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Quasi-invariance of the gamma process and multiplicative properties of the Poisson–Dirichlet measures

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Abstract. In this paper we describe new fundamental properties of the law P_Γ of the classical gamma process and related properties of the Poisson–Dirichlet measures $\mathrm{PD}(\theta)$. We prove the quasi-invariance of the measure P_Γ with respect to an infinite-dimensional multiplicative group (the fact first discovered in [4]) and the Markov–Krein identity as corollaries of the formula for the Laplace transform of P_Γ.

The quasi-invariance of the measure P_Γ allows us to obtain new quasi-invariance properties of the measure $\mathrm{PD}(\theta)$. The corresponding invariance properties hold for σ-finite analogues of P_Γ and $\mathrm{PD}(\theta)$. We also show that the measure P_Γ can be considered as a limit of measures corresponding to the α-stable Lévy processes when parameter α tends to zero.

Our approach is based on simultaneous considering the gamma process (especially its Laplace transform) and its simplicial part – the Poisson–Dirichlet measures. © Académie des Sciences/Elsevier, Paris

Quasi-invariance du processus gamma et propriétés multiplicatives des distributions de Poisson–Dirichlet

Résumé. *Dans cette Note, nous décrivons certaines propriétés fondamentales de la loi P_Γ du processus gamma classique et des propriétés correspondantes des mésures de Poisson–Dirichlet* $\mathrm{PD}(\theta)$. *Nous déduisons la propriété de quasi-invariance de la loi P_Γ par rapport à un « gros » groupe multiplicatif et l'identité de Markov–Krein directement de la transformation de Laplace de P_Γ.*

La quasi-invariance de la loi P_Γ permet d'obtenir des propriétés de quasi-invariance des lois de Poisson–Dirichlet $\mathrm{PD}(\theta)$. *On obtient les propriétés correspondantes d'invariance pour des analogues σ-finis de P_Γ et* $\mathrm{PD}(\theta)$. *Nous montrons enfin que la loi P_Γ peut être considerée comme une limite des lois de processus stables dont le paramètre α tend vers* 0.

Notre approche se fonde sur la considération simultanée du processus gamma (surtout sa transformation de Laplace) et sa partie simpliciale – des mesures de Poisson–Dirichlet. © Académie des Sciences/Elsevier, Paris

Note présentée par Marc YOR.

Version française abrégée

On définit un processus de Lévy sur un espace mesuré général $(X, \mathcal{X}, \nu)$, dont la mesure ν est bornée. On introduit les parties conique et simpliciale d'un processus de Lévy et formule un théorème de caractérisation pour les parties coniques.

Le processus gamma $\gamma(\cdot)$ sur $(X, \mathcal{X}, \nu)$ est un processus de Lévy qui satisfait la propriété suivante: si $A_1, \ldots, A_k$ sont des ensembles $\mathcal{X}$-mesurables disjoints de X, alors : $\gamma(A_1), \ldots, \gamma(A_k)$ sont des variables gamma indépendantes, de paramètres respectifs $\nu(A_1), \ldots, \nu(A_k)$. Le processus $\{\gamma(A)/\gamma[X]\}_{A\in\mathcal{X}}$ est indépendant de $\gamma(X)$. Ce processus gamma normalisé est souvent désigné dans la littérature sous le nom de processus (ou mesure) de Dirichlet. La partie simpliciale du processus gamma sur $(X, \mathcal{X}, \nu)$ avec $\nu(X) = \theta$ est la mesure de Poisson–Dirichlet $PD(\theta)$.

On montre que le processus gamma est quasi-invariant par rapport à un group multiplicatif $\mathcal{M} = \{a : X \to \mathbb{R}_+ : \int_X |\log a(x)|\, d\nu(x) < \infty\}$ et déduit la propriété correspondante de quasi-invariance des mesures de Poisson–Dirichlet. L'analyse de la transformation de Laplace du processus gamma permet d'obtenir la formule de Markov–Krein pour la distribution d'une fonctionnelle linéaire du processus gamma normalisé.

