Fock factorizations, and decompositions of the L^2 spaces over general Lévy processes

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Abstract

We explicitly construct and study an isometry between the spaces of square integrable functionals of an arbitrary Lévy process and a vector-valued Gaussian white noise. We obtain explicit formulas for this isometry at the level of multiplicative functionals and at the level of orthogonal decompositions. We consider in detail the central special case: the isometry between the L^2 spaces over a Poisson process and the corresponding white noise; in particular, we give an explicit combinatorial formula for the kernel of this isometry. The key role in our considerations is played by the notion of measure and Hilbert factorizations and related notions of multiplicative and additive functionals and logarithm. The obtained results allow us to introduce a canonical Fock structure (an analogue of the Wiener–Itô decomposition) in the L^2 space over an arbitrary Lévy process. An application to the representation theory of current groups is considered. An example of a non-Fock factorization is given.

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1 Introduction: setting of the problem and main results

1.1 Subject of the paper

This paper is a survey and an exposition of new results in the field, which has been for a long time relating the classical probability theory, functional and classical analysis, and combinatorics, as well as some areas of theoretical physics (the second quantization, Fock space). We mean the theory of random processes with independent values (or, in more traditional probabilistic setting, with independent increments) and decompositions of functional spaces over these processes. Such processes may be regarded as a *continual generalization of the notion of a sequence of independent random variables.* The theory of these processes is closely related to the theory of infinitely divisible distributions on the real line. It passed a long way from the original pioneering works by B. de Finetti and A. N. Kolmogorov, who suggested a formula for infinitely divisible distributions on the line with a finite variance, subsequent papers of the middle 30s by P. Lévy and A. Ya. Khintchin, who proved a general formula for these distributions, up to the notion of generalized random processes in the sense of Gelfand–Itô, which provided a solid base for understanding what is a process with independent values.

The central example is of course the Wiener process (or the Gaussian white noise if we consider generalized processes with independent values). The measure in the space $C([0,1])$ of realizations of this process was described by N. Wiener in the early 20s; in the sequel, this measure remained at the center of the whole stochastic analysis and the theory of stochastic differential equations. This theory was started already in the 50s in the works by K. Itô, which were continued by many other mathematicians, and by now it has a vast range of applications in the theory of random processes and other fields.

But there is another aspect, which we will consider below and which is directly related to another source of the theory of "continual products of independent random variables", which is less evident but perhaps the most important; we mean mathematical physics. It is worth recalling that the Wiener process is obviously a mathematical version of the Brownian motion and Einstein–Smolukhovsky process. However, it was not until the 50s that another remarkable fact became clear: the so-called Wiener–Itô–Cameron–Martin orthogonal decompositions in the Hilbert space of square integrable functionals of the Wiener processes, whose theory was constructed in [1, 2, 3], is nothing else but a reproduction of the second quantization scheme, which was first suggested by V. A. Fock in the early 30s and developed in dozens of mathematical and semimathematical papers of his colleagues; the so-called Fock space, which serves as a base for constructions of the quantum field theory, representation theory of many (especially infinite-dimensional) groups, many algebraic constructions, etc., has the "exponential" structure and orthogonal decompositions into multiparticle subspaces, exactly as the L^2 space over the white noise has the decomposition into "chaoses" of different orders; and the so-called Wick regularization is merely the process of orthogonalization of polynomial functionals of Hermite type (see, e.g., the monograph [4] or the survey [5]). Among numerous books related to the subject under consideration, we mention [6, 7, 8]. An important role in the popularization of the Fock space among the Russian mathematicians was played by F. A. Berezin's works (especially on the fermion Fock space), see, e.g., [9].

1.2 Structure of a factorization

A more careful analysis shows that both Hilbert spaces (the Fock space and the L^2 space over the Wiener measure) *have a structure of* the so-called *"factorization"*, or a continuous tensor product. It is this structure that corresponds to the intuitive notion of the "continual product of independent variables"; the existence of such structure means that the Hilbert space and the algebra of operators in this space admit infinitely divisible decompositions into tensor products. This structure of decompositions into tensor factors, or factorization, appears not only in the probability theory, but also in the representation theory of current groups and fields of C^*-algebras, models of the field theory, algebra, etc. It goes back to the pre-war works by von Neumann on tensor products [10, 11] and was investigated in the paper by Araki and Woods in the 60s [12]. Its metric (probability-theoretical) counterpart is more recent, it was suggested by Feldman [13]. Below we give the definition of a *measure factorization* and a short survey of few papers where it was considered. Roughly speaking, a continuous measure factorization of a measure space is a coherent family of decompositions of this space into the direct product of arbitrarily many measure spaces. The space of realizations of each process with independent values has such structure.

The main result of this work says that the Hilbert space of square integrable functionals over a random process with independent values in an arbitrary vector space (in short, Lévy process, though this term is not quite correct) has the structure of a Fock factorization. Thus, from this viewpoint, an arbitrary Lévy process has the same factorization structure as the Gaussian process of an appropriate dimension. This dimension is the only invariant of the factorization up to isomorphism, and it depends only on the number of points in the support of the Lévy measure, which implies that two Lévy processes with the same cardinality of the supports of the Lévy measures determine isomorphic factorizations. We present explicit formulas for the factorization-preserving isometry between the corresponding L^2 spaces. In particular, for any Lévy process, one can obtain an orthogonal decomposition, in the space of square integrable functionals, similar to the classical Wiener–Itô decomposition into "chaoses".

Though there are numerous (mostly technical) papers devoted to the construction of stochastic integrals and analogues of the Wiener–Itô decomposition for Lévy processes (see, e.g., [14, 15, 16], and also [17, 18, 19, 20, 21, 22, 23, 24, 25]), the crucial observation that the corresponding factorizations are isomorphic remained up to now in the background. This fact is closely related to the remarkable Araki–Woods theorem [12] (though does not follow from it), which claims that a factorization having sufficiently many multiplicative vectors is isomorphic to a Fock factorization, see [12, 26]. In probabilistic terms, this condition means that the above-mentioned existence of the canonical Fock structure (or Wiener–Itô decomposition) in the L^2 space over an arbitrary Lévy process follows from the totality of the set of multiplicative functionals of the process[1]. A multiplicative functional of a Gaussian process is the exponential of a linear functional, however, for general Lévy processes, the set of multiplicative functionals is much wider (see Sect. 3.3). The established Fock structure in the L^2 space over an arbitrary Lévy process and the isometry between these spaces and the L^2 spaces over Gaussian processes make unnecessary numerous special constructions of orthogonal decompositions (and stochastic integrals) in each particular case. Another important detail is that the base space over which the process is defined is irrelevant for these issues, it may be an arbitrary measure space rather than an interval or the line as usual. In particular, the isometry under consideration applies to random fields. The only advantage of the one-dimensional situation is that in this case one may argue in terms of processes with independent increments rather than independent values, and consider the Wiener process instead of the white noise, which is sometimes more convenient. However, we consider Lévy processes over an arbitrary base space.

1.3 Isomorphism of factorizations. Logarithmic operation in factorizations. Kernel

The simplest example of the isometry under consideration is the isometry between the L^2 spaces over the Poisson and Wiener processes. An analogy between the orthogonal structures in these spaces was observed in many papers; however, the existence of an isometry was not

[1] Recall that a subset of a Hilbert space is called total if its linear span is everywhere dense, and that a functional of a process defined on a set X is called multiplicative if its value at an arbitrary realization of the process is equal to the product of its values at the restrictions of this realization to subsets forming an arbitrary finite measurable partition of the whole set X (see Sect. 2.1 for a precise definition).

established even in this case. It is worth mentioning that the existence of this isometry was originally obtained in [27] from the equivalence of two realizations of the canonical representation of the groups of diffeomorphisms; later, this isometry was studied in [28] in terms of the so-called holomorphic model of the Fock space (i.e., the Fock space realized as the Hilbert space of holomorphic functionals rather than the L^2 space), which is more popular among physicists. Final explicit formulas for the isometry between the L^2 spaces over the Poisson and Gaussian processes are apparently new.

The general case of the isometry between an arbitrary Lévy process and a Gaussian process of an appropriate dimension can be reduced to the above-mentioned Poisson–Gauss case by means of the Lévy–Khintchin decomposition or Poissonian construction of Lévy processes. But in this case we need to consider vector-valued white noises. Namely, it is natural to take the Hilbert space $L^2(\mathbb{R}, \Pi)$, where Π is the Lévy measure of the Lévy process under consideration, as the space of values of the white noise. In particular, in the case of a Poisson process this space is one-dimensional, hence the Poissonian factorization is isomorphic to the factorization generated by the ordinary (one-dimensional) Wiener process.

As mentioned above, the only Hilbert invariant (i.e., invariant up to arbitrary isometries of the Hilbert space) of the factorizations arising in the theory of Lévy processes is the dimension of the space $L^2(\Pi)$, i.e., the number of points in the support of the Lévy measure. For example, the stable processes and the gamma processes, which are of importance for applications, generate the same factorization as the Gaussian white noise with values in an infinite-dimensional Hilbert space. At the same time, the metric invariants of factorizations, i.e., invariants up to measure-preserving transformations, are much more detailed; namely, as shown by Feldman [13], two measure factorizations generated by Lévy processes with Lévy measures Π_1 and Π_2 are isomorphic if and only if $(\mathbb{R}, \Pi_1)$ and $(\mathbb{R}, \Pi_2)$ are isomorphic as measure spaces. The metric classification of processes plays an important role in the theory of decreasing families of σ-fields (filtrations) (see [29]).

The explicit construction of the Fock–Wiener–Itô structure in the space of square integrable functionals over an arbitrary Lévy process, i.e., the decomposition of this space into the orthogonal sum of the symmetric tensor powers of the first chaos, is based on a kind of *"taking logarithm" of multiplicative functionals*, resulting in the space of additive functionals, i.e., the first chaos. However, the rule for calculating this "logarithm" substantially depends on the factorization, and in general it does not coincide with taking the ordinary logarithm (the coincidence takes place only for Gaussian processes; in this case the set of additive functionals coincides with the set of linear functionals of the process). The existence of this "logarithm" for general Lévy processes is not obvious; the proof of the Araki–Woods theorem consists essentially in constructing this logarithm (in a slightly more general context); a more explicit version of this construction can be found in [26, Appendix A]. Below (Sect. 3.3) we calculate the logarithm for the Gaussian and Poisson cases (which, as we have mentioned above, exhaust the general case of Lévy processes).

In order to determine uniquely the isometry, it suffices to establish a correspondence between the sets of multiplicative functionals or between the linear subspaces of additive functionals ("first chaoses"); if we fix a bijection between the canonical bases in these subspaces, then the isometry is unique. *The isometry can be also defined by a kernel*, i.e., a generalized function in realizations of the two processes. This kernel for the Poisson–Gauss case is computed in Sect. 3.5; remarkably, it is defined in purely combinatorial terms. The correspondence of successive chaoses in this case reduces to a correspondence between the Hermite and Charlier functionals. The kernel is not positive, hence the isometry is neither Markovian, nor multiplicative operator.

Let us say some words on the combinatorial and analytical aspects of the problem. The isometry under consideration, for a Lévy process with Lévy measure Π, takes a very explicit form if the moment problem for the measure $t^2 d\Pi(t)$ is definite. In this case we can consider the basis of orthogonal polynomials in $L^2(\mathbb{R}, t^2 d\Pi(t))$ and obtain interesting relations between orthogonal polynomials with respect to various measures on the real line (see, e.g., (15)). The most important role is played by the Hermite and Charlier polynomials. Combinatorial aspects of constructing stochastic integrals for a wide class of processes were most explicitly considered in [30].

1.4 Relation to the representation theory

One of the most important applications of the isometries between Hilbert spaces of functionals over various Lévy processes is the representation theory of infinite-dimensional groups, namely, of current and gauge groups, and of Kac–Moody algebras. Unitary representations of these groups naturally generate Hilbert factorizations. Usually, these representations are realized in the Fock space. The model of the Fock space as the L^2 space over the white noise or the close holomorphic model of this space as the space of holomorphic functions in infinitely many variables are only two of possible models. If we fix an arbitrary commutative subgroup (subalgebra) of the infinite-dimensional group (algebra), and construct the representation of this algebra where this subgroup (subalgebra) is diagonalized, then we obtain immediately one of these models. For instance, the commutative model of the canonical representation of the group of $\mathrm{SL}(2,\mathbb{R})$-currents (see [31]) with respect to the subgroup of unipotent matrices yields the isometry of the L^2 spaces over the infinite-dimensional white noise and the gamma process (see [32]). In particular, this observation led to discovering new symmetry properties of the gamma process. This example is considered in detail in [33].

1.5 Further development

Processes with independent values and factorizations appear in much wider context than discussed above. First of all, one may consider an analogue of Wiener and other processes on manifolds, groups, semigroups, and even more general systems; the only thing we need is a distribution that is infinitely divisible with respect to a composition of measures. Moreover, the notion of a composition of measures may be very general and even not related to a group or semigroup law.

For example, consider the simplest nonlinear case — the Brownian motion on the sphere, or the rotation group $SU(n)$. It was proved already in the 60s (see [34]) that this process is linearizable, i.e., it can be represented by means of the ordinary Wiener process; thus the corresponding factorization is a Fock factorization. The method of stochastic differential equations, used to obtain this result, does not give an answer to the question investigated in this paper: what is the decomposition into "chaoses" in this case, i.e., what is the explicit isomorphism of the corresponding spaces of functions. Note that an answer to this question would provide a direct proof of the linearizability of the Brownian motion. The strongest result on linearizability was recently obtained by B. Tsirelson [35]. It claims that a weakly continuous Brownian motion on the unitary group of an infinite-dimensional Hilbert space is linearizable; it follows that the same holds for all groups that have a faithful unitary representation. The question of finding an explicit isometry and an explicit decomposition of the L^2 space into chaoses remains open. But for a wider class of groups, for example, for the group of isometries of the universal Urysohn space or the group of homeomorphisms of a compact space, even the linearizability of the Brownian motion is still not proved. And it is absolutely unclear if this is true for

a) non-Gaussian processes with independent group values, for example, Lévy processes on finite-dimensional or infinite-dimensional groups.

b) any processes with independent semigroup and more general values.

Of great interest are questions concerning the *metric* classification of the factorizations generated by Lévy processes with arbitrary (nonlinear) values; the simplest of these questions is whether the factorization generated by the Brownian motion on the two-dimensional sphere is *metrically* isomorphic to the one-dimensional Gaussian factorization.

1.6 Non-Fock factorizations

The new stage of the development of the theory of factorizations is related to deeper questions.

In [13], the following question, which goes back to S. Kakutani, was discussed: whether it is true that every measure factorization is isomorphic to a Fock factorization, or, in another terminology, is linearizable? We will discuss this question in more detail at the end of the paper; here we only mention that, in view of a theorem similar to the remarkable Araki–Woods theorem on Hilbert factorizations ([12], see also Sect. 2.2.1 below), this question is equivalent to the question if there are sufficiently many factorizable (multiplicative) functionals in this measure space. It turns out that the cases are possible when there are no multiplicative functionals except constants, and such examples of non-Fock factorizations ("black noise") with characteristic strong nonlinearity were constructed in [26] for a base of dimension 0 and 1. The constructed examples are in no sense generalized random processes with group or semigroup values. We give a short version of the example for a base of dimension 0 in Appendix A.

There is another important difference of these factorizations from the Fock ones: unlike Fock factorizations, which are defined on the complete Boolean algebra of classes of mod0 coinciding measurable sets, these factorizations are defined on a more narrow Boolean algebra; it was this fact that caused the restriction on the dimension of the base. As shown by Tsirelson [36, Sect. 6c], on the complete Boolean algebra, every factorization satisfying certain continuity conditions is a Fock factorization.

The problem of applying non-Fock factorizations in the representation theory of current groups, fields, and C^*-algebras, and in the quantum theory is still actual; the existence of such factorizations apparently opens new possibilities in the representation theory of infinite-dimensional and field objects.

1.7 Structure of the paper

The paper is organized as follows.

