Functional Analysis and Its Applications, Vol. 40, No. 3, pp. 207–217, 2006
Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 40, No. 3, pp. 53–65, 2006

Quantum Inverse Scattering Method for the *q*-Boson Model and Symmetric Functions*

N. V. Tsilevich

Received August 10, 2005

Abstract. The purpose of this paper is to show that the quantum inverse scattering method for the so-called q-boson model has a nice interpretation in terms of the algebra of symmetric functions. In particular, in the case of the phase model (corresponding to $q = 0$) the creation operator coincides (modulo a scalar factor) with the operator of multiplication by the generating function of complete homogeneous symmetric functions, and the wave functions are expressed via the Schur functions $s_\lambda(x)$. The general case of the q-boson model is related in a similar way to the Hall–Littlewood symmetric functions $P_\lambda(x; q^2)$.

Key words: q-boson model, phase model, quantum inverse scattering method, symmetric functions, Hall–Littlewood functions, Schur functions.

1. Introduction

The *q-boson model* (e.g., see [2] and [4]) describes a strongly correlated exactly solvable one-dimensional boson system on a finite one-dimensional lattice. This system is of importance in several branches of modern physics such as solid state physics and quantum nonlinear optics. The corresponding *q-boson* (or *q-oscillator*) algebra [8] is closely related to the quantum algebra $sl_q(2)$ [7]. The special case $q = 0$ of the q-boson model, which is especially easy to investigate, is called the *phase model* ([3], [4], [1]).

The aim of this paper is to show that the *quantum inverse scattering method* [6] for the q-boson model has a nice and useful interpretation in terms of the *algebra of symmetric functions* [9]. The starting point for our approach was the paper [1] by Bogoliubov, who showed that the phase model is closely related to the enumeration of plane partitions.

Starting from the simpler case of the phase model, we construct a realization of this model in the algebra Λ of symmetric functions as follows: to each basis Fock vector $\psi_{n_0,\dots,n_M}$ with occupation numbers $n_0, \dots, n_M$, we assign the Schur function $s_\lambda(x)$ corresponding to the Young diagram λ with n_j rows of length j (for details, see Sec. 2.2). It turns out that under this realization the creation operator $B(u)$ of the quantum inverse scattering method coincides (up to a scalar factor) with the operator of multiplication by the (truncated) generating function $H_M(u^2) = \sum_{k=0}^{M} u^{2k} h_k$ of the complete homogeneous symmetric functions h_k (in what follows, for simplicity we denote the operator of multiplication by a function by the same symbol as the function itself), and the annihilation operator $C(u)$ is essentially the adjoint operator $H_M^{\perp}(u^{-2})$ with respect to the standard inner product in Λ. This allows us, in particular, to apply the machinery of symmetric functions and readily obtain the expansion of the wave function in the basis Fock vectors; the coefficients of this expansion are given by Schur functions. Furthermore, we can easily find the limit of the regularized creation and annihilation operators as $M \to \infty$.

On the other hand, we can use this interrelation between the phase model and symmetric functions in the opposite direction: for example, using the commutation relations for $B(u)$ and $C(u)$ given by the quantum inverse scattering method (i.e., the corresponding R-matrix), we can obtain commutation relations for $H_M(u)$ and $H_M^{\perp}(u)$ in the subspace Λ_M of Λ spanned by the Schur

*Supported by CRDF grant no. RUM1-2622-ST-04, RFBR grant no. 05-01-00899, and President of the Russian Federation grant for Support of Leading Scientific Schools no. NSh.-4329.2006.1.

functions whose diagrams have at most M columns. (They are more involved than the commutation relation for the operator of multiplication by the full generating function $H(u) = \sum_{k=0}^{\infty} u^k h_k$ and its adjoint $H^{\perp}(u)$ in the entire algebra Λ.)

We also establish a relation between the operators arising in the quantum inverse scattering method for the phase model and the vertex operator formalism used by Okounkov and Reshetikhin [10] for computing the correlation functions of three-dimensional Young diagrams (plane partitions). It turns out that the vertex operators in [10] are the same operators $H(u)$ and $H^{\perp}(u)$, i.e., the $M \to \infty$ limits of the regularized creation and annihilation operators of the phase model. However, if we wish to study three-dimensional Young diagrams contained in a *box*, then the approach in [10] fails, while the method of [1], based on the quantum inverse scattering method for the phase model, allows one to compute the partition and correlation functions for three-dimensional diagrams in a box, since, as was noted above, it allows one to obtain commutation relations for the "truncated" operators.

A similar scheme can be implemented for the general q-boson model. In this case, one should use a generalization of the Schur functions, namely, the *Hall–Littlewood functions* $P_\lambda(x; q^2)$ (which coincide with $s_\lambda(x)$ for $q = 0$). In particular, the wave functions of the q-boson model are expressed in terms of the Hall–Littlewood functions, and the creation operator coincides (up to a scalar factor) with the operator of multiplication by the generating function $Q_M(u^2) = \sum_{k=0}^{M} u^{2k} q_k$ (see Sec. 3.1).

