

# Markov measures on Young tableaux and induced representations of the infinite symmetric group

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## Abstract

We show that the class of so-called Markov representations of the infinite symmetric group  $\mathfrak{S}_{\mathbb{N}}$ , associated with Markov measures on the space of infinite Young tableaux, coincides with the class of simple representations, i.e., inductive limits of representations with simple spectrum. The spectral measure of an arbitrary representation of  $\mathfrak{S}_{\mathbb{N}}$  with simple spectrum is equivalent to a multi-Markov measure on the space of Young tableaux. We also show that the representations of  $\mathfrak{S}_{\mathbb{N}}$  induced from the identity representations of two-block Young subgroups are Markov and find explicit formulas for the transition probabilities of the corresponding Markov measures. The induced representations are studied with the help of the tensor model of two-row representations of the symmetric groups; in particular, we deduce explicit formulas for the Gelfand–Tsetlin basis in the tensor models.

Keywords: Markov measures, Young tableaux, induced representations, simple spectrum.

## 1 Introduction

Consider the Young graph  $\mathbb{Y}$ , i.e., the  $\mathbb{Z}_+$ -graded graph of Young diagrams: the vertices of its  $n$ th level  $\mathbb{Y}_n$  are Young diagrams with  $n$  cells, and edges connect two vertices of neighboring levels whenever the corresponding diagrams differ by one cell. The space  $T$  of Young tableaux, i.e., infinite paths in the Young graph, is a totally disconnected (nonstationary) Markov compactum, and Markov measures on  $T$  are defined in the usual way. In other words, for every fixed diagram  $\lambda_n \in \mathbb{Y}_n$ , the conditional measure on the set of tableaux “passing through” this diagram at level  $n$  is the direct product of the conditional measures on the “past” (before the moment  $n$ ) and “future” (after the moment  $n$ ), i.e., the past and the future of a path are independent under a fixed “present” diagram. Such a measure is determined by the conditional transition (or cotransition) probabilities. This in turn means that the sequence of random variables  $X_n$  (where  $X_n(t)$  is the content of the  $n$ th cell of a tableau  $t$ ) has a special structure: the distribution of  $X_{n+1}$  depends not on the all previous values  $X_k$ , but only on the “border” ones (i.e., only on the diagram).

The tail partition  $\xi$  on the space of paths  $T$  is the partition into classes of confinal paths. A measure  $\mu$  (not necessarily Markov) on  $T$  is called ergodic (with respect to the tail partition) if every measurable  $\xi$ -set has either zero or full  $\mu$ -measure. A measure  $\mu$  is called quasi-invariant if it is quasi-invariant in the ordinary sense under any transformations changing the beginnings of tableaux. In other words, for almost every tableau  $t$  and every  $n \in \mathbb{N}$ , the conditional measure on the space of tableaux coinciding with  $t$  starting from the moment  $n$  is strictly positive.

Recall (see [1, 2, 8]) that the infinite symmetric group  $\mathfrak{S}_{\mathbb{N}}$  is the inductive limit of the finite symmetric groups  $\mathfrak{S}_n$  with the natural embeddings; the Young diagrams with  $n$  cells parametrize

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the irreducible complex representations of  $\mathfrak{S}_n$ ; the Young graph is the branching graph of irreducible representations of the symmetric groups (or the Bratteli diagram of the group algebra of  $\mathfrak{S}_{\mathbb{N}}$ ). It is well known (see [8]) that each ergodic quasi-invariant measure on the space of infinite tableaux  $T$  and each 1-cocycle (on the tail equivalence relation in this space) with values in the group of complex numbers of modulus 1 determine an irreducible representation of  $\mathfrak{S}_{\mathbb{N}}$  with simple spectrum (with respect to the Gelfand–Tsetlin algebra) by the formula (2) (see §3). Conversely, each irreducible representation with simple spectrum can be realized in this form. Two such representations are equivalent if and only if the corresponding measures are mutually absolutely continuous and the cocycles are cohomological. Not all irreducible representations can be obtained in this way, there exist irreducible representations with nonsimple spectrum; but the representations in question form a natural and important class of irreducible representations, which is defined in invariant terms, i.e., closed under all automorphisms of the group  $\mathfrak{S}_{\mathbb{N}}$ .

A representation of the infinite symmetric group  $\mathfrak{S}_{\mathbb{N}}$  determined by a Markov measure is called a *Markov representation*; thus a Markov representation can be determined by the transition probabilities of adding a new cell to a Young diagram (or the cotransition probabilities of deleting a cell).

We define a *generalized Markov measure* on the space  $T$  as a measure for which there exists an increasing sequence  $N_1, N_2, \dots$  of positive integers such that for every  $n$  the conditional measures on the past and on the future are independent if one fixes diagrams at the interval  $N_n, \dots, N_{n+1} - 1$ .

By analogy with the corresponding theorem from [6] on quasi-invariant measures on the spaces of sequences, one can use the martingale convergence theorem to prove the following result: every quasi-invariant ergodic measure is mutually absolutely continuous with a generalized Markov quasi-invariant ergodic measure. Thus, for the study of representations with simple spectrum, it suffices to consider only quasi-invariant generalized Markov measures.

Let us say that a representation of  $\mathfrak{S}_{\mathbb{N}}$  is *simple* if it is the inductive limit of a sequence of representations of the finite symmetric groups  $\mathfrak{S}_n$ ,  $n = 1, 2, \dots$ , each having a simple spectrum, i.e., having no multiplicities in the decomposition into irreducible representations. Simple representations have a simple spectrum, but they do not exhaust all representations with simple spectrum. The class of simple representations essentially depends on the approximation of the group  $\mathfrak{S}_{\mathbb{N}}$  by finite groups (see the end of Sec. 3); but in this paper we consider only the standard approximation, so that we use the term “simple representation” omitting the reference to the approximation. It turns out (Theorem 1) that the class of simple representations coincides with the class of Markov representations. This simple yet important fact relates the notions that are quite different at first sight.

