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MATHEMATICS

Induced Representations of the Infinite Symmetric Group and Their Spectral Theory[1]

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We formulate the main facts on the representations of the infinite symmetric group induced from the identity representations of Young subgroups. In particular, we describe when these representations are irreducible or factor representations, and give examples of computing their spectral measures.

INTRODUCTION

In the classical representation theory of symmetric groups, the representations induced from Young subgroups (i.e., subgroups that leave some partition fixed) play an important role (see [5]). The decompositions of these representations into irreducible components contain, in a canonical way, all irreducible representations of the group, and this gives the traditional method for establishing a connection between Young diagrams and irreducible representations (Young–Frobenius correspondence). At present, there is an alternative approach to establishing this correspondence, which is based on the diagonalization of the group algebra and its representations with respect to the Gelfand–Tsetlin subalgebra [5, 3]. Nevertheless, the analysis of induced representations for finite and infinite symmetric groups is an important problem. Here, we present a brief list of properties of the representations of the infinite symmetric group induced from infinite Young subgroups. As was repeatedly observed, many properties of the infinite symmetric group are simpler and more natural compared with analogous properties of finite symmetric groups, but, of course, there are also new effects that are absent in the finite case.

We should mention another important fact. The representation theory of infinite discrete noncommutative groups (more exactly, nonvirtually commutative groups), i.e., groups that are not of type I, is considered unreasonable ("wild"), so that neither classification nor uniform models of representations can be obtained. The infinite symmetric group $\mathfrak{S}_{\mathbb{N}}$ is just one of the main nontrivial examples of nontype I groups. However, as regards this group, this point of view is completely unjustified; in fact, the structure of this group allows one to reduce its "wildness" to a standard model in the theory of dynamical systems, whereupon the study of its representations becomes quite a reasonable problem.

We mean the important fact that the group algebra of the group $\mathfrak{S}_{\mathbb{N}}$ (as well as any inductive limit of semisimple algebras with simple branching) has a natural structure of a skew product: the canonical commutative subalgebra is the so-called Gelfand–Tsetlin algebra, i.e., the algebra of functions on infinite Young tableaux, and an analog of its normalizer subgroup is the group of elements of the algebra that preserve the partition of the space of infinite Young tableaux into equivalence classes with respect to the "tail" equivalence relation (two tableaux are equivalent if they differ only by a finite beginning). Thus, every unitary representation of the group $\mathfrak{S}_{\mathbb{N}}$ is determined by a Borel measure on the space of infinite Young tableaux (it is natural to call it the spectral measure of the representation) and by a cocycle with values in the group of unitary operators of an auxiliary Hilbert space. This measure is quasi-invariant with respect to equivalence-preserving transformations, and its ergodicity, together with the indecomposability of the cocycle, is equivalent to the irreducibility of the representation. The analysis of the spectral measure of the representation, regarded as a measure on the space of infinite Young tableaux, is the most difficult and interesting part of the theory, and it is natural to call it the Fourier theory for the infinite symmetric group (see [3, 6]). The article [4] and subsequent papers contain a relatively detailed study of only the so-called central measures, i.e., the spectral measures of representations with traces, or, in our terms, the Fourier transforms of characters, e.g., the Plancherel measure. This program for other representations of $\mathfrak{S}_{\mathbb{N}}$ has not

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yet been carried out even for such natural representations as induced ones.

PARTITIONS, YOUNG SUBGROUPS, INDUCTION

The group $\mathfrak{S}_{\mathbb{N}}$ is the countable group of finite permutations of the set of positive integers; we regard it as the inductive limit of the increasing family of finite symmetric groups $\mathfrak{S}_n$, $n = 1, 2, \dots$, with natural embeddings. Consider partitions of the set of positive integers $\mathbb{N}$ and the corresponding Young subgroups. Let $\Pi = (A_1, A_2, \dots)$ be an arbitrary partition of $\mathbb{N}$. Denote the set of all partitions of $\mathbb{N}$ by $\mathcal{P}$. We will characterize a partition by its type, which is the vector of multiplicities of the cardinalities of its elements (blocks). Thus, the type of a partition is the sequence

$$\mathbf{r} = (r_0, r_1, r_2, \dots),$$

which consists of nonnegative integers or ∞, where r_0 is the number of infinite blocks, r_1 is the number of blocks consisting of a single element, etc., r_m is the number of blocks of cardinality m, $m = 1, 2, \dots$.