2. The Phase Model and Schur Functions

2.1. The phase model. Consider the algebra generated by three operators ϕ, $\phi^\dagger$, and N with the commutation relations

$$[N, \phi] = -\phi, \qquad [N, \phi^\dagger] = \phi^\dagger, \qquad [\phi, \phi^\dagger] = \pi, \tag{1}$$

where π is the vacuum projection. This algebra can be realized in the one-dimensional (i.e., having one-dimensional n-particle subspaces) Fock space, where the operators ϕ, $\phi^\dagger$, and N act as the phase operators and the number of particles operator, respectively:

$$\phi^\dagger |n\rangle = |n+1\rangle, \qquad \phi|n\rangle = |n-1\rangle, \ \ \phi|0\rangle = 0, \qquad N|n\rangle = n|n\rangle,$$

where $|n\rangle$ is the (normalized) n-particle Fock vector (in particular, $|0\rangle$ is the vacuum vector). Thus ϕ is an isometry (one-sided shift), and we have

$$\phi\phi^\dagger = 1, \qquad \phi^\dagger\phi = 1 - \pi,$$

where $\pi|0\rangle = |0\rangle$, $\pi|n\rangle = 0$ for $n \geqslant 1$.

Now fix a positive integer M (the number of sites) and consider the tensor product $\mathscr{F} = \mathscr{F}_0 \otimes \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_M$ of $M+1$ copies $\mathscr{F}_i$, $i = 0, \dots, M$, of the one-dimensional Fock space. Denote by $\phi_i, \phi_i^\dagger$, and N_i the operators that act as $\phi, \phi^\dagger$, and N, respectively, in the ith space and identically in the other spaces $\mathscr{F}_j$; for example,

$$\phi_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes \phi \otimes I_{i+1} \otimes \cdots \otimes I_M,$$

where I_j is the identity operator in $\mathscr{F}_j$. Thus the operators commute if they pertain to distinct sites and satisfy the commutation relations (1) if they pertain to the same site.

The Hamiltonian of the phase model has the form

$$H = -\frac{1}{2} \sum_{n=0}^{M} (\phi_n^\dagger \phi_{n+1} + \phi_n \phi_{n+1}^\dagger - 2N_n)$$

with the periodic boundary conditions, $M + 1 \equiv 1$.

Following the quantum inverse scattering method [6], consider the *L-matrix*

$$L_n(u) = \begin{pmatrix} u^{-1}I & \phi_n^\dagger \\ \phi_n & uI \end{pmatrix}, \qquad n = 0, 1, \dots, M,$$

where u is a scalar parameter and I is the identity operator in $\mathscr{F}$. For every $n = 0, 1, \ldots, M$, the matrix L_n satisfies the bilinear equation

$$R(u,v)(L_n(u) \otimes L_n(v)) = (L_n(v) \otimes L_n(u))R(u,v) \tag{2}$$

with the 4×4 R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{3}$$

where

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}\,. \tag{4}$$

The *monodromy matrix* is defined as

$$T(u) = L_M(u)L_{M-1}(u) \cdots L_0(u) \quad \text{(matrix product)}.$$

It satisfies the bilinear equation with the same R-matrix (3), (4):

$$R(u,v)(T(u) \otimes T(v)) = (T(v) \otimes T(u))R(u,v). \tag{5}$$

Let

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

The matrix entries $A(u), B(u), C(u), D(u)$ of the monodromy matrix $T(u)$ act in the space $\mathscr{F}$. Denoting by $\hat{N} = N_0 + \cdots + N_m$ the operator of the total number of particles, we have

$$\widehat{N}B(u) = B(u)(\widehat{N} + 1), \qquad \widehat{N}C(u) = C(u)(\widehat{N} - 1),$$

so that $B(u)$ is a creation operator and $C(u)$ is an annihilation operator. The operators $A(u)$ and $C(u)$ do not change the number of particles.

Denote by $|0\rangle_j$ the vacuum vector in $\mathscr{F}_j$ and by $|0\rangle = \bigotimes_{j=0}^{M} |0\rangle_j$ the total vacuum vector in $\mathscr{F}$. Consider N-particle state vectors of the form

$$|\Psi_N(u_1, \ldots, u_N)\rangle = \prod_{j=1}^{N} B(u_j)|0\rangle.$$

(According to the algebraic Bethe Ansatz (see [6]), the eigenfunctions of the Hamiltonian are precisely of this form.) We are interested in calculating the expansion of these vectors in the basis N-particle vectors

$$\psi_{n_0,\ldots,n_M} = \bigotimes_{j=0}^{M} |n_j\rangle_j, \qquad n_0 + \cdots + n_M = N, \tag{6}$$

where $|n_j\rangle_j = (\phi^\dagger)^n |0\rangle_j$ is the n_j-particle vector in the jth Fock space $\mathscr{F}_j$; the numbers n_k are called the *occupation numbers* of the vector (6).

2.2. Realization of the phase model in the algebra of symmetric functions. Necessary background on symmetric functions can be found in [9].

To a basis vector (6) with occupation numbers $n_0, \ldots, n_M$, we assign the Young diagram $\lambda = 1^{n_1}2^{n_2}\cdots$ with n_j rows of length j, $j = 1, \ldots, M$, and the corresponding Schur function* s_λ:

$$\bigotimes_{j=0}^{M} |n_j\rangle_j \longleftrightarrow s_\lambda, \qquad \lambda = 1^{n_1}2^{n_2}\cdots\,. \tag{7}$$

*Recall that the Schur functions form a basis in the algebra Λ of symmetric functions.

Remark. Bearing in mind this correspondence on the one hand and the standard realization of the Fock space in the algebra of symmetric functions (e.g., see [5, Ch. 14]) on the other hand, it is natural to refer to the jth Fock space $\mathscr{F}_j$ as the *space of particles of energy* j.