In this paper, we consider only two-row Young diagrams, which fill only a small part of the Young graph, namely, the “half” of the Pascal graph (the vertices are indexed by pairs of positive integers  $(n, k)$ , where  $n$  is the level number,  $k$  is the vertex number, and  $k \leq [n/2]$ ). A path in this graph is a trajectory of a random walk on the semilattice  $\mathbb{Z}_+$  with reflecting barrier at the origin. All previous definitions (Markov measures, representations, etc.) can be restricted to this case.

Representations of the finite symmetric groups determined by two-row diagrams are widespread in applications; they have a convenient model — the so-called tensor realization, where the representation space is a space of finite-dimensional tensors; the rank of tensors does not exceed half the dimension (see [4]). We use this model below.

Our main result (Theorem 5) claims that the representations of the infinite symmetric group induced from two-block Young subgroups are simple, and, consequently, Markov; we also find explicit formulas for the corresponding Markov spectral measures, which are the laws of remarkable and natural random walks on the semilattice.

This result illustrates the thesis from [7] on the contemporary fruitful rapprochement between algebraic and probabilistic constructions: the notion of induced representation is one of the central notions in representation theory, while that of Markov measures plays an important role in probability theory. In our construction, both notions are amazingly intertwined.

The paper is organized as follows. In Sec. 2, we recall basis notions and introduce notation related to the Young graph and representation theory of the symmetric groups, in particular, the notion of Markov measure on the space of Young tableaux, which plays a key role in our considerations. In Sec. 3, we introduce the notions of Markov vector and Markov representation and prove that the classes of Markov and simple representations coincide. Section 4 describes the so-called tensor model of two-row representations of the finite symmetric groups. In Sec. 5, we find explicit formulas for the Gelfand–Tsetlin basis in the tensor model. Finally, in Sec. 6 we prove that the representations of the infinite symmetric group induced from the identity representations of two-block subgroups are Markov and explicitly find the corresponding spectral measures.

## 2 The Young graph and Markov measures on the space of its paths

In this section, we recall necessary notions related to the Young graph and the representation theory of the symmetric groups (see, e.g., [1, 2] and [9]); in particular, the notion of a Markov measure on the space of Young tableaux.

We denote by  $\mathfrak{S}_n$  the symmetric group of degree  $n$  and by  $\mathbb{C}[\mathfrak{S}_n]$  the group algebra of  $\mathfrak{S}_n$ .

The irreducible representations of the symmetric group  $\mathfrak{S}_n$  are indexed by the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  cells. Let  $\pi_\lambda$  be the irreducible unitary representation of  $\mathfrak{S}_n$  corresponding to a diagram  $\lambda \in \mathbb{Y}_n$ , and let  $\dim \lambda$  be the dimension of  $\pi_\lambda$ .

The branching of irreducible representations of the symmetric groups is described by the Young graph  $\mathbb{Y}$ . The set of vertices of the  $\mathbb{Z}_+$ -graded graph  $\mathbb{Y}$  is  $\cup_n \mathbb{Y}_n$ , and two vertices  $\mu \in \mathbb{Y}_{n-1}$  and  $\lambda \in \mathbb{Y}_n$  are joined by an edge if and only if  $\mu \subset \lambda$ . By definition, the zero level  $\mathbb{Y}_0$  consists of the empty diagram  $\emptyset$ .

Denote by  $T_\lambda$  the set (consisting of  $\dim \lambda$  elements) of Young tableaux of shape  $\lambda \in \mathbb{Y}_n$ , or, which is the same, the set of paths in the Young graph from the empty diagram  $\emptyset$  to  $\lambda$ . Let  $T_n = \cup_{\lambda \in \mathbb{Y}_n} T_\lambda$  be the set of Young tableaux with  $n$  cells.

According to the branching rule for irreducible representations of the symmetric groups, the space  $V_\lambda$  of the irreducible representation  $\pi_\lambda$  decomposes into the sum of one-dimensional subspaces indexed by the tableaux  $u \in T_\lambda$ . The basis  $\{h_u\}_{u \in T_\lambda}$  consisting of vectors of these subspaces is called the *Gelfand–Tsetlin basis*. It is an eigenbasis for the *Gelfand–Tsetlin algebra*  $\text{GZ}_n$ , the subalgebra in  $\mathbb{C}[\mathfrak{S}_n]$  generated by the centers  $Z[\mathfrak{S}_1], Z[\mathfrak{S}_2], \dots, Z[\mathfrak{S}_n]$  (see [10]).

Now let  $\mathfrak{S}_\mathbb{N} = \cup_{n=1}^\infty \mathfrak{S}_n = \varinjlim \mathfrak{S}_n$  be the infinite symmetric group with the fixed structure of an inductive limit of finite groups.

Denote by  $T = \varprojlim T_n$  the space of infinite Young tableaux (the projective limit of  $T_n$  with respect to the natural projections forgetting the tail of a path). With the topology of coordinatewise convergence  $T$  is a totally disconnected metrizable compact space. The tail equivalence relation  $\sim$  on  $T$  is defined as follows: paths  $s = (\mu_1, \mu_2, \dots)$  and  $t = (\lambda_1, \lambda_2, \dots)$  are equivalent if and only if they are confinal, i.e.,  $\mu_k = \lambda_k$  for all sufficiently large  $k$ . Denote by  $[t]_n \in T_n$  the initial segment of length  $n$  of a tableau  $t \in T$ . Given a finite tableau  $u \in T_n$ , denote by  $C_u = \{t : [t]_n = u\}$  the corresponding cylinder set; for  $\lambda \in \mathbb{Y}_n$ , let  $C_\lambda = \{t : t_n = \lambda\} = \cup_{u \in T_\lambda} C_u$  be the set of all paths passing through  $\lambda$ .