§2 contains the necessary background on factorizations and processes with independent values. We also give the definition of the logarithm determined by a factorization. The main results are contained in §3, where we consider in detail the fundamental special case — the canonical isometry between the spaces of square integrable functionals over the Poisson and Gaussian processes. For both processes, we compute all the above-mentioned characteristics (multiplicative and additive functionals, logarithm, orthogonal decomposition into stochastic integrals). As model cases, we consider the finite-dimensional analogues corresponding to a finite base space. We would like to draw the reader's attention to formula (15) for the classical Hermite and Charlier orthogonal polynomials, which is apparently new. The problem of determining the kernel of the canonical isomorphism (formula (27)) is new both in setting and in suggested solution. There are several proofs of this formula including a purely combinatorial one. In §4, we study the isometry for a general Lévy process; this requires no substantially new ideas, since a well-known construction allows one to represent such process as a Poisson process on a wider space. Though this representation is well-known, nevertheless it was not realized that in this general case the L^2 space over an arbitrary Lévy process with Lévy measure Π is isometric to the L^2 space over a $L^2(\mathbb{R},\Pi)$-*valued* Wiener process. Thus all formulas in the general case merely reproduce the corresponding formulas for the Poisson–Gauss case. Finally, in §5, we consider an example of applying the isometry between the Fock space and the space of square integrable functionals over the gamma process. In fact, it was this example that gave rise to the series of papers [31, 37, 27, 38, 32, 33] resulting in the present understanding of the whole situation.

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2 Basic definitions

2.1 Factorizations

We will consider two types of factorizations: *Hilbert factorizations* and *measure factorizations.*

Given a Hilbert space $\mathcal{H}$, denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and let $\mathcal{R}(\mathcal{H})$ be the lattice of all von Neumann algebras on $\mathcal{H}$ (recall that the lattice operations in $\mathcal{R}(\mathcal{H})$ are defined as follows: $R_1 \wedge R_2 = R_1 \cap R_2$ and $R_1 \vee R_2 = (R_1 \cup R_2)''$, where $R' = \{a \in \mathcal{B}(\mathcal{H}) : ar = ra \ \forall r \in R\}$ is the commutant of R; for an exposition of the theory of von Neumann algebras, see, e.g., [39])). The following definition of a Hilbert factorization goes back to von Neumann [10, 11].

Definition 1. *A (type I)* Hilbert factorization *of a Hilbert space $\mathcal{H}$ over a Boolean algebra $\mathcal{A}$ is a map $\xi : \mathcal{A} \to \mathcal{R}(\mathcal{H})$ such that each algebra of operators $\xi(A)$ is a type I factor, and for all $A, A_1, A_2, \ldots \in \mathcal{A}$, the following conditions hold:*

- $\xi(A_1 \wedge A_2) = \xi(A_1) \wedge \xi(A_2)$;
- $\xi(A_1 \vee A_2) = \xi(A_1) \vee \xi(A_2)$;
- $\xi(A') = \xi(A)'$;
- $\xi(0_{\mathcal{A}}) = \{\alpha \cdot \mathrm{Id}_{\mathcal{H}},\, \alpha \in \mathbb{C}\} = 1_{\mathcal{H}}$[2], *where* $\mathrm{Id}_{\mathcal{H}}$ *is the identity operator in* $\mathcal{H}$;

[2]The algebra $\{\alpha \cdot \mathrm{Id}_{\mathcal{H}},\, \alpha \in \mathbb{C}\}$ is traditionally denoted by $1_{\mathcal{H}}$, though it is the *zero* of the lattice of von Neumann algebras in $\mathcal{H}$.

- $\xi(1_{\mathcal{A}}) = \mathcal{B}(\mathcal{H})$

A factorized Hilbert space $(\mathcal{H}, \xi)$ *is a Hilbert space* $\mathcal{H}$ *equipped with a Hilbert factorization* ξ *of its operator algebra. The Boolean algebra* $\mathcal{A}$ *is called the* base *of the factorization.*

Remark. If $(\mathcal{H}, \xi)$ is a factorized Hilbert space, then, as shown in [12], for each $A \in \mathcal{A}$, there is a subspace $\mathcal{H}_A \subset \mathcal{H}$ such that for each finite partition $A_1, \dots, A_n$ of the unity element $1_{\mathcal{A}}$ of the Boolean algebra $\mathcal{A}$, we have $\mathcal{H} = \mathcal{H}_{A_1} \otimes \dots \otimes \mathcal{H}_{A_n}$ and $\xi(A_k) = 1_{\mathcal{H}_{A_1}} \otimes \dots \otimes 1_{\mathcal{H}_{A_{k-1}}} \otimes \mathcal{B}(\mathcal{H}_{A_k}) \otimes 1_{\mathcal{H}_{A_{k+1}}} \otimes \dots \otimes 1_{\mathcal{H}_{A_n}}$.

The notion of a measure factorization was introduced by Feldman [13]. We follow the presentation adopted by Tsirelson and Vershik [26].

Definition 2. *Let* $(\Omega, \mathfrak{A}, \mathbb{P})$ *be a probability space (which is always assumed to be a continuous Lebesgue space). Denote by* $\Sigma(\mathbb{P})$ *the complete lattice of all sub-*σ*-fields (containing all negligible sets) of the* σ*-field* $\mathfrak{A}$. *A* measure factorization *of* $(\Omega, \mathfrak{A}, \mathbb{P})$ *over a Boolean algebra* $\mathcal{A}$ *is a map* $\zeta : \mathcal{A} \to \Sigma(\mathbb{P})$ *such that for all* $A, A_1, A_2, \dots \in \mathcal{A}$*, the following conditions*[3] *hold:*

- $\zeta(A_1 \wedge A_2) = \zeta(A_1) \wedge \zeta(A_2)$*;*
- $\zeta(A_1 \vee A_2) = \zeta(A_1) \vee \zeta(A_2)$*;*
- $\zeta(A')$ *is an independent complement*[4] *of the* σ*-field* $\zeta(A)$*, i.e.,* $\zeta(A) \wedge \zeta(A') = 0$*,* $\zeta(A) \vee \zeta(A') = 1$*, and the* σ*-fields* $\zeta(A)$ *and* $\zeta(A')$ *are independent (which means that* $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1)\mathbb{P}(E_2)$ *for all* $E_1 \in \zeta(A)$ *and* $E_2 \in \zeta(A')$*);*
- $\zeta(0_{\mathcal{A}}) = \mathfrak{A}_0$ *(the trivial* σ*-field);*
- $\zeta(1_{\mathcal{A}}) = \mathfrak{A}$.

A factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ *is a probability space equipped with a measure factorization* ζ *over some Boolean algebra* $\mathcal{A}$. *The Boolean algebra* $\mathcal{A}$ *is called the* base *of the factorization.*

In this paper, we will consider only *continuous* (Hilbert and measure) factorizations in the sense of the following condition (which is called "minimal up continuity condition" in [26]). In what follows, the term "factorization" means "continuous factorization", unless otherwise stated.

Definition 3. *A Hilbert factorization* ξ *(respectively, a measure factorization* ζ*) over a Boolean algebra* $\mathcal{A}$ *is called* continuous *if* $\bigvee_{A \in S} \xi(A) = \mathcal{B}(\mathcal{H})$ *(respectively,* $\bigvee_{A \in S} \zeta(A) = \mathfrak{A}$*) for every maximal ideal* $S \subset \mathcal{A}$.

The most important examples are factorizations over the Boolean algebra $\mathfrak{B}$ of *all Borel sets* of a standard Borel space $(X, \mathfrak{B})$ (which will be called *factorizations over the Borel space* $(X, \mathfrak{B})$), and factorizations over the Boolean algebra of mod0 *classes of measurable subsets* of a Lebesgue space (X, ν) (which will be called *factorizations over the Lebesgue space* (X, ν)). In this paper, we consider only factorizations of these two types. Moreover, in all our examples, a factorization over a Borel space can be correctly extended to a factorization over the corresponding Lebesgue space (in fact, this is a consequence of the fact that all factorizations considered in this paper turn out to be Fock factorizations, see below; in the case of non-Fock factorizations (i.e., of factorizations that are not isomorphic to Fock ones), the base Boolean algebra is more narrow than the algebra of all Borel sets, see Appendix A).

Definition 4. *1) Two* factorized Hilbert spaces $(\mathcal{H}_1, \xi_1)$ *and* $(\mathcal{H}_2, \xi_2)$ *over Boolean algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, respectively, are called* isomorphic *if there exists an isomorphism of Boolean algebras* $S : \mathcal{A}_1 \to \mathcal{A}_2$ *and an isometry of the Hilbert spaces* $T : \mathcal{H}_1 \to \mathcal{H}_2$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{S} & \mathcal{A}_2 \\ \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} \\ \mathcal{R}(\mathcal{H}_1) & \xrightarrow{\bar{T}} & \mathcal{R}(\mathcal{H}_2). \end{array}$$

Here $\bar{T}$ *is the operator from* $\mathcal{R}(\mathcal{H}_1)$ *to* $\mathcal{R}(\mathcal{H}_2)$ *generated by the isometry* T *of the Hilbert spaces.*

2) In a similar way, two factorized measure spaces $(\Omega_1, \mathfrak{A}_1, \mathbb{P}_1, \zeta_1)$ *and* $(\Omega_2, \mathfrak{A}_2, \mathbb{P}_2, \zeta_2)$ *over Boolean algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$*, respectively, are called* isomorphic *if there exists an isomorphism*

[3] These conditions are not independent.

[4] Such complement is not unique.

of Boolean algebras $S : \mathcal{A}_1 \to \mathcal{A}_2$ *and an isomorphism of measure spaces* $T : (\Omega_1, \mathfrak{A}_1, \mathbb{P}_1) \to (\Omega_2, \mathfrak{A}_2, \mathbb{P}_2)$ *such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{S} & \mathcal{A}_2 \\ \Big\downarrow \zeta_1 & & \Big\downarrow \zeta_2 \\ \Sigma(\mathbb{P}_1) & \xrightarrow{T} & \Sigma(\mathbb{P}_2). \end{array}$$

Definition 5. *If T is an isomorphism of (Hilbert or measure) factorizations defined over the same Boolean algebra $\mathcal{A}$, and the corresponding automorphism S is the identity automorphism of the Boolean algebra $\mathcal{A}$ (i.e., the base is fixed), then T is called a* special *isomorphism of factorizations.*

The following lemma is obvious.

Lemma 1. *Each measure factorization $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ over a Boolean algebra $\mathcal{A}$ generates a Hilbert factorization in the space $\mathcal{H} = L^2(\Omega, \mathbb{P})$ with the same base. Namely, for each $A \in \mathcal{A}$, let $H_A = L^2(\Omega, \zeta(A), \mathbb{P}|_{\zeta(A)}) \subset L^2(\Omega, \mathfrak{A}, \mathbb{P})$. Then the map $\xi : \mathcal{A} \to \mathcal{R}(\mathcal{H})$ given by*

$$\xi(A) = \mathcal{B}(H_A) \otimes 1_{H_{A'}} \tag{1}$$

is a Hilbert factorization in $\mathcal{H}$.

In this paper, we deal only with Hilbert factorizations of this type. Note that nonisomorphic measure factorizations may generate isomorphic Hilbert factorizations, since not every isometry of the L^2 spaces is generated by some isomorphism of the underlying measure spaces.

Remark. In fact, a measure factorization is a triple $(\xi, \mathcal{Z}, \psi)$, where $\xi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a Hilbert factorization, $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$ is a maximal commutative subalgebra, and $\psi \in \mathcal{H}$ is a factorizable vector (see Definition 6 below) such that $\mathcal{Z} \cap \xi(A)$ is a maximal commutative subalgebra of $\xi(A)$ for each $A \in \mathcal{A}$, and ψ is $\mathcal{Z}$-cyclic.

The key role in the study of factorizations is played by the notion of multiplicative and additive functionals.

Definition 6. *Let $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ be a factorized measure space over a Boolean algebra $\mathcal{A}$. A measurable function $F : \Omega \to \mathbb{C}$ is called an* additive *(respectively,* multiplicative*)* functional *if for every finite partition $A_1, \ldots, A_n$ of the unity element $1_{\mathcal{A}}$ of the Boolean algebra $\mathcal{A}$, there exist functions $F_{A_1}, \ldots, F_{A_n} : \Omega \to \mathbb{C}$ such that F_{A_k} is $\zeta(A_k)$-measurable and $F = F_{A_1} + \ldots + F_{A_k}$ (respectively, $F = F_{A_1} \cdot \ldots \cdot F_{A_k}$).*

Definition 7. *Let $(\mathcal{H}, \xi)$ be a Hilbert factorization over a Boolean algebra $\mathcal{A}$. A vector $h \in \mathcal{H}$ is called* factorizable *if for very finite partition $A_1, \ldots, A_n$ of the unity element $1_{\mathcal{A}}$, there exist operators $P_k \in \xi(A_k)$ such that the one-dimensional projection P_h to the vector h can be represented in the form $P_h = P_1 \otimes \ldots \otimes P_n$. Alternatively, there exist vectors $h_{A_i} \in H_{A_i}$ (where H_{A_i} are the subspaces from the remark after Definition 1) such that $h = h_{A_1} \otimes \ldots \otimes h_{A_n}$.*

Analogously, a vector $h \in \mathcal{H}$ is called additive *if for any finite partition $A_1, \ldots, A_n$ of the unity element $1_{\mathcal{A}}$, there exist vectors $h_{A_i} \in H_{A_i}$ such that $h = h_{A_1} + \ldots + h_{A_n}$.*

If a factorized Hilbert space $(L^2(\Omega, \mathbb{P}), \xi)$ is generated by a factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ as in Lemma 1, then the set of factorizable vectors in $L^2(\Omega, \mathbb{P})$ coincides with the set of square integrable multiplicative functionals in $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$. Since in this paper we consider only Hilbert factorizations of this type, we will use the term "multiplicative functionals" for factorizable vectors in $L^2(\Omega, \mathbb{P})$. The set of additive vectors in a factorized Hilbert space $\mathcal{H}$ is a linear subspace (maybe zero), and in the case $\mathcal{H} = (L^2(\Omega, \mathbb{P}), \xi)$ it coincides with the set of square integrable additive functionals in $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$.

2.2 First examples: Fock factorizations, Gaussian and Poisson processes

2.2.1 Fock spaces and Fock factorizations

The (boson) *Fock space* $\mathrm{EXP}\, H$ over a Hilbert space H is the symmetrized tensor exponential

$$\mathrm{EXP}\, H = S^0 H \oplus S^1 H \oplus \ldots \oplus S^n H \oplus \ldots,$$

where $S^n H$ is the nth symmetric tensor power of H. Given $h \in H$, let

$$\mathrm{EXP}\, h = 1 \oplus h \oplus \frac{1}{\sqrt{2!}} h \otimes h \oplus \frac{1}{\sqrt{3!}} h \otimes h \otimes h \oplus \dots.$$

The vectors $\{\mathrm{EXP}\, h\}_{h \in H}$ are linearly independent, and

$$(\mathrm{EXP}\, h_1, \mathrm{EXP}\, h_2)_{\mathrm{EXP}\, H} = \exp(h_1, h_2)_H,$$

where $(\cdot,\cdot)_{\mathrm{EXP}\, H}$ stands for the scalar product in $\mathrm{EXP}\, H$. In particular,

$$\| \mathrm{EXP}\, h\|^2 = \exp \|h\|^2.$$

The principal example of a Hilbert factorization is given by the following construction.

Definition 8. *Given a Lebesgue space (X, ν), consider the direct integral of Hilbert spaces $\mathcal{K} = \int^{\oplus} K(x) d\nu(x)$ and the corresponding Fock space $\mathcal{H} = \mathrm{EXP}\, \mathcal{K}$. For each measurable $A \subset X$, let $H_A = \mathrm{EXP}\, \mathcal{K}(A)$, where $\mathcal{K}(A) = \int_A^{\oplus} K(x) d\nu(x)$, and set $\xi(A) = \mathcal{B}(H_A) \otimes 1_{H_{A'}}$. The obtained Hilbert factorization $(\mathcal{H}, \xi)$ is called a* Fock factorization.

In particular, if $\dim K(x) \equiv 1$, then $\mathcal{H} = \mathrm{EXP}\, L^2(X, \nu)$, and $H_A = \mathrm{EXP}\, L^2(A, \nu_A)$, where ν_A is the restriction of the measure ν to the subset A. More generally, if $\dim K(x) \equiv n$ ($n = 1, 2, \dots, \infty$), then $\mathcal{H}$ can be identified with $\mathrm{EXP}\, L^2((X, \nu); H)$, where H is a Hilbert space of dimension n, and $L^2((X, \nu); H)$ is the space of square integrable H-valued functions on (X, ν). The corresponding factorization is called a *homogeneous Fock factorization of dimension n*.

