the free Abelian group of rank equal to the number of hyperplanes.

Now, if the complement were $K(\pi, 1)$, then it would have been of homotopy type of a torus of dimension $N = \binom{n}{k+1}$. In particular, the homology of the complement must be nonzero in dimension N. On the other hand, $N > n$, therefore, the rank of the intersection lattice is less than N. Thus the homology of the complement is 0 in dimension N, whence its homotopy type is not that of a torus. $\square$

Example. Let $n = 5$, $k = 2$, and $\lambda = (3, 1^2)$. There are $10 = \binom{5}{3}$ hyperplanes, given in the space $\Lambda^2(\mathbb{C}^5)$ of dimension 10 with coordinates t_{ij} by the following equations: for each triple $s = ijk$ $(i < j < k)$, the equation $E(s)$ is

$$t_{ij} - t_{ik} + t_{jk} = 0.$$

All the hyperplanes have the common subspace V_μ of dimension 4 where $\mu = (2, 1^3)$. Factoring out the common subspace, we get an essential arrangement in V_λ of dimension 6. The left-hand sides of the 10 equations above have 5 linear dependencies corresponding to the standard basis T_{ijkl} $(i < j < k < l)$ of $\Lambda^4(\mathbb{C}^5)$. For instance, the relation corresponding to the basic element T_{1234} is $E(123) - E(124) + E(134) - E(234) = 0$.

5 Representations of the partition lattice

Let $\Pi_n = (\mathbb{P}_n, \prec)$ be the partition lattice, i.e., the set of partitions of the set $[n]$ ordered by refinement. It is easy to see that the set YS_n of all Young subgroups in $\mathfrak{S}_n$ ordered by inclusion is a lattice, and the map

$$\psi : \Pi_n \to \mathrm{YS}_n, \qquad \psi(\alpha) = \mathfrak{S}_\alpha,$$

is a lattice isomorphism between YS_n and the partition lattice Π_n.

It is well known that the set of all subspaces of a vector space ordered by inclusion is also a lattice, where the greatest lower bound and the least upper bound are given by the intersection and the sum of subspaces, respectively. Now, for a fixed Young diagram λ of size n, denote by $\mathcal{S}(V_\lambda)$ the lattice of subspaces of V_λ. We have a map i_λ that associates with a subgroup $G \subset \mathfrak{S}_n$ its invariant subspace V_λ^G in V_λ. Considering the composite $\phi_\lambda = i_\lambda \circ \psi$, we obtain a map

$$\phi_\lambda : \Pi_n \to \mathcal{S}(V_\lambda)$$

that sends a partition α of $[n]$ to the invariant subspace $V_\lambda^{\mathfrak{S}_\alpha}$.

Given $\alpha \in \Pi_n$, denote by $\bar{\alpha} \in \mathbb{Y}_n$ the partition of the integer n determined by the sizes of blocks of α.

Proposition 3. *Fix $\lambda \in \mathbb{Y}_n$ and denote $\phi := \phi_\lambda$. Then*

- (i) $\phi(\alpha \vee \beta) = \phi(\alpha) \wedge \phi(\beta)$;
- (ii) $\phi(\alpha \wedge \beta) \supset \phi(\alpha) \vee \phi(\beta)$;
- (iii) $\dim \phi(\alpha) = K_{\lambda\bar{\alpha}}$;
- (iv) $\phi(\alpha) = 0$ *unless* $\lambda \trianglerighteq \bar{\alpha}$ *(where* $\trianglerighteq$ *stands for the dominance order on partitions).*

Proof. (i) Denote $V = V_\lambda$. We have $\phi(\alpha) \wedge \phi(\beta) = \phi(\alpha) \cap \phi(\beta) = V^{\mathfrak{S}_\alpha} \cap V^{\mathfrak{S}_\beta} = V^{\langle \mathfrak{S}_\alpha, \mathfrak{S}_\beta \rangle} = V^{\psi(\alpha \vee \beta)} = \phi(\alpha \vee \beta)$, where by $\langle \mathfrak{S}_\alpha, \mathfrak{S}_\beta \rangle$ we denote the subgroup in $\mathfrak{S}_n$ generated by $\mathfrak{S}_\alpha$ and $\mathfrak{S}_\beta$, which is exactly the Young subgroup corresponding to $\alpha \vee \beta$.