Note that the correspondence (7) does not take into account the number n_0 of zero energy particles. Thus (7) defines a realization of the positive energy subspace $\widehat{\mathscr{F}} = \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_M$ in the algebra Λ of symmetric functions, or, more precisely, in its subspace Λ_M spanned by the Schur functions s_λ whose diagrams have at most M columns (i.e., have rows of length at most M). In view of the Jacobi–Trudi identity [9, I.3.4], one can determine this subspace by choosing** the arguments of the symmetric functions so that

$$h_{M+1} = h_{M+2} = \cdots = 0. \tag{8}$$

(For details, see the proof of Proposition 2.)

On the other hand, if we know the total number N of particles, then we can recover the number of zero energy particles in the basis vector corresponding to s_λ as $n_0 = N - l(\lambda)$, where $l(\lambda)$ is the number of rows in the diagram λ. Consider the decomposition $\mathscr{F} = \mathscr{F}^0 \oplus \mathscr{F}^1 \oplus \cdots \oplus \mathscr{F}^N \oplus \cdots$ of the space $\mathscr{F}$ into N-particle subspaces $\mathscr{F}^N$, and let Λ_M^N be the space of symmetric functions corresponding to $\mathscr{F}^N$. Note that the space Λ_M^N is spanned by the Schur functions s_λ whose diagrams λ lie in the $M \times N$ box, i.e., have at most N rows and at most M columns. Thus the entire space $\mathscr{F}$ can be realized as the direct sum $\Lambda_M^0 \oplus \Lambda_M^1 \oplus \cdots \oplus \Lambda_M^N \oplus \cdots$.

Since $B(u)$ is a creation operator, i.e., increases the number of particles by one, it sends Λ_M^N to Λ_M^{N+1}. Thus it suffices to study its action on the space $\widehat{\mathscr{F}} \equiv \Lambda_M$, i.e., the operator $\mathscr{B}(u) := PB(u)P$, where P is the projection from $\mathscr{F}$ onto $\widehat{\mathscr{F}}$ ("forgetting the zero energy space").

Proposition 1. *Let* $\mathscr{B}(u) = u^{-M}\widetilde{\mathscr{B}}(u)$. *The operator* $\widetilde{\mathscr{B}}(u)$ *acts in* Λ_M *as the operator of multiplication by* $H_M(u^2)$, *where* $H_M(t) = \sum_{k=0}^M t^k h_k$ *is the (truncated) generating function of the complete homogeneous symmetric functions* h_k.

Proof. One can readily see that the operator $\widetilde{\mathscr{B}}$ has the form $\widetilde{\mathscr{B}} = \sum_{k=0}^M u^{2k}\mathscr{B}_k$. Thus it suffices to prove that $\mathscr{B}_k$ is the operator of multiplication by the kth complete symmetric function h_k. Set $\phi_j^{-1} = \phi_j$, $\phi_j^0 = 1$, and $\phi_j^1 = \phi_j^\dagger$. Since

$$B(u) = \sum_{j_M,\dots,j_1=1}^{2} (L_M(u))_{1j_M}(L_{M-1}(u))_{j_Mj_{M-1}} \cdots (L_0(u))_{j_12},$$

we have

$$\mathscr{B}_k = \sum_{\varepsilon_M,\dots,\varepsilon_0} \phi_M^{\varepsilon_M} \cdots \phi_1^{\varepsilon_1} = \sum_{\varepsilon_M,\dots,\varepsilon_0} \mathscr{B}_{\varepsilon_M,\dots,\varepsilon_0},$$

where the sum is over all sequences $\varepsilon_j \in \{-1,0,1\}$, $j = 0,\dots,M$, satisfying the following conditions: (a) let ε_l be the highest nonzero element; i.e., $\varepsilon_M = \cdots = \varepsilon_{l+1} = 0$ and $\varepsilon_l \neq 0$; then $\varepsilon_l = 1$; (b) $\varepsilon_0 \neq -1$; (c) adjacent elements do not have the same sign; i.e., $\varepsilon_{j+1}\varepsilon_j \neq 1$ for every j; (d) $\sum_{j=1}^M j\varepsilon_j = k$. Obviously, $\mathscr{B}_{\varepsilon_M,\dots,\varepsilon_0}$ takes a basis vector (6) to a basis vector.

In terms of Schur functions, we have $\phi_j^\dagger s_\mu = s_\lambda$, where the diagram λ is obtained from μ by inserting a row of length j, and $\phi_j s_\mu = s_\lambda$, where λ is obtained from μ by removing a row of length j (with $\phi_j s_\mu = 0$ if μ does not contain a row of length j). Denote by ν_i' the length of the ith column of a diagram ν and by $n_i(\nu)$ the number of rows of length i in ν. Then $\nu_i' - \nu_{i+1}' = n_i(\nu)$. Now let $\mathscr{B}_{\varepsilon_M,\dots,\varepsilon_0} s_\mu = s_\lambda$ and set $\theta_i' = \lambda_i' - \mu_i'$. Then $n_i(\lambda) = n_i(\mu) + \varepsilon_i$, so that $\theta_i' = \theta_{i+1}' + n_i(\lambda) - n_i(\mu) = \theta_{i+1}' + \varepsilon_i$, whence

$$\theta_M' = \varepsilon_M, \qquad \theta_{M-1}' = \varepsilon_M + \varepsilon_{M-1}, \quad \dots, \quad \theta_1' = \varepsilon_M + \cdots + \varepsilon_1.$$