The set of multiplicative (factorizable) vectors in a Fock space $\mathcal{H} = \mathrm{EXP}\, \mathcal{K}$ is

$$\mathcal{M} = \{c \cdot \mathrm{EXP}\, h,\ h \in \mathcal{K}, c \in \mathbb{C}\}.$$

The linear subspace of additive vectors in the Fock space $\mathcal{H} = \mathrm{EXP}\, \mathcal{K}$ can be identified with the space $\mathcal{K} = \int^{\oplus} K(x) d\nu(x)$, where $\mathcal{K}$ is embedded in $\mathcal{H} = \mathrm{EXP}\, \mathcal{K}$ as the subspace of the first chaos: $\mathcal{K} \ni h \mapsto 0 \otimes h \otimes 0 \otimes \ldots \in \mathrm{EXP}\, \mathcal{K}$. Thus the space of the first chaos, as well as the set of multiplicative functionals, is defined in invariant terms; it has the structure of the direct integral of Hilbert spaces, the base of the integral coinciding obviously with the base of the factorization.

Fock factorizations are characterized by the following important theorem.

Theorem 1 (Araki–Woods [12])**.** *1) A Hilbert factorization $(\mathcal{H}, \xi)$ over a nonatomic Boolean algebra $\mathcal{A}$ is a Fock factorization if and only if the set of factorizable vectors is total in $\mathcal{H}$.*

2) The complete invariant of a Fock factorization is the set of values assumed by the dimension function $\dim K(x)$ at sets of positive measure: $\{n \in \{0, 1, \dots; \infty\} :\ \nu(\{x : \dim K(x) = n\}) > 0\}$. Thus two Fock factorizations are isomorphic if and only if their dimension functions are equivalent in the following sense: the set of values that they assume at sets of positive measure coincide.

2.2.2 Gaussian white noise and Gaussian factorizations

Consider the *Gaussian white noise* α on a Lebesgue space (X, ν), i.e., the generalized random process[5] on the Hilbert space $L^2(X, \nu)$ with the characteristic functional given by the formula

$$\mathbb{E} e^{i\langle h, \cdot\rangle} = e^{-\frac{1}{2}\|h\|^2}, \quad h \in L^2(X, \nu), \tag{2}$$

where $\|h\|$ is the norm of a vector h in the space $L^2(X, \nu)$. (This process can also be defined explicitly in the nuclear extension $\hat{H}$ of the Hilbert space $L^2(X, \nu)$ corresponding to the quadratic form $B(\cdot, \cdot) = (\cdot, \cdot)$. In particular, the space $L^2(\alpha)$ of square integrable functionals of the white noise can be identified with $L^2(\hat{H}, \mu)$, where μ is the standard Gaussian measure in $\hat{H}$.)

It is well-known (see, e.g., [4]) that the space $L^2(\alpha)$ of square integrable functionals of the white noise is canonically isomorphic to the Fock space $\mathrm{EXP}\, H$, where $H = L^2(X, \nu)$. This isomorphism is given by the formula

$$\mathrm{EXP}\, h \leftrightarrow e^{-\frac{\|h\|^2}{2} + \langle h, \cdot\rangle}. \tag{3}$$

[5] The notion of a generalized random process was developed by I. M. Gelfand [40] (see also [41]) and K. Itô [42]. Concerning the Gaussian white noise, see also [43] and [44].

In particular, the vacuum vector $\mathrm{EXP}\,0$ corresponds to the unity function $1 \in L^2(\alpha)$, and the n-particle subspace $S^n H$ is identified with the subspace of $L^2(\alpha)$ spanned by the n-multiple stochastic integrals, i.e., by the generalized Hermite functionals of order n, see Sect. 3.4. The structure of a unitary ring in $L^2(\alpha)$ was described axiomatically in [45], see also [46].

For each measurable set $A \subset X$, denote by $\zeta_\alpha(A)$ the σ-field generated by the restriction of the process α to A. It is easy to check that we obtain a measure factorization over (X, ν) called a *Gaussian* factorization. According to the general construction of Lemma 1, the white noise α determines also a Hilbert factorization ξ_α in the space $L^2(\alpha)$. The following well-known proposition is a consequence of the isomorphism (3) between $L^2(\alpha)$ and $\mathrm{EXP}\, L^2(X, \nu)$.

Proposition 1. *The Gaussian white noise on an arbitrary Lebesgue space (X, ν) generates a Fock factorization.*

In particular, the set of multiplicative functionals in the space $L^2(\alpha)$ is $\{c \cdot e^{\langle h, \cdot \rangle},\ h \in L^2(X, \nu),\ c \in \mathbb{C}\}$, and the set of additive functionals is $\{c \cdot \langle h, \cdot \rangle,\ h \in L^2(X, \nu),\ c \in \mathbb{C}\}$ (see Sect. 3.3).

Remark. One may consider the Gaussian process on (X, ν) with an arbitrary variance $\sigma^2 > 0$, i.e., the generalized random process on $L^2(X, \nu)$ with the characteristic functional

$$\mathbb{E} e^{i\langle h, \cdot \rangle} d\mu(\cdot) = e^{-\frac{\sigma^2}{2}\|h\|^2}, \quad h \in L^2(X, \nu).$$

Clearly, this case reduces to the standard white noise by means of the map $\langle h, \cdot \rangle \mapsto \sigma^{-1}\langle h, \cdot \rangle$, and the factorization determined by such process is also a Fock factorization.

2.2.3 Poisson process and Poissonian factorization

A standard reference on the theory of Poisson processes is the book [47].

By definition, a point *configuration* in the space X is a (non-ordered) empty, finite, or countable set of points of X with positive (integral) multiplicities. Denote by $\mathcal{E} = \mathcal{E}(X)$ the set of all point configurations on X.

The *point Poisson process* on the space X with mean measure ν (in short, the Poisson process on (X, ν)) is a random configuration $\pi \in \mathcal{E}$ such that for each measurable subset $A \subset X$, the random variable $\#\{\pi \cap A\}$ has the Poisson distribution with parameter $\nu(A)$, i.e., $\mathrm{Prob}\{\#\{\pi \cap A\} = n\} = \frac{\nu(A)^n}{n!} e^{-\nu(A)}$; and for any disjoint measurable subsets $A_1, \dots, A_n \subset X$, the random variables $\#\{\omega \cap A_k\}$, $k = 1, \dots, n$, are independent.

Note that a Poisson process can be regarded as a random measure $\tau(A) = \#\{\pi \cap A\}$ on the space X. Denote by $\mathcal{P}$ the distribution of the Poisson process in the space of configurations $\mathcal{E}$.

As in the Gaussian case, for each measurable subset $A \subset X$, denote by $\zeta_\pi(A)$ the σ-field generated by the restriction of the Poisson process π to A. We obtain a *Poissonian* measure factorization over (X, ν). According to the general construction of Lemma 1, the Poisson process π determines also a Hilbert factorization ξ_π in the space $L^2(\pi)$.

2.3 Logarithm

In this section, we describe the operation of "logarithm", introduced in [26], which allows one to construct the space of additive functionals of a factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$, given the set of multiplicative functionals. For example, if the set of multiplicative functionals is total in $L^2(\Omega, \mathbb{P})$, then, by the Araki–Woods theorem, the corresponding Hilbert factorization of $L^2(\Omega, \mathbb{P})$ is a Fock factorization, and the logarithmic operation allows one to construct the space of the first chaos, and hence to recover the whole Fock structure in $L^2(\Omega, \mathbb{P})$ by means of the standard orthogonalization process (see Sect. 2.6). The logarithmic operation depends substantially on the factorization. In general, it does not coincide with the ordinary logarithm of a multiplicative functional.

Let $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ be a factorized measure space over a Lebesgue space (X, ν). Denote by $\mathcal{A}$ the space of all square integrable additive functionals, and by $\mathcal{M}$ the space of all square integrable multiplicative functionals on this factorized space. Given $F \in \mathcal{M}$, denote by $F_A(\cdot)$ the $\zeta(A)$-measurable function from the definition of a multiplicative functional (it is defined uniquely up to scalar factor). For each measurable subset $A \subset X$, let $m_F(A) = \log \mathbb{E}|F_A(\cdot)|^2 - \log |\mathbb{E} F_A(\cdot)|^2$. Then m_F is a nonatomic measure on X ([26, Lemma A2]).

Theorem 2 ([26], Theorem A6)**.** *There is a natural one-to-one correspondence* $\mathrm{LOG}_\varsigma : \mathcal{M} \to \mathcal{A}$*. Given* $F \in \mathcal{M}$ *with* $\mathbb{E}F = 1$*,*

$$\mathrm{LOG}_\varsigma F(\cdot) = \lim_{\max m_F(A_k)\to 0} \sum_k (F_{A_k}(\cdot) - 1) \qquad \text{in} \quad L^2(\Omega, \mathbb{P}), \tag{4}$$

where $A_1, \dots, A_k$ *is a measurable partition of* X*, and each* F_{A_k} *is assumed to be normalized so that* $\mathbb{E}F_{A_k}(\cdot) = 1$*.*

In Sect. 3.3, we apply this theorem to compute the spaces of additive functionals for the Gaussian and Poisson processes.

Note that multiplicative and additive functionals can be defined in an obvious way for general processes. However, if the process is not a process with independent values, then it does not determine a factorization, so that formula (4) does not make sense, and the logarithm is not defined.

Thus, if we have two factorized measure spaces with isomorphic Hilbert factorizations of the corresponding L^2 spaces, then this isomorphism can be determined by indicating the correspondence either between all chaoses, either between additive functionals, or between multiplicative functionals. We will construct all these correspondences for the Poisson–Gauss and Lévy–Gauss isomorphisms.

2.4 General Lévy processes

The notion of a generalized random process was developed by I. M. Gelfand [40] (see also [41]) and K Itô [42]. Generalized processes with independent values are considered in detail in [41], see also [13]. A standard reference on Lévy processes in $\mathbb{R}^n$ is the monograph [48].

Let (X, ν) be a standard Borel space with a continuous finite measure ν. Denote by $\mathcal{F}$ the linear space of (mod 0 classes of) bounded measurable functions on X. A generalized random process[6] on the space $\mathcal{F}$ is called a *process with independent values (Lévy process)* if for any functions $a_1, a_2 \in \mathcal{F}$ such that $a_1(x)a_2(x) = 0$ a.e., the random variables $\langle a_1, \cdot\rangle$ and $\langle a_1, \cdot\rangle$ are independent. A process is called *homogeneous* if it is invariant under measure-preserving transformations of the space (X, ν).

It is well-known (see, e.g., [13]) that homogeneous processes with independent values are described by the Lévy–Khintchin theorem. For each homogeneous process with independent values on (X, ν), there exists a Borel measure Π on $\mathbb{R}$ such that $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}} \frac{t^2}{1+t^2} d\Pi(t) < \infty$ (the Lévy–Khintchin measure), a nonnegative number $\sigma^2 \in \mathbb{R}_+$ (the Gaussian variance), and a number $c \in \mathbb{R}$ (the drift) such that for every function $a \in \mathcal{F}$,

$$\mathbb{E}e^{i\langle a,\cdot\rangle} = \exp\left(\int_X \log\phi(a(x))d\nu(x)\right),$$

where

$$\log\phi(y) = icy - \frac{\sigma^2 y^2}{2} + \int_{\mathbb{R}} \left(e^{ity} - 1 - \frac{ity}{1+t^2}\right) d\Pi(t),$$

and the parameters Π, σ^2, and c are uniquely determined by the process. In particular, Π is the measure of jumps of the process.

The following lemma is well-known.

Lemma 2. *Let* η *be an arbitrary process with independent values on a standard Borel space* $(X, \mathfrak{B})$ *with a continuous finite measure* ν*. For each* $A \in \mathfrak{B}$*, let* $\zeta_\eta(A)$ *be the* σ*-field generated by the restriction of* η *to* A*. Then* ζ_η *is a measure factorization over the Borel space* $(X, \mathfrak{B})$*, which can be extended to a measure factorization over the algebra of* mod 0 *classes of measurable functions (i.e., over the corresponding Lebesgue space).*

According to the general construction of Lemma 1, a process η with independent values defines also a Hilbert factorization ξ_η in the space of square integrable functionals $\mathcal{H} = L^2(\eta)$ of the process; namely, $\xi_\eta(A) = \mathcal{B}(H_A) \otimes 1_{H_{X\setminus A}}$, where H_A is the set of square integrable functionals that depend only on the restriction of η to A.

A very important problem is to classify processes according to the factorizations they generate. It turns out that all processes with independent values generate Fock factorizations,

[6] Recall that a generalized random process on a real topological vector locally convex space $\mathcal{L}$ is a continuous linear map $a \mapsto \langle a, \cdot\rangle$ from $\mathcal{L}$ to the space $L_0(\Omega, \mathfrak{A}, \mathbb{P})$ of random variables on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. A generalized random process induces a weak distribution (a measure on cylinder sets) on the conjugate space L^*, which sometimes can be extended to a probability measure.

and the complete invariant of this Fock factorization is the number of points in the support $\operatorname{supp}\Pi$ of the Lévy–Khintchin measure Π (Theorem 7).

Note that the measure factorizations generated by processes with independent values with the same cardinality of the Lévy–Khintchin measure need not be isomorphic. Feldman [13] showed that *the measure factorizations generated by processes with independent values without Gaussian component are isomorphic if and only if the measure spaces* $(\mathbb{R}, \Pi_1)$ *and* $(\mathbb{R}, \Pi_2)$, *where* Π_1 *and* Π_2 *are the corresponding Lévy–Khintchin measures, are isomorphic.* The Gaussian factorization is not isomorphic to a non-Gaussian one.

The Gaussian and Poisson processes described above are processes with independent values ($\Pi = 0$, $c = 0$ and $\sigma = 0$, $\Pi = \delta_1$, $c = \frac{1}{2}$, respectively[7]). Moreover, each process η with independent values can be uniquely decomposed into the sum $\eta = \eta_0 + \eta_1 + \eta_2$, where η_0 is a deterministic component: $\langle \eta_0, a\rangle = c \cdot \int_X a(x)d\nu(x)$, and η_1 and η_2 are *independent* processes with independent values, η_1 being a Gaussian process, and η_2 being a purely jump (i.e., having no Gaussian component) process with zero drift; thus, in order to study the spaces of functionals of a Lévy process, it suffices to consider separately Gaussian processes and purely jump processes without drift. The Gaussian case was considered above. If η is a Lévy process without Gaussian component and drift, then it can be uniquely recovered from the measure of jumps, which is the Poisson process $\pi_{\nu\times\Pi}$ on the space $X \times \mathbb{R}$ with the product mean measure $\nu \times \Pi$. Thus the space $L^2(\eta)$ of square integrable functionals of a Lévy process without Gaussian component can be identified with the space $L^2(\pi_{\nu\times\Pi})$ of square integrable functionals of the Poisson process on the space $(X \times \mathbb{R}, \nu \times \Pi)$. Clearly, this identification preserves the structure of a factorization over (X, ν). Thus the *study of the factorizations generated by an arbitrary process with independent values reduces to the case of the Gaussian and Poisson processes.*

Remark. In the case when the Gaussian component and drift are zero, and the Lévy–Khintchin measure is concentrated on $\mathbb{R}_+$ and satisfies the condition

$$\int_0^\infty (1 - e^{-s})d\Pi(s) < \infty, \tag{5}$$

it is more convenient to define the Lévy process by the Laplace transform

$$\mathbb{E}e^{-\langle a,\eta\rangle} = \exp\left(\int_X \log \psi_\Pi(a(x))d\nu(x)\right), \qquad a \ge 0,$$

where ψ_Π is the Laplace transform of the infinitely divisible distribution F_Π with the Lévy–Khintchin measure Π:

$$\psi_\Pi(t) = \exp\left(-\int_0^\infty (1 - e^{-ts})d\Pi(s)\right).$$

If $X = \mathbb{R}$, then such processes are *subordinators*[8], thus a Lévy processes that satisfies the above conditions will be called a *generalized subordinator.*

In the case of subordinators, the explicit construction of the process by means of the corresponding Poisson process looks as follows (see, e.g., [47, Ch. 8]). Consider a Poisson point process on the space $X \times \mathbb{R}_+$ with the mean measure $\nu \times \Pi$. We associate with a realization $\pi = \{(x_i, z_i)\}$ of this process the measure

$$\eta = \sum_{(x_i,z_i)\in\pi} z_i\delta_{x_i}. \tag{6}$$

Then η is the generalized subordinator with the Lévy measure Λ. In particular, it follows that the distribution of a generalized subordinator is concentrated on the cone D^+, where

$$D = \left\{\sum z_i\delta_{x_i},\ x_i \in X,\ z_i \in \mathbb{R}, \sum |z_i| < \infty\right\}$$

is the real linear space of all finite real discrete measures on X, and $D^+ = \{\sum z_i\delta_{x_i} \in D : z_i > 0\} \subset D$ is the cone in D consisting of all positive measures.