(ii) We have $\phi(\alpha) \vee \phi(\beta) = V^{\mathfrak{S}_\alpha} + V^{\mathfrak{S}_\beta} \subset V^{\mathfrak{S}_\alpha \cap \mathfrak{S}_\beta} = V^{\mathfrak{S}_{\alpha \wedge \beta}} = \phi(\alpha \wedge \beta)$.

(iii) Follows from Lemma 2.

(iv) Follows from (iii) and the upper triangularity of Kostka numbers (see, e.g., [6, Section I.6]). □

Remark. As one can easily see, the equality in (ii) does not hold in general.

Corollary 1. *The Kostka numbers satisfy the following inequality: for every $\lambda \in \mathbb{Y}_n$ and any $\alpha, \beta \in \Pi_n$,*

$$K_{\lambda,\overline{\alpha\vee\beta}} + K_{\lambda,\overline{\alpha\wedge\beta}} \geq K_{\lambda,\bar{\alpha}} + K_{\lambda,\bar{\beta}}.$$

Proof. By Proposition 3, the right-hand side is equal to

$$\dim \phi(\alpha) + \dim \phi(\beta) = \dim(\phi(\alpha) \wedge \phi(\beta)) + \dim(\phi(\alpha) \vee \phi(\beta)).$$

But $\dim(\phi(\alpha) \wedge \phi(\beta)) = \dim \phi(\alpha \vee \beta) = K_{\lambda,\overline{\alpha\vee\beta}}$, while

$$\dim(\phi(\alpha) \vee \phi(\beta)) \leq \dim \phi(\alpha \wedge \beta) = K_{\lambda,\overline{\alpha\wedge\beta}}.$$

□

Thus, in the lattice $\mathcal{S}(V_\lambda)$ of all subspaces of V_λ we have the subset $\mathcal{S}^{\mathrm{Y}}(V_\lambda) = \phi(\Pi_n)$ of subspaces invariant with respect to Young subgroups of $\mathfrak{S}_n$. It follows from Proposition 3 that the intersection $V_1 \wedge V_2$ of two subspaces $V_1, V_2 \in \mathcal{S}^{\mathrm{Y}}(V_\lambda)$ also lies in $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$. However, their sum $V_1 \vee V_2$ does not necessarily lie in $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$. Nevertheless, the following statement holds.

Lemma 7. *The set $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ of subspaces in V_λ invariant with respect to Young subgroups of $\mathfrak{S}_n$ ordered by inclusion is a coatomistic lattice.*

Proof. As we have observed, the intersection of two subspaces from $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ belongs to $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$, which implies that $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ is a meet-semilattice. Besides, it contains the greatest element $\hat{1} = V_\lambda = \phi(\varepsilon)$ where ε is the partition into separate points: $\varepsilon = \{\{1\}, \{2\}, \ldots, \{n\}\}$. But it is well known (see, e.g., [10, Proposition 3.3.1]) that a meet-semilattice with $\hat{1}$ is a lattice.

Obviously, for every $\alpha \in \Pi_n$ with $\bar{\alpha} = (2, 1^{n-2})$, the corresponding subspace $V_\lambda^{\mathfrak{S}_\alpha}$ is a coatom of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$. The fact that every element V_λ^β of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ is the greatest lower bound of atoms follows from the considerations of Section 2: it suffices to take a collection of transpositions that generate $\mathfrak{S}_\beta$; each transposition (ij) gives rise to a partition $\alpha_{ij} \in \Pi_n$ with $\bar{\alpha}_{ij} = (2, 1^{n-2})$ where α_{ij} consists of the 2-block $\{ij\}$ and $n-2$ singletons, and V_λ^β is the greatest lower bound of the corresponding coatoms of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$. □

We emphasize that the meet of two elements in $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ coincides with their meet in $\mathcal{S}(V_\lambda)$, but for the join this is, in general, not the case. So, $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ is *not* a sublattice of $\mathcal{S}(V_\lambda)$.

