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STATIONARY RANDOM PARTITIONS OF POSITIVE INTEGERS*

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(*Translated by A. Rukhin*)

Abstract. This paper gives a description of stationary random partitions of positive integers (equivalently, stationary coherent sequences of random permutations) under the action of the infinite symmetric group. Equivalently, all stationary coherent sequences of random permutations are described. This result gives a new characterization of the Poisson–Dirichlet distribution PD(1) with the unit parameter, which turns out to be the unique invariant distribution for a family of Markovian operators on the infinite-dimensional simplex. This result also provides a new characterization of the Haar measure on the projective limit of finite symmetric groups.

Key words. random partitions, random permutations, stationary distribution, Markovian operator, Poisson–Dirichlet distribution

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1. The setting of the problem and the formulation of the results. The classical de Finetti theorem describes all *exchangeable* sequences of scalar random variables, i.e., random sequences whose joint distributions are invariant under finite permutations. According to this theorem any random exchangeable sequence (or *a class of equivalent random variables* in the terminology of [4]) can be represented as a mixture of sequences of independent identically distributed (i.i.d.) random variables. There are a number of different proofs of this important result; we mention only papers [1], [4], [9] out of many. It is convenient to formulate the de Finetti theorem as follows: Any ergodic measure defined on the infinite product $\prod_1^\infty [0,1]$ which is invariant under the group of finite permutations is a Bernoulli distribution.

The same group of finite permutations of positive integers, i.e., the infinite symmetric group $\mathcal{S}_\infty$, acts in the natural fashion on *the space of all partitions* Π_∞ of the set of all positive integers $\mathbf{N}$. Namely, the image of the partition $\xi \in \Pi_\infty$ under the permutation $g \in \mathcal{S}_\infty$ is the partition $g\xi$ such that elements i and j belong to the same component of the partition $g\xi$ if and only if elements $g^{-1}i$ and $g^{-1}j$ belong to the same component of the partition ξ. Kingman [13], (see also [12]) posed and solved the question for exchangeable random partitions of positive integers similar to that for random exchangeable sequences.

Endow the space Π_∞ with the topology of the projective limit of spaces of partitions of finite sets $[n] = \{1, \dots, n\}$. Let $\mathcal{M}(\Pi_\infty)$ denote the collection of all Borel distributions on the space Π_∞. The distribution $\eta \in \mathcal{M}(\Pi_\infty)$ is called *exchangeable* and the random partition $\xi \in \Pi_\infty$ with distribution η is called *exchangeable* if η is invariant under the described action of finite permutations.

According to Kingman's result [13] any ergodic exchangeable partition of positive integers can be parametrized by the elements of the simplex $\Sigma = \{x = (x_1, x_2, \dots)$:

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$x_1 \geqq x_2 \geqq \cdots \geqq 0, \sum_{i=1}^{\infty} x_i \leqq 1\}$ and can be constructed as follows. For $x \in \Sigma$ consider the distribution α over the interval $[0,1]$ which has a countable number of different atoms with the weights $x_1, x_2, \ldots$ and an absolutely continuous component with the weight $1-x_1-x_2-\cdots$. Let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. random variables with common distribution α. This sequence defines a random equivalence relation on $\mathbf{N}$ (leading to a random partition ξ_x of $\mathbf{N}$ into equivalence classes). Namely, $i \sim j$ if and only if $Z_i = Z_j$. By construction, this random partition ξ_x is exchangeable. Denote its distribution by $\overline{P}^x$.

THEOREM 1 (see [13]). *Let ξ be an exchangeable random partition of positive integers. Denote by ξ_n its restriction onto the set $[n] \subset \mathbf{N}$ and let $L_1(\xi_n) \geqq L_2(\xi_n) \geqq \cdots$ be the sizes of components of the partition ξ_n written in nonincreasing order. Then the limits*

$$X_i(\xi) = \lim_{n\to\infty} \frac{L_i(\xi_n)}{n}, \quad i = 1, 2, \ldots,$$

exist with probability one. For any $x \in \Sigma$ the conditional distribution of ξ under the condition $(X_1, X_2, \ldots) = x$ is $\overline{P}^x$. Thus, the distribution $\overline{\mu}$ of an exchangeable sequence ξ has the form

$$\overline{\mu} = \int_{\Sigma} \overline{P}^x \, d\nu(x),$$

where ν is the distribution of the vector $X = (X_1, X_2, \ldots)$ on the simplex Σ.

