Записки научных
семинаров ПОМИ
Том 378, 2010 г.

N. V. Tsilevich

SPECTRAL PROPERTIES OF THE PERIODIC COXETER LAPLACIAN IN THE TWO-ROW FERROMAGNETIC CASE

Abstract. This paper is a part of the project suggested by A. M. Vershik and the author and aimed to combine the known results on the representation theory of finite and infinite symmetric groups and a circle of results related to the quantum inverse scattering method and Bethe ansatz. In this first part, we consider the simplest spectral properties of a distinguished operator in the group algebra of the symmetric group, which we call the periodic Coxeter Laplacian. Namely, we study this operator in the two-row representations of symmetric groups and in the "ferromagnetic" asymptotic mode.

1. Introduction

This paper is a part of the project suggested by A. M. Vershik and the author and aimed to combine the known results on the representation theory of finite and infinite symmetric groups and a circle of results related to the quantum inverse scattering method and Bethe ansatz.

In this first part, we consider the simplest spectral properties of a distinguished operator in the group algebra of the symmetric group, which we call the periodic Coxeter Laplacian. Namely, we study this operator in the two-row representations of symmetric groups and in the "ferromagnetic" asymptotic mode. In subsequent papers, we will consider the analogous problems for the "antiferromagnetic mode" and for other representations. The importance of this operator is due to its close relation to one of the classical integrable models of statistical physics, namely, the XXX Heisenberg model of spins. On the other hand, this is the ordinary Laplace operator for the Cayley graph of the symmetric group

Key words and phrases: Coxeter Laplacian, representations of symmetric groups, Bethe ansatz.

Supported by the grants RFBR 08-01-00379-a, 09-01-12175-ofi_m, and 10-01-90411-Ukr_a.

corresponding to the *periodic* Coxeter system of generators, i.e., the classical Coxeter generators with one additional transposition $(N,1)$ imposed by the periodic conditions.

The importance of studying the spectral properties of the periodic Coxeter Laplacian in various representations of the symmetric group, though obvious enough, became clear quite recently. A significant role in our understanding of this connection was played by discussions with P. P. Kulish and V. O. Tarasov, whom we would like to thank. The author is also grateful to P. P. Kulish for comments that have led to improvement of the presentation, and to A. M. Vershik for numerous fruitful discussions of various problems related to this project.

The paper is organized as follows. In Sec. 2, we recall the basic facts related to the XXX Heisenberg model and Bethe ansatz, and introduce our main object of study, the periodic Coxeter Laplacian. Section 3 is devoted to the description of the quantum inverse scattering method for the model under consideration and the related subalgebra of the group algebra of the symmetric group. In Sec. 4, we present some results concerning the spectra of the periodic Coxeter Laplacian in finite cases. Finally, Sec. 5, which is the main section of the paper, deals with asymptotic results on the spectrum of the periodic Coxeter Laplacian in the "ferromagnetic" mode.

2. Bethe ansatz and the periodic Coxeter Laplacian

The XXX Heisenberg model describes a chain of N interacting spins with quantum number $s = 1/2$ on a one-dimensional lattice. The Hamiltonian of this model acts in the Hilbert space $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ of dimension 2^N spanned by the orthogonal basis vectors $|\varepsilon_1 \ldots \varepsilon_N\rangle$, where $\varepsilon_n = \uparrow$ represents an up spin and $\varepsilon_n = \downarrow$ represents a down spin at site n, and is given by the formula

$$\begin{aligned} H &= -\frac{J}{4}\sum_{n=1}^{N}\left[\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z\right] \\ &= -\frac{J}{4}\sum_{n=1}^{N}\left[\frac{1}{2}(\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+) + \sigma_n^z\sigma_{n+1}^z\right]; \end{aligned}$$

here

$$\sigma_n^a = I \otimes \ldots \otimes I \otimes \sigma^a \otimes I \otimes \ldots \otimes I, \quad a = x, y, z,$$

with σ^a acting in the nth space and I being the identity matrix in $\mathbb{C}^2$,

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices, $\sigma_n^{\pm} \equiv \sigma_n^x \pm i\sigma_n^y$ are spin flip operators, and J is a parameter ($J > 0$ corresponding to the ferromagnetic case, and $J < 0$ to the antiferromagnetic one). We assume the periodic boundary conditions, that is, $\sigma_{N+1}^a \equiv \sigma_1^a$, $a = x, y, z$.

The Bethe ansatz is a method for calculating the eigenvalues and eigenfunctions of H.

Observe that the symmetric group $\mathfrak{S}_N$ acts in the space $\mathcal{H}$ by permuting the factors in the tensor product $(\mathbb{C}^2)^{\otimes N}$. Denote this representation of $\mathfrak{S}_N$ by π. By the Schur–Weyl duality, we have

$$\pi = \bigoplus_{\substack{|\mu|=N \\ \ell(\mu)\leqslant 2}} \pi^\mu \otimes \rho^\mu, \tag{1}$$

where π^μ is the irreducible representation of $\mathfrak{S}_N$ corresponding to a Young diagram μ of size $|\mu| = N$ and length (number of rows) $\ell(\mu) \leqslant 2$, and ρ^μ is the irreducible representation of $\mathrm{GL}(2, \mathbb{C})$ corresponding to μ.

Now let s_k be the transposition $(k, k+1)$ (hereafter we always adopt the convention that $N + 1 \equiv 1$), and put

$$L_N = Ne - (s_1 + \ldots + s_N),$$

where e is the identity permutation. Note that the sum contains the $N-1$ Coxeter transpositions s_k, $k = 1, \ldots, N-1$, and the additional transposition $s_N = (N, 1)$ imposed by the periodic boundary conditions. We will call L_N the *periodic Coxeter Laplacian* for the symmetric group $\mathfrak{S}_N$.

Now fix N and denote by L the operator corresponding to L_N in the representation π: $L = \pi(L_N)$. It is easy to check that

$$H = \frac{J}{4}(2L - N).$$

That is, the operators H and L have the same eigenfunctions, and the eigenvalues E_j of H are related to the eigenvalues λ_j of L by

$$E_j = \frac{J}{4}(2\lambda_j - N),$$

or, denoting by $E_0 = -JN/4$ the smallest eigenvalue (the ground energy of the ferromagnetic chain),

$$E_j - E_0 = \frac{J}{2} \cdot \lambda_j.$$

Thus the original problem *reduces to finding the eigenvalues and eigenfunctions of the periodic Coxeter Laplacian in the representation* π.