Also, observe that, as we have mentioned in the proof of Lemma 7, the greatest element of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ is $\hat{1} = V_\lambda$, while the smallest element $\hat{0}$ is the zero subspace $\{0\}$ for all $\lambda \neq (n)$.

Summarizing, we obtain the following.

Theorem 4. *Denote by $\mathcal{S}^{\mathrm{Y}}(V_\lambda)^*$ the order dual of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)$ (which is an atomistic lattice). Then the map $\phi : \Pi_n \to \mathcal{S}^{\mathrm{Y}}(V_\lambda)^*$ is a join homomorphism of lattices.*

If $\pi_{\mathrm{nat}} = \pi_{(n-1,1)} + \pi_{(n)}$ is the natural representation of $\mathfrak{S}_n$ in $\mathbb{C}^n$, then $\mathcal{S}^{\mathrm{Y}}(V_{\mathrm{nat}})^*$ is exactly the intersection lattice of the braid arrangement Br_n, which is well known to be isomorphic to the partition lattice Π_n (see, e.g, [9]); thus, in this case $\phi = \phi_{\mathrm{nat}}$ is in fact an isomorphism of lattices.

Theorem 5. *The lattice $\mathcal{S}^{\mathrm{Y}}(V_\lambda)^*$ is embedded into the intersection lattice $L(\mathcal{A}_\lambda)$ of the hyperplane arrangement $\mathcal{A}_\lambda$, which is a minimal hyperplane arrangement satisfying this property.*

Proof. Recall that $\mathcal{S}^{\mathrm{Y}}(V_\lambda)^*$ is an atomistic lattice, the atoms being the subspaces V_λ^σ for all transpositions $\sigma \in \mathfrak{S}_n$. Thus, to show that $\mathcal{S}^{\mathrm{Y}}(V_\lambda)^* \subset L(\mathcal{A}_\lambda)$, it suffices to prove that every such atom can be obtained as the intersection of a collection of hyperplanes H_α. For $i, j \in [n]$ and $\alpha \in \mathbb{P}_n(\lambda')$, we write $i \sim_\alpha j$ if i and j lie in the same block of α. Let us show that for every transposition $\sigma = (ij)$,

$$V_\lambda^\sigma = \bigcap_{\alpha \in \mathbb{P}_n(\lambda') :\, i \sim_\alpha j} H_\alpha. \tag{5}$$

Obviously (since everything is invariant under $\mathfrak{S}_n$), it suffices to prove this for $\sigma = (12)$. Let α_0 be the distinguished partition (see Sec. 2). Any other partition α of type λ' such that $1 \sim_\alpha 2$ is obtained from α_0 by conjugation by some element $g \in \mathfrak{S}_{\{3,\ldots,n\}}$, and then $H_\alpha = gH_{\alpha_0}$. It follows that the orthogonal complement to the right-hand side of (5) is the subspace spanned by $\mathfrak{S}_{\{3,\ldots,n\}} n_{\alpha_0}$. But this is exactly the subspace in V_λ spanned by the Gelfand–Tsetlin vectors e_t indexed by Young tableaux t such that 2 lies in the first column of t, while V_λ^σ is the subspace in V_λ spanned by the vectors e_t indexed by Young tableaux t such that 2 lies in the first row of t. Thus, these are the orthogonal complements to each other, and the first assertion of the theorem follows.

On the other hand, each hyperplane H_α of $\mathcal{A}_\lambda$ is, by construction, obtained as the join of elements of $\mathcal{S}^{\mathrm{Y}}(V_\lambda)^*$, which implies the minimality. □

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