Exchangeable random partitions are used in many branches of mathematics and its applications such as combinatorics, the theory of symmetric groups, and population genetics. The most important is *Ewens' random partition,* closely related to Haar measures over symmetric groups and the Poisson–Dirichlet distribution on the simplex (see Example 1). This partition admits many additional symmetries. The goal of this paper is to derive a new characterization of this random partition.

To relate this problem to the theory of representations of the infinite symmetric group it is convenient to reformulate Theorem 1 as follows: Let $\mathcal{S}_n$ be the symmetric group of degree n, i.e., the group of all permutations of the set $[n]$. The *canonical projection* $\pi_n\colon \mathcal{S}_{n+1} \to \mathcal{S}_n$ puts into correspondence to a permutation $w \in \mathcal{S}_{n+1}$ the permutation of the set $[n]$ obtained by deletion of the element $n+1$ from the cycle of permutation w which contains this element.

The sequence $(w_1, w_2, \ldots)$, where $w_i \in \mathcal{S}_i$, of permutations is called *coherent* if $\pi_n w_{n+1} = w_n$ for all $n = 1, 2, \ldots$. Denote by $\mathcal{S}^\infty$ the space of all coherent sequences of permutations endowed with the topology of the projective limit of finite groups $\mathcal{S}_n$. This space is introduced in [11] and is called the *space of virtual permutations.* Let $\mathcal{M}(\mathcal{S}^\infty)$ denote the collection of all Borel probability distributions on $\mathcal{S}^\infty$.

An action of the group $G = \mathcal{S}_\infty \times \mathcal{S}_\infty$ is defined on the space $\mathcal{S}^\infty$ as follows. Let $\omega = (w_1, w_2, \ldots)$ be a coherent sequence of permutations and fix $(g_1, g_2) \in \mathcal{S}_n \times \mathcal{S}_n$. Put

$$\big((g_1, g_2)\,\omega\big)_i = \begin{cases} g_1^{-1} w_i g_2 & \text{if } i \geqq n; \\ \pi_i(g_1^{-1} w_n g_2) & \text{if } i < n. \end{cases} \tag{1}$$

Denote by $K = \{(h,h),\ h \in \mathcal{S}_\infty\}$ the diagonal subgroup of the group G which is isomorphic to $\mathcal{S}_\infty$. The distribution $\mu \in \mathcal{M}(\mathcal{S}^\infty)$ is called *central* if it is invariant under the action of the diagonal subgroup K.

Each coherent sequence of permutations $\omega \in \mathcal{S}^\infty$ defines a partition $\xi(\omega)$ of the set of all positive integers. Namely, the elements i and j belong to the same component of $\xi(\omega)$ if i and j belong to the same cycle of permutation w_n for sufficiently large n. Denote by $\Phi\colon \mathcal{S}^\infty \to \Pi_\infty$ the mapping $\omega \to \xi(\omega)$. The action of the group K, which is isomorphic to $\mathcal{S}_\infty$, on the space $\mathcal{S}^\infty$ agrees with that of the group $\mathcal{S}_\infty$ on the space Π_∞, i.e., $g\Phi(\omega) = \Phi(g^{-1}\omega g)$. For this reason the mapping Φ transforms central distributions on $\mathcal{S}^\infty$ into exchangeable distributions on Π_∞. This mapping is one-to-one since a central measure on the symmetric group $\mathcal{S}_n$ is completely determined by the distribution of its cycle partition.

Let P^x be the central distribution on the space $\mathcal{S}^\infty$ which corresponds to an ergodic distribution $\overline{P}^x$ on the space Π_∞. Kingman's theorem can be formulated in terms of random permutations as follows.

THEOREM 1$'$. *Let $\omega = (w_1, w_2, \ldots)$ be a central sequence of random permutations. Denote by $l_1(w_n) \geqq l_2(w_n) \geqq \cdots$ the sizes of cycles of the permutation $w_n \in \mathcal{S}_n$ written in nonincreasing order. Then the limits*

$$X_i(\omega) = \lim_{n\to\infty} \frac{l_i(w_n)}{n}, \quad i = 1, 2, \ldots, \tag{2}$$

exist with probability one. For any $x \in \Sigma$ the conditional distribution of ω given $(X_1, X_2, \ldots) = x$ is P^x. Thus, the distribution μ of the central sequence ω has the form

$$\mu = \int_\Sigma P^x \, d\nu(x), \tag{3}$$

where ν is the distribution of the vector $X = (X_1, X_2, \ldots)$ on the simplex Σ.