On montre enfin que la loi P_Γ du processus gamma peut être considerée comme une limite des lois de processus α-stables dont le paramètre α tend vers 0.

1. Introduction: definition of the gamma process

In this section we present a definition of the gamma process on an arbitrary space. This general definition turns out to be more convenient for our purposes than the process on the interval.

Let (X, ν) be a standard Borel space with a non-atomic finite non-negative measure ν, and let $\nu(X) = \theta$ be the total charge of ν. We denote by $D = \{\sum z_i \delta_{x_i},\ x_i \in X,\ z_i \in \mathbb{R},\ \sum |z_i| < \infty\}$ a real linear space of all finite real atomic measures on X.

DEFINITION 1. –The *gamma process* on the space X with parameter measure ν is a generalized process on the space D with the law $P_\Gamma = P_\Gamma(\nu)$ (called the *gamma measure* on the space (X, ν)) given by the characteristic functional (Laplace transform):

$$\mathbb{E}_{P_\Gamma}\left[\exp\left(-\int_X a(x)\, d\eta(x)\right)\right] = \exp\left(-\int_X \log(1 + a(x))\, d\nu(x)\right), \tag{1.1}$$

where a is an arbitrary non-negative bounded Borel function on the space X.

The correctness of this definition is guaranteed by the following explicit construction (*see* [7], chapter 8). Consider a Poisson point process on the space $X \times \mathbb{R}_+$ with mean measure $\nu \times \Lambda$, where Λ is the Lévy measure of the gamma process, that is $d\Lambda(z) = z^{-1} e^{-z}\, dz$, $z \in \mathbb{R}^+$. We associate with a realization $\Pi = \{(x_i, z_i)\}$ of this process an element $\eta = \sum z_i \delta_{x_i} \in D$. Then η is a random atomic measure obeying the law P_Γ. So the gamma measure is concentrated on the cone $D^+ \subset D$ consisting of all finite positive atomic measures on X.

Let $\mathcal{M} = \mathcal{M}(X, \nu)$ be the set of (classes mod 0 of) non-negative measurable functions on the space X with ν-summable logarithm, $\mathcal{M} = \{a : X \to \mathbb{R}_+ : \int_X |\log a(x)|\, d\nu(x) < \infty\}$. It follows from the above Poisson construction and formula (1.1) that each function $a \in \mathcal{M}$ correctly defines a measurable linear functional $\eta \mapsto f_a(\eta) = \int_X a(x)\, d\eta(x)$ on D, and formula (1.1) holds for all $a \in \mathcal{M}$.

Denote by $D_1^+ \subset D^+$ the simplex of all normalized atomic measures. Then $D^+ = D_1^+ \times [0, \infty)$, that is each $\eta \in D^+$ can be represented as

$$\eta = \big(\eta/\eta(X), \eta(X)\big). \tag{1.2}$$

The second coordinate in this decomposition is the total charge of the measure η, and the first one is called the *normalization* of the measure η.

The following lemma presents a well-known independence property of the gamma process.

LEMMA 1. – *In representation* (1.2) *the gamma measure is a product measure $P_\Gamma = \mathcal{G}_\theta \times \overline{P}_\Gamma$, that is the total charge $\gamma(X)$ of the gamma process and the normalized gamma process $\bar\gamma = \gamma/\gamma(X)$ are independent. The distribution $\mathcal{G}_\theta$ of the total charge is the gamma distribution on $\mathbb{R}_+$ with shape parameter θ and scale parameter* 1, *i.e.* $\mathrm{d}\mathcal{G}_\theta = \frac{1}{\Gamma(\theta)} t^{\theta-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$, $t > 0$.

Remarks. – 1. This independence property characterizes the gamma process in the class of Lévy processes (*see* [9]).

2. The random probability measure $\bar\gamma = \gamma/\gamma(X)$ is known in the literature as the *Dirichlet process* on the space X with parameter measure ν (*see* [2]).

3. Our definition of the gamma process on an arbitrary space is closely related to a particular case of the completely random measure considered in [7], chapter 8.