The Young subgroup $\mathfrak{S}_\Pi$ corresponding to a partition $\Pi = (A_1, A_2, \dots)$ consists of all elements of the group $\mathfrak{S}_{\mathbb{N}}$ that leave the blocks $A_1, A_2, \dots$ fixed. The representation of $\mathfrak{S}_{\mathbb{N}}$ by right shifts of the argument in the space $l^2(\mathfrak{S}_{\mathbb{N}}/\mathfrak{S}_\Pi)$ of functions on the space of left cosets of this subgroup is the representation induced from the identity representation of the subgroup $\mathfrak{S}_\Pi$, denoted by $\mathrm{ind}_{\mathfrak{S}_\Pi}^{\mathfrak{S}_{\mathbb{N}}}\mathbf{1}$ or, in short, ind_Π. Note that $\mathfrak{S}_{\mathbb{N}}$ acts in a natural way on the set of all partitions $\mathcal{P}$, and the orbits of this action correspond precisely to the classes of conjugate Young subgroups: $\mathfrak{S}_{g\Pi} = g\mathfrak{S}_\Pi g^{-1}$. Obviously, induction from conjugate Young subgroups yields equivalent representations. The converse assertion is also true.

Theorem 1. *The representations induced from two Young subgroups that are not conjugate in $\mathfrak{S}_{\mathbb{N}}$ are not equivalent.*

It is convenient to divide all partitions into two classes: the class $\mathcal{P}_l$ of large partitions that contain finitely many finite blocks $\left(\sum_{i>0} r_i < \infty\right)$ and, hence, at least one infinite block ($r_0 > 0$) and the class $\mathcal{P}_s$ of small partitions that contain infinitely many finite blocks and an arbitrary (possibly zero) number of infinite blocks $\left(\sum_{i>0} r_i = \infty\right)$; obviously, $\mathcal{P} = \mathcal{P}_l \cup \mathcal{P}_s$.

The Young subgroup associated with a partition will be called large or small depending on the type of the partition. If the number of finite blocks is finite, then they form a (possibly empty) Young diagram; denote it by $\lambda(\Pi)$.

Theorem 2. *The representations induced from large Young subgroups are representations of type* I (*see* [8]) (*i.e., they have a unique decomposition into the direct integral of irreducible representations*). *If the partition Π contains at most one finite block $\left(\sum_{i>0} r_i \le 1\right)$, then the representation is irreducible; otherwise it is a finite sum of irreducible representations indexed by Young diagrams with $|\lambda(\Pi)| = \sum_{i>0} r_i$ cells majorizing $\lambda(\Pi)$ in the dominance ordering; in general, these representations are not induced from any Young subgroups.*

The above decomposition into irreducible components can be described explicitly. In a sense, it reproduces the decomposition of the representation of a finite symmetric group from the Young subgroup corresponding to the diagram $\lambda(\Pi)$. The part of this theorem concerning irreducibility was proved in [1].

Let us turn to small Young subgroups; here, we have a completely different picture. Recall that induction from the identity subgroup (i.e., the Young subgroup corresponding to the partition into separate points: $r_1 = \infty$, $r_i = 0$ for $i \ne 1$) is a factor representation of type II_1. Hence, it is not surprising that a similar assertion holds for other small subgroups. In a small partition Π, consider finite blocks of finite multiplicities ($\{i > 0 \colon r_i < \infty\}$). They form a Young diagram $\nu(\Pi)$.

Theorem 3. *The representation induced from the Young subgroup $\mathfrak{S}_\Pi$, where Π is a small partition in which at least one multiplicity r_i, $i > 0$, is infinite and the diagram $\nu(\Pi)$ is finite, is a representation of type* II. *It is a factor if the diagram $\nu(\Pi)$ is either empty or consists of a single row; otherwise, it decomposes into a finite sum of factor representations indexed as in the previous case.*

Here, the most interesting phenomenon is the appearance of coupled factors of different types: II_1 and II_∞.

Theorem 4 (on the infinite hook) *Consider small partitions Π of one of the types $r_0 = 1$, $r_1 = n$, $r_i = 0$, $i > 1$, where n can be an arbitrary positive integer greater than one or infinity.*

Then the algebra generated by the operators of the representation ind_Π is a factor of type II, *and its commutant is a factor of type* II_1.

Recall that there are examples of factor representations of the infinite symmetric group with a coupling constant not equal to one (see [2]). In our case, this constant is equal to infinity; i.e., we have naturally coupled factors of types II_1 and II_∞. This means, in particular, that the group representation has a cyclic vector, whereas its commutant does not. Note that in this example, the commutant is also generated by the regular representation of a subgroup of the original group isomorphic to $\mathfrak{S}_{\mathbb{N}}$. The type of the representation ind_Π, where Π is a partition without infinite blocks in which all finite blocks are of finite multiplicities, i.e., $r_0 = 0$, $r_i < \infty$, $i > 0$, remains an open problem.

As is known, it is natural to classify factor representations up to quasiequivalence [8] rather than ordinary (spatial) equivalence. For factor representations, quasiequivalence is reduced to the algebraic equivalence of representations. The problem of quasiequivalence of induced representations is more complicated. In any case, one can assuredly assume that, for induced representations, quasiequivalence is much rougher than the conjugacy of the corresponding subgroups.