The limit $X_i(\omega)$ is called the *normalized length* of the ith cycle of the sequence ω. The central distribution μ is called *saturated* if it is supported by the set of sequences whose normalized cycles lengths add up to one. Denote by $\phi\colon \mathcal{S}^\infty \to \Sigma$ the mapping sending $\omega \to (X_1(\omega), X_2(\omega), \ldots)$.

Example 1. Let $w_1 = e \in \mathcal{S}_1$ be the identical permutation. If w_n has been defined, put $w_{n+1} = w_n(n+1, k)$ with probability $1/(n+1)$ for $k = 1, \ldots, n+1$ (here (i, j) denotes the transposition which interchanges elements i and j). The obtained random sequence $\omega = (w_1, w_2, \ldots)$ is coherent, central, and the permutation w_n has the uniform distribution on the symmetric group $\mathcal{S}_n$. The distribution of this sequence $\omega \in \mathcal{S}^\infty$ is called the *Haar measure on the space of virtual permutations.* The corresponding random exchangeable partition of positive integers is called the *Ewens random partition.* The distribution of the normalized cycles lengths of the sequence with the Haar measure is the known *Poisson–Dirichlet distribution* PD(1) [14]. (An independent proof is given in [8].) Thus the Haar measure admits the representation (3) with $\nu = \mathrm{PD}(1)$. In particular, the Haar measure is not ergodic. The distribution PD(1) is supported by the sequences with the unit sum, so that the Haar measure is saturated.

The Haar measure on $\mathcal{S}^\infty$ is the only distribution which is invariant under the right (left) action of the group $\mathcal{S}_\infty$. Indeed, for any $n \in \mathbf{N}$ the finite-dimensional projection of an invariant measure on $\mathcal{S}_n$ is invariant under all shifts, and therefore it must be the Haar measure on $\mathcal{S}_n$. It turns out that the condition of $\mathcal{S}_\infty$-invariance is too strong, and the Haar measure on $\mathcal{S}^\infty$ is the only distribution which satisfies a weaker condition of *stationarity* (in the class of all saturated distributions). We describe now this more general setting of the problem suggested by A. M. Vershik. Note that the image $R_g\mu$ of a central distribution μ on $\mathcal{S}^\infty$, under the right multiplication R_g

by a permutation $g \in \mathcal{S}_\infty$, is not a central measure. However, one can determine its projection $PR_g\mu$ onto the space of all central distributions (see section 3.) The following problem appears to be a natural one: *Describe all central distributions μ such that for any finite permutation $g \in \mathcal{S}_\infty$ the measure μ is preserved by the composition of the shift R_g and the projection P*, i.e., for all $g \in \mathcal{S}_\infty$, $PR_g\mu = \mu$. We will call such distributions *stationary*. In other words, stationary distributions are central and invariant under a family of Markovian operators.

The main result of this paper is the description of all stationary central distributions.

Theorem 3. *Ergodic central stationary distributions on the space $\mathcal{S}^\infty$ are parametrized by the elements of the unit interval $[0,1]$. For an ergodic measure M^t corresponding to $t \in [0,1]$ the sum of the normalized cycle lengths is equal to t with probability one, and the finite-dimensional distributions of this measure have the form*

$$M_n^t(h) = \sum_{k=0}^{s} \frac{1}{(n-k)!} \binom{n}{k} t^{n-k}(1-t)^k,$$

where s denotes the number of fixed points of permutation $h \in \mathcal{S}_n$.

It follows that in the class of all saturated distributions the Haar measure is the unique stationary distribution.

Without going into detail, we mention that the shift R_g can be projected onto the space Π_∞ via the mapping Φ. The obtained projection $\overline{R}_g$ is not a one-to-one mapping but is a Markovian operator (or a polymorphism in the terminology of [2]). The distribution $\overline{\mu} \in \mathcal{M}(\Pi_\infty)$ is called *stationary* if for any finite permutation g the measure $\overline{\mu}$ is preserved by the composition of the shift $\overline{R}_g$ and the projection onto the space $\mathcal{M}(\Pi_\infty)$ of exchangeable distributions. Theorem 3 reformulated in terms of all exchangeable partitions gives the description of all stationary distributions on Π_∞. We will not give here any exact formulations.