4. It is clear that the ordinary definition of the gamma subordinator on $\mathbb{R}_+$ is obtained for $X = \mathbb{R}_+$ and ν equal to the Lebesgue measure.

2. Multiplicative quasi-invariance of the gamma process

It follows immediately from the formula (1.1) that the measure $P_\Gamma(\nu)$ is invariant under all ν-preserving transformations of the space (X,ν). More exactly, let $T : X \to X$ be a ν-preserving transformation, then the operator U_T, which acts on D by substituting the coordinates, preserves the measure P_Γ. Now we present a large group of linear transformations of the space D (preserving the cone D^+) for which P_Γ is a quasi-invariant measure.

Consider the defined above class $\mathcal{M}$ of non-negative functions on X with ν-summable logarithm. Each function $a \in \mathcal{M}$ defines not only a linear functional f_a on D but also a multiplicator $M_a : D \to D$ by $(M_a\eta)(x) = a(x)\eta(x)$, that is $M_a\eta = \sum a(x_i) z_i \delta_{x_i}$ for $\eta = \sum z_i \delta_{x_i}$. Note that the set $\mathcal{M}$ is a commutative group with respect to pointwise multiplication of functions, and M_a is a group action of $\mathcal{M}$. Denote by $\tilde a$ the function $\tilde a(x) = 1/a(x) - 1$.

THEOREM 1. – 1) *For each $a \in \mathcal{M}$, the measure P_Γ is quasi-invariant under M_a, and the corresponding density is given by the following formula,*

$$\frac{\mathrm{d}(M_a P_\Gamma)}{\mathrm{d}P_\Gamma}(\eta) = \exp\left(-\int_X \log a(x)\,\mathrm{d}\nu(x)\right)\cdot \exp\left(-\int_X \tilde a(x)\,\mathrm{d}\eta(x)\right). \tag{2.1}$$

2) *The action of the group $\mathcal{M}$ on the space (D^+, P_Γ) is ergodic.*

Remarks. – 1. This property of the gamma process was first discovered in [4], [5] in quite different terms; it plays an important role in the representation theory of the current group $\mathrm{SL}(2,F)$, where F is the space of functions on a manifold.

2. As shown in [12], there exist processes that are invariant under all transformations M_a with *step* functions a, but are not equivalent to any scaled gamma process.

The analysis of the formula for the Laplace transform of the gamma process leads to a simple proof of the following result that was first obtained in [1] in the context of Dirichlet processes by hard analytic arguments. Let μ_a denote the distribution of the functional $\eta \mapsto f_a(\eta) = \int_X a(x) d\eta(x)$ on D with respect to the law $\overline{P}_\Gamma$ of the normalized gamma process, and let ν_a be the distribution of the function a with respect to the normalized measure $\bar\nu$. Then the measures μ_a and ν_a are related by the following *Markov–Krein* identity:

$$\int_{\mathbb{R}} \frac{1}{(1+zu)^\theta}\,\mathrm{d}\mu_a(u) = \exp\left(-\int_X \log(1+zu)^\theta\,\mathrm{d}\nu_a(u)\right).$$

3. Decomposition of the laws of general Lévy processes

Let (X,ν) be a standard Borel space, $\nu(X)=\theta$ and $\bar\nu=\nu/\theta$. Consider a measure Λ on $\mathbb{R}_+$ satisfying the following properties:

$$\Lambda(0,\infty)=\infty,\quad \Lambda(1,\infty)<\infty,\quad \int_0^1 s\,\mathrm{d}\Lambda(s)<\infty,\quad \Lambda(\{0\})=0. \tag{3.1}$$

Let F_Λ be the infinitely divisible distribution with Lévy measure Λ, i.e. the Laplace transform ψ_Λ of F_Λ is given by $\psi_\Lambda(t)=\exp\big(\int_0^\infty(1-\mathrm{e}^{-ts})\,\mathrm{d}\Lambda(s)\big)$.