Note that in the case of subordinators, it is easy to present explicitly the isomorphism of factorizations from Feldman's theorem (see above in this section). Namely, consider two

[7] It is sometimes convenient to think that in the Gaussian case the Lévy–Khintchin measure Π is concentrated at 0.

[8] Recall (see, e.g., [48, 47]) that a subordinator is a homogeneous process on $\mathbb{R}$ with independent positive increments.

generalized subordinators η_1 and η_2 with Lévy measures Π_1 and Π_2, respectively, and let $T:\mathbb{R}_+\to\mathbb{R}_+$ be the map that realizes the isomorphism of measure spaces $(\mathbb{R},\Pi_1)$ and $(\mathbb{R},\Pi_2)$. Then the map

$$T(\sum_{(x_i,z_i)\in\pi} z_i\delta_{x_i}) = \sum_{(x_i,z_i)\in\pi} T(z_i)\delta_{x_i}$$

realizes the isomorphism of the measure factorizations generated by the generalized subordinators η_1 and η_2.

2.5 On groups of automorphisms of factorizations

According to Definition 4, the group $\mathrm{AUT}(\mathcal{H},\xi)$ of automorphisms of a factorized Hilbert space $(\mathcal{H},\xi)$ over a Boolean algebra $\mathcal{A}$ consists of isometries T of the space $\mathcal{H}$ such that $\bar{T}\circ\xi=\xi\circ S$ for some automorphism S of the Boolean algebra $\mathcal{A}$ (recall that $\bar{T}$ stands for the operator in the lattice $\mathcal{R}(\mathcal{H})$ generated by T). The subgroup $\mathrm{SAUT}(\mathcal{H},\xi)$ of *special* automorphisms consists of automorphisms that leave the base unchanged: $\bar{T}\xi(A)=\xi(A)$ for all $A\in\mathcal{A}$ (i.e., S is the identity automorphism of $\mathcal{A}$). The subgroup $\mathrm{SAUT}_\psi(\mathcal{H},\xi)\subset\mathrm{SAUT}(\mathcal{H},\xi)$ consists of special automorphisms of $(\mathcal{H},\xi)$ that leave a multiplicative vector $\psi\in\mathcal{H}$ (the vacuum) unchanged. Thus there is the following natural hierarchy of groups of automorphisms of a factorized Hilbert space $(\mathcal{H},\xi)$:

$$\mathrm{SAUT}_\psi(\mathcal{H},\xi)\subset\mathrm{SAUT}(\mathcal{H},\xi)\subset\mathrm{AUT}(\mathcal{H},\xi).$$

Denote by $\mathrm{Aut}(X,\nu)$ the group of automorphisms of the measure space (X,ν).

For a Fock factorization, the groups $\mathrm{SAUT}(\mathcal{H},\xi)$ and $\mathrm{SAUT}_\psi(\mathcal{H},\xi)$ were computed in [49].

Proposition 2. *Let ξ be a homogeneous Fock factorization in the space $\mathcal{H}=\mathrm{EXP}\,L^2((X,\nu);H)$. The group $\mathrm{AUT}(\mathcal{H})$ consists of operators of the form*

$$\mathrm{EXP}\,h\mapsto e^{ib-\frac{\|\psi\|}{2}-(\psi,Uh(S^{-1}\cdot))}\cdot\mathrm{EXP}(Uh(S^{-1}\cdot)+\psi),\tag{7}$$

where $b\in\mathbb{R}$, $\psi\in L^2((X,\nu);H)$, $S\in\mathrm{Aut}(X,\nu)$, and U is a unitary operator in $L^2((X,\nu);H)$ that commutes with all projections P_A to subspaces of the form $L^2((A,\nu_A);H)$, where A is a measurable subset of X and ν_A is the restriction of the measure ν to A.

The subgroup $\mathrm{SAUT}(\mathcal{H})$ of special automorphisms consists of operators (7) with $S=\mathrm{Id}$ (the identity map in X), and the subgroup $\mathrm{SAUT}_1(\mathcal{H})$ of vacuum-preserving special automorphisms consists of operators (7) with $S=\mathrm{Id}$ and $\psi=0$.

In particular, in the case of a one-dimensional Fock factorization in the space $\mathcal{H}=\mathrm{EXP}\,L^2(X,\nu)$, it follows from the Spectral Theorem that U is a multiplicator $h(\cdot)\mapsto a(\cdot)h(\cdot)$ by a measurable function $a:X\to\mathbb{C}$ with $|a|\equiv1$. Thus $\mathrm{AUT}(\mathcal{H})$ is isomorphic to the semidirect product $(\mathrm{Aut}(X,\nu)\leftthreetimes\mathbb{T}^X)\leftthreetimes(L^2(X,\nu)\leftthreetimes\mathbb{T})$, where the semidirect product of the additive group of the Hilbert space $L^2(X,\nu)$ with the unit circle $\mathbb{T}=\{e^{ib},\,b\in\mathbb{R}\}$ is determined by the cocycle $c((\psi_1,b_1),(\psi_2,b_2))=b_1+b_2-\mathrm{Im}(\psi_2,\psi_1)$; the group $\mathrm{Aut}(X,\nu)$ of automorphisms of the measure space (X,ν) acts on $\mathbb{T}^X=\{a:X\to\mathbb{C}:\,|a(x)|\equiv1\}$ as $Sa(\cdot)=a(S^{-1}\cdot)$, and the pair (S,a) sends (ψ,b) to $(a(\cdot)\psi(S^{-1}\cdot),b)$.

If a Hilbert factorization is generated by a measure factorization as in Lemma 1, then we may consider also the group $\mathrm{AUT}(\zeta)$ of automorphisms of the underlying measure factorization, which is obviously a subgroup of $\mathrm{AUT}(\mathcal{H},\xi)$, since each automorphism of the measure space induces an isometry in the corresponding L^2 space. This group may be different for different measure factorizations generating the same Hilbert factorization.

Let we are given a measure factorization ζ of a Lebesgue space $(\mathcal{X},\mu)$ with a finite measure over the base Lebesgue space (X,ν), i.e., over the Boolean algebra of mod0 classes of measurable sets in (X,ν). Assume that each automorphism T of the base space (X,ν) induces an automorphism V_T of the factorized space $(\mathcal{X},\mu)$, in other words, the factorization is invariant with respect to the group $\mathrm{Aut}(X,\nu)$. These conditions are satisfied if the measure factorization generates a *Fock* Hilbert factorization in $L^2(\mathcal{X},\mu)$ and, as follows from Tsirelson's theorem (see Sect. 1.6 of the Introduction), only in this case. Thus we have a nontrivial monomorphism

$$\mathrm{Aut}(X,\nu)\to\mathrm{AUT}(\zeta),$$

i.e., a monomorphic embedding of the group of all automorphisms of a Lebesgue spaces with a finite or σ-finite measure into the group of automorphisms of a Lebesgue spaces with a finite measure; V. A. Rokhlin called this a "dynamical system over a dynamical system".

Thus we obtain the problem of finding the (operator and metric) classification of such systems over systems that arise from the factorizations generated by Lévy processes. It seems that a particular case of this problem was first considered in 1956 by K. Itô [50]. Namely, in our terms his result can be stated as follows. Let (X, ν) be the real line $\mathbb{R}$ with the Lebesgue measure, and let $(\mathcal{X}, \mu)$ be the space of realizations of a Lévy process (understood as a process with independent values) on $\mathbb{R}$. Consider the action of the one-parameter group of shifts on $\mathbb{R}$ on the space $(\mathcal{X}, \mu)$; then for each nondegenerate Lévy process, these actions are spectral isomorphic; more precisely, the corresponding one-parameter groups always have the Lebesgue spectrum of infinite multiplicity. The proof in [50] uses arguments similar to the Wiener–Itô decomposition for Lévy processes, which makes it close to our considerations. In the same paper, a problem of metric isomorphism of these actions for different Lévy processes was posed. Now, using the achievements of the ergodic theory, one can answer this question in the affirmative.

Theorem 3. *The action of the one-parameter group of shifts on the space of realizations of any nondegenerate Lévy process is a Bernoulli action of the group $\mathbb{R}^1$ with infinite entropy. Hence all these actions are metrically isomorphic.*

Note that this isomorphism does not preserve the type of the factorization.

Consider the action of the whole group $\mathrm{Aut}(X, \nu)$ on the space of realizations of a Lévy process. It is not difficult to show that the spectral or metric isomorphism of two such actions implies the corresponding (Hilbert or metric) isomorphism of the factorizations. It will follow from our main result that the actions are spectral isomorphic for Lévy processes of the same dimension (i.e., with the same cardinality of the support of the Lévy measure) and nonisomorphic for Lévy processes of different dimensions. The metric classification of actions reduces to the metric classification of factorizations, see Sect. 2.4 above.

Note that the metric theory of actions of groups of automorphisms of the Gaussian process is well developed starting from the works by A. N. Kolmogorov, S. V. Fomin, I. V. Girsanov, Ya. G. Sinai, G. Maruyama, A. M. Vershik, and others (see, e.g., [51, 52]), however, for other processes with independent values, much less is known. For example, for the gamma process, this theory must be of interest (see, e.g., [33]).

2.6 On orthogonal decompositions

By an orthogonal decomposition in the space of square integrable functionals of a process ϕ with independent values we mean the result of the standard orthogonalization process in $L^2(\phi)$ applied to the symmetric tensor powers of the subspace of additive functionals (the first chaos). The problem of constructing such a decomposition for an arbitrary process with independent values can be addressed within the following general scheme.

Let ϕ be an arbitrary process with independent values on the space X, which can be regarded as a random measure on X. We will construct an orthogonal decomposition in the space $L^2(\phi)$ of square integrable functionals of this process. Set $\mathcal{H}_0 = \mathbb{C}$. Let $\mathcal{H}_1$ be the subspace of centralized (i.e., orthogonal to constants) *additive* functionals in the factorization generated by η (if we are given the set $\mathcal{M}$ of square integrable multiplicative functionals in this factorization, then the space $\mathcal{H}_1$ can be obtained by the logarithm construction described in Theorem 2). Assuming that we have already constructed $\mathcal{H}_1, \dots, \mathcal{H}_{n-1}$, the next space $\mathcal{H}_n$ is defined as the orthogonal complement to $\mathcal{H}_1 \oplus \dots \oplus \mathcal{H}_{n-1}$ in the subspace spanned by the functionals of order n, that is, by products of n additive functionals:

$$\mathcal{H}_n = \overline{\{F_1(\phi) \cdot \ldots \cdot F_n(\phi),\ F_k \in \mathcal{H}_1\}} \ominus (\mathcal{H}_0 \oplus \dots \oplus \mathcal{H}_{n-1}).$$

The space $\mathcal{H}_n$ is called the *nth chaos* of the process ϕ. We have $L^2(\phi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$.

Note that for general Lévy processes, the space of *additive* functionals $\mathcal{H}_1$ is wider than the space of *linear* functionals L_1. If we apply the above orthogonalization process to L_1 instead of $\mathcal{H}_1$, then the nth subspace L_n will be the space of n-multiple stochastic integrals of ϕ, but the sum of this subspaces will not exhaust the whole space $L^2(\phi)$ (see §4). However, for the Gaussian and Poisson processes, the spaces of linear and additive functionals coincide, hence $\mathcal{H}_n = L_n$ for all n.

The spaces L_n can be described explicitly using the general combinatorial approach to stochastic integrals suggested by Rota and Wallstrom [30]. Denote by Δ_n the nth diagonal measure of ϕ, that is,

$$\Delta_n(A) = \phi^{\otimes n}\{(x, \dots, x) : x \in A\}$$

for each measurable A. Then it follows easily from the results of [30] (see also [15]) that L_n is spanned by the n-multiple stochastic integrals of the form

$$I^{(n)}_{f_1,\dots,f_n}(\phi) = \int f_1(x_1)\dots f_n(x_n)\, d\phi(x_1,\dots,x_n)$$
$$= \sum_{g\in\mathfrak{S}_n(x_1,\dots,x_n)} (-1)^{n-c(g)} \prod_{(x_{i_1}\dots x_{i_k})\in C(g)} \int f_{i_1}(x)\dots f_{i_k}(x) d\Delta_k(x), \quad (8)$$

where $\mathfrak{S}_n(x_1,\dots,x_n)$ is the symmetric group of degree n realized as the group of permutations of the set $\{x_1,\dots,x_n\}$, $C(g)$ is the set of cycles of a permutation g, and $c(g) = \#C(g)$ is the number of cycles in g.

In Sect. 3.4, we will consider orthogonal decompositions for the Poisson and Gaussian processes, and in §4, for general Lévy processes.

3 Canonical isomorphism between the factorizations generated by the Gaussian and Poisson processes

In this section, we study the isometry between the spaces of square integrable functionals of the Gaussian and Poisson processes over the same base space. Many authors observed that these spaces have many common features. However, the existence of a natural isometry between these spaces was first established by Vershik, Gelfand, and Graev [27] from considerations related to the representation theory of groups of diffeomorphisms; this isometry was explicitly described by Neretin [28] as an isometry between the space of square integrable functionals of the Poisson process and the so-called holomorphic model of the boson Fock space.

From now on we fix the base space (X,ν), which is a continuous Lebesgue space. According to the general construction of the factorization determined by a process with independent values, the spaces $L^2(\alpha)$ and $L^2(\pi)$ of square integrable functionals of the Gaussian white noise and the Poisson process on (X,ν), respectively, are equipped with Hilbert factorizations over (X,ν).

Theorem 4. *There exists a unique unity-preserving special real*[9] *isomorphism of the Hilbert factorizations in the spaces $L^2(\alpha)$ and $L^2(\pi)$. The corresponding isometry*

$$\Phi : L^2(\alpha) \to L^2(\pi)$$

of the Hilbert spaces is given by the following formula on the set of multiplicative functionals: for each $h \in L^2(X,\nu)\cap L^1(X,\nu)$,

$$\Phi : \quad e^{<h,\cdot> - \frac{\|h\|^2}{2}} \mapsto \prod_{x\in\omega}(1+h(x))\cdot e^{-\int h(x)d\nu(x)}, \qquad \omega\in\mathcal{E}. \quad (9)$$

Remarks. 1. Since the map $L^2(X,\nu) \ni h \mapsto \Phi h \in L^2(\pi)$, given by (9), is continuous in the topology of $L^2(X,\nu)$ (since the norm of the functional determined by the right-hand side equals, as can be easily seen, $e^{\|h\|^2}$), it extends by continuity to the whole space $L^2(X,\nu)$.

2. Formula (9) itself appeared, e.g., in [7], however, it was not apparently observed that this formula determines an isometry of Hilbert spaces and an isomorphism of factorizations.

Proof. Formula (9) follows from the formula for the boson–Poisson correspondence described in [28] in terms of the holomorphic model of the Fock space and formula (3) for the isomorphism between $L^2(\alpha)$ and the boson Fock space. On the other hand, it is not difficult to check directly that the map defined by (9) is an isomorphism with desired properties.

Obviously, any two isomorphisms with desired properties differ from each other by a vacuum-preserving special automorphism of the Fock factorization in $\mathrm{EXP}\, L^2(X,\nu)$, i.e., by an element of the group $\mathrm{SAUT}_1(\mathrm{EXP}\, L^2(X,\nu))$. Thus the uniqueness part follows from Proposition 2. □

[9] That is, sending the real subspace of real-valued functionals of one process to the similar subspace for the other process.

Remark. In particular, we obtain that the set of square integrable multiplicative functionals of the Poisson process is $\{c \cdot \prod_{x\in\omega}(1+h(x)),\ h \in L^2(X,\nu)\}$.

Our purpose is to study the isometry (9) in more detail. In particular, we would like to find its *kernel*, that is, a (generalized) function $K(\omega, f)$ on $\mathcal{E} \times \hat{H}$ such that for every $F \in L^2(\pi)$,

$$(\Phi^{-1}F)(\cdot) = \int_{\mathcal{E}} K(\omega,\cdot)F(\omega)d\mathcal{P}(\omega). \tag{10}$$

Let η be a process with independent values on (X,ν). A functional $F \in L^2(\eta)$ is called *singly generated* if it depends only on the integral $\langle \eta, 1\rangle$ of the process η over the whole space X. Similarly, F is called *finitely generated* (n-generated) if there is a finite measurable partition $X = A_1 \cup \ldots \cup A_n$ of the space X such that F depends only on the integrals $\langle \eta, \chi_{A_1}\rangle, \ldots, \langle \eta, \chi_{A_n}\rangle$ of the process η over the subsets $A_1, \ldots, A_n$. Note that the space $L^2(\eta)$ is the projective limit of the subspaces of finitely generated functionals with respect to refinement of partitions. Clearly, each isomorphism of the factorizations generated by two processes η_1 and η_2 must send the set of n-generated functionals of η_1 to the same set for η_2.