The very important property of the operators H and L is that they commute with the periodic shift T that sends the kth factor in the tensor product $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ to the $(k+1)$th one, $k = 1, 2, \ldots, N$ (as usual, this means, in particular, that the Nth factor goes to the first one):

$$HT = TH, \qquad HL = LH.$$

Obviously, the eigenvalues of the operator T are the N roots of unity of degree N:

$$\alpha_k = e^{2\pi ik/N}, \qquad k = 0, 1, \ldots, N-1.$$

Let $S^z = \sum_{n=1}^{N} \sigma_n^z$. We have $[H, S^z] = 0$, so this spin is conserved, and we decompose the whole space $\mathcal{H}$ into the sum of subspaces according to the quantum number $S^z = N/2 - r$, where r is the number of down spins. In the decomposition (1), the parameter r corresponds to the length of the second row of the diagram μ.

The case $r = 0$ corresponds to the identity representation π^μ with $\mu = (N)$, so that the only eigenvalue of L is 0 and the corresponding eigenvalue of H is $E_0 = -JN/4$.

3. Quantum inverse scattering method

The quantum inverse scattering method for the XXX Heisenberg model proceeds as follows (see, e.g., [8]).

The local transition matrix is the 2×2 operator-valued matrix of the form

$$\mathcal{L}_n(\lambda) = \begin{pmatrix} \lambda I + \frac{i}{2}\sigma_n^z & \frac{i}{2}\sigma_n^- \\ \frac{i}{2}\sigma_n^+ & \lambda I - \frac{i}{2}\sigma_n^z \end{pmatrix},$$

where λ is a spectral parameter. It satisfies the basic relation

$$R(\lambda - \mu)\,(\mathcal{L}_n(\lambda) \otimes \mathcal{L}_n(\mu)) = (\mathcal{L}_n(\mu) \otimes \mathcal{L}_n(\lambda))\,R(\lambda - \mu) \tag{2}$$

with the R-matrix

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{i}{\lambda+i} & \frac{\lambda}{\lambda+i} & 0 \\ 0 & \frac{\lambda}{\lambda+i} & \frac{i}{\lambda+i} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The monodromy matrix is defined as

$$\mathcal{T}_N(\lambda) = \mathcal{L}_N(\lambda)\ldots\mathcal{L}_1(\lambda).$$

It satisfies the same relation (2). Set

$$\mathcal{T}_N(\lambda) = \begin{pmatrix} A_N(\lambda) & B_N(\lambda) \\ C_N(\lambda) & D_N(\lambda) \end{pmatrix},$$

where the matrix elements $A_n(\lambda), B_N(\lambda), C_N(\lambda), D_N(\lambda)$ of the monodromy matrix act in the quantum space $\mathcal{H}$.

Now set

$$\overline{T}_N(\lambda) = A_N(\lambda) + D_N(\lambda).$$

Our goal in this section is to study the algebra $\mathfrak{T}_N$ generated by the operators $\{\overline{T}_N(\lambda)\}$.

First of all, as follows from the quantum inverse scattering method, all these operators commute:

$$[\overline{T}_N(\lambda), \overline{T}_N(\mu)] = 0.$$

Further, each of them also commutes with the action of $\mathrm{GL}(2,\mathbb{C})$. Hence, by the Schur–Weyl duality, it is an element of the group algebra of the symmetric group $\mathfrak{S}_N$. It is more convenient to introduce the parameter $\mu = -i(\lambda - i/2)$ and consider the operators

$$T_N(\mu) = (-i)^N \overline{T}_N(i\mu + i/2). \tag{3}$$

Lemma 1. *The explicit form of the operators $T_N(\mu)$ is as follows:*

$$T_N(\mu) = \sum_{k=0}^{N} \mu^k T_{N,k}, \tag{4}$$

where

$$T_{N,k} = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_{N-k} \leqslant N} (i_1 i_2 \ldots i_{N-k}) \quad \textit{for } k = 0, \ldots, N-1 \qquad (5)$$

and

$$T_{N,N} = 2e.$$

Proof. We have

$$T_N(\mu) = \operatorname{tr}\left((-i)L_N(i\mu + i/2) \ldots (-i)L_1(i\mu + i/2)\right),$$

and, as can easily be seen,

$$(-i)L_n(i\mu + i/2) = \mu I + \begin{pmatrix} p_n^+ & q_n^- \\ q_n^+ & p_n^- \end{pmatrix},$$

where

$$p_n^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad p_n^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_n^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad q_n^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Denote $M^n = \begin{pmatrix} p_n^+ & q_n^- \\ q_n^+ & p_n^- \end{pmatrix}$.

Let us first consider the case $k = 0$, i.e., prove that $T_{N,0}$ is the periodic shift $(1 \quad 2 \quad \ldots \quad N)$. We have

$$T_{N,0} = \operatorname{tr}\left(M^N \ldots M^1\right) = \sum_{j_1, \ldots, j_{N-1} = 1,2} \Big(M^N_{1 j_{N-1}} M^{N-1}_{j_{N-1} j_{N-2}} \ldots M^2_{j_2 j_1} M^1_{j_1 1}$$
$$+ M^N_{2 j_{N-1}} M^{N-1}_{j_{N-1} j_{N-2}} \ldots M^2_{j_2 j_1} M^1_{j_1 2} \Big).$$

The required assertion now follows from the following simple observation: if j_k is equal to 1 (say), then, on the one hand, the corresponding term applied to a basis vector $|\varepsilon_1 \ldots \varepsilon_N >$ does not vanish only if the kth spin is up (since both operators p_k^+ and q_k^- in the first row of M^k do), and, on the other hand, the $(k+1)$th spin of the *image* of this term is necessarily up, too (since this is so for both operators p_{k+1}^+ and q_{k+1}^+ in the first column of M^{k+1}).