A measure  $M$  on the space  $T$  is called *central* if for any  $n \in \mathbb{N}$ ,  $\lambda \in \mathbb{Y}_n$  and any tableaux

$u, v \in T_\lambda$  of shape  $\lambda$ , the measures of the corresponding cylinder sets coincide:  $M(C_u) = M(C_v)$  (equivalently, a measure  $M$  is central if it is invariant under any transformations changing the beginnings of tableaux). One can easily see that the cotransition probabilities  $\frac{M(C_\lambda \cap C_\Lambda)}{M(C_\Lambda)}$  of a central measure depend only on the pair of diagrams  $\lambda \subset \Lambda$  (and do not depend on the measure) and are equal to  $\frac{\dim \lambda}{\dim \Lambda}$  (the relative dimension of the representation  $\pi_\lambda$  in  $\pi_\Lambda$ ). Central measures play a very important role in the representation theory of the symmetric groups.<sup>1</sup>

A measure  $M$  on the space  $T$  is called *Markov* if for every  $n \in \mathbb{N}$  the following condition holds: for any diagrams  $\lambda \in \mathbb{Y}_n$  and  $\Lambda \in \mathbb{Y}_{n+1}$  such that  $\Lambda \subset \lambda$  and for any path  $u \in T_\lambda$ , the events  $C_u$  (“the past”) and  $C_\Lambda$  (“the future”) are independent given  $C_\lambda$  (“the present”). In other words, a random tableau  $t = (\lambda_1, \lambda_2, \dots)$ , regarded as a sequence of random variables  $\lambda_n$ , where  $\lambda_n$  takes values in the set  $\mathbb{Y}_n$  of Young diagrams with  $n$  cells, is a Markov chain in the ordinary sense. In terms of transition probabilities, this means that the transition probability  $\frac{M(C_\Lambda \cap C_u)}{M(C_u)}$  depends only on the form  $\lambda$  of a tableau  $u$ , but not on the tableau itself. Note that the “forward” and “backward” Markov properties are equivalent, so that the definition of a Markov measure can be formulated in a similar way in terms of cotransition probabilities.

It is easy to see that every central measure is Markov. Thus central measures form a class of Markov measures with fixed cotransition probabilities equal to  $\frac{\dim \lambda}{\dim \Lambda}$ .

### 3 Markov vectors and Markov representations

Consider a cyclic representation  $\pi$  of the group  $\mathfrak{S}_n$  in a space  $V$  that has a simple spectrum (i.e., decomposes into the sum of pairwise nonequivalent irreducible representations) and a unit cyclic vector  $\xi$  in this representation. Consider a pair of diagrams  $\mu \supset \lambda$  with  $n$  and  $n-1$  cells, respectively, and assume that  $\pi$  contains a subrepresentation equivalent to  $\pi_\mu$  (for convenience, we will denote it by the same symbol). Project the vector  $\xi$  to the space  $V_\mu$  of the representation  $\pi_\mu$  and denote the obtained vector by  $\xi_\mu$ ; then project  $\xi_\mu$  to the space (contained in  $V_\mu$ ) of the representation  $\pi_\lambda$  of the group  $\mathfrak{S}_{n-1}$  and denote the obtained vector by  $\xi_{\mu,\lambda}$ . Let us call the ratio of the squared norms

$$(\|\xi_{\mu,\lambda}\|/\|\xi_\mu\|)^2 \tag{1}$$

the cotransition probability of the pair  $\mu, \lambda$ .

Let us define a measure on the space of all tableaux  $t$  with diagrams corresponding to the representations occurring in the representation  $\pi$  of the group  $\mathfrak{S}_n$  as follows: the probability of a tableau (i.e., a path in the Young graph) is equal to the product of cotransition probabilities along the whole path. It is easy to see that this measure is well defined and coincides with the spectral measure of the vector  $\xi$  with respect to the Gelfand–Tsetlin algebra; and the values (1) are exactly the cotransition probabilities of this spectral measure in the ordinary sense.

Note also that in our considerations the cyclic vector can be multiplied by a scalar factor of modulus 1 — this does not change the spectral measure; any other change of the cyclic vector changes the measure.

**Definition 1.** *We will say that a cyclic vector  $\xi$  is Markov if its spectral measure with respect to the Gelfand–Tsetlin algebra is Markov.*

Thus a cyclic vector is Markov if for every  $k < n$  and every diagram  $\lambda \in \mathbb{Y}_k$ , the probability of any tableau with this diagram does not depend on the continuation of this tableau to the level

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<sup>1</sup>In the theory of dynamical systems, such measures are called measures of maximum entropy.

$n$ . In terms of representations and cyclic vectors, this means that the norm of the projection of the cyclic vector to the subspace of the representation of the group  $\mathfrak{S}_k$  equivalent to  $\pi_\lambda$  does not depend on the way in which we have arrived at this subspace.

Now we use the following simple lemma from representation theory.

**Lemma 1.** *Assume that in a finite-dimensional Hilbert space  $H$  there is a unitary representation of a group  $G$  that is primary, i.e., decomposes into the direct (not necessarily orthogonal) sum  $H = H_1 \oplus H_2 \oplus \dots \oplus H_n$  of equivalent irreducible representations, and in each of them there is a cyclic vector  $v_i \in H_i$ ,  $i = 1, \dots, n$ . Then the following two assertions are equivalent:*

1. *For any  $i, j$ , there exists an isometry  $T_{i,j} : H_i \rightarrow H_j$  intertwining the corresponding representations such that  $T_{i,j}v_i = v_j$ .*
2. *In the cyclic hull of the vector  $v = \sum v_i$ , the representation is irreducible.*

Applying this lemma to the direct sum of the subrepresentations of  $\pi$  equivalent to  $\pi_\lambda$ , we see that the norms of the projections of the cyclic vector to the corresponding subspaces coincide if and only if  $\pi_\lambda$  has multiplicity 1 in the decomposition of the representation of  $\mathfrak{S}_k$  in the cyclic hull  $\mathfrak{S}_k\xi$  of  $\xi$ . Since, as shown above, the coincidence of these norms is in turn equivalent to the Markov property, we have proved the following lemma on characterization of Markov vectors.

**Lemma 2.** *Let  $\pi$  be a unitary representation of the group  $\mathfrak{S}_n$  with simple spectrum. A cyclic vector  $\xi$  of the representation  $\pi$  is Markov if and only if for every  $k < n$ , the representation of the group  $\mathfrak{S}_k$  in the cyclic hull  $\mathfrak{S}_k\xi$  of  $\xi$  with respect to  $\mathfrak{S}_k$  has a simple spectrum.*

Note that the described procedure of constructing the spectral measure does not determine a measure on diagrams, because in general the probabilities of the same diagram in different tableaux do not coincide. If they do coincide, then the spectral measure is central.