A homogeneous Lévy process on (X,ν) with Lévy measure Λ satisfying (3.1) is a process on D^+ whose law $P_\Lambda=P_\Lambda(\nu)$ has the Laplace transform: $\mathbb{E}\big[\exp\big(-\int_X a(x)\,\mathrm{d}\eta(x)\big)\big]=\exp\big(\int_X\ln\psi_\Lambda(a(x))\,\mathrm{d}\nu(x)\big)$. One may obtain this process explicitly via the Poisson process construction similar to that for the gamma process (*see* Section 1).

Consider the cone $C=\{y=(y_1,y_2,\ldots):\ y_1\geq y_2\geq\cdots\geq 0,\ \sum y_i<\infty\}\subset\ell^1$. We define a class of measures on C as follows. Let $n\in\mathbb{N}$ and $p_1,\ldots,p_n>0$, $p_1+\cdots+p_n=1$. Consider a sequence ξ_i of i.i.d. variables such that $P(\xi_i=k)=p_k$ for $k=1,\ldots,n$. For $(Q_1,Q_2,\ldots)\in C$, denote by Σ_k the sum $\Sigma_k=\sum_{i:\xi_i=k}Q_i$. Let $\varkappa$ be a measure on the cone C such that the distribution F of the sum $\sum Q_i$ with respect to $\varkappa$ is infinitely divisible. We say that a measure $\varkappa$ is *product-type*, if for each $n\in\mathbb{N}$ and each sequence $p_1,\ldots,p_n>0$, $p_1+\cdots+p_n=1$, the sums $\Sigma_1,\ldots,\Sigma_n$ are independent and Σ_k obeys the law F^{*p_k}. Note that a product-type measure on the cone is defined by just one distribution on the half-line. In case of distributions concentrated on sequences with a bounded number of elements, a product-type measure is just an ordinary product measure.

We define a map $T:D^+\to C\times X^\infty$ as $T\eta=\big((Q_1,Q_2,\ldots),\ (X_1,X_2,\ldots)\big)$ if $\eta=\sum Q_i\delta_{X_i}$. Let P be a distribution on the space D^+, and let η be a random process obeying the law P. Then the random sequence of charges $Q_1,Q_2,\ldots$ is called the *conic part* of the process, and its distribution on the cone C is called the *conic part* of the law P.

The first part of the following theorem is a fundamental property of homogeneous Lévy processes that was first proved in [3]. We will present a simpler proof in the detailed version of this paper. The second part of the theorem seems to be new.

THEOREM 2. – 1) *Let $\eta=\sum Q_i\delta_{X_i}$ be a homogeneous Lévy process on the space (X,ν) with Lévy measure Λ. Then $TP_\Lambda=\varkappa_\Lambda\times\bar\nu^\infty$, that is $X_1,X_2,\ldots$ is a sequence of i.i.d. random variables with common distribution $\bar\nu$, and this sequence is independent of the conic part $\{Q_i\}_{i\in\mathbb{N}}$.*

2) *The measure $\varkappa$ on the cone C is a conic part of some Lévy process P_Λ if and only if it is product-type with $F=F_\Lambda$.*

One may also consider a map $T':D^+\to\mathbb{R}_+\times\Sigma\times X^\infty$, where $\Sigma=\{y=(y_1,y_2,\ldots):\ y_1\geq y_2\geq\cdots\geq 0,\ y_1+y_2+\cdots=1\}$ is the infinite-dimensional simplex, and $T'\eta=\big(\eta(X),\ (Q_1/\eta(X),Q_2/\eta(X),\ldots),(X_1,X_2,\ldots)\big)$ if $\eta=\sum Q_i\delta_{X_i}$.

The normalized sequence of charges $Q_1/\eta(X)$, $Q_2/\eta(X),\ldots$ is called the *simplicial part* of the process and its distribution σ_Λ is called the *simplicial part* of the law P_Λ.

THEOREM 3. – *The simplicial part of the gamma measure $P_\Gamma(\nu)$ with $\nu(X)=\theta$ is the Poisson–Dirichlet distribution* PD(θ) *with parameter θ.*

This fact was mentioned in [6] but it seems that the advantages of simultaneous studying both measures P_Γ and PD(θ) were not used systematically before.