Now, to prove the assertion for arbitrary k, observe that

$$T_{N,k} = \sum_{1 \leqslant j_1 < j_2 < \ldots < j_{N-k} \leqslant N} \operatorname{tr} \left(\begin{pmatrix} p^+_{j_1} & q^-_{j_1} \\ q^+_{j_1} & p^-_{j_1} \end{pmatrix} \cdots \begin{pmatrix} p^+_{j_{N-k}} & q^-_{j_{N-k}} \\ q^+_{j_{N-k}} & p^-_{j_{N-k}} \end{pmatrix} \right).$$

For $k > 0$, each term in this sum is exactly the operator $T_{N-k,0}$ for the model restricted to the sites $i_1, \ldots, i_{N-k}$, so, as we have already proved, it is equal to $(i_1 i_2 \ldots i_{N-k})$. The lemma follows. □

Thus, in particular,

$$T_{N,0} = T_N(0) = (\,1 \quad 2 \quad \ldots \quad N\,) = T$$

is the periodic shift,

$$\begin{aligned} T_{N,1} = (\,1 \quad 2 \quad \ldots \quad N-1\,) + (\,1 \quad 2 \quad \ldots \quad N-2 \quad N\,) \\ + \ldots + (\,2 \quad 3 \quad \ldots \quad N\,) \end{aligned}$$

is the sum of the N cycles of length $N-1$ obtained by deleting one element from the cycle $T = (\,1 \quad 2 \quad \ldots \quad N\,)$. Note that

$$T_{N,1} = T \cdot (s_1 + \cdots + s_N), \tag{6}$$

where $s_1 + \cdots + s_N = Ne - L_N$ is the (periodic) sum of Coxeter transpositions. Also, we have that

$$T_{N,N-2} = \sum_{i<j} (ij)$$

is the sum of all transpositions in $\mathfrak{S}_N$ and

$$T_{N,N-1} = Ne.$$

Finally, the periodic Coxeter Laplacian L_N is essentially the logarithmic derivative of $T_N(\mu)$ at $\mu = 0$:

$$\frac{d}{d\mu} \log T_N(\mu) \bigg|_{\mu=0} = T_{N,1} \cdot T_{N,0}^{-1} = T \cdot (s_1 + \ldots + s_N) \cdot T^{-1} = s_1 + \ldots + s_N,$$

so that

$$L_N = Ne - \frac{d}{d\mu} \log T_N(\mu) \bigg|_{\mu=0}. \tag{7}$$

Thus the algebra $\mathfrak{T}_N$ *is the commutative subalgebra of* $\mathbb{C}[\mathfrak{S}_N]$ *generated by the elements* $e, T_{N,0}, T_{N,1}, \ldots, T_{N,N-2}$. Alternatively, we can consider the generators

$$R_{N,k} = T_{N,k} \cdot T_{N,0}^{-1} = \sum_{i_1 < \ldots < i_k} s_{i_1} \ldots s_{i_k}, \quad k = 0, 1, \ldots, N-1,$$

where $s_i = (i, i+1)$ is a Coxeter generator (with $s_N = (N, 1)$) and the inequalities are understood with respect to the cyclic order (which does not lead to any ambiguity, since nonneighbor Coxeter generators commute).

The algebra $\mathfrak{T}_N$ was also independently studied in [7].

Now let $p_{N+1,N}$ be the canonical (virtual) projection from the group algebra of $\mathfrak{S}_{N+1}$ to the group algebra of $\mathfrak{S}_N$; recall that for $g \in \mathfrak{S}_{N+1}$ the projection $p_{N+1,N}g$ is obtained by deleting the element $N+1$ from the corresponding cycle of g. The following lemma can be proved directly using the explicit form of the operators $T_N(\mu)$.

Lemma 2. *We have*

$$p_{N+1,N} T_{N+1}(\mu) = (1+\mu) T_N(\mu) - \mu^N e. \tag{8}$$

In other words, if we introduce the operators $\tilde{T}_N(\mu) = T_N(\mu) - \mu^N e$ *(taking* $T_{N,n} = e$ *instead of* $2e$*), then*

$$p_{N+1,N} \widetilde{T}_{N+1}(\mu) = (1+\mu) \widetilde{T}_N(\mu).$$

4. Exact solutions

4.1. The case $r = 1$

Here the problem is to find the eigenvalues of the peridodic Coxeter Laplacian in the irreducible representation π^μ of $\mathfrak{S}_N$ for the Young diagram $\mu = (N-1, 1)$. It is more convenient to turn to the representation ϱ^1 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_1 \times \mathfrak{S}_{N-1}$, which, as is well known, is the sum of π^μ and the identity representation.

The representation ϱ^1 is the natural representation of $\mathfrak{S}_N$ in $\mathbb{C}^N$, which we realize as the space of functions $f : \{1, \ldots, N\} \to \mathbb{C}$, so that the matrix of L has the form

$$-A = -A_N^{(1)} = -\begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 \end{bmatrix} \tag{9}$$

(an almost Jacobi matrix, with two extra 1's at the north-east and south-west corners). The eigenvalues of L can be found in several ways.

The first method is to express the characteristic polynomial $P_N(x)$ of the matrix $-A+2I$ through Chebyshev polynomials U_N of the second kind, using the well-known formula

$$U_N(x) = \det \begin{bmatrix} 2x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & 2x & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2x \end{bmatrix} \quad (N \times N \text{ matrix}).$$

In this way, we obtain

$$P_N(-x) = U_N(x/2) - U_{N-2}(x/2) - 2(-1)^N = 2(T_N(x/2) - (-1)^N),$$

where T_N is a Chebyshev polynomial of the first kind, i.e., using the fact that T_N is odd for odd N and even for even N,

$$P_N(x) = (-1)^N \cdot 2(T_N(x/2) - 1).$$

The roots of P_N are thus $2\cos(2\pi k/N)$, $k = 0, 1, \dots, N-1$, and the eigenvalues of L are $2(1-\cos(2\pi k/N))$, $k = 0, 1, \dots, N-1$.

But the easiest way is to use the invariance of L under the shift T, which means that $\mathbb{C}^N$ decomposes into the eigenspaces H_α indexed by the eigenvalues α of T. If $v \in H_\alpha$ is an eigenvector from H_α, then, denoting $f(1) = x$, we have $f(k) = \alpha^{k-1}x$, and the equation for the eigenvalues of L takes the form

$$\lambda = 2 - \alpha - \alpha^{-1},$$

whence

$$\lambda_k = 2(1-\cos(2\pi k/N)), \quad k = 0, 1, \dots, N-1. \tag{10}$$

In particular, we see that all eigenvalues lie in the interval $[0,4]$ for every N, and it is easy to show by standard arguments that the limiting distribution of the eigenvalues on this interval has the density

$$p^{(1)}(x) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{x(4-x)}}, \tag{11}$$

which is just a linear transformation of the so-called Chebyshev density $\frac{1}{\pi} \cdot \frac{1}{\sqrt{1-x^2}}$, $x \in [-1,1]$, which is the limiting density of the roots of a large class of orthogonal polynomials including the Chebyshev ones.