Now let us consider representations of the infinite symmetric group  $\mathfrak{S}_\mathbb{N}$ .

If we are given a quasi-invariant measure  $\mu$  on the space of Young tableaux  $T$  and a 1-cocycle  $c$  on pairs of confinal paths taking values in the group of complex numbers of modulus 1, then we can construct a unitary representation of the group  $\mathfrak{S}_\mathbb{N}$  in the space  $L^2(T, \mu)$  as follows (see, e.g., [8]). Recall that the Fourier transform allows one to realize the group algebra  $\mathbb{C}[\mathfrak{S}_\mathbb{N}]$  of the infinite symmetric group as the cross product constructed from the commutative algebra of functions on the space of tableaux  $T$  (Gelfand–Tsetlin algebra) and the tail equivalence relation. The desired representation is given by

$$L_g h(s) = \sum_{t \sim s} \sqrt{\frac{d\mu(s)}{d\mu(t)}} \hat{g}(s, t) c(s, t) h(t), \quad h \in L^2(T, \mu), \quad (2)$$

where  $\hat{g}$  is the function on pairs of confinal paths corresponding to an element  $g \in \mathfrak{S}_\mathbb{N}$  (the Fourier transform of  $g$ ). Note that the cocycle is trivial on the space of finite tableaux.

**Definition 2.** A representation of the infinite symmetric group  $\mathfrak{S}_\mathbb{N}$  is called *simple* if it is the inductive<sup>2</sup> limit of representations of the finite symmetric groups  $\mathfrak{S}_n$  with simple spectrum.

**Remarks. 1.** Strictly speaking, we should say “a simple representation with respect to the approximation of the group  $\mathfrak{S}_\mathbb{N}$  by the sequence of finite groups  $\mathfrak{S}_n$ ,  $n = 1, 2, \dots$ , with standard

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<sup>2</sup>Recall that the inductive limit of unitary representations  $\pi_k$  of finite groups  $G_k$  forming an inductive chain  $G_1 \subset G_2 \subset \dots$  is the representation of the group  $G = \cup G_k$  in the Hilbert space that is the completion of the chain of spaces of the representations  $\pi_k$  with (equivariant) isometric embeddings.

embeddings.” Another approximation will give another class of simple representations. But since in this paper we use only the standard approximation, we omit this specification (see also the remark below on generalized Markov measures).

**2.** We use the term “simple representation” in a much wider sense than in the paper [8], where a representation was called simple if it is the inductive limit of irreducible representations; we will call such representations *elementary*. Of course, elementary representations are simple in our sense.

Note that a representation with simple spectrum is cyclic, i.e., contains a cyclic vector.

**Definition 3.** A representation  $\pi$  of the group  $\mathfrak{S}_{\mathbb{N}}$  with simple spectrum is called *Markov* if the space of  $\pi$  contains a cyclic vector whose spectral measure (with respect to the Gelfand–Tsetlin algebra) is Markov. Note that a representation with simple spectrum is Markov if and only if the measure  $\mu$  in its realization (2) is Markov.

**Theorem 1.** *A representation of the infinite symmetric group is Markov if and only if it is simple.*

*Proof.* If  $\xi$  is a Markov cyclic vector, then for every  $n$  the representation of the group  $\mathfrak{S}_n$  in the cyclic hull  $\mathfrak{S}_n\xi$  is Markov and hence, by Lemma 2, has a simple spectrum. Thus the representation is the inductive limit of representations with simple spectrum. Conversely, if we have the inductive limit of representations with simple spectrum, then, by Lemma 2, the vector obtained by the successive embeddings from the unit vector in the original one-dimensional representation of  $\mathfrak{S}_1$  is a Markov cyclic vector.  $\square$

Consider a Markov representation  $\pi$  and a Markov cyclic vector  $\xi$ . The spectrum of the representation of  $\mathfrak{S}_n$  in its cyclic hull  $\mathfrak{S}_n\xi$  is simple (by Lemma 2), and we obtain an approximation of  $\pi$  by representations with simple spectrum, but the action of  $\mathfrak{S}_n$  in this representation may differ from the standard Young form by a factor of modulus 1. Of course, correcting the basis by such a factor (introducing the “phase”), we can obtain the standard action, but it may happen that there is no convergent system of factors and, consequently, the limit action involves a cocycle  $c$ , as in (2). The question of convergence of factors is precisely the question of whether the cocycle is cohomological to the identity one. It is well known that there are many cocycles that are not cohomological to the identity one. Thus, for a given Markov measure, there are many nonequivalent representations differing by a cocycle. The cocycle “measures the deviation” of the given realization of a representation from its standard realization. We do not dwell on this interesting questions, because in all realizations considered below (tensor model, induced representations), the cocycle is trivial.

It is not difficult to see how Theorem 1 can be extended to the case of generalized Markov measures. Assume that the spectral measure of the representation is a generalized Markov measure with Markov intervals  $N_1, N_2, \dots$  (i.e., for every  $k$ , if we fix the diagrams with  $N_{k-1}, \dots, N_k - 1$  cells, then the diagrams preceding  $N_{k-1}$  and those following  $N_k - 1$  are independent); then the corresponding representation is simple, but with respect to a sparse chain of subgroups  $\mathfrak{S}_1, \mathfrak{S}_{N_1}, \mathfrak{S}_{N_2}, \dots$ . In other words, it is the limit of representations with simple spectrum of the sequence of groups  $\mathfrak{S}_{N_k}$ ,  $k \rightarrow \infty$ . As mentioned in the introduction, every quasi-invariant measure on the space of tableaux is equivalent to a generalized Markov measure — the proof of this fact is the same as for a similar assertion for quasi-invariant measures in the space of one-sided sequences (see [6]). Therefore, *each irreducible representation of the group  $\mathfrak{S}_{\mathbb{N}}$  with simple spectrum (with respect to the Gelfand–Tsetlin algebra) is the limit of finite-dimensional representations with simple spectrum of a certain sequence of groups  $\mathfrak{S}_{N_k}$ ,  $k = 1, 2, \dots$*

## 4 The tensor model of two-row representations

In this section, we describe the so-called tensor model of two-row representations of the symmetric groups, which was suggested by the first author and studied in [4] (see also [5]).