3.1 The restriction of the canonical isomorphism to the subspace of singly generated functionals

Consider the restriction of the isomorphism (9) to the subspaces of singly generated functionals (see the definition above). If η is the Gaussian white noise on the space (X,ν) with $\nu(X) = a$, then $\eta(X) = \langle \eta, 1\rangle$ is the Gaussian random variable with zero mean and variance a, so it is natural to identify the space of singly generated functionals of η with the space $L^2(\mathbb{R}, N(0,a))$ of square integrable functions with respect to the normal distribution (which can be also regarded as the L^2 space over the Gaussian process on the space $X = \{x\}$ that consists of a single point of weight a). Similarly, the space of singly generated functionals of the Poisson process π on (X,ν) is identified with the space $L^2(\mathbb{Z}_+, P_a)$ of sequences $b = \{b_n\}_{n\geq 0}$ with the scalar product

$$(b, b') = e^{-a}\sum_{n=0}^{\infty}\frac{a^n}{n!}b_n b'_n$$

(i.e., the L^2 space over the Poisson process on a single-point space). Consider this situation in more detail.

Formula (9) takes the following form: for all $t \in \mathbb{R}$,

$$e^{tx-\frac{at^2}{2}} \leftrightarrow \left\{e^{-at}(1+t)^k\right\}_{k=0}^{\infty}. \tag{11}$$

Note that the left-hand side of (11) is the generating function for the Hermite polynomials $H_n^a(x)$ and the right-hand side is the generating function for the Charlier polynomials $C_n^a(k)$ (see Appendix D). These polynomials constitute orthogonal families in $L^2(\mathbb{R}, N(0,a))$ and $L^2(\mathbb{Z}_+, P_a)$, respectively, and $\|H_n^a\|^2_{L^2(\mathbb{R},N(0,a))} = \|C_n^a\|^2_{L^2(\mathbb{Z}_+,P_a)} = a^n n!$, hence

$$\Phi(H_n^a) = C_n^a. \tag{12}$$

We see that formula (11) is precisely the expression of (12) in terms of generating functions.

Proposition 3. *The kernel* (10) *of the unitary isomorphism between* $L^2(\mathbb{R}, N(0,a))$ *and* $L^2(\mathbb{Z}_+, P_a)$ *is given by the formula*

$$K^a(k,x) = e^{-\frac{a}{2}-x}\frac{H_k^a(x+2a)}{a^k}. \tag{13}$$

Proof. Observe that if ν_1 and ν_2 are arbitrary measures on $\mathbb{R}$ with finite moments, and $P_n^{(j)}$ are the orthonormalized polynomials with respect to ν_j, $j = 1,2$, then the kernel of the unitary isomorphism between $L^2(\mathbb{R},\nu_1)$ and $L^2(\mathbb{R},\nu_2)$ is given by the formula

$$K(x,y) = \sum_{n=0}^{\infty} P_n^{(1)}(x)P_n^{(2)}(y),$$

provided that the series converges in $L^2(\mathbb{R},\nu_1)$ for a.e. y.

Thus let us consider the series

$$\sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k)}{a^n n!}.$$

We would like to prove that this series converges in $L^2(\mathbb{R}, N(0,a))$ for each $k \in \mathbb{Z}_+$. Since H_n^a are orthogonal polynomials in $L^2(\mathbb{R}, N(0,a))$, and $\|H_n^a\|^2 = n! a^n$, it suffices to check that

$$\sum_{n=0}^{\infty} \frac{|C_n^a(k)|^2}{a^n n!} < \infty. \tag{14}$$

But in view of (57), $\frac{|C_n^a(k)|^2}{a^n} = |C_k^a(n)|^2 a^{n-2k} \leq \text{const} \cdot n^{2k} a^{n-2k}$, since C_k^a is a polynomial of degree k, and (14) follows immediately.

Thus we have

$$K(k,x) := K^a(k,x) = \sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k)}{a^n n!}.$$

Consider the generating function

$$\mathcal{K}(t,x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} K(k,x).$$

Changing the order of summation yields

$$\mathcal{K}(t,x) = \sum_{n=0}^{\infty} \frac{H_n^a(x)}{n!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{C_n^a(k)}{a^n}.$$

In view of (57) and (55), the internal sum equals

$$\sum_{k=0}^{\infty} (-1)^{n+k} \frac{t^k}{a^k k!} C_k^a(n) = (-1)^n e^t \left(1 - \frac{t}{a}\right)^n.$$

Thus

$$\mathcal{K}(t,x) = \sum_{n=0}^{\infty} \frac{H_n^a(x) a^n}{n!} \left(\frac{t}{a} - 1\right)^n e^t = \exp\left(2t + \frac{tx}{a} - x - \frac{t^2}{2a} - \frac{a}{2}\right),$$

which coincides, in view of (51), with the generating function for $e^{-\frac{a}{2}-x} \frac{H_k^a(x+2a)}{a^k}$. □

Remark. One can also check formula (13) directly, using a known formula (see, e.g., [53, formula 7.377 of the fourth Russian edition])

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x+a) H_m(x+b) dx = m!(-b)^{n-m} L_m^{n-m}(-ab), \quad m \leq n,$$

where $L_n^\alpha(x)$ is the Laguerre polynomial with parameter α, and the formula $C_n^a(x) = n! L_n^{x-n}(a)$ relating the Charlier and Laguerre orthogonal polynomials.

Corollary 1. *We have proved the following identity relating the Hermite and Charlier orthogonal polynomials*

$$\sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k) a^n}{n!} = e^{-\frac{a}{2}-x} \frac{H_k^a(x+2a)}{a^k}. \tag{15}$$

In particular, denoting $C_n(\cdot) = C_n^1(\cdot)$, we obtain

$$\sum_{n=0}^{\infty} \frac{H_n(x) C_n(k)}{n!} = e^{-\frac{1}{2}-x} H_k(x+2). \tag{16}$$

Thus in the case of a single-point space X, we have the description of the Gauss–Poisson isomorphism at three levels:

- correspondence of multiplicative functionals: (11);
- correspondence of orthogonal polynomials: (12);
- explicit kernel: (13).

Our purpose is to obtain the formulas of the second and third level for general spaces.

3.2 The restriction of the canonical isomorphism to the subspace of finitely generated functionals

Let $X = A_1 \cup \ldots \cup A_m$ be a measurable partition of the space X with $\nu(A_j) = a_j$. Consider the corresponding subspaces of finitely generated functionals in $L^2(\alpha)$ and $L^2(\pi)$. Like in the case of singly generated functionals, these subspaces can be identified with the L^2 spaces over the corresponding processes on a finite space $X = \{s_1, \ldots, s_m\}$ that consists of m points with weights $\nu(s_j) = a_j$, $j = 1, \ldots, m$. Using the results of the previous section, we obtain that

$$\begin{aligned} L^2(\alpha) &= \bigotimes_{j=1}^m L^2(\mathbb{R}, N(0, a_j)), \\ L^2(\pi) &= \bigotimes_{j=1}^m L^2(\mathbb{Z}_+, P_{a_j}), \\ \Phi\left(\prod_{j=1}^m H_{k_j}^{a_j}(\cdot)\right) &= \prod_{j=1}^m C_{k_j}^{a_j}(\cdot), \end{aligned}$$

and for $k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^n$ and $x = (x_1, \ldots, x_m) \in \mathbb{R}^n$,

$$K(k, x) = \prod_{j=1}^m e^{-\frac{a_j}{2} - x_j} \frac{H_{k_j}^{a_j}(x_j + 2a_j)}{a_j^{k_j}}. \tag{17}$$

3.3 Logarithm

In this section, we apply the logarithmic construction described in Theorem 2 to computing the sets of additive functionals for the Gaussian and Poisson processes. Note that the canonical isomorphism Φ sends the logarithmic operation in the Poissonian factorization to the logarithmic operation in the Gaussian factorization, and hence sends additive functionals to additive functionals.

Poisson process

In this case $(\Omega, \mathbb{P}) = (\mathcal{E}(X), \mathcal{P})$, and normalized multiplicative functionals are given by the formula

$$F_A(\omega) = \prod_{x \in \omega \cap A} (1 + h(x)) e^{-\int_A h(x) d\nu(x)}, \qquad h \in L^2(X, \nu),$$

for each measurable subset $A \subset X$.

Lemma 3. $\mathrm{LOG}\, F_A(\omega) = \sum_{x \in \omega \cap A} h(x) - \int_A h(x) d\nu(x)$.

Proof. Denote the right-hand side by $G_A(\omega)$. Note that

$$\begin{aligned} \|\sum_k (F_{A_k}(\cdot) - 1) - G(\cdot)\|^2 &= \|\sum_k (F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot))\|^2 \\ &= \sum_k \|F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot)\|^2, \end{aligned}$$

since the restrictions of the Poisson process to disjoint subsets are independent. Using Campbell's theorem for sums and products over Poisson processes (see, e.g., [47], Sect. 3.2, 3.3), it is not difficult to compute that in our case

$$m_F(A) = \int_A h^2(x) d\nu(x),$$

and

$$\|F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot)\|^2 = e^{m_F(A_k)} - 1 - m_F(A_k).$$

Assume that $m_F(A_k) < \delta$ for all k. Then for sufficiently small δ we have

$$\|F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot)\|^2 < C \cdot m_F(A_k)^2.$$

Hence

$$\begin{aligned}\|\sum_k (F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot))\|^2 &\le C \cdot \sum_k m_F(A_k)^2 \\ &\le C \cdot \sum_k \delta \cdot \int_{A_k} h^2(x) d\nu(x) \le C \cdot \|h\|^2 \cdot \delta,\end{aligned}$$

which is arbitrarily small for sufficiently small δ, and we are done. □

Thus we obtain that the space of additive functionals ("the first chaos") of the Poisson process is

$$\left\{ \sum_{x \in \omega} h(x) - \int_X h(x) d\nu(x), \quad h \in L^2(X, \nu) \right\}. \tag{18}$$

Gaussian process

Let us now compute the logarithm for the Gaussian processes. In this case a normalized multiplicative functional is given by the formula

$$F_A(\eta) = e^{\langle h, \eta_A \rangle - \frac{1}{2} \int_A h^2(x) d\nu(x)}, \qquad h \in L^2(X, \nu),$$

where η_A is the restriction of η to $A \subset X$.

Lemma 4. $\operatorname{LOG} F(\eta) = \langle h, \eta \rangle$.

Proof. Let $G_A(\eta) = \langle h, \eta_A \rangle$. It is easy to compute that in this case

$$m_F(A) = \log \mathbb{E} \| e^{\langle h, \eta_A \rangle - \frac{1}{2} \int_A h^2(x) d\nu(x)} \|^2 = \int_A h^2(x) d\nu(x),$$

and

$$\begin{aligned}\|F_A(\cdot) - 1 - G_A(\cdot)\|^2 &= \mathbb{E} \| e^{\langle h, \eta_A \rangle - \frac{1}{2} \int_A h^2(x) d\nu(x)} - 1 - \langle h, \eta_A \rangle \|^2 \\ &= e^{m_F(A)} - 1 - m_F(A),\end{aligned}$$

exactly as in the Poissonian case, so the proof just reproduces the proof of Lemma 3. □

3.4 Correspondence of orthogonal decompositions (chaoses)

In this section, we will state the canonical isomorphism (9) in terms of orthogonal decompositions. Recall that by the orthogonal decomposition we mean the result of the standard orthogonalization process in the Hilbert space L^2 applied to the symmetric tensor powers of the subspace of additive functionals (the first chaos). The general scheme for constructing such decomposition is described in Sect. 2.6. Recall also that for the Gaussian and Poisson processes, the spaces of additive and linear functionals coincide, hence the space of the nth chaos $\mathcal{H}_n$ coincides with the space of n-multiple stochastic integrals L_n, and the construction of the orthogonal decomposition can be performed using the combinatorial scheme described in Sect. 2.6.

The orthogonal decomposition

$$L^2(\alpha) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{19}$$

in the space of square integrable functionals of the Gaussian white noise is the well-known Wiener–Itô–Cameron–Martin decomposition. Though the corresponding formulas are classical, it is instructive to observe how they can be obtained in the general combinatorial scheme. If ϕ is the Gaussian white noise on (X, ν), then $\Delta_2 = \nu$, $\Delta_3 = \Delta_4 = \ldots = 0$ ([30], Example G), whence $\mathcal{H}_n = L_n$ is spanned by the functionals of the form

$$\mathfrak{H}^{(n)}_{f_1, \ldots, f_n}(\cdot) = \sum_{g \in \operatorname{Inv}_n} \prod_{i \in C_1(g)} \langle f_i, \cdot \rangle \prod_{\{j,k\} \in C_2(g)} (-(f_j, f_k)), \tag{20}$$

where Inv_n is the set of all involutions in $\mathfrak{S}_n$, $C_k(g)$ is the number of cycles of length k of a permutation $g \in \mathfrak{S}_n$, and $(f_j, f_k) = \int_X f_j(x) f_k(x) d\nu(x)$ is the scalar product in $L^2(X, \nu)$.

Definition 9. *The functional $\mathfrak{H}^{(n)}_{f_1, \ldots, f_n}$ is called the nth generalized Hermite functional.*

In particular, for $f_1 = f_2 = \ldots = f_n = f$, we obtain (see Appendix D, (52))

$$\mathfrak{H}^{(n)}_{f,\ldots,f}(\cdot) = \sum_{g\in\mathrm{Inv}_n} \langle f,\cdot\rangle^{c_1(g)} \cdot (-\|f\|^2)^{c_2(g)} = H^\sigma_n(\langle f,\cdot\rangle), \tag{21}$$

that is, $I^g_n(f,\ldots,f)$ is the nth ordinary Hermite polynomial in $\langle f,\cdot\rangle$ with parameter $\sigma = \|f\|^2$.

Note that in terms of the Fock space $\mathrm{EXP}\, H$, the subspace $\mathcal{H}_n$ is precisely the n-particle subspace $S^n H$.