An equivalent setting of the problem is to describe the distributions on the simplex Σ which are invariant under a family of Markovian operators. Namely, consider the operator $\widetilde{T}_g$ which transforms an element x, $x \in \Sigma$, into $\phi(\omega_x g)$. Here ω_x is the central sequence of random permutations with the distribution P^x.

Theorem 2. *Ergodic measures on the simplex Σ for the family of operators $\{\widetilde{T}_g\}_{g\in\mathcal{S}_\infty}$ are parametrized by the points of the unit interval $[0,1]$. The ergodic distribution corresponding to the point $t \in [0,1]$ is supported by the simplex Σ_t of all monotone sequences with the sum equal to t and coincides with the image of the Poisson–Dirichlet distribution* PD(1) *under the homothety action $x \to tx$.*

In particular, the Poisson–Dirichlet distribution PD(1) is the only invariant measure on the simplex of all monotone sequences with the unit sum. Thus, a new characterization of the Poisson–Dirichlet distribution PD(1) is obtained.

The paper is organized as follows. Section 2 contains necessary background on the space of virtual permutations and central measures on this space. In section 3 different formulations of the main result are offered. The proof of the main theorem is given in three steps. The first step (Lemmas 1, 2) consists in applying the ergodic method [1] to reduce the original problem to a collection of conditions formulated in terms of distributions on finite symmetric groups. Section 5 contains the proof of the principal particular case of Theorem 3. Namely, the uniqueness of the stationary distribution in the class of saturated measures is established. The main idea of the proof here is considering the shift by a *random* permutation $g \in \mathcal{S}_n$. This additional randomization provides with good uniformity properties of the obtained sequence of permutations, which allows showing that its distribution is arbitrarily close to the Haar measure. Finally, in section 6 Theorem 3 is reduced to this particular case.

2. The space of virtual permutations. In this section we give all necessary background on the space of virtual permutations and central measures on this space.

DEFINITION. *For a given subset $J \subset [n]$ and a given permutation $w \in \mathcal{S}_n$ denote by $\pi_{n,J}w$ the permutation of the set J which is obtained by deletion from the cycles of permutation w all elements not belonging to the set J. The permutation $\pi_{n,J}w$ is called the* induced permutation *of w on the set J.*

The induced permutation on the set $J = [m]$ will be denoted by $\pi_{n,m}w$. The index n, if determined by the context, will be usually omitted. For example, if $w = (6351)(42)(7)$, then $\pi_4 w = (31)(42)$.

DEFINITION. *The* space of virtual permutations $\mathcal{S}^\infty$ *is defined as the projective limit $\mathcal{S}^\infty = \varprojlim \mathcal{S}_n$ of finite symmetric groups $\mathcal{S}_n$ with regard to the canonical projections $\pi_{n+1,n}\colon \mathcal{S}_{n+1} \to \mathcal{S}_n$.*

Thus, the virtual permutation is a sequence $\omega = (w_1, w_2, \ldots) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \cdots$ such that w_n is the induced permutation of w_{n+1} for all $n \in \mathbf{N}$.

Note that the projection π_n commutes with the shifts by elements of the group $\mathcal{S}_n$. In other terms, for any $N > n$, one has $\pi_n(g_1^{-1} h g_2) = g_1^{-1}\pi_n(h) g_2$ for all $h \in \mathcal{S}_N$ and $g_1, g_2 \in \mathcal{S}_n$. Thus, the action of the group $G = \mathcal{S}_\infty \times \mathcal{S}_\infty$ on the space of virtual permutations $\mathcal{S}^\infty$ is well defined by (1).

The sequence $\{\mu_n\}$ of distributions on finite symmetric groups $\mathcal{S}_n$ is called *coherent* if it agrees with the action of the projections, $\pi_{n+1,n}$, i.e., if for any $n \in \mathbf{N}$, $\pi_{n+1,n}\mu_{n+1} = \mu_n$. Any coherent sequence of distributions defines a Borel measure $\mu = \varprojlim \mu_n$ on the space of virtual permutations $\mathcal{S}^\infty$, and every Borel measure on $\mathcal{S}^\infty$ admits such a representation. *Further, all measures under consideration are assumed to be Borel probability measures.* If all distributions μ_n are central, i.e., are invariant under inner automorphisms of $\mathcal{S}_n$, then the limiting distribution μ is invariant under the action of the diagonal subgroup $K = \{(g_1, g_2) \in G,\ g_1 = g_2\}$. A distribution on the space of virtual permutations $\mathcal{S}^\infty$ is called *central* if it is invariant under the action of the diagonal subgroup K. These measures correspond to the central measures of the branching graph of conjugacy classes of symmetric groups; see [3], [5]. We denote by $\mathcal{M}^K(\mathcal{S}^\infty)$ the set of all central measures.