For  $0 \leq k \leq n$ , denote by  $F_{n,k}$  the set of  $k$ -element subsets in  $\{1, \dots, n\}$ . Given  $I = \{i_1, \dots, i_k\} \in F_{n,k}$ , let  $x_I = x_{i_1} \cdots x_{i_k}$ .

Denote by  $A_{n,k} = \{\sum_{I \in F_{n,k}} c_I x_I\}$  the vector space of square-free homogeneous forms of degree  $k$  in  $n$  variables. This space can also be identified with the space of symmetric tensors of rank  $k$  with zero diagonal components over the  $n$ -dimensional space (a form  $f = \sum_{I \in F_{n,k}} c_I x_I$  is identified with the tensor  $\{T_{j_1, \dots, j_k}\}_{j_1, \dots, j_k=1}^n$ , where  $T_{j_1, \dots, j_k} = c_{\{j_1, \dots, j_k\}}$  if the indices  $j_1, \dots, j_k$  are pairwise distinct and  $T_{j_1, \dots, j_k} = 0$  otherwise).

Denote by  $\|\cdot\|$  the norm determined by the standard scalar product in the space of forms (tensors):

$$\|f\|^2 = \sum_{I \in F_{n,k}} |c_I|^2, \quad f = \sum_{I \in F_{n,k}} c_I x_I \in A_{n,k}. \quad (3)$$

Let  $A_{n,k}^0$  be the subspace of  $A_{n,k}$  defined as

$$A_{n,k}^0 = \left\{ \sum c_I x_I \in A_{n,k} \mid \sum_{j \notin J} c_{J \cup j} = 0 \text{ for every } J \in F_{n,k-1} \right\}.$$

There is a natural action of the symmetric group  $\mathfrak{S}_n$  on the space  $A_{n,k}$  by substitutions of indices: given  $\sigma \in \mathfrak{S}_n$ ,

$$\sigma \cdot \sum_{I \in F_{n,k}} c_I x_I = \sum_{I \in F_{n,k}} c_I x_{\sigma I}, \quad \text{where } \sigma\{i_1, \dots, i_k\} = \{\sigma(i_1), \dots, \sigma(i_k)\},$$

or, in tensor form,  $\sigma\{T_{j_1, \dots, j_k}\} = \{T'_{j_1, \dots, j_k}\}$ , where  $T'_{j_1, \dots, j_k} = T_{\sigma^{-1}(j_1), \dots, \sigma^{-1}(j_k)}$ . It is easy to see that the subspace  $A_{n,k}^0$  is invariant under this action. Note that the spaces  $A_{n,k}^0$  and  $A_{n,n-k}^0$  (as well as  $A_{n,k}$  and  $A_{n,n-k}$ ) are naturally isometric and the corresponding representations of  $\mathfrak{S}_n$  are equivalent.

Realizations of representations of the symmetric groups in spaces of tensors (square-free homogeneous forms) are called *tensor realizations*. The following theorem is a well-known fact, which in terms of the tensor model was proved in [4].

**Theorem 2.** *Let  $k \leq n/2$ .*

(1) *The representation of the symmetric group  $\mathfrak{S}_n$  in the space  $A_{n,k}^0$  (and in the space  $A_{n,n-k}^0$ ) is equivalent to the irreducible representation  $\pi_{n-k,k}$  corresponding to the two-row diagram  $\lambda_{n,k} = (n-k, k)$  with rows of lengths  $n-k$  and  $k$ .*

(2) *The representation of the symmetric group  $\mathfrak{S}_n$  in the space  $A_{n,k}$  is equivalent to the multiplicity-free direct sum of  $\pi_{n-l,l}$  over all  $l = 0, 1, \dots, k$ . In particular, the representation of  $\mathfrak{S}_n$  in  $A_{n, \lfloor n/2 \rfloor}$  is equivalent to the multiplicity-free direct sum of all two-row representations:*

$$A_{n, \lfloor n/2 \rfloor} \simeq \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \pi_{n-k,k}.$$

## 5 The Gelfand–Tsetlin basis in tensor realizations

In this section, we find explicit formulas for the Gelfand–Tsetlin basis in the tensor realizations of two-row representations of the symmetric groups.

First note that the space  $A_{n,k}^0$  can be also defined as follows.

**Lemma 3.** *The space  $A_{n,k}^0$  is the subspace of  $A_{n,k}$  consisting of forms that are invariant under simultaneous shifts of all variables by a constant. It is spanned by functions of the form  $(x_{i_1} - x_{j_1}) \dots (x_{i_k} - x_{j_k})$ , where all indices  $i_1, \dots, i_k, j_1, \dots, j_k$  are pairwise distinct.*

Of course, functions of the form  $(x_{i_1} - x_{j_1}) \dots (x_{i_k} - x_{j_k})$  are linearly dependent, so that they form an overfull system in  $A_{n,k}^0$ . We will call them “pseudo-monomials.”

Note that a two-row Young tableau  $u$  is uniquely determined by the sequence  $p_1 < p_2 < \dots < p_k$  of the elements of its second row.

**Theorem 3.** *Let  $u \in T_n$  be a two-row Young tableau, and denote by  $p_1 < p_2 < \dots < p_k$  the elements of its second row. Then the element  $h_u^0$  of the Gelfand–Tsetlin basis in  $A_{n,k}^0$  corresponding to the tableau  $u$  is given by the formula*

$$h_u^0 = c_u^0 \sum_{i_1, \dots, i_k} (x_{i_1} - x_{p_1}) \dots (x_{i_k} - x_{p_k}), \quad (4)$$

where  $c_u^0$  is a normalizing constant and the sum runs over all indices  $i_1, \dots, i_k$  from 1 to  $n$  such that  $i_j < p_j$  for all  $j = 1, \dots, k$  and all indices  $i_1, \dots, i_k, p_1, \dots, p_k$  are pairwise distinct.