The corresponding orthogonal decomposition for the Poisson process was first discussed by Itô [2] and explicitly constructed by Ogura [14]. Within the combinatorial approach of [30], it is obtained as follows. In this case ϕ is the centralized Poisson process on (X,ν), that is, $\phi = \sum_{x\in\omega}\delta_x - \nu$, where ω is the Poisson process on (X,ν), and the diagonal measures equal $\Delta_2 = \Delta_3 = \ldots = \omega$ ([30], Example CP). Thus we have

$$L^2(\mathcal{E},\mathcal{P}) = \bigoplus_{n=0}^{\infty} V_n, \tag{22}$$

where $V_0 = \mathbb{C}$ and V_n is spanned by the n-multiple stochastic integrals given by the formula

$$\mathfrak{C}^{(n)}_{f_1,\ldots,f_n} = \sum_{g\in\mathfrak{S}_n} (-1)^{n-c(g)} \prod_{i\in C_1(g)} \left(\sum_{x\in\omega} f_i(x) - \int_X f_i(x)d\nu(x)\right) \cdot \prod_{(x_{i_1},\ldots,x_{i_k})\in C(g)} \sum_{x\in\omega} f_{i_1}(x)\ldots f_{i_k}(x). \tag{23}$$

Definition 10. *The functional* $\mathfrak{C}^{(n)}_{f_1,\ldots,f_n}$ *is called the nth generalized Charlier functional.*

If $f_1 = \ldots = f_n = \chi_A$, where χ_A is the characteristic function of a measurable set $A \subset X$, then

$$\mathfrak{C}^{(n)}_{\chi_A,\ldots,\chi_A} = \sum_{g\in\mathfrak{S}_n} (\#(\omega\cap A) - \nu(A))^{c_1(g)} \cdot \#(\omega\cap A)^{-c_2(g)+c_3(g)-\cdots} = C^\sigma_n(\#(\omega\cap A)), \tag{24}$$

that is, $\mathfrak{C}^{(n)}_{\chi_A,\ldots,\chi_A}$ is the nth ordinary Charlier polynomial with parameter $\sigma = \nu(A)$ (see Appendix D, (56)). (Note that in the Poissonian case, unlike the Gaussian one, the functional $\mathfrak{C}^{(n)}_{f,\ldots,f}$ with an arbitrary function f is not an ordinary Charlier polynomial. The reason is that all linear functionals of the Gaussian process have Gaussian distributions, while in the Poissonian case only integrals over subsets of X have Poisson distributions.) In particular, it follows from (23) that the first chaos of the Poisson process consists of functionals of the form $\mathfrak{C}^{(1)}_f(\pi) = \sum_{x\in\pi} f(x) - \int_X f(x)d\nu(x)$ (cf. (18)), and the second chaos is generated by the functionals of the form

$$\mathfrak{C}^{(2)}_{f,g}(\pi) = \mathfrak{C}^{(1)}_f(\pi)\mathfrak{C}^{(1)}_g(\pi) - \sum_{x\in\omega} f(x)g(x). \tag{25}$$

Corollary 2. *The canonical isomorphism* Φ *sends the generalized Hermite functional to the corresponding generalized Charlier functional:*

$$\Phi\mathfrak{H}^{(n)}_{f_1,\ldots,f_n} = \mathfrak{C}^{(n)}_{f_1,\ldots,f_n}. \tag{26}$$

3.5 Kernel

Let T be an isometry of the Hilbert spaces $L^2(A,\mu)$ and $L^2(B,\nu)$. In some cases this isometry can be represented in the integral form

$$(TF)(\cdot) = \int_A K(x,\cdot)F(x)d\mu(x), \quad F \in L^2(A,\mu),$$

where K is a (perhaps, generalized in some sense) function of two variables on the space $A\times B$ called the kernel of the isometry. In this section, we will find the kernel (10) of the Poisson–Gauss isometry Φ, assuming, for the sake of simplicity, that $X = [0,1]$ and ν is the Lebesgue measure on $[0,1]$, i.e., the kernel of the isometry between the spaces of square integrable functionals of the standard white noise on the interval $[0,1]$ and the homogeneous Poisson process on $[0,1]$ with unit rate. The case of an arbitrary continuous Lebesgue space

(X,ν) is completely analogous. In our case the kernel turns out to be "almost" ordinary function, namely, for any measurable sets $A \subset \mathcal{E}$ and $B \subset \hat{H}$ (recall that $\mathcal{E}$ is the set of configurations in the space X, i.e., the space of realizations of the Poisson process, and $\hat{H}$ is the nuclear extension of the space $L^2(X,\nu)$, which is the space of realizations of the Gaussian white noise), set $\rho(A,B) = \int_A \int_B K(\omega,\eta) d\mathcal{P}(\omega) d\mu(\eta)$. Then ρ is an additive set function on $\mathcal{E} \times \hat{H}$. Thus K can be regarded as the density of a signed measure (of infinite variation) on $\mathcal{E} \times \hat{H}$.

Let us introduce the following notation. Given a point configuration $\omega \in \mathcal{E}$ (since the parameter measure ν is continuous, all configurations of the Poisson process are simple, i.e., each point has multiplicity one; thus we may consider only simple (multiplicity-free) configurations), denote by $\Pi_{\le 2}(\omega)$ the set of partitions of the set ω into subsets consisting of at most two points.

For example, if $\omega = \{x,y,z\}$, then

$$\Pi_{\le 2}(\omega) = \{\{\{x\},\{y\},\{z\}\},\ \{\{x,y\},\{z\}\},\ \{\{x\},\{y,z\}\},\ \{\{x,z\},\{y\}\}\}\,.$$

For each partition $R \in \Pi_{\le 2}$, let $C_k(R)$ be the set of k-point subsets in R, $k = 1,2$, and $|R| = \#C_2(R)$.

Theorem 5. *The kernel (10) of the isomorphism (9) between $L^2(\alpha)$ and $L^2(\pi)$ is given by the following formula:*

$$K(\omega,\eta) = e^{-\frac{1}{2}-\langle \eta,1\rangle} \sum_{R\in\Pi_{\le 2}(\omega)} (-1)^{|R|} \prod_{z\in C_1(R)} (\eta+2)(z) \prod_{\{x,y\}\in C_2(R)} \delta(x-y) \tag{27}$$

for almost all ω, η.

(Here $\eta+2$ is the generalized function $\langle \eta+2,h\rangle = \langle \eta,h\rangle + 2\int h(t)dt$, and the product is the direct product of generalized functions.)

Proof. It suffices to check that (10) holds for multiplicative functionals F, that is, to show that

$$\mathbb{E}\left(\prod_{x\in\omega}(1+h(x))K(\omega,\eta)\right) = \exp\left(-\frac{\|h\|^2}{2} + \langle \eta+1,h\rangle\right) \tag{28}$$

for all $h \in L^2(X,\nu)$. It is more convenient to rewrite the sum in (27) over *permutations* rather than partitions. Given an n-point configuration $\omega = \{x_1,\ldots,x_n\}$, let $\mathfrak{S}_n(\omega)$ be the symmetric group of degree n realized as the group of all permutations of the set $\{x_1,\ldots,x_n\}$. Let $C_i(g)$ be the set of all cycles of length i in a permutation $g \in \mathfrak{S}_n$ and set $c_i(g) = \#C_i(g)$. Finally, denote by $\mathrm{Inv}(\omega)$ the subset of $\mathfrak{S}_n(\omega)$ consisting of all involutions (i.e., permutations with cycles of length at most two). Recall that the number of points of the homogeneous Poisson process on $[0,1]$ obeys the Poisson distribution with parameter one, and the conditional distribution of these points, given that the number of points is equal to n, coincides with the distribution of n i.i.d. variables with the uniform distribution on $[0,1]$. Then the left-hand side of (28) equals

$$e^{-3/2-\langle\eta,1\rangle} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \ldots \int_0^1 (1+h(x_1))\ldots(1+h(x_n)) \cdot \sum_{g\in\mathrm{Inv}(x_1,\ldots,x_n)} \prod_{\{x_i,x_j\}\in C_2(g)} (-\delta(x_i-x_j)) \cdot \prod_{x_k\in C_1(g)} (\eta(x_k)+2)\, dx_1\ldots dx_n.$$

The contribution of each pair $\{x,y\} \in C_2(g)$ is equal to

$$-\int_0^1\int_0^1 (1+h(x))(1+h(y))\delta(x-y)dxdy = -\int_0^1 (1+h(x))^2 dx,$$

and the contribution of each element $x \in C_1(g)$ is equal to

$$\int_0^1 (1+h(x))(\eta(x)+2)dx = \langle \eta+2, 1+h\rangle.$$

Thus the sum under consideration equals

$$e^{-3/2-\langle\eta,1\rangle} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{g\in\mathrm{Inv}_n} t_1^{c_1(g)} t_2^{c_2(g)}, \tag{29}$$

where

$$t_1 = \langle \eta+2, 1+h\rangle,$$
$$t_2 = -\int_0^1 (1+h(x))^2 dx.$$

But the sum in (29) is just the augmented cycle index $\tilde{Z}(\mathfrak{S}_n)[t_1, t_2, 0, 0, \ldots]$ (see Appendix C, (47)). Thus applying (48) with $z = 1$ we obtain that (29) is equal to

$$\exp\left(-3/2 - \langle \eta, 1\rangle + \langle \eta+2, 1+h\rangle - \frac{1}{2}\int (1+h(x))^2 dx\right),$$

and (28) follows by trivial computations. □

Remark. There is another proof of Theorem 5 which allows one to derive formula (27) rather than to check it. The idea of this proof is as follows. Observe that

$$L^2(\alpha) = \varprojlim A_n,$$

where A_n is the subspace consisting of functionals $F(\eta)$ depending only on $\langle \eta, \chi_{[0,\frac{1}{n}]}\rangle, \ldots, \langle \eta, \chi_{[\frac{n-1}{n},1]}\rangle$. Obviously, A_n is isometric to

$$\bigotimes_{j=1}^{m} L^2(\mathbb{R}, N(0, 1/n)).$$

Then one should apply (17) and pass to the limit.

Example 1. Let A_n be the subset in $L^2(\pi)$ consisting of functions supported by n-point configurations, $n = 1, 2, \ldots$. Then it follows from (27) that the image of A_1 under the canonical isomorphism is the subspace of functions of the form

$$e^{-\frac{3}{2}-\langle \eta, 1\rangle}\langle \eta+2, f\rangle, \quad f \in L^2([0,1], \nu);$$

the image of A_2 consists of functions of the form

$$e^{-\frac{3}{2}-\langle \eta, 1\rangle}\frac{1}{2!}\left[\int_0^1\int_0^1 f(x,y)(\eta+2)(x)(\eta+2)(y)dxdy - \int_0^1 f(x,x)dx\right],$$

where $f \in L^2([0,1]\times[0,1], \nu\times\nu)$; and the image of L_3 is

$$\begin{aligned} e^{-\frac{3}{2}-\langle \eta, 1\rangle} \quad \cdot \quad & \frac{1}{3!}\left[\int_0^1\int_0^1\int_0^1 f(x,y,z)(\eta+2)(x)(\eta+2)(y)(\eta+2)(z)dxdydz\right. \\ - \quad & \int_0^1\int_0^1 f(x,x,z)(\eta+2)(z)dxdz - \int_0^1\int_0^1 f(x,y,y)(\eta+2)(x)dxdy \\ - \quad & \left.\int_0^1\int_0^1 f(z,y,z)(\eta+2)(y)dydz\right], \end{aligned}$$

$f \in L^2([0,1]\times[0,1]\times[0,1], \nu\times\nu\times\nu)$.

Example 2. Each function $h \in L^2(X, \nu)$ determines a "linear" functional $F_h(\omega) = \sum_{x\in\omega} h(x)$ of the Poisson process. Let us compute its image in $L^2(\alpha)$. We have

$$\begin{aligned} \mathbb{E}K(\omega,\eta)F_h(\omega) = e^{-3/2-\langle \eta,1\rangle}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1\ldots\int_0^1\sum_{l=1}^{n}h(x_l) \\ \sum_{g\in \mathrm{Inv}(x_1,\ldots,x_n)}\prod_{\{x_i,x_j\}\in C_2(g)}(-\delta(x_i-x_j))\cdot\prod_{x_k\in C_1(g)}(\eta+2)(x_k)dx_1\ldots dx_n. \end{aligned}$$

(Recall that $C_k(g)$ is the set of cycles of length k in a permutation g.) It is easy to check that each summand $h(x_l)$ contributes

$$\begin{cases} \langle h, \eta+2\rangle(\langle 1, \eta\rangle + 2)^{c_1(g)-1}(-1)^{c_2(g)}, & \text{if } x_l \in C_1(g) \\ \langle h\rangle(\langle 1, \eta\rangle + 2)^{c_1(g)}(-1)^{c_2(g)}, & \text{if } x_l \in C_2(g). \end{cases}$$

Thus the sum under consideration equals $e^{-3/2-\langle\eta,1\rangle}(S_1+S_2)$, where

$$S_1 = \langle h,\eta+2\rangle \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{g\in\mathrm{Inv}(x_1,\dots,x_n)} c_1(g)(\langle 1,\eta\rangle+2)^{c_1(g)-1}(-1)^{c_2(g)},$$

$$S_2 = 2\langle h\rangle \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{g\in\mathrm{Inv}(x_1,\dots,x_n)} c_2(g)(\langle 1,\eta\rangle+2)^{c_1(g)}(-1)^{c_2(g)}.$$

Note that the sum in S_1 is the derivative of the augmented cycle index $\tilde{Z}(\mathfrak{S}_n)$ in t_1 calculated at $t=(\langle 1,\eta\rangle+2,-1,0,0,\dots)$, hence $S_1=\langle h,\eta+2\rangle e^{\langle 1,\eta\rangle+3/2}$. Similarly, S_2 is the derivative of the same cycle index in t_2, thus $S_2=-\langle h\rangle e^{\langle 1,\eta\rangle+3/2}$, where $\langle h\rangle=\int_X h(x)d\nu(x)$, and simple computations show that the image of the functional F_h in $L^2(\alpha)$ equals

$$F_h(\eta)=\langle h,\eta\rangle+\int_X hd\nu(x),$$

in agreement with (18).

4 Isomorphism of the factorizations generated by general Lévy processes

The purpose of this section is to apply the results on the Poisson–Gauss isomorphism to general Lévy processes. Recall that, as was mentioned in Sect. 2.4, the study of the factorizations generated by general Lévy processes reduces to the study of the Poissonian and Gaussian factorizations. We emphasize that we reduce the general case to the Poisson–Gauss one using the universality of the isomorphism with respect to the base.

As mentioned in Sect. 2.4, the space $L^2(\eta_\Pi)$ of square integrable functionals of a Lévy process η_Π with Lévy–Khintchin measure Π can be identified with the space $L^2(\pi_{\nu\times\Pi})$ of square integrable functionals of the Poisson process $\pi_{\nu\times\Pi}$ on the direct product $X\times\mathbb{R}$ with the mean measure $\nu\times\Pi$. Thus it is natural to introduce the white noise $\alpha_{X\times\mathbb{R}}$ on the space $(X\times\mathbb{R},\nu\times\Pi)$. Note that this process may be also regarded as the $L^2(\mathbb{R},\Pi)$-valued white noise $\alpha^{L^2(\mathbb{R},\Pi)}$ on the space (X,ν), i.e., one may identify $L^2(\alpha_{X\times\mathbb{R}})$ with the homogeneous Fock space $\mathrm{EXP}\,L^2((X,\nu);L^2(\mathbb{R},\Pi))$. The spaces $L^2(\alpha^{L^2(\mathbb{R},\Pi)})$ and $L^2(\eta_\Pi)$ are equipped with natural Hilbert factorizations over (X,ν).

Theorem 6. *There exists a unity-preserving isometry (which is an isomorphism of Hilbert factorizations)*

$$\Phi: L^2(\alpha^{L^2(\mathbb{R},\Pi)})\to L^2(\eta_\Pi).$$

On the set of multiplicative functionals, it is given by the following formula: for each $h\in L^2(X\times\mathbb{R},\nu\times\Pi)\cap L^1(X\times\mathbb{R},\nu\times\Pi)$,

$$\Phi:\ e^{<h,\cdot>-\frac{\|h\|^2}{2}}\mapsto\prod_i(1+h(x_i,t_i))\cdot e^{-\int\int h(x,t)d\nu(x)d\Lambda(t)},\quad \eta=\sum_i t_i\delta_{x_i}\in D. \tag{30}$$

This is the unique real special vacuum-preserving automorphism of Hilbert factorizations that acts identically on the space of values $L^2(\mathbb{R},\Pi)$.

Remark. When we say that the isomorphism acts identically on the space of values, we mean that it is an isomorphism of factorizations over the space $(X\times\mathbb{R},\nu\times\Pi)$. In other words, consider the restriction of the isomorphism to the first chaos, which can be identified with $L^2((X,\nu);L^2(\mathbb{R},\Pi))$ for both processes. Then for an arbitrary set $A\subset L^2(\mathbb{R},\Pi)$, the isomorphism preserves the subset of $L^2((X,\nu);L^2(\mathbb{R},\Pi))$ that consists of functions whose values lie in A.

Proof. Follows immediately from Theorem 4 and the above observations. □

Without assuming that the isomorphism acts identically on the space of values, the above isomorphism is not unique. Indeed, apply Proposition 2 and observe that the set of operators in $L^2((X,\nu);\,L^2(\mathbb{R},\Pi))$ that commute with all projections P_A for $A\subset X$ is $L^\infty(X,\nu)\otimes\mathcal{B}(L^2(\mathbb{R},\Pi))$. Thus in this case the group $SAUT_1$ is generated by operators of the form $\mathrm{EXP}(h_1(\cdot)\otimes h_2(\cdot))\mapsto\mathrm{EXP}((a(\cdot)h_1(\cdot))\otimes(Uh_2(\cdot)))$, where $h_1\in L^2(X,\nu)$, $h_2\in L^2(\mathbb{R},\Pi)$, a

is a complex-valued measurable function on X with $|a| \equiv 1$, and U is an arbitrary unitary operator in $L^2(\mathbb{R}, \Pi)$.

Theorem 6 implies immediately

Theorem 7. *Let η be a Lévy process on the space (X, ν) with Lévy measure Π. Then the Hilbert factorization determined by η is a homogeneous Fock factorization, and its dimension is equal to the number of points in* $\operatorname{supp} \Lambda$.