4.2. The case $r = 2$

As in the previous section, we will consider the representation ϱ^2 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_2 \times \mathfrak{S}_{N-2}$, which is the sum of $\pi^{(N-2,2)}$, $\pi^{(N-1,1)}$, and the identity representation $\mathbf{1}$:

$$\varrho^2 = \pi^{(N-2,2)} + \pi^{(N-1,1)} + \mathbf{1} = \pi^{(N-2,2)} + \varrho^1. \tag{12}$$

This representation is realized in the linear space $\mathcal{H}^{(2)}$ with basis consisting of all (unordered) pairs (kl) of distinct numbers with $k, l = 1, \ldots, N$, $k \neq l$. Again, this space decomposes into the sum of eigenspaces of T:

$$\mathcal{H}^{(2)} = \sum_{\alpha:\, \alpha^N = 1} H_\alpha,$$

and we will find the eigenvalues of L in each H_α.

First assume for simplicity that $N = 2m+1$ is odd, $m \geqslant 2$. Under the action of the shift T, the basis of $\mathcal{H}^{(2)}$ splits into m orbits of length N:

$$\begin{array}{lccccccccc}
\Omega_1: & 12 & \to & 23 & \to & \ldots & \to & N-1, N & \to & 1N, \\
\Omega_2: & 13 & \to & 24 & \to & \ldots & \to & 1, N-1 & \to & 2N, \\
 & & & & & \ldots & & & & \\
\Omega_m: & 1, m+1 & \to & 2, m+2 & \to & \ldots & \to & m-1, N-1 & \to & mN,
\end{array}$$

and for each eigenfunction $f \in H_\alpha$, the value of f at the jth element of Ω_k is equal to $\alpha^{j-1} f(1, k+1)$, $k = 1, \ldots, m$, $j = 1, \ldots, N$. Writing down the equations for an eigenvalue λ of L, we obtain

$$\begin{aligned}
\lambda x_1 &= 2x_1 - (1+\alpha^{-1})x_2, \\
\lambda x_k &= -(1+\alpha)x_{k-1} + 4x_k - (1+\alpha^{-1})x_{k+1}, \quad k = 2, \ldots, m-1, \\
\lambda x_m &= -(1+\alpha)x_{m-1} + (4 - \alpha^m - \alpha^{-m})x_m.
\end{aligned}$$

Thus the eigenvalues of L in H_α are the eigenvalues of the $m \times m$ three-diagonal matrix

$$A_m = A_m(\alpha) = \begin{bmatrix}
2 & a & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
b & 4 & a & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & b & 4 & a & 0 & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & b & 4 & a \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & b & c
\end{bmatrix},$$

where $a = -(1+\alpha^{-1})$, $b = -(1+\alpha)$,

$$\begin{aligned} c &= 4 - \alpha^m - \alpha^{-m} = 4 - 2\cos(2\pi km/N) = 4 - 2\cos(2\pi k(N-1)/(2N)) \\ &= 4 - 2(-1)^k \cos(\pi k/N). \end{aligned}$$

Denote by $P_m(x)$ the characteristic polynomial of the matrix A_m. By the standard methods we can deduce a recurrence relation for $P_m(x)$:

$$P_m(x) = (4-x)P_{m-1}(x) - abP_{m-2}(x), \tag{13}$$

where

$$ab = (1+\alpha^{-1})(1+\alpha) = 2(1+\cos(2\pi k/N)) = 4\cos^2(\pi k/N).$$

Originally, the polynomials P_m are defined for $m \geqslant 2$, but we can define them for $m = 0, 1$ so as to satisfy the recurrence relation (13). Namely,

$$\begin{aligned} P_0(x) &= 1 + \frac{2(c-4)}{ab}, \\ P_1(x) &= c - 2 - x. \end{aligned}$$

For $m = 2, 3$, we have

$$\begin{aligned} P_2(x) &= (x-2)(x-c) - ab, \\ P_3(x) &= -(x-2)(x-4)(x-c) + ab(2x-2-c). \end{aligned}$$

It is not difficult to compute, using the recurrence relation (13), the generating function for $P_m(x)$:

$$\sum_{k=0}^{n} P_k(x)t^k = \frac{1 + \frac{2(c-4)}{ab} - t\left(-\frac{2(c-4)}{ab}x + 6 - c + \frac{8(c-4)}{ab}\right)}{1 - (4-x)t + t^2 ab}.$$

This allows us to write $P_m(x)$ in terms of Chebyshev polynomials of the second kind:

$$\frac{P_m(4 - 2x\sqrt{ab})}{(\sqrt{ab})^m} = U_m(x) + \frac{c-6}{\sqrt{ab}}U_{m-1}(x) + \frac{2(4-c)}{ab}U_{m-2}(x).$$

Since

$$\frac{c-6}{\sqrt{ab}} = -\frac{1+(-1)^k\cos(\pi k/N)}{\cos(\pi k/N)}, \qquad \frac{2(4-c)}{ab} = \frac{(-1)^k}{\cos(\pi k/N)},$$

we obtain

$$\begin{aligned}\frac{P_m(4(1-\cos\theta\cdot x))}{(2\cos\theta)^m} =& U_m(x)+(-1)^{k+1}U_{m-1}(x)\\ &-\frac{1}{\cos\theta}\left(U_{m-1}(x)+(-1)^{k+1}U_{m-2}(x)\right),\end{aligned} \tag{14}$$

where $\theta = \pi k/N$. Thus

$$\frac{P_m(4(1-\cos\theta\cdot x))}{(2\cos\theta)^m} = \begin{cases} V_m(x)-\frac{1}{\cos\theta}V_{m-1}(x), & k \text{ is even,}\\ W_m(x)-\frac{1}{\cos\theta}W_{m-1}(x), & k \text{ is odd,}\end{cases} \tag{15}$$

where $V_k(x)$ and $W_k(x)$ are Chebyshev polynomials of the third and fourth kind, respectively (see [6]).