Theorem $1'$ gives a complete description of all central measures on the space of virtual permutations. In particular, this result implies that for any measure ν on the simplex Σ there exists a unique central measure μ on $\mathcal{S}^\infty$, such that ν is its distribution of the vector of normalized cycle lengths. Let $\mathcal{M}(\Sigma)$ denote the set of all Borel distributions on the simplex Σ. Let ρ be the operator which puts in correspondence to the distribution $\nu \in \mathcal{M}(\Sigma)$ the central measure $\mu = \int P^x \, d\nu(x)$.

Example 2. The simplest description of the Poisson–Dirichlet distribution PD(1), which is the distribution of the normalized cycle lengths of virtual permutations with regard to the Haar measure (see Example 1), can be derived from the following model: Let $U_1, U_2, \ldots$ be a sequence of i.i.d. random variables with the uniform distribution on the interval $[0, 1]$. Put $V_n = U_n \prod_{i=1}^{n-1}(1 - U_i)$. The Poisson–Dirichlet measure PD(1) represents the distribution of order statistics $V_{(1)} \geqq V_{(1)} \geqq \cdots$ of the sequence $V_1, V_2, \ldots$.

Now we describe a modification of Kingman's construction for ergodic exchangeable partitions, which allows the construction of a random virtual permutation with the distribution P^x. Put $x_0 = 1 - x_1 - x_2 - \cdots$. We distribute stepwise elements

$1, 2, \ldots$ into cycles and assign to these cycles random labels, so that at each stage a new element with probability x_0 will form a *special* cycle of length 1, which has no label and to which addition of any new elements is forbidden. Thus,

(1) at the first stage element 1 with probability x_0 forms a special cycle and with probability x_j $(j = 1, 2, \ldots)$ forms a regular cycle of length 1 with the label j;

(2) if the elements $1, \ldots, m$ have been allocated so that there are several special cycles and k regular cycles with labels $i_1, \ldots, i_k$, then the element $m+1$ is allocated into one of the possible positions in the jth regular cycle with probability x_{i_j}, $j = 1, \ldots, k$. With probability x_i, $i \neq i_1, \ldots, i_k$, it forms a new regular cycle of length 1 having the label i, and with probability x_0 it forms a new special cycle.

After the nth step of this procedure one obtains a random permutation $w_n \in \mathcal{S}_n$. According to the construction the sequence w_n is coherent. It is easy to see that its distribution is given by P^x. In particular, when $x = (0, 0, \ldots)$, the distribution P^0 is supported by the sequence of identity permutations.

Example 3. The projection of an ergodic measure P^x onto the group $\mathcal{S}_2$ is given by the formula

$$P_2^x\big((1)\,(2)\big) = 2 \sum_{1 \leqq i < j < \infty} x_i x_j + x_0^2 + 2x_0(1 - x_0), \qquad P_2^x((12)) = \sum_{i=1}^{\infty} x_i^2.$$

Notice that the sum of normalized cycle lengths of a virtual permutation may not be equal to one. The simplest illustration of this fact is provided by the virtual permutation $e = (e_1, e_2, \ldots)$ with $e_i \in \mathcal{S}_i$ being identity permutations. Clearly the lengths of all cycles are equal to zero.

DEFINITION. *A central distribution μ on the space of virtual permutations is called* saturated *if the sum of normalized cycle lengths equals one almost surely with regard to μ. A coherent family $\{\mu_n\}$ of central distributions on symmetric groups is called* saturated *if the limiting distribution $\mu = \varprojlim \mu_n$ on the space of virtual permutations is saturated.*

The problem with general central distributions on the space of virtual permutations in many cases can be reduced to that with saturated distributions.