It follows from Lemma 1 that the conic part of the gamma measure is a product measure $\mathcal{G}_\theta\times\mathrm{PD}(\theta)$, where $\mathcal{G}_\theta$ is the gamma distribution on $\mathbb{R}_+$ with parameter θ, that is $T'P_\Gamma=\mathcal{G}_\theta\times\mathrm{PD}(\theta)\times\bar\nu^\infty$. This fact, characterizing the gamma process in the class of Lévy processes, is a key point for many important properties of P_Γ.

The simplicial part of the stable Lévy process with parameter α is the measure $\mathrm{PD}(\alpha,0)$ on the simplex Σ as defined in [10]. This is the most natural definition of these measures. Some modification of this construction gives the whole two-parameter family $\mathrm{PD}(\alpha,\theta)$ from [10].

4. Equivalent σ-finite extensions of the measures P_Γ and $\mathrm{PD}(\theta)$

In this section we define, following [4], [5], a σ-finite measure on D^+ which is equivalent to P_Γ and invariant under M_a for all functions $a \in \mathcal{M}$ such that $\int_X \log a(x)\,\mathrm{d}\nu(x) = 0$. Namely, consider a σ-finite measure $\widetilde{P}_\Gamma$ on D^+ defined by $\frac{\mathrm{d}\widetilde{P}_\Gamma}{\mathrm{d}P_\Gamma}(\eta) = \exp(\eta(X))$.

Obviously, $T'\widetilde{P}_\Gamma = m_\theta \times \mathrm{PD}(\theta) \times \bar{\nu}^\infty$ and $T\widetilde{P}_\Gamma = \widetilde{\mathrm{PD}}(\theta) \times \bar{\nu}$, where m_θ has density $t^{\theta-1}/\Gamma(\theta)$, $t>0$ (in particular, m_1 is just the Lebesgue measure on the half-line), and $\widetilde{\mathrm{PD}}(\theta) = m_\theta \times \mathrm{PD}(\theta)$.

Theorem 1 implies:

THEOREM 4. – *For each $a \in \mathcal{M}$, the measure $\widetilde{P}_\Gamma$ is quasi-invariant under M_a with a constant density* $\frac{\mathrm{d}M_a(\widetilde{P}_\Gamma)}{\mathrm{d}\widetilde{P}_\Gamma} = \exp\left(-\int_X \log a(x)\,\mathrm{d}\nu(x)\right)$.

COROLLARY 1. – *If $\int_X \log a(x)\,\mathrm{d}\nu(x) = 0$, then $\widetilde{P}_\Gamma$ is invariant with respect to M_a.*

5. Quasi-invariance of the Poisson–Dirichlet distributions

Let $a \in \mathcal{M}$. According to the general theory of polymorphisms (*see* [11]), the transformation M_a induces a Markovian operator R_a on the cone C. Namely, let $y = (y_1, y_2, \ldots) \in C$. Consider the conditional distribution $\overline{P}^y_\Gamma$ of the gamma process on (X,ν), given the conic part equal $(y_1, y_2, \ldots)$. Then the random image of the point y under R_a is the conic part of the process $M_a\eta$, where η obeys the law $\overline{P}^y_\Gamma$. It follows from Theorem 2 that $R_a y = V(a(X_1)y_1, a(X_2)y_2, \ldots)$, where $(X_1, X_2, \ldots)$ is a sequence of i.i.d. random variables on X with common distribution $\bar{\nu}$, and V denotes a map that arranges the coordinates in non-increasing order.

In a similar way, the transformation M_a induces a Markovian operator S_a on the simplex Σ, $S_a y = V\left(\frac{a(X_1)y_1}{\sigma}, \frac{a(X_2)y_2}{\sigma}, \ldots\right)$, where the sequence $(X_1, X_2, \ldots)$ is as before, and $\sigma = a(X_1)y_1 + a(X_2)y_2 + \cdots$.