Remark. For an odd k, setting $x = \cos t$ and using the trigonometric definition of Chebyshev polynomials, we obtain

$$\frac{\cos\theta\cdot\sin((2m+1)t/2)-\sin((2m-1)t/2)}{\sin(t/2)} = 0. \tag{16}$$

It is not difficult to transform this equation into

$$\frac{\operatorname{tg}(mt)}{\operatorname{tg}(t/2)} = \operatorname{ctg}^2(\theta/2). \tag{17}$$

In a similar way, for an even k we obtain

$$\frac{\cos\theta\cdot\cos((2m+1)t/2)-\cos((2m-1)t/2)}{\cos(t/2)} = 0, \tag{18}$$

or

$$\operatorname{tg}(mt)\operatorname{tg}(t/2) = -\operatorname{tg}^2(\theta/2). \tag{19}$$

Thus the eigenvalues of the periodic Coxeter Laplacian L in the sector H_α corresponding to an eigenvalue $\alpha = e^{2\pi ik/N}$ of the shift T are the solutions of (17) for even k and the solutions of (19) for odd k.

Obviously, the solutions for k and $-k$ coincide, so it suffices to consider $k = 0, 1, \ldots, m$, keeping in mind that the eigenvalues obtained for $k = 1, \ldots, m$ should be counted twice.

Example. Consider the simplest case of $k = 0$, i. e., $\alpha = 1$. In this case, $a = b = c = 2$, so that we obtain that the characteristic polynomial $P_m(x)$ of the matrix $A_m(0)$ is a multiple of the Chebyshev polynomial:

$$P_m(x) = -2^{m-1} x U_{m-1}\left(1 - \frac{x}{4}\right).$$

The corresponding eigenvalues of L are

$$0 \quad \text{and} \quad 4(1 - \cos\frac{\pi j}{m}), \quad j = 1, \ldots, m-1.$$

Example. Since the representation ϱ^2 contains ϱ^1 (see (12)), the eigenvalues of L in $\mathcal{H}^{(2)}$ must contain, in particular, the eigenvalues (10). More precisely, λ_k must be an eigenvalue of L in the sector H_α with $\alpha = e^{2\pi i k/N}$. Let us check that λ_k is indeed a root of P_m given by (14) for $\theta = \pi k/N$. From the equation $4 - 4\cos\theta \cdot x = 2 - 2\cos 2\theta$ we obtain $x = \cos\theta$, so that in (17) and (19) we have $t = \theta$ and these equations are easy to verify.

Since V_j and W_j form families of orthogonal polynomials on the interval $[-1, 1]$, we easily obtain from the properties of roots of orthogonal polynomials that the right-hand side of (15) has $m-1$ roots inside the interval $[-1, 1]$ and one root lying outside this interval. More precisely, if k is odd and $\cos\theta > 0$, there is a root of the right-hand side of (15) between the jth root $x_j = \cos\frac{\pi(m-j+1)}{m+1/2}$ of W_m and the jth root $y_j = \cos\frac{\pi(m-j)}{m-1/2}$ of W_{m-1}, $j = 1, \ldots, m-1$, and similarly in the other cases.

Using all these facts and the properties of orthogonal polynomials, with some tedious calculations one can obtain the limiting density of eigenvalues of the Coxeter Laplacian in the representation ϱ^2:

$$p^{(2)}(u) = \frac{1}{2\pi^2} K'\left(1 - \frac{u}{4}\right), \quad u \in [0, 8], \tag{20}$$

where $K'(v) = K(\sqrt{1 - v^2})$ is the complete elliptic integral of the first kind.

It is not difficult to check that the "two-magnon" limiting density $p^{(2)}(u)$ is the convolution of two "one-magnon" densities $p^{(1)}(u)$:

$$p^{(2)}(u) = \int p^{(1)}(u - x) p^{(1)}(x)\, dx.$$

In the next section we will prove that this convolution formula holds for all r, that is, the limiting density of eigenvalues of the periodic Coxeter Laplacian in the representation induced from the identity representation of $\mathfrak{S}_r \times \mathfrak{S}_{N-r}$ is the r-fold convolution of $p^{(1)}$.

5. Asymptotic results. Ferromagnetic case

In this section, we find the limiting distribution of eigenvalues of the periodic Coxeter Laplacian L_N in the representation induced from the identity representation of $\mathfrak{S}_r \times \mathfrak{S}_{N-r}$ when r is fixed and N goes to infinity. In the framework of the XXX Heisenberg model, this asymptotic mode corresponds to considering excitations of the *ferromagnetic* chain ($J > 0$).

5.1. Some facts on the limiting distribution of eigenvalues

Given an $n \times n$ matrix $A = \{a_{ij}\}$, denote by $\|A\|$ its spectral norm

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

(here $\|x\|$ is the Euclidean norm of a vector x) and by $|A|$ its normalized Frobenius (Hilbert–Schmidt) norm

$$|A|^2 = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2.$$

Recall the useful inequality

$$\|A\| \leqslant \sqrt{\|A\|_1 \|A\|_\infty}, \tag{21}$$

where $\|A\|_1 = \max_j \sum\limits_{i=1}^{n} |a_{ij}|$ and $\|A\|_\infty = \max_i \sum\limits_{j=1}^{n} |a_{ij}|$.

Denote by $\lambda_k(A)$ the kth largest eigenvalue of A.

Now, let C_n and D_n be $n \times n$ matrices. Let us say that the sequences $\{C_n\}$ and $\{D_n\}$, $n = 1, 2, \ldots$, are equivalent if the following two conditions hold:

$$\exists M < \infty \text{ such that } \forall n,\ \|C_n\|, \|D_n\| \leqslant M; \tag{22}$$

$$\lim_{n\to\infty} |C_n - D_n| = 0. \tag{23}$$

We denote this fact by $\{C_n\} \sim \{D_n\}$. Note that (22) implies that the spectra of all the matrices C_n and D_n are uniformly bounded, i.e., there exists a bounded interval $[m, M]$ such that $\lambda_k(C_n), \lambda_k(D_n) \in [m, M]$ for all n, k.

The following result borrowed from [4] essentially relies on the well-known Wielandt–Hoffman theorem (see, e.g., [11]).