** This can be done, since, as is well known, the complete symmetric functions h_k are algebraically independent in Λ.

Now it follows from (a) that $\theta'_M = \dots = \theta'_{l+1} = 0$ and $\theta'_l = 1$. Further, in view of (c), we have $\theta'_i \in \{0, 1\}$, which means that $\lambda \supset \mu$ and the skew diagram $\lambda \setminus \mu$ contains at most one cell in each column, i.e., is a horizontal strip. Moreover, $\sum \theta'_i = k$, so that $\lambda \setminus \mu$ contains k cells. Denoting by $\mathscr{H}_k$ the set of horizontal k-strips, we obtain

$$\mathscr{B}_k s_\mu = \sum_{\lambda \colon \lambda \setminus \mu \in \mathscr{H}_k} s_\lambda,$$

whence, in view of the Pieri formula [9, I.5.16], $\mathscr{B}_k s_\mu = h_k s_\mu$. □

Remark. The truncated generating function $H_M(t) = \sum_{k=0}^M t^k h_k$ can also be viewed as the image of the full generating function $H(t) = \sum_{k=0}^\infty t^k h_k$ under the specialization (8), $H_M(t) = H(t)|_{h_{M+1}=h_{M+2}=\dots=0}$.

Corollary 1. *The $M \to \infty$ limit of the regularized creation operator $\widetilde{\mathscr{B}}(u) = u^M \mathscr{B}(u)$ on the positive energy subspace $\widehat{\mathscr{F}}$ is just the operator of multiplication by $H(u^2)$ in the entire algebra Λ of symmetric functions.*

Using the interpretation of $\mathscr{B}(u)$ obtained in Proposition 1, we can readily find the desired expansion of the N-particle vector in basis vectors.

Proposition 2. *The expansion of the N-particle vector in basis vectors is given by the formula*

$$|\Psi_N(u_1, \dots, u_N)\rangle = \sum_\lambda s_\lambda(u_1^2, \dots, u_N^2) \bigotimes_{j=0}^M |n_j\rangle_j,$$

where the sum is over Young diagrams λ with at most N rows and at most M columns.

Proof. By the formula for the generating function of complete symmetric functions [9, I.2.5],

$$H(u^2) = \prod_i \frac{1}{1 - u^2 x_i}.$$

Note that $\Psi_N(u_1, \dots, u_N) \in \mathscr{F}^N$ and identify $\mathscr{F}^N$ with Λ_M^N as described above. Observing that the vacuum vector corresponds to the unit function $s_\emptyset \equiv 1$ and using Proposition 1, we obtain

$$|\Psi_N(u_1, \dots, u_N)\rangle = \prod_{j=1}^N B(u_j)|0\rangle = (u_1 \cdots u_M)^{-M} \prod_{j=1}^N \tilde{B}(u_j)|0\rangle = (u_1 \cdots u_M)^{-M} \prod_j \prod_i \frac{1}{1 - u_j^2 x_i}.$$

The well-known Cauchy identity [9, I.4.3] yields

$$|\Psi_N(u_1, \dots, u_N)\rangle = (u_1 \dots u_M)^{-M} \sum_\lambda s_\lambda(u_1^2, \dots, u_N^2) s_\lambda(x),$$

which gives the desired formula in view of (7). The restrictions on λ are obtained as follows. First, a Schur function vanishes if the number of nonzero arguments is less than the number of its rows. Thus $s_\lambda(u_1^2, \dots, u_N^2) = 0$ if $l(\lambda) > N$. On the other hand, by the Jacobi–Trudi identity [9, I.3.4], $s_\lambda = \det(h_{\lambda_i - i + j})_{i,j=1}^n$, where $n \geqslant l(\lambda)$. Thus we see that the first row of this determinant, and hence s_λ, vanishes for $\lambda_1 > M$ under the specialization (8). □

Lemma 1. *The matrix entries of the monodromy matrix $T(u)$ are related by the formulas*

$$B(u) = uA(u)\phi_0^\dagger, \qquad C(u) = u^{-1}\phi_0 A^\dagger(u^{-1}), \qquad D(u) = \phi_0 A^\dagger(u^{-1})\phi_0^\dagger.$$

The proof is by easy induction on M. □

In particular, setting $\mathscr{A}(u) = PA(u)P$, $\mathscr{C}(u) = PC(u)P$, and $\mathscr{D}(u) = PD(u)P$, we have

$$\mathscr{A}(u) = u^{-1}\mathscr{B}(u), \qquad \mathscr{C}(u) = \mathscr{B}^\dagger(u^{-1}), \qquad \mathscr{D}(u) = u\mathscr{B}^\dagger(u^{-1}).$$

It follows, for example, that in the realization of the phase model in the algebra of symmetric functions the annihilation operator has the representation

$$\mathscr{C}(u) = u^M \widetilde{\mathscr{C}}(u), \qquad \widetilde{\mathscr{C}}(u) = \widetilde{\mathscr{B}}^\dagger(u^{-1}) = H_M^\perp(u^{-2}) = \sum_{n=0}^{M} u^{-2n} h_n^{\perp,M},$$

where $h_n^{\perp,M}$ is the adjoint of the operator of multiplication by h_n in the space Λ_M with the standard inner product (with respect to which the Schur functions form an orthonormal basis). Note that $h_n^{\perp,M}$ substantially depends on M. In the $M \to \infty$ limit, we have

$$\widetilde{\mathscr{C}}(u) = H^\perp(u^{-2}) = \sum_{n=0}^{\infty} u^{-2n} h_n^\perp, \tag{9}$$

where $h_n^\perp$ is the adjoint of the operator of multiplication by h_n in the entire space Λ (cf. [9, Ex. I.5.3, Ex. I.5.29]).