Note that the definitions of the operators S_a and R_a depend only on the distribution of the function a. Thus when studying the Poisson–Dirichlet distributions we may assume that $X = [0,1]$ and $\nu = \theta\lambda$, where λ is the Lebesgue measure on the interval.

Theorems 1 and 4 immediately imply:

THEOREM 5. – 1) *The Poisson–Dirichlet distribution* $\mathrm{PD}(\theta)$ *is quasi-invariant under S_a for all $a \in \mathcal{M}$, and* $\frac{\mathrm{d}S_a\mathrm{PD}(\theta)}{\mathrm{d}\,\mathrm{PD}(\theta)}(y) = \exp\left(-\theta\int_0^1 \log a(s)\,\mathrm{d}s\right) \cdot \int_0^\infty \frac{\sigma^{\theta-1}}{\Gamma(\theta)}\left(\prod_{i=1}^\infty L_{1/a}(\sigma y_i)\right)\mathrm{d}\sigma$, *where $L_{1/a}(\cdot)$ is the Laplace transform of the distribution of the function $1/a(t)$ with respect to the uniform distribution on the interval* $[0,1]$.

2) *The Poisson–Dirichlet distribution* $\mathrm{PD}(\theta)$ *is ergodic with respect to* $\{S_a\}_{a\in\mathcal{M}}$.

3) *The σ-finite measure $\widetilde{\mathrm{PD}}(\theta)$ on the cone C is invariant under R_a for all $a \in \mathcal{M}$.*

6. The gamma measure as a weak limit of laws of α-stable processes when $\alpha \to 0$

For $\alpha > 0$, let P^α denote the law of the standard α-stable process on (X,ν), i.e. a homogeneous process with Lévy measure $\frac{\alpha}{\Gamma(1-\alpha)}\,s^{-\alpha-1}\,\mathrm{d}s$, $s>0$. The Laplace transform of this process equals

$$\mathbb{E}^\alpha\left[\exp\left(-\int_X a(x)\,\mathrm{d}\eta(x)\right)\right] = \exp\left(-\int_X a(x)^\alpha\,\mathrm{d}\nu(x)\right). \tag{6.1}$$

It is natural to consider the law P_Γ of the gamma process (as well as its σ-finite extension $\widetilde{P}_\Gamma$) as a limit of the laws of α-stable processes when $\alpha \to 0$. Note that this limit does not exist in a usual sense. However, there are several ways to give sense to this statement. The key point is the following. If we consider the conditional measures $\varkappa_s^\alpha$ of the conic part of P^α on the simplex of sequences with sum s, then $\varkappa_s^\alpha$ converge to the corresponding conditional measure of the gamma process. Thus we only have to handle the factor measure, and this can be done, for example, by introducing an appropriate density. Namely, consider the measure P_k^α on D with

$$\frac{\mathrm{d}P_k^\alpha}{\mathrm{d}P^\alpha}(\eta) = \frac{\exp\left(-\frac{k}{\beta}\,\eta(X)\right)}{\mathbb{E}^\alpha\left[\exp\left(-\frac{k}{\beta}\,\eta(X)\right)\right]}, \quad \text{where} \quad \beta = \alpha^{\frac{1}{\alpha}}.$$

Consider a function $k(\alpha)$ such that $\lim\limits_{\alpha\to 0} \frac{k(\alpha)^\alpha}{\Gamma(1-\alpha)} = 1$. Then the measures $P_{k(\alpha)}^\alpha$ weakly converge to P_Γ when $\alpha \to 0$ (*see* [4], [8], [12]) (another density allows to obtain the σ-finite measure $\widetilde{P}_\Gamma$). This important result is a key point of the construction of two-parameter Poisson–Dirichlet distributions $\mathrm{PD}(\alpha,\theta)$ (*see* [10]). In particular, one obtains as a corollary that for a fixed $\theta \neq 0$, the distributions $\mathrm{PD}(\alpha,\theta)$ converge to $\mathrm{PD}(0,\theta) = \mathrm{PD}(\theta)$ when $\alpha \to 0$.

A full version of this Note will be published elsewhere.

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