Lemma 3. *Let $\{C_n\}$ and $\{D_n\}$ be two equivalent sequences of Hermitian matrices. Then the sequences of their eigenvalues are asymptotically absolutely equally distributed, i.e., for an arbitrary continuous function $f(x)$ on $[m, M]$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |f(\lambda_k(C_n)) - f(\lambda_k(D_n))| = 0.$$

In particular, it follows that if either of the limits exists separately, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\lambda_k(C_n)) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\lambda_k(D_n)). \tag{24}$$

5.2. The $r = 1$ case

Now consider the $r = 1$ case, i.e., the representation ϱ^1 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_1 \times \mathfrak{S}_{N-1}$. Note that the matrix (9) of the Coxeter Laplacian L in this representation is almost a three-diagonal Toeplitz matrix, namely, differs from the three-diagonal Toeplitz matrix

$$B_N = \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \tag{25}$$

only by two extra minus ones in the north-east and south-west corners. However, as long as we are interested in the limiting distribution of eigenvalues, this small difference is irrelevant, because it is easy to check that the sequences of real symmetric matrices $\{A_N\}$ and $\{B_N\}$ are equivalent (indeed, (22) easily follows from the inequality (21), and, obviously, $|A_N - B_N|^2 \leqslant \frac{2}{N}$). Thus Lemma 3 applies, and we deduce that these sequences of matrices have the same limiting distribution of eigenvalues. But for B_N this distribution is given by a well-known result from the theory of Toeplitz matrices. Namely, given a sequence $\{t_k\}$ (for our purposes, it suffices to consider only finite sequences, $t_k = 0$ for $|k| > m$, corresponding to banded Toeplitz matrices), consider a sequence of Toeplitz matrices

$T_n = (t_{j-k})_{k,j=1,\dots,n}$. Let $a(\lambda) = \sum_{k=-\infty}^{\infty} t_k e^{ik\lambda}$ be the corresponding symbol. If a is real, i.e., the matrices $T_n = T_n(a)$ are symmetric, then for any continuous function f,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\lambda_k(T_n)) = \frac{1}{2\pi} \int_0^{2\pi} f(a(\lambda))\, d\lambda. \tag{26}$$

In our case, $B_N = T_N(a)$, where

$$a(\lambda) = 2 - 2\cos\lambda, \tag{27}$$

so that applying (26) and making an appropriate change of variables, we recover the Chebyshev density (11).

5.3. General case

For clarity, consider first the $r = 2$ case, i.e., the representation ϱ^2 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_2 \times \mathfrak{S}_{N-2}$. Denote by e_{kl}, $1 \leqslant k < l \leqslant N$, the natural basis of the space $\mathcal{H}^{(2)}$ of this representation. It is easy to see that the action of the periodic Coxeter Laplacian L in terms of this basis reads as follows (with $N+1 \equiv 1$):

$$Le_{kl} = \begin{cases} 4e_{kl} - e_{k-1,l} - e_{k+1,l} - e_{k,l-1} - e_{k,l+1}, & l \neq k+1, \\ 2e_{k,k+1} - e_{k-1,k+1} - e_{k,k+2}, & l = k+1. \end{cases}$$

Denote the corresponding matrix by $A_N^{(2)}$.

Let $\mathcal{H}^{(1)}$ be the space of the representation ϱ^1, consider the space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$, and the operator $B_N^{(2)} = A_N^{(1)} \otimes I + I \otimes A_N^{(1)}$ in this space. Clearly, the eigenvalues of $B_N^{(2)}$ are exactly the sums $\lambda + \mu$, where λ, μ are the eigenvalues of $A_N^{(1)}$, so that the limiting distribution of eigenvalues of $B_N^{(2)}$ as $N \to \infty$ is the convolution of two copies of $p^{(1)}$.

Now we may assume that $\mathcal{H}^{(2)}$ is the subspace of $\mathcal{H}$ spanned by e_{kl} with $k < l$. We may also extend the action of L to the whole $\mathcal{H}$ putting $Le_{kl} = Le_{lk}$ for $k > l$ and $Le_{kk} = 0$. Denote the corresponding matrix by $\tilde{A}_N^{(2)}$. Obviously, the limiting distribution of eigenvalues for $\tilde{A}_N^{(2)}$ is the same as for $A_N^{(2)}$.

Lemma 4. *The sequences of matrices* $\{\tilde{A}_N^{(2)}\}$ *and* $\{B_N^{(2)}\}$ *are equivalent, i.e., satisfy the conditions of Lemma* 3.

Proof. Obviously, $\|\tilde{A}_N^{(2)}\| \leqslant (\|\tilde{A}_N^{(2)}\|_1 \|\tilde{A}_N^{(2)}\|_\infty)^{1/2} \leqslant 8$ and a similar bound holds for $B_N^{(2)}$, so that the first condition of equivalence is satisfied. Now, the order of these matrices is N^2 and

$$(B_N^{(2)} - \tilde{A}_N^{(2)}) e_{kl} = \begin{cases} 0, & |l-k|>1, \\ 2e_{kl} - e_{kk} - e_{ll}, & l = k \pm 1, \\ 4e_{kk} - e_{k-1,k} - e_{k+1,k} - e_{k,k-1} - e_{k+1}, & l = k, \end{cases}$$

so that $B_N^{(2)} - \tilde{A}_N^{(2)}$ has at most $O(N)$ nonzero entries, each being $O(1)$. It follows that $|B_N^{(2)} - \tilde{A}_N^{(2)}| = \frac{1}{N^2} \cdot O(N) \cdot O(1) \to 0$ as $N \to \infty$, and the second condition is also satisfied. □

Corollary 1. *The limiting density* $p^{(2)}$ *of eigenvalues of the periodic Coxeter Laplacian* L *in the* $r = 2$ *sector is the convolution of two copies of the limiting density* $p^{(1)}$ *of the eigenvalues of* L *in the* $r = 1$ *sector:*

$$p^{(2)}(u) = (p^{(1)} * p^{(1)})(u), \qquad u \in [0, 8].$$

In exactly the same way one can prove the following general result.