*Proof.* It suffices to check that for every  $l = 1, \dots, n$ , the form  $h_u^0$  is an eigenfunction for the action of the YJM-element<sup>3</sup>  $X_l$  with eigenvalue equal to  $c_l(u)$ , the contents of the cell of  $u$  containing  $l$ . This can be done by direct combinatorial calculations.  $\square$

One can prove that the normalizing constant in (4) equals

$$c_u^0 = \frac{1}{\left(\prod_{j=1}^k (p_j - 2j + 1)(p_j - 2j + 2)\right)^{1/2}}. \quad (5)$$

**Example 1.** Consider the tableau  $u$  of shape  $\lambda = \lambda_{n-k,k}$  in which the second row contains the numbers  $2, 4, \dots, 2k$ . Then

$$h_u^0 = c_u^0 \cdot (x_1 - x_2)(x_3 - x_4) \dots (x_{2k-1} - x_{2k}), \quad (c_u^0)^2 = \frac{1}{2^k}.$$

This is the only tableau of shape  $\lambda$  such that the corresponding Gelfand–Tsetlin element  $h_u^0$  is a pseudo-monomial; we will call it the *good* tableau of shape  $\lambda$ . For all other tableaux of the same shape,  $h_u^0$  is a linear combination of pseudo-monomials.

Denote by  $H_{n,m}^k$  the subspace in  $A_{n,m}$  corresponding to the representation  $\pi_{n-k,k}$ . Thus

$$A_{n,m} = \bigoplus_{k=0}^m H_{n,m}^k. \quad (6)$$

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<sup>3</sup>Recall that the  $l$ th Young–Jucys–Murphy (YJM) element is defined as  $X_l = (1, l) + (2, l) + \dots + (l-1, l) \in \mathbb{C}[\mathfrak{S}_n]$ , where, as usual,  $(i, j)$  stands for the transposition permuting the elements  $i$  and  $j$ . The YJM-elements form a multiplicative basis of the Gelfand–Tsetlin algebra  $\text{GZ}_n$ . Concerning the Gelfand–Tsetlin algebras, YJM-elements, and their role in the representation theory of the symmetric groups, see [10].



Let  $\psi_n^l$  be the linear operator that acts on monomials as follows:

$$\psi_n^l x_I = x_I \sum_{\substack{j_1, \dots, j_l \notin I \\ \text{distinct}}} x_{j_1} \dots x_{j_l}.$$

As follows from the results of [4],

$$H_{n,m}^k = \psi_n^{m-k} A_{n,k}^0, \quad (7)$$

and  $\psi_n^{m-k}$  is an isomorphism of  $H_{n,m}^k$  and  $A_{n,k}^0$  intertwining the corresponding representations.

Using (7), we can obtain the following assertions from Lemma 3 and Theorem 3.

**Lemma 4.** *The space  $H_{n,m}^k$  is spanned by functions of the form  $(x_{i_1} - x_{j_1}) \dots (x_{i_k} - x_{j_k}) x_{s_1} \dots x_{s_{m-k}}$ , where all indices  $i_1, \dots, j_k, j_1, \dots, j_k, s_1, \dots, s_{m-k}$  are pairwise distinct.*

**Theorem 4.** *Let  $u \in T_n$  be a two-row Young tableau, and denote by  $p_1 < p_2 < \dots < p_k$  the elements of its second row. Then the element  $h_u$  of the Gelfand–Tsetlin basis in  $H_{n,m}^k$  corresponding to the tableau  $u$  is given by the formula*

$$h_u = c_u \sum (x_{i_1} - x_{p_1}) \dots (x_{i_k} - x_{p_k}) x_{s_1} \dots x_{s_{m-k}}, \quad (8)$$

where  $c_u$  is a normalizing constant and the sum runs over all indices  $i_1, \dots, i_k, s_1, \dots, s_{m-k}$  from 1 to  $n$  such that  $i_j < p_j$  for all  $j = 1, \dots, k$  and all indices  $i_1, \dots, i_k, p_1, \dots, p_k, s_1, \dots, s_{m-k}$  are pairwise distinct.

**Example 2.** For the good tableau  $u$  of shape  $\lambda = \lambda_{n-k,k}$  considered in Example 1, we have

$$h_u = c_u \cdot (x_1 - x_2)(x_3 - x_4) \dots (x_{2k-1} - x_{2k}) \sum_{\substack{s_1, \dots, s_{m-k} \in \{2k+1, \dots, n\} \\ \text{distinct}}} x_{s_1} \dots x_{s_{m-k}}, \quad c_u^2 = \frac{1}{2^k C_{n-2k}^{m-k}}. \quad (9)$$

## 6 Induced representations

In this section, we consider the class of irreducible representations of the infinite symmetric group  $\mathfrak{S}_{\mathbb{N}}$  induced from two-block Young subgroups. Namely, consider a partition  $\mathbb{N} = A \cup B$  of the set of positive integers into two subsets and the representation of  $\mathfrak{S}_{\mathbb{N}}$  induced from the identity representation of the subgroup  $\mathfrak{S}_A \times \mathfrak{S}_B$ . It is not difficult to prove that this representation is irreducible. It can also be described in terms of the tensor model as follows.

A partition  $\mathbb{N} = A \cup B$  of the set of positive integers into two subsets is uniquely determined by an infinite sequence  $\xi = \xi_1 \xi_2 \dots$  of 0's and 1's (an “infinite tensor”), where  $\xi_i = 1$  if  $i \in A$ , and  $\xi_i = 0$  if  $i \in B$ . Then the induced representation in question is equivalent to the natural substitutional representation of  $\mathfrak{S}_{\mathbb{N}}$  on infinite sequences in the cyclic hull of the sequence  $\xi$ , which we will denote by  $\pi_{\xi}$ . Note that the orbit of  $\xi$  is the discrete set  $O_{\xi}$  of infinite sequences of 0's and 1's eventually coinciding with  $\xi$ , and  $\pi_{\xi}$  is a unitary representation of  $\mathfrak{S}_{\mathbb{N}}$  in the space  $l^2(O_{\xi})$ .