SPECTRAL THEORY

One of the most interesting problems is the spectral theory of induced representations. We mean a decomposition of representation operators in spaces of functions on infinite Young tableaux integrable with respect to some (spectral) measure quasiinvariant with respect to the tail partition; in other words, we mean the diagonalization of representation operators with respect to the Gelfand–Tsetlin algebra. In [6, 11], we call this circle of problems the Fourier theory for $\mathfrak{S}_{\mathbb{N}}$.

Let us define several classes of representations of the group $\mathfrak{S}_{\mathbb{N}}$. Note that the space T of infinite Young tableaux, i.e., infinite paths in the Young graph, is a totally disconnected (nonstationary) Markov compactum, so that we have a standard notion of a Markov measure on this space. A representation of $\mathfrak{S}_{\mathbb{N}}$ with a simple spectrum is called Markov if in the representation space there is a cyclic vector whose central measure with respect to the Gelfand–Tsetlin algebra is Markov. Now let f be a state on the group $\mathfrak{S}_{\mathbb{N}}$, i.e., a positive definite complex-valued function normalized by the condition $f(e) = 1$ (in what follows, we consider states that are the characteristic functions of Young subgroups). The restrictions of f to the subgroups $\mathfrak{S}_n$ determine a coherent system of states on $\mathfrak{S}_n$ and the monotone limit of cyclic representations. If these states determine irreducible representations (respectively, representations with a simple spectrum) of the groups $\mathfrak{S}_n$ starting from some n, then the cyclic representation determined by the state f is called elementary (respectively, simple). In these cases, the spectrum of the representation of $\mathfrak{S}_\infty$ determined by this state is simple.

For large partitions of type $r_0 = 1$, $\sum_{i>0} r_i < \infty$ (i.e., for partitions with one infinite block and finitely many finite blocks), the spectral measure is obviously discrete and concentrated on one class or finitely many classes of tail-equivalent Young tableaux; in particular, if this representation is irreducible, then it is elementary.

The first unobvious result on the spectral measures of induced representations (see [7]) is as follows.

Theorem 5. *Let Π be a partition of the set of positive integers $\mathbb{N}$ into two infinite blocks ($r_0 = 2$, $r_i = 0$, $i > 0$), and let $\mathfrak{S}_\Pi$ be the corresponding Young subgroup. The spectral measure of the induced representation* ind_Π *is a Markov measure, and the representation itself is simple and irreducible.*

The irreducibility was already mentioned above, and the simplicity is a consequence of the following fact, which is important in itself and connects two notions that are a priori far from each other:

Theorem 6. *A representation of the infinite symmetric group is Markov if and only if it is simple.*

In particular, for the induced representation associated with a two-block partition $\Pi = (A_1, A_2)$, the transition probabilities of the spectral measure are given by the following formula. Let $m(n) = |A_1 \cap \{1, 2, \ldots, n\}|$. If $n + 1 \notin A_1$, then

$$\mathrm{Prob}((n-k, k), (n+1-k, k)) = \frac{n - m(n) - k + 1}{n - 2k + 1},$$

$$\mathrm{Prob}((n-k, k), (n-k, k+1)) = \frac{m(n) - k}{n - 2k + 1}.$$

If $n + 1 \in A_1$, then

$$\mathrm{Prob}((n-k, k), (n+1-k, k)) = \frac{m(n) - k + 1}{n - 2k + 1},$$

$$\mathrm{Prob}((n-k, k), (n-k, k+1)) = \frac{n - m(n) - k}{n - 2k + 1}.$$

Another nontrivial example of spectral analysis is the computation of the spectral measure of the induced representation ind_Π, where Π is a small partition of type $r_0 = 0$, $r_1 = \infty$, $\sum_{i \ge 2} r_i < \infty$. Let $\nu = \nu(\Pi) = (\nu_1, \nu_2, \ldots)$. Denote by P the Plancherel measure (= the spectral measure of the regular representation) on the space T of infinite Young tableaux $t = (t_1, t_2, \ldots)$ and by $K_{\nu,\mu}$ the Kostka numbers.

Theorem 7. *The spectral measure of the representation* ind_Π *for the above partition Π is a convex combination of conditional Plancherel measures*:

$$M = \sum_{\mu \trianglerighteq \nu} c_\mu \cdot \mathrm{P}(\cdot | t_n = \mu),$$

where $c_\mu = \dfrac{K_{\nu,\mu} \dim\mu \prod \nu_i!}{n!}$ and $\mathrm{P}(\cdot | t_n = \mu)$ is the conditional distribution of the Plancherel measure given that the path goes through the vertex μ at the nth level. In particular, the spectral measure is absolutely continuous with respect to the Plancherel measure and has a piecewise constant (cylinder) density.

A detailed exposition of these and other results on induced representations of the infinite symmetric group will be published in *Pure and Applied Mathematics Quarterly.*

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