More precisely, let $t \in [0, 1]$. Denote by $\Sigma_t = \{x \in \Sigma \colon \sum_{i=1}^{\infty} x_i = t\}$ the simplex of monotone sequences with the sum t. When $t \neq 0$, the standard simplex Σ_1 is identified with Σ_t via homothety $\Gamma_t \colon x \to xt$. Denote by Γ_0 the constant mapping of Σ_1 into $\Sigma_0 = \{0\}$. Let a distribution ν be given on the simplex Σ and denote by τ the image of the measure ν under the mapping $x \to x_1 + x_2 + \cdots$ and by ν^t the conditional distribution of ν on the simplex Σ_t, so that $\nu = \int_0^1 \nu^t \, d\tau(t)$. Let $\overline{\nu}^t = \Gamma_t^{-1} \nu^t$. One can show that the finite-dimensional distributions of the central measure μ which corresponds to ν are given by the formula

$$\mu_n(h) = \int_0^1 \sum_{k=0}^{s} \binom{n}{k} t^{n-k} (1-t)^k \mu_{n-k}^t(h^k) \, d\tau(t), \tag{4}$$

where $\mu^t = \int_{\Sigma_1} P^y \, \overline{\nu}^t(y)$ is the saturated central distribution corresponding to $\overline{\nu}^t$, s denotes the number of fixed points of permutation $h \in \mathcal{S}_n$, and h^k is the permutation obtained from h by deletion of k fixed points.

Remark. The space $\mathcal{S}^\infty$ of virtual permutations has been introduced in [11]. This paper gives a family of quasi-invariant distributions on the space $\mathcal{S}^\infty$ under the described action of the group $G = \mathcal{S}_\infty \times \mathcal{S}_\infty$ and studies the related family of unitary representations of the infinite symmetric group. The subsequent papers [6], [7], and [8] deal with different properties of the space of virtual permutations.

3. The main theorem. The infinite symmetric group $\mathcal{S}_\infty$ acts on the space of virtual permutations by two-sided shifts (1). Denote as $R_g\colon \mathcal{S}_\infty \to \mathcal{S}_\infty$ the right shift by a permutation $g \in \mathcal{S}^\infty$.

It is not difficult to show that for the image $\mu^g = R_g\mu$ of a central distribution μ the normalized cycle lengths (2) exist under almost all with respect to μ^g virtual permutations. Therefore one can define the projection of the shift R_g onto the simplex Σ which must be a Markovian operator. This operator transforms $x \in \Sigma$ into the element $\phi(\omega_x g)$, where ω_x is a random virtual permutation with distribution P^x. Thus the corresponding operator $\widetilde{T}_g$ completes the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}^K(\mathcal{S}^\infty) & \xrightarrow{R_g} & \mathcal{M}(\mathcal{S}^\infty) \\ \rho\uparrow & & \phi\downarrow \\ \mathcal{M}(\Sigma) & \xrightarrow{\widetilde{T}_g} & \mathcal{M}(\Sigma) \end{array}$$

It is easy to see that the operator $\widetilde{T}_g$ depends only on the conjugacy class of the permutation g, i.e., only on the cycle structure of g.

Example 4. Obviously $\widetilde{T}_e$ is the identity operator. The simplest nontrivial example is the operator corresponding to a transposition. The construction of the ergodic measure P^x implies that the restriction $T_{(1,2)}$ of the operator $\widetilde{T}_{(1,2)}$ onto the simplex Σ_1 can be obtained as follows: Let V be the operator which puts the coordinates in nonincreasing order. Then

$$T_{(1,2)}x = \begin{cases} V(x_i + x_j, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{j-1}, x_{j+1}, \dots), \quad i < j, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{with probability } 2x_ix_j; \\ V(t, x_i - t, x_1, \dots, x_{i-1}, x_{i+1}, \dots), \quad i = 1, 2, \dots, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{with probability } dt,\ t \in [0, x_i]. \end{cases}$$

It is clear that if the measure μ is invariant with respect to the shift R_g, then the distribution $\nu = \phi\mu$ is invariant under the operator $\widetilde{T}_g$. Therefore the Poisson–Dirichlet distribution PD(1) is $\widetilde{T}_g$-invariant for any $g \in \mathcal{S}_\infty$.

It is easy to show that for any $t \in [0, 1]$ the set $\Sigma_t = \{x \in \Sigma\colon \sum x_i = t\}$ is invariant under $\widetilde{T}_g$. Indeed, a shift by a finite permutation does not change the sum of normalized cycle lengths of a virtual permutation. It turns out that these sets are ergodic components, and the ergodic measure supported by Σ_t coincides with the image of the Poisson–Dirichlet distribution PD(1) under the homothety $\Gamma_t\colon \Sigma \to \Sigma_t$.

Theorem 2. *Ergodic measures on the simplex Σ with respect to the family of operators $\{\widetilde{T}_g\}_{g\in\mathcal{S}_\infty}$ are parametrized by the points of the unit interval* $[0, 1]$. *The ergodic distribution corresponding to the point $t \in [0, 1]$ is supported by the simplex Σ_t and is equal to* $\mathrm{PD}_t(1) = \Gamma_t(\mathrm{PD}(1))$.