2.3. Vertex operators and enumeration of plane partitions.

Lemma 2. *In the $M \to \infty$ limit, the operator $\widetilde{\mathscr{B}}(u)$ has the vertex operator representation*

$$\widetilde{\mathscr{B}}(u) = \exp\biggl(\sum_{k=1}^{\infty} \frac{u^{2k}}{k}\, \alpha_{-k}\biggr), \tag{10}$$

where α_{-k}, $k = 1, 2, \dots$, are the free boson operators.

Proof. By the well-known formula [9, I.2.10],

$$\frac{d}{dt} \ln H(t) = P(t),$$

where $P(t) = \sum_{k=1}^\infty t^{k-1} p_k$ is the generating function of the Newton power sums p_k. Thus

$$H(t) = \exp\biggl(\int P(t)\biggr) = \exp\biggl(\sum_{k=1}^{\infty} \frac{t^k}{k}\, p_k\biggr).$$

On the other hand, it is well known that in the realization of the Fock space as the algebra of symmetric functions, the free boson operator α_{-k} corresponds to the multiplication by p_k, and so (10) follows by Proposition 1. □

Note that the vertex operator on the right-hand side in (10) is exactly the operator used by Okounkov and Reshetikhin [10] in connection with the computation of the correlation functions for plane partitions. Namely, in the notation of [10],

$$\tilde{B}_0(q^{\frac{j}{2}}) = \Gamma_+(\phi_j), \quad \text{where } \phi_j(z) = \phi_{3\mathrm{D}}[j](z) = \frac{1}{1 - q^j z}\,.$$

In particular, in the symmetric function realization of the Fock space, the vertex operator associated with the Schur process describing plane partitions is just the operator of multiplication by the generating function of complete symmetric functions,

$$\Gamma_+(\phi_{3\mathrm{D}}[j]) = H(q^j).$$

In view of (9), the $M \to \infty$ limit of the regularized annihilation operators $\widetilde{\mathscr{C}}(v) = v^{-M}\mathscr{C}(v)$ has the vertex operator representation

$$\exp\biggl(\sum_{k=1}^{\infty} \frac{v^{-2k}}{k}\, \alpha_k\biggr).$$

2.4. Commutation relations for the "truncated" operators. Using the commutation relations for vertex operators (e.g., see [5, (14.10.12)] or [10, (11)]), one can readily obtain the well-known commutation relation [9, Ex. I.5.29, (2)] for the operators H and $H^{\perp}$ in the entire algebra Λ of symmetric functions:

$$H^{\perp}(u)H(v) = \frac{1}{1-uv}\, H(v)H^{\perp}(u). \tag{11}$$

However, the vertex representation (10) and the commutation relation (11) are no longer valid in the subspace Λ_M generated by the Schur functions whose diagrams have at most M columns. Nevertheless, we can obtain the commutation relation for $H_M(v) = \sum_{k=0}^{M} v^k h_k$ and $H_M^{\perp}(v)$ in Λ_M using the machinery of the quantum inverse scattering method [6] and the above interpretation of the phase model in terms of symmetric functions. Namely, the bilinear equation (5) implies, in particular, that

$$D(u)B(v) = \frac{u^2}{u^2-v^2}\, B(v)D(u) - \frac{uv}{u^2-v^2}\, B(u)D(v).$$

Using Proposition 1 and Lemma 1, we obtain

$$H_M^{\perp}(u)H_M(v) = \frac{1}{1-uv}\, [H_M(v)H_M^{\perp}(u) - (uv)^{M+1}H_M(u^{-1})H_M^{\perp}(v^{-1})]. \tag{12}$$

We see that in the formal $M \to \infty$ limit with $|uv| < 1$, relation (12) is reduced to (11).

Expanding both sides of (12) into power series in u, v and matching the coefficients of like powers of u, v, we obtain the following commutation relations in Λ_M:

$$h_m^{\perp,M} h_n = \sum_{i=0}^{\min\{m,n\}} h_{n-i}h_{m-i}^{\perp,M} - \sum_{i=0}^{\min\{m,n\}-1} h_{M+1-m+i}h_{M+1-n+i}^{\perp,M}.$$

Examples. For $M = 1$, we obtain the relation $h_1^{\perp,1}h_1 = 1$ in Λ_1. Indeed, Λ_1 is generated by Schur functions with one-column diagrams, so that the operators h_1 and $h_1^{\perp,1}$ correspond to adding and removing one cell, respectively, i.e., are one-sided shifts.

For $M = 2$, we obtain

$$h_1^{\perp,2}h_1 = h_1h_1^{\perp,2} + 1 - h_2h_2^{\perp,2}, \quad h_1^{\perp,2}h_2 = h_1, \quad h_2^{\perp,2}h_1 = h_1^{\perp,2}, \quad h_2^{\perp,2}h_2 = 1.$$

3. The *q*-Boson Model and Hall–Littlewood Functions

3.1. The *q*-Boson model. The phase model considered in the previous section is a special case of the so-called *q-boson* model ([2], [4]).