Theorem 1. *The limiting density* $p^{(k)}$ *of eigenvalues of the periodic Coxeter Laplacian* L_N *in the representation* ϱ_N^k *induced from the identity representation of the Young subgroup* $\mathfrak{S}_k \times \mathfrak{S}_{N-k}$ *as* $N \to 0$ *is the convolution of* k *copies of the limiting density* $p^{(1)}$ *of the eigenvalues of* L_N *in the representation* ϱ_n^1*:*

$$p^{(k)}(u) = (\underbrace{p^{(1)} * \ldots * p^{(1)}}_{k})(u), \qquad u \in [0, 4k]. \tag{28}$$

Corollary 2. *For a fixed* $k = 1, 2, \ldots$, *the limiting distribution of eigenvalues of the periodic Coxeter Laplacian* L_N *in the* irreducible *representation corresponding to the two-row Young diagram* $\mu_N^k = (N-k, k)$ *as* $N \to \infty$ *is equal to* (28).

Proof. Follows from the fact that the relative dimension of μ_N^k in ϱ^k tends to 1 as $N \to \infty$:

$$\frac{\dim \mu_N^k}{\dim \varrho_N^k} = \frac{N!(N-2k+1)}{k!(N-k+1)!} \cdot \frac{k!(N-k)!}{N!} = \frac{N-2k+1}{N-k+1} \to 1. \qquad \square$$

Remarks. 1. In terms of the XXX Heisenberg model, the result of Theorem 1 means that in the limit under consideration, only eigenvalues corresponding to "independent magnons" survive. The number of eigenvalues corresponding to "bound magnons" is asymptotically negligible and does not affect the limiting density.

2. As follows from the proof, the limiting density of eigenvalues for the periodic Coxeter Laplacian is the same as for the *ordinary* (nonperiodic) Coxeter Laplacian $Ne-(1,2)-(2,3)-\ldots-(N-1,N)$.

Taking into account the convolution formula (28), it is natural to use the Fourier transform for finding the limiting density $p^{(k)}$. Let $F(t)$ be the Fourier transform of $p^{(1)}$. Then $p^{(k)}$ is the inverse Fourier transform of $F(t)^k$. But (see, e.g., [3, 3.387.2])

$$F(t)=\frac{1}{\pi}\int\limits_0^4 \frac{e^{-itx}}{\sqrt{x(4-x)}}\,dx=e^{-2it}J_0(-2t),$$

where J_0 is the Bessel function of the first kind. Thus we are interested in the inverse Fourier transforms of powers of Bessel functions. For $k=2$, using the known formulas for integral transforms of the product of tho Bessel functions, we can recover the density (20). Unfortunately, for $k>2$, the corresponding integrals are not known. For Taylor series expansions of powers of Bessel functions, see [2].

5.4. Limiting operators

Denote by $A_N^{(k)}$ the operator of the Coxeter Laplacian L in the representation $\varrho_N^{(k)}$ of the symmetric group $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_k\times\mathfrak{S}_{N-k}$. The inductive limit of the representations $\varrho_N^{(k)}$ as $N\to\infty$ is the irreducible representation $\rho^{(k)}$ of the infinite symmetric group $\mathfrak{S}_\infty$ induced from the identity representation of the Young subgroup $\mathfrak{S}_{\{1,\ldots,k\}}\times\mathfrak{S}_{\{k+1,k+2,\ldots\}}$ (see [9]).

Lemma 5. *The operators $A_N^{(k)}$ weakly converge as $N\to\infty$ to some operator $A^{(k)}$ in the space of the representation $\rho^{(k)}$. In particular, $A^{(1)}=T(a)$ is the infinite Toeplitz matrix with the symbol $a(p)=2-2\cos p$.*

Proof. The space of the representation $\rho^{(k)}$ is $l^2(\Pi_k)$, where Π_k is the set of k-tuples of distinct positive integers. Denote by $H_m\subset l^2(\Pi_k)$ the subset of functions supported by k-tuples of integers $\leqslant m$. Then the set $\cup_{m=1}^\infty H_m$

is dense in $l^2(\Pi_k)$, and it suffices to check that the limits $\lim_{N\to\infty}(A_N^{(k)} f, g)$ exist for $f, g \in H_m$ for all m. But it is obvious that for such f, g we have $((k, k+1)f, g) = (f, g)$ for $k > m$ and $((1, N)f, g) = 0$. Thus the required limits exist, and we see that the operators $A_N^{(k)}$ weakly converge to some operator $A^{(k)}$ such that for $f \in H_m$,

$$A^{(k)} f = \big((m+1)E - \big((1,2) + \ldots + (m, m+1)\big)\big) f. \qquad \square$$

Remark. We have shown that the weak limit of the operators $A_N^{(1)}$ of the periodic Coxeter Laplacian of $\mathfrak{S}_N$ in the "one-row" representation ϱ_N^1 is the infinite Toeplitz operator with the symbol $a(p) = 2 - 2\cos p$. It follows from the spectral theory of Toeplitz operators that this operator is unitarily equivalent to the multiplication by x in the space

$$L^2\left([0,4],\, \frac{1}{4}\sqrt{x(4-x)}\right).$$

Recall that the limiting density of eigenvalues in this asymptotic mode is

$$\frac{1}{\pi}\frac{1}{\sqrt{x(4-x)}}, \quad x \in [0,4].$$

5.5. Antiferromagnetic case

We may also consider another asymptotic mode, namely, assuming for simplicity that $N = 2n$ is even, consider the operator L_N in the representation ϱ_N^n induced from the identity representation of $\mathfrak{S}_n \times \mathfrak{S}_n$ as $N \to \infty$ and in the irreducible representation π_{μ_N} with diagram $\mu_N = (n, n)$. This mode corresponds to considering the antiferromagnetic XXX chain. The results for this case will be presented in a subsequent paper.

5.6. Applications: characters of the symmetric groups

Let π be a representation of the symmetric group $\mathfrak{S}_N$ (and its group algebra), χ be its character, and $M = \dim \pi$ be its dimension. Let $\lambda_1, \ldots, \lambda_M$ be the eigenvalues of the periodic Coxeter Laplacian in the representation π. Then it follows from the results of [1] that for every positive integer k

$$\sum_{j=1}^{M} \lambda_j^k = \sum_{j_1=1}^{N} \cdots \sum_{j_k=1}^{N} \chi\big((e - \sigma_{j_1})\ldots(e - \sigma_{j_k})\big). \tag{29}$$

Combining (29) with the results on the limiting densities of eigenvalues obtained in the previous sections, we can derive some interesting asymptotic formulas.