Theorem 6 states the Lévy–Gauss isomorphism at the level of multiplicative functionals. Let us now describe it in terms of orthogonal decompositions, applying the general scheme described in Sect. 2.6 and assuming for simplicity that the process is a generalized subordinator. By (22) we have

$$L^2(\eta_\Pi) = \bigoplus_{n=0}^{\infty} V_n,$$

where V_n is the space generated by the generalized Charlier functionals of order n (23) in the space $(X \times \mathbb{R}, \nu \times \Pi)$. In particular, additive functionals of the Lèvy process are of the form

$$\sum_i h(x_i, t_i), \qquad h \in L^2(X \times \mathbb{R}_+, \nu \times \Lambda).$$

As we have seen above, in the case of Gaussian and Poisson processes, all additive functionals are linear. However, in the case when $\operatorname{supp} \Lambda$ consists of more than one point, that is, the Lévy process is neither Gaussian nor Poisson, *the space of additive functionals does not coincide with the space of linear functionals*, which are given by

$$\langle a, \eta \rangle = \int_X a(x) d\eta(x) = \sum_i a(x_i) t_i, \qquad a \in L^2(X, \nu). \tag{31}$$

This is exactly the reason of the well-known fact (see, e.g., [54]) that the only Lévy processes with the so-called chaotic representation property (which means that the L^2 space can be decomposed into the direct sum of the subspaces spanned by ordinary multiple stochastic integrals) are the Gaussian and Poisson processes.

In the rest of this section we assume that the measure Π satisfies the condition (5), i.e., the Lévy process is a generalized subordinator, and moreover the measure $t^2\Pi(t)$ has finite moments of all orders, and the moment problem for this measure is definite. In this case we can obtain a more detailed description of the orthogonal decomposition.

Let $\{P_k(t)\}_{k=0}^{\infty}$ be the family of orthogonal polynomials on $\mathbb{R}_+$ with respect to the measure $t^2\Pi(t)$, and let $V_{n,k}$ be the subspace spanned by the generalized Charlier functionals $\mathfrak{C}^{(n)}_{f_1,\dots,f_n}$ of order n (see Definition 10) corresponding to functions of the form $f(x,t) = a(x) t P_{k-1}(t)$, i.e., to functions of the *kth power* in t.

It is not difficult to see that

$$V_n = \sum_{\lambda \vdash n} \bigoplus_{\substack{(i_1,\dots,i_k) \\ \text{distinct}}} \bigotimes_{j=1}^{k} V_{\lambda_j, i_j},$$

where $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ is a partition of n, and the tensor product is symmetric. Thus, rearranging the summands, we obtain the following orthogonal decomposition for the general Lévy process:

$$\begin{aligned} L^2(D, P_\Lambda) &= \bigoplus_n \bigoplus_{\lambda \vdash n} \otimes_s V_{n_k, k} \\ &= \mathbb{C} \oplus V_{1,1} \oplus (V_{1,2} \oplus V_{2,1}) \oplus (V_{3,1} \oplus (V_{1,1} \otimes_s V_{1,2}) \oplus V_{1,3}) \oplus \ldots, \end{aligned}$$

where $\lambda = 1^{n_1} 2^{n_2} \ldots$ is a partition of n with n_k parts equal to k, and $\otimes_s$ is the symmetric tensor product.

The same decomposition can be described in another way (cf. [20]). Given the Lévy process $\eta = \sum t_i \delta_{x_i}$, consider the processes $\eta_k = \sum_i t_i P_{k-1}(t_i) \delta_{x_i}$ (in particular, $\eta_1 = \eta$). Then $V_{n,k}$ is the space of n-multiple stochastic integrals of the process η_k.

The corresponding decomposition for the L^2 space over the vector-valued Gaussian process is obtained in a similar way. Namely,

$$\begin{aligned} L^2(\widehat{H^\Lambda}, \mu^\Lambda) &= \bigoplus_n \bigoplus_{\lambda \vdash n} \otimes_s \mathcal{H}_{n_k, k} \\ &= \mathbb{C} \oplus \mathcal{H}_{1,1} \oplus (\mathcal{H}_{1,2} \oplus \mathcal{H}_{2,1}) \oplus (\mathcal{H}_{3,1} \oplus (\mathcal{H}_{1,1} \otimes_s \mathcal{H}_{1,2}) \oplus \mathcal{H}_{1,3}) \oplus \ldots, \end{aligned}$$

where $\mathcal{H}_{n,k}$ is the subspace spanned by the generalized Hermite functionals of order n (see Definition 9) corresponding to functions of the form $f(x,t) = a(x)tP_{k-1}(t)$.

Corollary 3. *In terms of orthogonal decompositions, the canonical isomorphism (30) takes the form*

$$\Phi\mathfrak{H}^{(n)}_{f_1,\dots,f_n} = \mathfrak{C}^{(n)}_{f_1,\dots,f_n},$$

where $f_k(x,t) = a_k(x)tP_{k-1}(t)$.

Example. Gamma processes. The standard gamma process on the space (X,ν) is the generalized subordinator γ with the Lévy measure

$$\Lambda_\Gamma(t) = \frac{e^{-t}}{t}dt, \qquad t > 0. \tag{32}$$

Thus the Laplace transform of the gamma process is given by

$$\mathbb{E}e^{-\langle a,\gamma\rangle} = \exp\left(-\int_X \log\left(1 + a(x)\right) d\nu(x)\right), \tag{33}$$

where a is an arbitrary nonnegative measurable function on X such that $\int_X \log(a(x) + 1)d\nu(x) < \infty$.

To the space $L^2(\gamma)$, we can apply all considerations of this section; observe that the orthogonal polynomials with respect to $t^2\Lambda_\Gamma(t) = te^{-t}dt$ are the Laguerre polynomials with parameter one: $P_n(t) = L_n^{(1)}(t)$. Note that the Laguerre polynomials appear in this example not because of the well-known fact that they are orthogonal with respect to the gamma distribution, which is the infinitely divisible distribution corresponding to the gamma process, but since they are orthogonal with respect to $t^2\Lambda_\Gamma(t)$, i.e., to the Lévy measure of the gamma process with the density t^2.

5 Representations of the current group $\mathrm{SL}(2,\mathbb{R})^X$

As an application of the obtained results, we consider the isomorphism between the Fock space and the L^2 space over the gamma process (and the isomorphic L^2 space over the "infinite-dimensional Lebesgue measure") and apply this construction to representations of the current groups over $\mathrm{SL}(2,\mathbb{R})$.

5.1 The canonical state on $\mathrm{SL}(2,\mathbb{R})^X$

Let (X,ν) be a standard Borel space with a fixed finite measure ν. The *current group* $G^X = \mathrm{SL}(2,\mathbb{R})^X$ on (X,ν) is the group of Borel bounded $\mathrm{SL}(2,\mathbb{R})$-valued functions on X. In other words, G^X consists of 2×2-matrices whose elements are bounded measurable real functions on X.

The *canonical representation* of the current group G^X is a unitary irreducible representation with spherical function given by the formula

$$\Omega(g(\cdot)) = C\exp\left(-\int_X \log\left(2 + \mathrm{Tr}(g(x)g^*(x))\right)d\nu(x)\right), \qquad g(\cdot)\in G^X. \tag{34}$$

The restriction of this spherical function to the subgroup of constant functions (isomorphic to $G = \mathrm{SL}(2,\mathbb{R})$) equals $\Omega_0(g) = \frac{C}{2+\mathrm{Tr}\,gg^*}$, the so-called *canonical state* of $\mathrm{SL}(2,\mathbb{R})$, see [31].

Consider an infinitely divisible positive definite function on a group, in other words, a continuous one-parameter semigroup of positive definite functions. The only interesting case is when the generator of this semigroup is not positive definite, but only conditionally positive definite. Then this generator, as a function on the group, is not bounded, and it is the norm of a nontrivial cocycle of the group with values in the space of some irreducible representation of this group, see [31, 55, 56, 49, 6, 57], and others. The existence of such cocycle is possible only if the identity representation is not isolated in the space of all irreducible representations (i.e., if the group does not satisfy Kazhdan's property (T) [58]). Among classical groups, only $SO(n,1)$ and $SU(n,1)$, $n = 1,2,\dots$, do have this property, and the corresponding cocycle and state were found in [31, 37]. It is this state that is called canonical. It allows one to define a representation of the current group in the Fock space. Formula (34) above determines the positive definite function on the current group $\mathrm{SL}(2,\mathbb{R})^X$ generated by the canonical state; it is the spherical function generated by the vacuum vector of the corresponding representation realized in the Fock space. Below we give another realization of this representation (see Sect. 5.3).

Note that restrictions of the canonical state to different subgroups (or commutative subalgebras of the group algebra) determine different infinitely divisible measures on the dual subgroup (respectively, the dual space to the algebra), thus diagonalization of different subgroups or subalgebras generates different infinitely divisible measures, Lévy processes, and hence models of the Fock space.

5.2 The Fock model of the canonical representation

Let us describe the Fock model of the canonical representation of $\mathrm{SL}(2,\mathbb{R})^X$. Consider the so-called *special* representation of the group $G = \mathrm{SL}(2,\mathbb{R})$, which is realized in the Hilbert space $H = L^2(\mathbb{R}, \frac{dt}{|t|})$ and is given by the following formulas:

$$\begin{aligned}
\left(T\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\phi\right)(t) &= e^{ibt}\phi(t), \\
\left(T\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\phi\right)(t) &= \phi(a^2 t), \\
\left(T\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\phi\right)(t) &= \int_{-\infty}^{\infty} K_0(t,s)\phi(s)ds,
\end{aligned} \tag{35}$$

where

$$K_0(t,s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{|u|^2} e^{-i(tu+su^{-1})}du. \tag{36}$$

This is an irreducible unitary representation of discrete series. (Note that considered over $\mathbb{C}$ this representation is reducible: it decomposes into the sum of two irreducible subrepresentations.) However, this representation is distinguished as the only representation having a nontrivial cocycle.

Let $\phi_0(t) = e^{-|t|}$ and fix a cocycle $\beta : G \times H \to H$ given by

$$\beta(g,t) = T_g\phi_0(t) - \phi_0(t). \tag{37}$$

Consider the Hilbert space

$$H^X = L^2\left(X \times \mathbb{R},\, \nu \times \frac{dt}{|t|}\right)$$

and the corresponding Fock space $\mathrm{EXP}\, H^X$. The realization of the canonical representation of $G^X = \mathrm{SL}(2,\mathbb{R})^X$ in $\mathrm{EXP}\, H^X$ is given by the formula

$$U_{g(\cdot)}\,\mathrm{EXP}\, h(\cdot,\cdot) = \lambda(g(x), h(x,t)) \cdot \mathrm{EXP}(T_{g(x)}h(x,t) + \beta(g(x),t)), \tag{38}$$

where

$$\begin{aligned}
\lambda(g,h) &= \exp\left(-\frac{1}{2}\|\beta\|^2 - \langle T_g h, \beta\rangle\right) \\
&= \exp\left(-\frac{1}{2}\int_X\int_{\mathbb{R}} \frac{|\beta(g(x),t)|^2}{|t|}dt\, d\nu(x) - \int_X\int_{\mathbb{R}} \frac{T_{g(x)}h(x,t)\cdot\bar{\beta}(g(x),t)}{|t|}dt\, d\nu(x)\right).
\end{aligned}$$

The vacuum vector in the Fock realization is $\mathrm{EXP}\,0$, and the corresponding spherical function equals (34).

Using the isomorphism (3) between the Fock space and the L^2 space over the Gaussian white noise, one can obtain the Gaussian realization of the canonical representation. We do not reproduce here the corresponding formulas, which can be found in [32].

5.3 The Lebesgue model of the canonical representation

The commutative model of the canonical representation of G^X with respect to the unipotent subgroup was given in [32]. Another realization of this model, in the L^2 space over the so-called infinite-dimensional Lebesgue measure, was constructed in [33]. Let us describe this model.

The *Lebesgue measure* $\mathcal{L}^+$ on the space $D^+(X,\nu)$ is a σ-finite measure equivalent to the law $\mathcal{G}$ of the gamma process (see the example at the end of §4) with the density given by

$$\frac{d\mathcal{L}^+}{d\mathcal{G}}(\eta) = \exp(\eta(X)). \tag{39}$$

It follows from (33) and (39) that the Laplace transform of $\mathcal{L}^+$ equals

$$\int_{D^+}\left[\exp\left(-\int_X a(x)d\eta(x)\right)\right]d\mathcal{L}^+(\eta) = \exp\left(-\int_X \log a(x)d\nu(x)\right). \tag{40}$$

The *Lebesgue measure on* $D(X)$ is the convolution $\mathcal{L}^+ * \mathcal{L}^-$, where $\mathcal{L}^-$ is the image of $\mathcal{L}^+$ under the mapping $\eta \to -\eta$.

An arbitrary measurable function $a : X \to \mathbb{R}_+$ with $\int_X |\log a(x)|d\nu(x) < \infty$ defines a multiplicator $M_a : D \to D$ by the formula

$$M_a : \eta = \sum_i t_i\delta_{x_i} \mapsto \sum_i a(x_i)t_i\delta_{x_i}.$$

As shown in [33], the Lebesgue measure is projective invariant with respect to the group of multiplicators, namely,

$$\frac{dM_a(\mathcal{L})}{d\mathcal{L}} = \exp\left(-\int_X \log a(x)d\nu(x)\right).$$

This key property of the infinite-dimensional Lebesgue measure is a consequence of a remarkable quasi-invariance property of the gamma process (see [59, 33]). In particular, it makes it possible to construct a representation of the current group in the L^2 space over the Lebesgue measure. Note also that though there exist other subordinators quasi-invariant with respect to the group of multiplicators (see [60]), however, the gamma process is the only subordinator that admits an equivalent measure that is projective invariant with respect to this group ([33]).

Consider the triangular subgroup $\mathcal{T}$ of $\mathrm{SL}(2,\mathbb{R})^X$:

$$\mathcal{T} = \left\{T_{a,b} = \begin{pmatrix} a(\cdot)^{-1} & 0 \\ b(\cdot) & a(\cdot)\end{pmatrix}\right\}.$$

Theorem 8 ([33]). *The formula*

$$\mathcal{U}(T_{a,b})F(\eta) = \exp\left(\int_X \log|a(x)|d\nu(x) + i\int_X a(x)b(x)d\eta(x)\right)F(M_{a^2}\eta) \tag{41}$$

defines a unitary irreducible representation of the triangular subgroup $\mathcal{T}$ in the space $L^2(D,\mathcal{L})$, which is extendable to a unitary irreducible representation of the whole group $\mathrm{SL}(2,\mathbb{R})^X$.

5.4 Isomorphism of the Fock and Lebesgue models of the canonical representation

The Fock model (38) and the Lebesgue model (41) define isomorphic representations, since their spherical functions coincide. However, now we can use the canonical isomorphism between the space of square integrable functionals of the gamma process and the Fock space to construct explicitly the isomorphism of these realizations.

Theorem 9. *The isometry of the spaces $\mathrm{EXP}\,H^X$ and $L^2(D,\mathcal{L})$ that intertwines the Fock realization U and the Lebesgue realization $\mathcal{U}$ of the canonical representation of the current group $\mathrm{SL}(2,\mathbb{R})^X$ is given by*

$$\mathrm{EXP}\,h \leftrightarrow \Psi_h, \qquad h \in L^2(X\times\mathbb{R}, \nu\times\frac{dt}{|t|}),$$

where

$$\Psi_h(\eta) = \prod_i \left(h(x_i,t_i) + e^{-|t_i|/2}\right)\cdot\exp\left(-\int_X\int_{\mathbb{R}} \frac{h(x,t)\cdot e^{-|t|/2}}{|t|}dt\,d\nu(x)\right) \tag{42}$$

for $\eta = \sum t_i\delta_{x_i} \in D$.

Proof. Formula (42) defines an isometry between $\operatorname{EXP} H^X$ and $L^2(D, \mathcal{L})$, as follows from Theorem 6 in the special case of the gamma process and the obvious isometry between $L^2(D, \mathcal{G})$ and $L^2(D, \mathcal{L})$ given by $F(\eta) \leftrightarrow F(\eta)e^{-\eta(X)/2}$. It is not difficult to verify by direct calculation that this isomorphism intertwines the representations U and $\mathcal{U}$. $\square$

Note that the vacuum vector in the Lebesgue realization is $\Psi_0(\eta) = e^{-\frac{|\eta|(X)}{2}}$, where $|\eta| = \sum |t_i|$ is the total charge of the (signed) measure η.

Corollary 4. *The action of the involution* $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *in the Lebesgue realization of the canonical representation of* G^X *is given by the formula*

$$U_\sigma \Psi_{f(\cdot,\cdot)} = \Psi_{T_\sigma f(\cdot,\cdot)}, \tag{43}$$

where

$$T_\sigma f(x,t) = \int_{-\infty}^{\infty} K_0(t,s) f(s) ds, \tag{44}$$

with the kernel K_0 *given by (36).*

Proof. Follows from Theorem 9 and (38), since $\beta(\sigma, t) \equiv 0$. $\square$

Appendix

A. An example of a zero-dimensional non-Fock factorization (a model of hierarchical voting [26])

In this appendix, we reproduce the example of a non-Fock zero-dimensional factorization constructed in [26].