Let q be a nonnegative parameter. Consider the *q-boson algebra* generated by three operators B, $B^{\dagger}$, and N with the commutation relations

$$[N, B] = -B, \qquad [N, B^{\dagger}] = B^{\dagger}, \qquad [B, B^{\dagger}] = q^{2N}.$$

We set

$$[n] = \frac{1-q^{2n}}{1-q^2}, \qquad [n]! = \prod_{j=1}^{n}[j].$$

The standard realization of the q-boson algebra in the Fock space $\mathscr{F}$ has the form

$$B^{\dagger}|n\rangle = [n+1]^{1/2}|n+1\rangle, \qquad B|n\rangle = [n]^{1/2}|n-1\rangle,\ B|0\rangle = 0, \qquad N|n\rangle = n|n\rangle.$$

However, it will be more convenient to use another realization, namely,

$$B^{\dagger}|n\rangle = [n+1]|n+1\rangle, \qquad B|n\rangle = |n-1\rangle,\ B|0\rangle = 0, \qquad N|n\rangle = n|n\rangle. \tag{13}$$

For the operators B and $B^{\dagger}$ to be the adjoints of each other, we should normalize the Fock vectors so that

$$\langle n|n\rangle^2 = \frac{1}{[n]!}. \tag{14}$$

Yet another realization of the q-boson model in the Fock space is given by the formula

$$B^\dagger|n\rangle = |n+1\rangle, \qquad B|n\rangle = [n]|n-1\rangle,\ B|0\rangle = 0, \qquad N|n\rangle = n|n\rangle \tag{15}$$

with the normalization

$$\langle n|n\rangle^2 = [n]!. \tag{16}$$

One can readily see that the phase model is the special case of the q-boson model corresponding to $q = 0$. As $q \to 1$, the operators B and $B^\dagger$ turn into the canonical free boson operators b and $b^\dagger$, respectively, satisfying the commutation relation $[b, b^\dagger] = 1$.

Now we apply the same scheme as we have used for the phase model in Sec. 2: fix the number of sites M, consider the tensor product $\mathscr{F} = \mathscr{F}_0 \otimes \mathscr{F}_1 \otimes \cdots \otimes \mathscr{F}_M$ of $M+1$ copies $\mathscr{F}_i$, $i = 0, \ldots, M$, of the one-dimensional Fock space, and denote by B_i, $B_i^\dagger$, and N_i the operators that act as B, $B^\dagger$, and N, respectively, in the ith space and identically in the other spaces $\mathscr{F}_j$. It will be convenient to use the realization (13) of the q-boson algebra for $i = 1, \ldots, M$ and the realization (15) for $i = 0$.

Note that in view of (14) and (16) the squared norms of the basis N-particle vectors (6) are equal to

$$\|\psi_{n_0,\ldots,n_M}\|^2 = \frac{[n_0]}{\prod_{j=1}^M [n_j]!}\,. \tag{17}$$

The Hamiltonian of the q-boson model has the form

$$H = -\frac{1}{2}\sum_{n=0}^{M}(B_n^\dagger B_{n+1} + B_n B_{n+1}^\dagger - 2N_n)$$

with the periodic boundary conditions, $M+1 \equiv 1$.

The L*-matrix* for the q-boson model is given by

$$L_0(u) = \begin{pmatrix} u^{-1}I & B_0^\dagger \\ (1-q^2)B_0 & uI \end{pmatrix}, \quad L_n(u) = \begin{pmatrix} u^{-1}I & (1-q^2)B_n^\dagger \\ B_n & uI \end{pmatrix}, \qquad n = 1, \ldots, M. \tag{18}$$

This L-matrix satisfies the bilinear equation (2) with the R-matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & q^{-1} & 0 \\ 0 & q & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{19}$$

where

$$f(v,u) = \frac{q^{-1}u^2 - qv^2}{u^2 - v^2}, \qquad g(v,u) = \frac{uv}{u^2 - v^2}(q^{-1} - q). \tag{20}$$

Note that the R-matrix (3), (4) of the phase model is obtained from the R-matrix of the q-boson model as the renormalized $q \to 0$ limit,

$$R_{\text{phase}} = \lim_{q\to 0} qR_{q\text{-boson}}.$$

Denote by

$$T(u) = L_M(u)\cdots L_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

the monodromy matrix of the q-boson model. It satisfies the bilinear equation (5) with the R-matrix (19), (20).

3.2. The Hall–Littlewood functions. In this section, we give a brief account of basic facts related to the Hall–Littlewood symmetric functions; for details, see [9, Ch. III].

The Hall–Littlewood symmetric functions with parameter $t \geqslant 0$, indexed by Young diagrams λ, can be defined, for example, as follows. First, for a finite number $n \geqslant l(\lambda)$ of variables $x_1, \ldots, x_n$, set

$$P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in \mathfrak{S}_n} w\biggl(x_1^{\lambda_1}\cdots x_n^{\lambda_n} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j}\biggr),$$

where $\mathfrak{S}_n$ is the symmetric group of degree n acting by permutations of variables and

$$v_\lambda(t) = \prod_{i \geqslant 0} v_{n_i(\lambda)}(t), \qquad v_n(t) = \prod_{i=1}^{n} \frac{1-t^i}{1-t}\,.$$