First let π be the representation ϱ^1 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_1\times\mathfrak{S}_{N-1}$. It is easy to see that its dimension is equal to N and the value of its character at a permutation $g\in\mathfrak{S}_N$ equals the number $r_1(g)$ of fixed points of g. On the other hand, from the results of Sec. 4.1 we have

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\lambda_j^k=\frac{1}{\pi}\int\limits_0^4 x^k\frac{dx}{\sqrt{x(4-x)}}=\frac{2^k(2k-1)!!}{k!}.$$

Thus for every positive integer k,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j_1=1}^{N}\cdots\sum_{j_k=1}^{N} r_1\big((e-\sigma_{j_1})\ldots(e-\sigma_{j_k})\big)=\frac{2^k(2k-1)!!}{k!}. \tag{30}$$

Examples. Let $k=1$. Then $r_1(e-\sigma)=r_1(e)-r_1(\sigma)=n-(n-2)=2$, and the left-hand side of (30) equals $\frac{1}{N}\cdot N\cdot 2=2$, which is the right-hand side for $k=1$.

Now let $k=2$. Then $r_1((e-\sigma_{j_1})(e-\sigma_{j_2}))=r_1(e)-2r_1(\sigma)+r_1(\sigma_{j_1}\sigma_{j_2})$. But $r_1(\sigma_{j_1}\sigma_{j_2})$ is equal to N if $j_1=j_2$, equal to $N-4$ if the (periodic) distance between j_1 and j_2 is at least two, and equal to $N-3$ if j_1 and j_2 are neighbors, whence the left-hand side of (30) equals

$$\frac{1}{N}\left[N^2(N-2(N-2))+N^2+N(N-3)(N-4)+2N(N-3)\right]=6,$$

again in accordance with the right-hand side for $k=2$.

Consider now the representation ϱ^2 of $\mathfrak{S}_N$ induced from the identity representation of the Young subgroup $\mathfrak{S}_2\times\mathfrak{S}_{N-2}$. In this case, the dimension is equal to $N(N-1)/2$ and the value of the corresponding character at a permutation $g\in\mathfrak{S}_N$ equals $\binom{r_1(g)}{2}+r_2(g)$ where $r_1(g)$ and $r_2(g)$ are the number of fixed points of g and the number of cycles of length 2 in g, respectively. On the other hand, from the results of Sec. 4.2 we have

$$\begin{aligned}\lim_{N\to\infty}\frac{2}{N(N-1)}\sum_{j=1}^{N}\lambda_j^k&=\frac{1}{2\pi^2}\int\limits_0^8 x^kK'(1-x/4)\,dx\\&=\frac{2\cdot 4^k}{\pi^2}\int\limits_{-1}^{1}(1-y)^kK'(y)\,dy.\end{aligned} \tag{31}$$

Examples. Let $k = 1$. Then the integral in the right-hand side of (31) equals

$$\int_{-1}^{1} (1-y)K'(y)\,dy = \int_{-1}^{1} K'(y)\,dy - \int_{-1}^{1} yK'(y)\,dy,$$

the first summand being equal to $\pi^2/2$ by [3, 6.141.2] and the second one vanishing since $K'(y)$ is an even function of y. That is, the right-hand side is equal to 4. Now, the left-hand side equals

$$\frac{2}{N(N-1)} \cdot N \cdot \left[\frac{N(N-1)}{2} - \left(\frac{(N-2)(N-3)}{2} + 1\right)\right] = \frac{4N-8}{N(N-1)} \to 4.$$

For $k = 2$, using again the evenness of K', we have

$$\int_{-1}^{1} (1-y)^2 K'(y)\,dy = \int_{-1}^{1} K'(y)\,dy + \int_{-1}^{1} y^2 K'(y)\,dy = \frac{\pi^2}{2} + \frac{\pi^2}{8},$$

where the second integral was find using [3, 6.147 and 6.148.2]. Thus the right-hand side of (31) is equal to 20. Now, the left-hand side equals

$$\begin{aligned} \frac{2}{N(N-1)} \biggl[\frac{N^2 \cdot N(N-1)}{2} &- 2N^2 \left(\frac{(N-2)(N-3)}{2} + 1\right) \\ &+ N \cdot \frac{N(N-1)}{2} + N(N-3)\left(\frac{(N-4)(N-5)}{2} + 2\right) \\ &\qquad + N \cdot 2 \cdot \frac{(N-3)(N-4)}{2} \biggr] = \frac{20N-48}{N(N-1)} \to 20. \end{aligned}$$

References

1. L. Babai, *Spectra of Cayley graphs.* — J. Combin. Theory B **27** (1979), 180–189.
2. C. M. Bender, D. C. Brody, B. K. Meister, *Powers of Bessel functions.* — J. Math. Phys. **44** (2003), No. 1, 309–314.
3. I. S. Gradshtein, I. M. Ryzhik, *Table of Integrals, Series, and Products.* Academic Press, New York–London, 1965.
4. R. M. Gray, *Toeplitz and Circulant Matrices: A Review.* Now Publishers, Norwell, Massachusetts.
5. Yu. A. Izyumov, Yu. N. Skryabin, *Statistical Mechanics of Magnetically Ordered Systems.* Consultants Bureau, New York, 1988.
6. J. C. Mason, D. C. Handscomb, *Chebyshev Polynomials.* Chapman & Hall/CRC, Boca Raton, Florida, 2003.

7. E. Mukhin, V. Tarasov, A. Varchenko, *Bethe algebra of homogeneous XXX Heisenberg model has simple spectrum.* — Comm. Math. Phys. **288** (2009), No. 1, 1–42.
8. L. A. Takhtadzhyan, L. D. Faddeev, *The spectrum and scattering of excitations in the one-dimensional isotropic Heisenberg model.* — Zap. Nauchn. Semin. LOMI **109** (1981), 134–178.
9. N. V. Tsilevich, A. M. Vershik, *Induced representations of the infinite symmetric group.* — Pure Appl. Math. Quart. **3** (2007), No. 4, 1005–1026.
10. A. M. Vershik, S. V. Kerov, *Asymptotic theory of the characters of a symmetric group.* — Funkts. Anal. i Prilozhen. **15** (1981), No. 4, 15–27.
11. J. H. Wilkinson, *The Algebraic Eigenvalue Problem.* Clarendon Press, Oxford, 1965.

St. Petersburg Department of Steklov
Institute of Mathematics, Fontanka 27,
191023 St. Petersburg, Russia

E-mail: natalia@pdmi.ras.ru

Поступило 12 сентября 2010 г.