For simplicity, it is convenient to assume that the number of 1's among the first  $n$  elements of  $\xi$  does not exceed  $n/2$ . It is not difficult to see that an arbitrary case can be reduced to this one, but we omit the corresponding technical details.

Consider the cyclic hull  $\mathfrak{S}_n \xi$  of the sequence  $\xi$  with respect to the finite symmetric group  $\mathfrak{S}_n$ . It can be naturally identified with the space  $A_{n,m}$ , where  $m = m(n)$  is the number of 1's among the first  $n$  elements of  $\xi$ . Moreover, the scalar product induced from  $l^2(O_{\xi})$  coincides with the

standard scalar product in  $A_{n,m}$ . Thus the representation  $\pi_\xi^n$  of the group  $\mathfrak{S}_n$  in  $\mathfrak{S}_n\xi$  is unitarily equivalent to the tensor representation in the space  $A_{n,m}$ , the cyclic vector  $\xi$  corresponding to the monomial  $x_{i_1}\dots x_{i_m} \in A_{n,m}$ , where  $i_1, \dots, i_m$  are the numbers of positions from 1 to  $n$  at which  $\xi$  has 1's. We also have a natural embedding  $\iota_n : \pi_\xi^n \hookrightarrow \pi_\xi^{n+1}$  defined as follows: if  $\xi_{n+1} = 0$ , then  $\iota_n$  is the identical embedding  $A_{n,m} \hookrightarrow A_{n+1,m}$ ; and if  $\xi_{n+1} = 1$ , then  $\iota_n : A_{n,m} \rightarrow A_{n+1,m+1}$  is the multiplication by  $x_{n+1}$ , i.e.,  $\iota_n f = x_{n+1}f$ . The following lemma is obvious.

**Lemma 5.** *The induced representation  $\pi_\xi$  of the group  $\mathfrak{S}_\mathbb{N}$  is the inductive limit of the tensor representations  $\pi_\xi^n$  of the groups  $\mathfrak{S}_n$ .*

The main result of this section is the following Theorem 5. Note that it easily follows from the tensor realization, Theorem 2, and Lemma 2 that the representation  $\pi_\xi$  is Markov, so that the main part of the theorem is the computation of the transition probabilities of the corresponding spectral measure.

**Theorem 5.** *The spectral measure  $\mu_\xi$  of the cyclic vector  $\xi$  in the representation  $\pi_\xi$  with respect to the Gelfand–Tsetlin algebra is a Markov measure on the space of infinite Young tableaux  $T$ , and its transition probabilities are given by the following formula. Denote by  $m(n)$  the number of 1's among the first  $n$  elements of  $\xi$ .*

*If  $\xi_{n+1} = 0$ , then*

$$\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k}) = \frac{n - m(n) - k + 1}{n - 2k + 1}, \quad \text{Prob}(\lambda_{n,k}, \lambda_{n+1,k+1}) = \frac{m(n) - k}{n - 2k + 1}. \quad (10)$$

*If  $\xi_{n+1} = 1$ , then*

$$\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k}) = \frac{m(n) - k + 1}{n - 2k + 1}, \quad \text{Prob}(\lambda_{n,k}, \lambda_{n+1,k+1}) = \frac{n - m(n) - k}{n - 2k + 1}. \quad (11)$$

*Proof.* As mentioned above, for all  $n$  the representation  $\pi_\xi^n$  is realized in the space  $A_{n,m}$ , where  $m = m(n)$  is the number of 1's among the first  $n$  elements of  $\xi$ . In particular, for all  $l \leq n$ , the representation of  $\mathfrak{S}_l$  in the corresponding cyclic hull  $\mathfrak{S}_l\xi$  of  $\xi$  has a simple spectrum by Theorem 2. It follows from Lemma 2 that  $\mu_\xi$  is a Markov measure.

Let us find explicit formulas for the transition probabilities of  $\mu_\xi$ . As follows from the branching rule for irreducible representations of the symmetric groups, for each vector  $f \in H_{n,m}^k$ , we have  $\iota_n f = f_{n+1,k} + f_{n+1,k+1}$ , where  $f_{n+1,k} \in H_{n+1,m'}^k$ ,  $f_{n+1,k+1} \in H_{n+1,m'}^{k+1}$ , and  $m' = m(n+1) = m + \xi_{n+1}$ .

**Lemma 6.** *Let  $f \in H_{n,m}^k$ . By (7),  $f = \psi_n^{m-k} f_0$ , where  $f_0 \in A_{n,k}^0$ . Then*

• *if  $\xi_{n+1} = 0$ , then*

$$\begin{aligned} f_{n+1,k} &= \frac{n - m - k + 1}{n - 2k + 1} (f + x_{n+1} \psi_n^{m-k-1} f_0), \\ f_{n+1,k+1} &= \frac{1}{n - 2k + 1} ((m - k)f - (n - m - k + 1)x_{n+1} \psi_n^{m-k-1} f_0); \end{aligned}$$

• *if  $\xi_{n+1} = 1$ , then*

$$\begin{aligned} f_{n+1,k} &= \frac{m - k + 1}{n - 2k + 1} (x_{n+1}f + \psi_n^{m-k+1} f_0), \\ f_{n+1,k+1} &= \frac{1}{n - 2k + 1} ((n - m - k)x_{n+1}f - (m - k + 1)\psi_n^{m-k+1} f_0). \end{aligned}$$

*Proof.* One can easily check that the forms  $f_{n+1,k}$  and  $f_{n+1,k+1}$  lie in the appropriate spaces. Obviously, their sum equals  $\iota_n f$ .  $\square$

**Corollary 1.** *Let  $f \in H_{n,m}^k$ . Then*

- if  $\xi_{n+1} = 0$ , then

$$\|f_{n+1,k}\|^2 = \frac{n-m-k+1}{n-2k+1} \|f\|^2, \quad \|f_{n+1,k+1}\|^2 = \frac{m-k}{n-2k+1} \|f\|^2.$$