Corollary. *Any distribution on the simplex Σ which is invariant under the family of operators $\{\widetilde{T}_g\}$, $g \in \mathcal{S}_\infty$, has the form $\nu = \int_0^1 \mathrm{PD}_t(1)\, d\tau(t)$, with some Borel distribution τ on the interval* $[0, 1]$.

The main problem admits the following reformulation: Let $\nu \in \mathcal{M}(\Sigma)$, and let $\rho(\widetilde{T}_g\nu)$ denote the central measure corresponding to the distribution $\widetilde{T}_g\nu$. According to the definition of the operator $\widetilde{T}_g$ one has $\widetilde{T}_g\nu = \phi\mu^g$, where $\mu = \rho\nu$ is the central measure on $\mathcal{S}^\infty$, which corresponds to ν, and $\mu^g = R_g\mu$. Therefore,

$\rho(\widetilde{T}_g\nu) = \int_\Sigma P^x \, d(\phi\mu^g)(x)$. Thus, the distribution ν is invariant under $\widetilde{T}_g$ if and only if μ satisfies the condition

$$\mu = \int_\Sigma P^x \, d(\phi\mu^g)(x). \tag{5}$$

A central measure μ on the space of virtual permutations is called *stationary* if it satisfies (5) for all $g \in \mathcal{S}_\infty$.

The right-hand side of (5) defines the projection $P\mu^g$ of the measure μ^g onto the space $\mathcal{M}^K(\mathcal{S}^\infty)$ of all central (i.e., K-invariant) measures. Namely, $P\mu^g$ is the unique central measure which has the same distribution of normalized cycle lengths as μ^g. Notice that the group K, which is isomorphic to $\mathcal{S}_\infty$, is locally finite. A. M. Vershik [1] suggested using the following *ergodic method* to determine the projection P.

Lemma 1. *Let η be a distribution on the space of virtual permutations such that the normalized cycle lengths exist almost surely with regard to η. Then the projection of the distribution η onto the space of central measures $\mathcal{M}^K(\mathcal{S}^\infty)$ can be found from the formula*

$$P\eta = \lim_{N\to\infty} \frac{1}{N!} \sum_{u\in\mathcal{S}_N} u^{-1}\eta u. \tag{6}$$

Remark. Here and further the weak convergence of distributions is meant.

Proof. Let $\eta^N = (N!)^{-1} \sum_{u\in\mathcal{S}_N} u^{-1}\eta u$. The space $\mathcal{M}(\mathcal{S}^\infty)$ is compact. Thus one can find a weakly convergent subsequence $\{\eta^{N_k}\}$ of the sequence $\{\eta^N\}$. Let η_0 be the limit of this subsequence. Since the operation of conjugation does not change the cycles lengths, one has $\phi(u^{-1}\eta u) = \phi(\eta)$ for any $u \in \mathcal{S}_\infty$. Thus $\phi(\eta_0) = \phi(\eta)$.

Let $K_n = K \cap (\mathcal{S}_n \times \mathcal{S}_n)$ so that $K = \cup K_n$. Clearly for $N \geqq n$, the measure η^N is invariant under K_n. Therefore for any $n \in \mathbf{N}$ the measure η_0 is invariant under K_n, i.e., $\eta_0 \in \mathcal{M}^K(\mathcal{S}^\infty)$.

Thus, the measure η_0 is central and has the same distribution of the normalized cycle lengths as η, i.e., $\eta_0 = P\eta$. We have shown that any convergent subsequence of the sequence $\{\eta^N\}$ converges to $P\eta$, which proves (6).

Remark. One can define a stationary measure by the formula

$$\mu = \lim_{N\to\infty} \frac{1}{N!} \sum_{u\in\mathcal{S}_N} u^{-1}\mu^g u, \quad g \in \mathcal{S}_\infty. \tag{7}$$

Recall the classical notion of a stationary distribution under a group action for a locally compact group (cf. [10]). Let $\mathcal{G}$ be a locally compact group acting on a measurable space X, and let κ be a distribution over $\mathcal{G}$. A distribution η on X is called stationary (with respect to κ) if, with η^α denoting the image of the measure η under the element $\alpha \in \mathcal{G}$, one has $\eta = \int_\mathcal{G} \eta^\alpha \, d\kappa(\alpha)$. Formula (7) provides a natural generalization of this definition for locally finite groups.