We consider the simplest, in fact purely combinatorial, model of a Hilbert and measure factorization, over a Cantor compactum, that is not isomorphic to a Fock factorization (i.e., is not linearizable). This model is determined by two positive integers $m, r > 1$ and a symmetric map $\phi : X_r^m \to X_r$, where X_r is a set consisting of r elements (it is convenient to enumerate them by the numbers $0, 1, \dots, r-1$) and X_r^m is its mth power, and the number of points in the preimage of each point $x \in X_r$ is the same, i.e., $\#(\phi^{-1}(x)) = m^{r-1}$. The latter condition implies that the ϕ-image of the uniform measure on X_r^m is the uniform measure on X_r.

For each such triple (m, r, ϕ), we will construct a factorization; under very wide assumptions on ϕ, these factorizations are not isomorphic to a Fock factorization and have a large group of symmetries.

The map ϕ is called *antiadditive* (respectively, *antimultiplicative*) if it satisfies the following condition. If for a function $g : X_r \to \mathbb{C}$, there exists a function $f : X_r \to \mathbb{C}$ such that the following relation holds identically (i.e., for any $a_1 \in X_r, \dots, a_m \in X_r$):

$$f(\phi(a_1, \dots a_m)) = g(a_1) + \dots + g(a_m) \tag{45}$$

(respectively,

$$f(\phi(a_1, \dots a_m)) = g(a_1) \cdot \dots \cdot g(a_m)), \tag{46}$$

then the function f (and hence g) is a constant.

For example, if X_r is an additive or multiplicative group, and ϕ is the group operation, then nonconstant solutions of these equations are additive or multiplicative characters. Here are examples of antiadditive maps.

Examples. 1. The model of voting by majority: $m = 3$, $r = 2$, and ϕ is given by

$$\phi(a, b, b) = b, \qquad a, b = 0, 1.$$

2. Let $m = 2$, $r = 3$ (i.e., $X_r = \{0, 1, 2\}$), and let ϕ be given by the table of values

$$\phi = \begin{matrix} 2 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{matrix}$$

It is easy to see that in both examples (45) and (46) have no nonconstant solutions.

The paper [26] contains a convenient criterion for the solutions of (45) and (46) to be constant functions (see below). It turns out that this case is generic, only in degenerate cases (similar to group laws) nonconstant solutions appear. In our construction, the absence of such solutions will guarantee the absence of additive and multiplicative vectors in the constructed factorization. The key role in the sequel is played by the following condition on the map ϕ.

Abundance condition. Let we are given a map $\phi : X_r^m \to X_r$. Fix $m-1$ arguments in an arbitrary way (due to the symmetry, it does not make difference what arguments we choose), take all maps of the set X_r into itself obtained in this way: $a_m \mapsto \phi_{a_1,\dots a_{m-1}}(a_m) \equiv \phi(a_1,\dots a_m)$, and consider the subsemigroup generated by all these maps in the semigroup of all maps of the set X_r into itself.

Definition 11. *The map ϕ is called* abundant *if the obtained subsemigroup contains at least one constant map.*

It is easy to check that abundance is a generic condition. For example, for $m=2$, it does not hold only for those maps ϕ that determine a semigroup law on the set X_r; in this case formula (46) defines a multiplicative character of the group or semigroup. The abundance condition also appears in the theory of Markov chains.

Before constructing a Hilbert factorization for an arbitrary triple (m,r,ϕ), let us describe the corresponding probability space.

Let T_m be the infinite rooted m-ary tree, and assume that each its vertex is assigned a random variable that takes r values with equal probabilities, the random variables of the same level (i.e., at the same distance from the root) being independent and the random variable ξ_v corresponding to a vertex v being equal to $\phi(\xi_{v_1},\dots,\xi_{v_m})$, where $v_1,\dots,v_m$ are the sons of v, and ϕ is the map ("voting") defined above.

The probability space $\Omega=\Omega(m,r,\phi)$ is the space of realizations of this family of random variables, i.e., the space of all functions f on the set of vertices of the tree T_m with values in the set $X_r=\{1,2,\dots,r\}$ that satisfy the above condition: $\phi(f(v_1),f(v_2),\dots,f(v_m))=f(v)$, where $v_1,\dots,v_m$ are the sons of the vertex v. (Following our analogy, one may call this space the space of ballot-papers).[10] By the properties of the map ϕ, this space is equipped with a well-defined uniform measure, and the values of functions at different vertices of the same level (the voters of the same level) are independent with respect to this measure. Note that the space Ω with the uniform measure is the inverse limit of the finite spaces X_r^m with the uniform measures with respect to the projections defined by ϕ.

It is useful to give another interpretation of the space Ω. Let K_m be the Cantor compactum of all infinite paths in the tree T_m endowed with the natural totally disconnected topology. Each function $f\in\Omega$ determines a *pseudomeasure* ν_f on the cylinder sets of the space K_m. Namely, by definition, the value of the pseudomeasure ν_f on the cylinder C_v of all paths going through a vertex v is equal to the value of the function f at the vertex v. By a pseudomeasure, we mean a function ν defined on the algebra of cylinder sets of the compactum K_m and satisfying a unique condition on the values at elementary cylinders[11], which reproduces the condition on the functions of the space Ω: $\nu(C_v)=\phi(\nu(C_{v_1}),\dots,\nu(C_{v_m}))$; one may call this condition ϕ-additivity. Thus we have described the space Ω as a space of pseudomeasures. The measure on this space allows us to speak about random pseudomeasures.

The space Ω has a natural measure factorization in the sense of Definition 2 over the Boolean algebra of cylinder sets in the space K_m. Note that the map ζ from the definition of a measure factorization is defined in our case only on elementary cylinders, however, it can be correctly extended to the Boolean algebra (but not the σ-algebra!) generated by cylinders, since every cylinder can be uniquely decomposed into elementary ones. However, the ϕ-additivity condition must hold only for the decomposition of an elementary cylinder into elementary ones.

Now we are in a position to define a Hilbert factorization, which will be non-Fock under a certain condition on the map ϕ. But first let us give the following analogy. The above description is similar to the following nonconventional simple description of processes with independent values, namely, the approximative description. For simplicity, we will speak only about the white noise and use the same notation as before. In our example, replace a finite space X_r by the real line $\mathbb{R}$ with the standard Gaussian measure, and let $\phi:\mathbb{R}^m\to\mathbb{R}$ be the

[10] Note that in the "model of voting by majority" from example 1 above, 2^n voters of the nth level can legally defeat all 3^n voters participating in the vote; thus already for the two-level system ($n=2$, the total number of voters is 9), four voters can defeat the remaining five voters, though the probability of this event is small.

[11] An elementary cylinder of order k is the set of paths in the tree with a given initial segment of length k.

normalized sum: $\phi(\xi_1, \ldots, \xi_m) = \frac{\xi_1+\ldots+\xi_m}{\sqrt{m}}$ (ϕ sends the standard Gaussian measure on $\mathbb{R}^m$ to the Gaussian measure on $\mathbb{R}$). It is not difficult to see that in this case our construction leads to a space Ω whose elements are ordinary additive measures on the same Cantor compactum K_m, and the Gaussian probability measure on Ω is defined by the condition that the value of a (random) additive real-valued measure on every cylinder is the integral of a realization of the standard white noise with the base space K_m over this cylinder. In other words, we have represented the Gaussian measure determined by the white noise as the inverse limit (in the sense of linear spaces) of the Gaussian measures on $\mathbb{R}^n$.

In some sense, in our example, the value of a pseudomeasure on a cylinder of the set K_m can be also regarded as the result of measuring a certain nonlinear noise ("black noise", though this term looks too gloomy) on this cylinder; but instead of additivity we have only the "ϕ-additivity" defined above.

Now it is not difficult to describe the Hilbert space $L^2(\Omega)$ and explain the appearance of a non-Fock factorization. The Hilbert space $H = L^2(\Omega)$ is of course the inductive limit in k of the finite-dimensional spaces $H_k = L^2(X_r^{m^k})$ with respect to the embeddings determined by the map ϕ; here H_k is the space of functions on the product of the spaces X_r over the vertices of the kth level. But the embedding

$$\bar{\phi}_k : H_k \to H_{k+1}$$

is the tensor product of k copies of the embedding $\bar{\phi}_1 : L^2(X_r) \to L^2(X_r^m)$, thus it suffices to define only the latter embedding; it is defined on the basis $e_1, \ldots, e_r$ of the space $L^2(X_r)$ by the formula $\bar{\phi}_1(e_i) = \chi_{\phi^{-1}(i)}$, where χ_E is the characteristic function of the set E, and $\phi^{-1}(i)$ is the preimage of an element $i \in X_r$ under the map ϕ. The constructed embeddings are obviously isometric, and they define a scalar product in the inductive limit of spaces. Recall that the inductive limit in the category of Hilbert spaces is the completion of the algebraic limit with respect to the Hilbert norm. It follows immediately from the above considerations that the constructed inductive limit can be naturally identified with the space $L^2(\Omega)$. It also follows from construction that the following proposition holds.

Proposition 4. *The space $L^2(\Omega)$ has a factorization over the Boolean algebra of cylinder sets of the space K_m.*

Recall that the continuity of a factorization at a point means the following: for each decreasing sequence of cylinders whose intersection is a point, the corresponding sequence of operator algebras converges to the algebra consisting of scalar operators.

Theorem 10. *The constructed factorization is continuous at a point if and only if the map ϕ is abundant. Under this condition, the factorization is non-Fock.*

Proof. Indeed, the abundance condition implies the antiadditivity, i.e., the absence of additive, and hence (by Theorem 2) multiplicative vectors. □

Corollary 5. *The constructed factorization cannot be extended to the Boolean algebra of* mod 0 *classes of measurable subsets of the space K_m with the natural (Lebesgue) product measure μ.*

Indeed, by Tsirelson's theorem, such extension must be a Fock factorization. On the other hand, a direct calculation shows that not for every convergent sequence of cylinder sets, the limit of the corresponding random variables does exist.

It follows from the definition of the constructed factorization that its group of symmetries includes the whole group of automorphisms of the tree and, moreover, contains also other transformations, see [26]. The study of properties of these factorization, in particular, the structure of the Hilbert space equipped with such factorization is of great interest. It is very important to consider operators similar to the canonical operators in a Fock factorization. This space may be eventually useful for the representation theory of groups and C^*-algebras. From the probabilistic viewpoint, the example under consideration is apparently related to the theory of branching processes.

The constructed factorization has a zero-dimensional base (the Cantor compactum). The construction of a non-Fock factorization with a one-dimensional base is more complicated, see [26]. The difficulty appears since we must coordinate the factorization at different representations of the interval as the union of intervals (while the decomposition of elementary cylinders of the Cantor compactum is unique). However, in the one-dimensional case, the factorization is also constructed by means of the inverse limit of Gaussian measures with nonlinear, as above, projections. Examples of non-Fock factorizations in dimensions greater than one are not yet constructed.

B. On the spectrum of a Fock factorization

Consider an arbitrary factorization over a Boolean algebra of sets. Associate with each measurable set B of the "base" the projection (conditional expectation) P_B onto the subspace of functionals that depend only on the restriction of a realization of the process to this set. Obviously, these projections commute with each other. Thus they generate a commutative C^*-algebra $\mathfrak{M}$. The following notion is introduced by Tsirelson [35].

Definition 12. *The spectrum of the commutative C^*-algebra $\mathfrak{M}$ is called the* spectrum of the factorization.

By definition, the spectrum is an invariant of a factorization. Consider the spectrum of the Fock factorization generated by the one-dimensional Gaussian process. Let us prove that in this case the algebra $\mathfrak{M}$ is maximal in the algebra of all operators, i.e., each operator that commutes with all these projections belongs to the same algebra. First of all, the projections to the "chaos" of a given order commute with all P_B, thus we may restrict the subalgebra $\mathfrak{M}$ to the subspace of the given chaos, and this restriction is a maximal commutative subalgebra in the subalgebra of operators of this chaos. Finally, it is obvious that two different chaoses are separated by the algebra $\mathfrak{M}$.

Theorem 11. *The spectrum of the one-dimensional Fock factorization is the disconnected union of symmetrized n-tuples from the base equipped with the symmetrized powers of the measure on the base.*

For a multidimensional Fock factorization, the algebra $\mathfrak{M}$ is not maximal, but the spectrum is the same.

C. Cycle index

Let $\mathfrak{S}_n$ be the symmetric group of degree n. Given a permutation $g \in \mathfrak{S}_n$, denote by $c_k(g)$ the number of cycles of length k in g. Let $t = (t_1, t_2, \ldots)$ be a sequence of indeterminates. The (augmented) cycle index of the symmetric group $\mathfrak{S}_n$ is

$$\tilde{Z}(\mathfrak{S}_n) = \tilde{Z}(\mathfrak{S}_n)[t] = \sum_{g \in \mathfrak{S}_n} t_1^{c_1(g)} t_2^{c_2(g)} \cdot \ldots . \tag{47}$$

A well-known formula (see, e.g., [61, (5.30)]) claims that

$$\sum_{n=0}^{\infty} \tilde{Z}(\mathfrak{S}_n)[t] \frac{z^n}{n!} = \exp \sum_{i=1}^{\infty} t_i \frac{z^i}{i}. \tag{48}$$

D. Orthogonal polynomials

For convenience, in this Appendix we reproduce necessary formulas concerning the classical orthogonal polynomials of Hermite and Charlier, which play an important role in the theory of orthogonal decompositions for Lévy processes. A standard reference on the theory of orthogonal polynomials is the monograph [62]. For combinatorial aspects of orthogonal polynomials, see also [61].

D.1. Hermite polynomials

Definition:

$$\begin{aligned} H_n(x) &= H_n^1(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \\ H_n^a(x) &= a^{\frac{n}{2}} H_n(x/\sqrt{a}). \end{aligned} \tag{49}$$

Orthogonality: orthogonal on $\mathbb{R}$ with respect to the normal law $N(0,a)$ with zero mean and variance a:

$$\frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} H_n^a(x) H_m^a(x) e^{-\frac{x^2}{2a}} \, dx = \delta_{nm} a^n n!. \tag{50}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{H_n^a(x)t^n}{n!} = e^{tx-\frac{t^2a}{2}}. \tag{51}$$

Combinatorial description:

$$H_n^a(x) = \sum_{g\in \mathrm{Inv}_n} x^{c_1(g)}(-a)^{c_2(g)} = \tilde{Z}_{\mathfrak{S}_n}[x, -a, 0, 0, \ldots], \tag{52}$$

where Inv_n is the set of all involutions in $\mathfrak{S}_n$.

D.2. Charlier polynomials

Definition:

$$C_n^a(x) = a^n \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} a^{-j} \binom{x}{j} j!. \tag{53}$$

Orthogonality: orthogonal on $\mathbb{Z}_+$ with respect to the Poisson law P_a with parameter a:

$$e^{-a} \sum_{k=0}^{\infty} C_n^a(k) C_m^a(k) \frac{a^k}{k!} = \delta_{nm} a^n n!. \tag{54}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{C_n^a(y)t^n}{n!} = (1+t)^y e^{-ta}. \tag{55}$$

Combinatorial description:

$$C_n^a(x) = \sum_{g\in\mathfrak{S}_n} (x-a)^{c_1(g)} x^{-c_2(g)+c_3(g)-c_4(g)+\dots} = \tilde{Z}_{\mathfrak{S}_n}[x-a, -x, x, -x, \ldots]. \tag{56}$$

The Charlier polynomials satisfy the following convenient formula:

$$\frac{(-1)^n}{a^n} C_n^a(k) = \frac{(-1)^k}{a^k} C_k^a(n). \tag{57}$$

Notation

(X, ν)	standard Borel space with a continuous finite measure
α	standard Gaussian white noise on (X, ν)
π	Poisson process on (X, ν)
Π	Lévy–Khintchin measure of a Lévy process
Φ	canonical Poisson–Gauss isomorphism
$\mathfrak{H}^{(n)}_{f_1,\ldots,f_n}$	generalized Hermite functional
$\mathfrak{C}^{(n)}_{f_1,\ldots,f_n}$	generalized Charlier functional
D	space of finite real discrete measures on X
$\mathfrak{S}_n$	symmetric group of degree n
Inv_n	the set of all involutions in $\mathfrak{S}_n$

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