- if  $\xi_{n+1} = 1$ , then

$$\|f_{n+1,k}\|^2 = \frac{m-k+1}{n-2k+1} \|f\|^2, \quad \|f_{n+1,k+1}\|^2 = \frac{n-m-k}{n-2k+1} \|f\|^2.$$

*Proof.* The desired formulas follow from Lemma 6 by straightforward calculations taking into account that  $(f_{n+1,k}, f_{n+1,k+1}) = 0$ ;  $\|f_{n+1,k}\|^2 + \|f_{n+1,k+1}\|^2 = \|f\|^2$ ;  $(f, x_{n+1}\psi_n^{m-k-1}f_0) = 0$  in the case  $\xi_{n+1} = 0$ , and  $(x_{n+1}f, \psi_n^{m-k+1}f_0) = 0$  in the case  $\xi_{n+1} = 1$  (the first two relations are valid by definitions, and the last one holds because one of the vectors consists of monomials containing  $x_{n+1}$  and the other one consists of monomials not containing  $x_{n+1}$ ).  $\square$

As follows from the proof of Lemma 2,  $\text{Prob}(\mu | \lambda) = \left(\frac{\|f_\lambda\|}{\|f_\mu\|}\right)^2$ , so that formulas (10), (11) follow from Corollary 1. Theorem 5 is proved.  $\square$

Note that all spectral measures of induced representations considered in Theorem 5 are not central (except for the trivial case when one of the sets in the partition is empty).

**Corollary 2.** *The operator  $\psi_n^{m-k} : A_{n,k}^0 \mapsto H_{n,m}^k$  is an isometry up to a constant. Namely, for  $f_0 \in A_{n,k}^0$ ,*

$$\|\psi_n^{m-k} f_0\|^2 = C_{n-2k}^{m-k} \cdot \|f_0\|^2. \quad (12)$$

*Proof.* Follows by simple calculations based on Lemma 6 and Corollary 1.  $\square$

In particular, we obtain from (5), (7), and (12) that the coefficients  $c_u$  in (8) satisfy

$$\frac{1}{c_u^2} = C_{n-2k}^{m-k} \cdot \prod_{j=1}^k (p_j - 2j + 1)(p_j - 2j + 2).$$

**Example 3.** Let  $\xi = 0101\dots$ . Then  $m(n) = \lfloor n/2 \rfloor$ , and the formulas for transition probabilities take the following form:

- if  $n$  is odd,

$$\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k}) = \frac{n-2k+2}{2(n-2k+1)}, \quad \text{Prob}(\lambda_{n,k}, \lambda_{n+1,k+1}) = \frac{n-2k}{2(n-2k+1)}; \quad (13)$$

- if  $n$  is even,

$$\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k}) = \text{Prob}(\lambda_{n,k}, \lambda_{n+1,k+1}) = \frac{1}{2}. \quad (14)$$

These formulas can also be obtained in another way. Since  $\mu_\xi$  is a Markov measure, it suffices to find the transition probabilities for one tableau  $u$  of each shape  $\lambda = \lambda_{n-k,k}$ . Let  $u$  be the good tableau of shape  $\lambda$  (see Examples 1 and 2). Note that the tableau  $v$  obtained from a good tableau  $u$  by adding the element  $n+1$  to the first row is also good. Thus, using formula (9) and observing

that in our case  $\xi$  is the monomial  $x_2x_4\dots x_{2n}$ , we see that this monomial occurs exactly once in the right-hand side of (9), so that  $(\xi, h_u) = c_u$ . Hence

$$\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k}) = \frac{(\xi, h_v)^2}{(\xi, h_u)^2} = \frac{C_{n-2k}^{m(n)-k}}{C_{n+1-2k}^{m(n+1)-k}},$$

which implies the required formula for  $\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k})$ , taking into account that  $m(n) = [n/2]$ . Obviously,  $\text{Prob}(\lambda_{n,k}, \lambda_{n+1,k+1}) = 1 - \text{Prob}(\lambda_{n,k}, \lambda_{n+1,k})$ .

It is interesting to compare formulas (13), (14) with the transition probabilities of the ergodic central measure  $\mu_\alpha$  on  $T$  corresponding to the Thoma parameters  $\alpha = (1/2, 1/2, 0, \dots)$ ,  $\beta = 0$  (see, e.g., [9]).

**Lemma 7.** *The transition probabilities of the central measure  $\mu_\alpha$  are given by (13) for all  $n$ .*

*Proof.* Let  $u \in T_\lambda$ ,  $\lambda \in \mathbb{Y}_n$ . Then

$$\mu_\alpha(C_u) = s_\lambda(1/2, 1/2) = \frac{1}{2^n} s_\lambda(1, 1) = \prod_{\square \in \lambda} \frac{2 + c(\square)}{h(\square)},$$

where  $s_\lambda$  is a Schur function,  $c(\square)$  and  $h(\square)$  are the contents and the hook length of a square  $\square \in \lambda$ , respectively, and we have used the well-known formula for  $s_\lambda(1, \dots, 1)$  (see [3, Example I.3.4]). The lemma follows by elementary calculations.  $\square$

It is convenient to rewrite formulas (13), (14) introducing the change of indices  $j = n - 2k$ . In these terms, a Young tableau is determined by a sequence  $(j_1, j_2, \dots)$ , where  $j_n$  takes the values  $0, 1, \dots, n$ , and the transition probabilities of the measure  $\mu_\xi$  are equal to

$$\text{Prob}(j, j+1) = \frac{j+2}{2(j+1)}, \quad \text{Prob}(j, j-1) = \frac{j}{2(j+1)}$$

at an odd moment of time; and

$$\text{Prob}(j, j+1) = \text{Prob}(j, j-1) = \frac{1}{2}$$

at an even moment of time. We see that a random Young tableau governed by the measure  $\mu_\xi$  is a trajectory of a nonhomogeneous (in time as well as in space) random walk on  $\mathbb{Z}_+$ . Thus the induced representations of the infinite symmetric group considered in this paper act in spaces of functions over trajectories of natural random walks. Explicit formulas for this action are given by Young's orthogonal form (see, e.g., [2]).

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