Then observe that for each diagram λ with $l(\lambda) \leqslant n$ we have $P_\lambda(x_1, \ldots, x_n; t) = P_\lambda(x_1, \ldots, x_n, 0; t)$, so that we can define a symmetric function $P_\lambda(x;t)$ of infinitely many variables with coefficients in $\mathbb{Z}[t]$ as the inductive limit of $P_\lambda(x_1, \ldots, x_n; t)$ with respect to the projections sending the last variable to 0. The functions $P_\lambda(x;t)$ form a $\mathbb{Z}[t]$-basis of the algebra $\Lambda[t]$ of symmetric functions with coefficients in $\mathbb{Z}[t]$. They interpolate between the Schur functions s_λ and the monomial symmetric functions m_λ,

$$P_\lambda(x;0) = s_\lambda(x), \qquad P_\lambda(x;1) = m_\lambda(x). \tag{21}$$

It is convenient to introduce another family of symmetric functions $Q_\lambda(x;t)$ that are scalar multiples of $P_\lambda(x;t)$. Namely, we set

$$Q_\lambda(x;t) = b_\lambda(t) P_\lambda(x;t),$$

where

$$b_\lambda(t) = \prod_{i \geqslant 1} \phi_{n_i(\lambda)}(t), \qquad \phi_n(t) = (1-t)(1-t^2)\cdots(1-t^n).$$

For $q = 0$, we have $Q_\lambda(x;0) = P_\lambda(x;0) = s_\lambda(x)$.

Now set

$$q_r(x;t) = Q_{(r)}(x;t) = (1-t)P_{(r)}(x;t), \quad r \geqslant 1, \qquad q_0(x,t) = 1.$$

The generating function for q_r is equal to

$$Q(u) = \sum_{r=0}^{\infty} q_r(x;t) u^r = \prod_i \frac{1 - x_i t u}{1 - x_i u} = \frac{H(u)}{H(tu)}, \tag{22}$$

where $H(u)$ is the generating function of complete symmetric functions. In particular,

$$q_r(x,0) = h_r(x), \tag{23}$$

$$q_r(x;1) = 0 \quad \text{for } r \geqslant 1. \tag{24}$$

Let

$$q_\lambda(x;t) = \prod_{i \geqslant 0} q_{\lambda_i}(x;t). \tag{25}$$

The symmetric functions $q_\lambda(x;t)$ form a $\mathbb{Q}[t]$-basis of $\Lambda[t]$.

For $t \neq 1$, we introduce an inner product in $\Lambda[t]$ by requiring that the bases $\{q_\lambda\}$ and $\{m_\lambda\}$ be dual to each other:

$$\langle q_\lambda(x;t), m_\mu(x) \rangle = \delta_{\lambda\mu}.$$

Then the bases $\{P_\lambda\}$ and $\{Q_\lambda\}$ are also dual,

$$\langle P_\lambda(x;t), Q_\mu(x;t) \rangle = \delta_{\lambda\mu},$$

so that the squared norm of the Hall–Littlewood function $P_\lambda(x;t)$ is equal to

$$\langle P_\lambda(x;t), P_\lambda(x;t) \rangle = \frac{1}{b_\lambda(t)}\,. \tag{26}$$

Note that for $t = 0$ this inner product is reduced to the standard inner product in Λ (with respect to which the Schur functions form an orthonormal basis).

The generalization of the Cauchy identity to the case of Hall–Littlewood functions reads

$$\prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j} = \sum_\lambda P_\lambda(x;t) Q_\lambda(y;t) = \sum_\lambda b_\lambda(t) P_\lambda(x;t) P_\lambda(y;t). \tag{27}$$

There is also a generalization of the Pieri formula. Namely,

$$P_\mu q_r = \sum_{\lambda:\lambda\setminus\mu\in\mathscr{H}_k} \phi_{\lambda\setminus\mu}(t) P_\lambda \tag{28}$$

with

$$\phi_{\lambda\setminus\mu}(t) = \prod_{i\in I}(1 - t^{n_i(\lambda)}), \tag{29}$$

where $\theta = \lambda \setminus \mu$ and $I = \{i : \theta'_i = 1,\ \theta'_{i+1} = 0\}$. (Recall that θ'_i is the length of the ith column of the skew diagram θ, which in the case of a horizontal strip can be equal to 0 or 1.)

3.3. Realization of the q-boson model in the algebra of symmetric functions. We follow the same scheme as was used for the phase model in Sec. 2.2.

To each basis vector (6), we assign the Hall–Littlewood function $P_\lambda(x; q^2)$ with diagram determined by the occupation numbers,

$$\bigotimes_{j=0}^{M} |n_j\rangle_j \leftrightarrow P_\lambda(x; q^2), \qquad \lambda = 1^{n_1} 2^{n_2} \cdots .$$

Note that in view of (17) and (26) this correspondence is not an isometry.

Set $\mathscr{B}(u) = PB(u)P$, where P is the projection onto the positive energy subspace. Denote by $\Lambda_M[q^2]$ the subspace in $\Lambda[q^2]$ spanned by the Hall–Littlewood functions $P_\lambda(x; q^2)$ with diagrams having at most M columns.

Proposition 3. *Let $\mathscr{B}(u) = u^{-M}\widetilde{\mathscr{B}}(u)$. The operator $\widetilde{\mathscr{B}}(u)$ acts in $\Lambda_M[q^2]$ as the operator of multiplication by $Q_M(u^2)$, where $Q_M(t) = \sum_{k=0}^{M} t^k q_k(x; q^2)$.*

Proof. Arguing as in the proof of Proposition 1, we see that

$$\mathscr{B}_k(u) P_\mu(x; q^2) = \sum_{\lambda:\, \lambda\setminus\mu\in\mathscr{H}_k} c(\mu, \lambda) P_\lambda(x; q^2)$$

but the coefficient $c(\mu, \lambda)$ is no longer equal to 1. However, we can readily compute it. Indeed, the coefficient arises from applying the creation operators $B_j^\dagger$ with $j \geqslant 1$. Namely, if $\mathscr{B}_{\varepsilon_M,\dots,\varepsilon_0} s_\mu = s_\lambda$, then $c(\mu, \lambda)$ is the product of the factors $(1 - q^2)[n_i(\mu) + 1] = 1 - q^{2(n_i(\mu)+1)}$ over all $i \geqslant 1$ such that $\varepsilon_i = 1$. But the latter condition is equivalent to $n_i(\lambda) = n_i(\mu) + 1$, or $\theta'_i = 1$ and $\theta'_{i+1} = 0$; i.e., the product is over all i belonging to the set I in the notation of (29). Thus

$$c(\mu, \lambda) = \prod_{i\in I}(1 - q^{2n_i(\lambda)}) = \phi_{\lambda\setminus\mu}(q^2),$$

and the proposition follows by the Pieri type formula (28) for the Hall–Littlewood functions. □

Remark. Just as in the case of complete symmetric functions, we can treat the truncated generating function $Q_M(t)$ as the full generating function $Q(t)$ under an appropriate specialization,

$$Q_M(t) = Q(t)|_{q_{M+1}=q_{M+2}=\cdots=0}. \tag{30}$$

Corollary 2. *There is a well-defined $M \to \infty$ limit of the operator $\widetilde{\mathscr{B}}(u)$. In the realization of the q-boson model in the algebra of symmetric functions, it is the operator of multiplication by $Q(u^2) = \frac{H(u^2)}{H(q^2u^2)}$.*

Proposition 4. *Let $|\Psi_N(u_1, \dots, u_N)\rangle = \prod_{j=1}^{N} B(u_j)|0\rangle$. Then*

$$|\Psi_N(u_1, \dots, u_N)\rangle = \sum_{\lambda} Q_\lambda(u_1^2, \dots, u_N^2; q^2) \bigotimes_{j=0}^{M} |n_j\rangle_j,$$

where the sum is over all Young diagrams λ with at most N rows and at most M columns.

Proof. The proof is similar to that of Proposition 2 and uses Proposition 3, formula (22) for the generating function of $q_k(x; t)$, and the Cauchy type identity (27) for the Hall–Littlewood functions.

The restriction on the diagrams λ is a consequence of the following facts: (a) $P_\lambda(x;t)=0$ if the number of nonzero variables x_i is less than $l(\lambda)$ (this readily follows from the definition of the Hall–Littlewood functions), and (b) the transition matrix from the basis $\{Q_\lambda\}$ to the basis $\{q_\lambda\}$ is strictly lower triangular [9, III.2.16], so that the specialization (30) implies, in view of (25), that $Q_\lambda = 0$ unless $\lambda_1 \leqslant M$. $\square$

Remark. For $q=0$, the results of this section reproduce those of Sec. 2.2 in view of (21) and (23).

For $q=1$, the L-matrix (18) degenerates into a lower triangular matrix for $n=1,\dots,M$ and an upper triangular matrix for $n=0$, so that $B(u)=u^{-M}B_0^\dagger$, whence $\widetilde{\mathscr{B}}(u)=1$, in accordance with (24).

The author is grateful to N. M. Bogoliubov for introducing into the quantum inverse scattering method and q-boson model and to A. M. Vershik for numerous useful discussions.

References

[1] N. M. Bogoliubov, Boxed plane partitions, bosons, and the quantum inverse scattering method, PDMI Preprint 1/2004, 2004.

[2] N. M. Bogoliubov, R. K. Bullough, and J. Timonen, "Critical behavior for correlated strongly coupled boson systems in $1+1$ dimensions," Phys. Rev. Lett., **25** (1994), 3933–3936.

[3] N. M. Bogoliubov, A. G. Izergin, and N. A. Kitanine, "Correlators of the phase model," Phys. Lett. A, **231** (1997), 347–352.

[4] N. M. Bogoliubov, A. G. Izergin, and N. A. Kitanine, "Correlation functions for a strongly correlated boson system," Nucl. Phys. B, **516 [FS]** (1998), 501–528.

[5] V. Kac, Infinite-Dimensional Lie Algebras, 3rd ed., Cambridge Univ. Press, Cambridge, 1990.

[6] V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge Univ. Press, Cambridge, 1993.

[7] P. P. Kulish, "Contraction of quantum algebras and q-oscillators," Teoret. Mat. Fiz., **86** (1991), 157–160; Theoret. Math. Phys., **86** (1991), 108–110.

[8] P. P. Kulish and E. V. Damaskinsky, "On the q-oscillator and the quantum algebra $\mathrm{su}_q(1,1)$," J. Phys. A, **23** (1990), L415–L419.

[9] I. Macdonald, Symmetric Functions and Hall Polynomials, 2nd ed., Clarendon Press, Oxford, 1995.

[10] A. Okounkov and N. Reshetikhin, "Correlation function of Schur process with application to local geometry of a random 3-dimensional Young diagram," J. Amer. Math. Soc., **16** (2003), 581–603.

St. Petersburg Department of Steklov Institute of Mathematics
e-mail: natalia@pdmi.ras.ru

Translated by N. V. Tsilevich