According to formula (4), Theorem 2 is tantamount to the following theorem.

Theorem 3. *Any stationary probability distribution $\mu = \varprojlim \mu_n$ on the space of virtual permutations has the form*

$$\mu_n(h) = \sum_{k=0}^{s} \frac{1}{(n-k)!} \binom{n}{k} \int_0^1 t^{n-k}(1-t)^k \, d\tau(t),$$

with some Borel distribution τ on the interval $[0,1]$ *and s denoting the number of fixed points of permutation $h \in \mathcal{S}_n$.*

In the class of all saturated distributions (corresponding to the measure τ concentrated at $t = 1$) the unique stationary distribution is the Haar measure. Thus, the class of all saturated distributions does not contain noninvariant stationary distributions.

Remark. As the family $\{\widetilde{T}_g\}$ does not provide a group action of $\mathcal{S}_\infty$, it is natural to pose a question about the description of all distributions which are invariant under merely one operator $\widetilde{T}_{(1,2)}$ arising from transpositions. This question is more interesting, as this operator $\widetilde{T}_{(1,2)}$ has a very simple form (see Example 4). We conjecture that the set of all invariant distributions for the operator $\widetilde{T}_{(1,2)}$ is the same as for the whole family $\{\widetilde{T}_g\}$, $g \in \mathcal{S}_\infty$, but this remains an open problem.

4. Reduction to the finite-dimensional conditions. The first part of the proof of the main theorem consists in reducing our infinite-dimensional problem to an (infinite) number of conditions which are formulated in terms of finite symmetric groups.

LEMMA 2. *Let μ be a central distribution on the space of virtual permutations and let $\{\mu_n\}$ be the corresponding coherent family of distributions on symmetric groups $\mathcal{S}_n$. Then the finite-dimensional projections of the distribution $P\mu^g$, $g \in \mathcal{S}_k$, can be found from the formula*

$$(P\mu^g)_n(u) = \sum_{\substack{w \in \mathcal{S}_{n+k} \\ \pi_n(w)=u}} \mu_{n+k}(wg^n), \quad u \in \mathcal{S}_n,$$

where $g^n \in \mathcal{S}_{n+k}$ is a permutation such that $g^n(i) = i$ for $i = 1, \dots, n$; $\pi_{\{n+1,\cdots,n+k\}} g^n$ has the same cycle structure as g.

Proof. Let $\eta = P\mu^g$ and $u \in \mathcal{S}_n$. Then

$$\eta_n(u) = \lim_{N\to\infty} \frac{1}{N!} \sum_{h \in \mathcal{S}_N} \sum_{\substack{w \in \mathcal{S}_N \\ \pi_n(w)=u}} \mu_N(h^{-1}whg).$$

Denote by Π_n the class of all partitions of an integer n. The conjugacy classes of the symmetric group $\mathcal{S}_n$ can be parametrized by elements $\lambda \in \Pi_n$. Thus there exists a one-to-one correspondence between central measures on $\mathcal{S}_n$ and distributions on Π_n. Let $[\lambda]$ denote the conjugacy class corresponding to $\lambda \in \Pi_n$, i.e., the set of all permutations whose cycle lengths form a partition λ, and let $|\lambda|$ be the number of elements in this class. The distribution corresponding to the measure η has the form

$$\overline{\eta}_n(\lambda) = \sum_{v \in [\lambda]} \eta_n(v) = |\lambda|\, \eta_n(u),$$

where u is an arbitrary representative of the class $[\lambda]$. Thus

$$\overline{\eta}_n(\lambda) = \lim_{N\to\infty} \frac{1}{N!} \sum_{h \in \mathcal{S}_N} \sum_{\substack{w \in \mathcal{S}_N \\ \pi_n(w)\in[\lambda]}} \mu_N(h^{-1}whg).$$

Denote by σ_N the expression within the limit sign in the right-hand side of this identity. This quantity can be rewritten in the following form:

$$\sum_{v \in \mathcal{S}_N} \mu_N(v) \frac{1}{N!} \# \bigl\{ h \in \mathcal{S}_N \colon \pi_n(hvg^{-1}h^{-1}) \in [\lambda] \bigr\} = \sum_{v \in \mathcal{S}_N} \mu_N(v)\, q(v,\lambda).$$

Here $q(v,\lambda)$ are the transition probabilities of the Markovian operator acting from $\mathcal{S}_N$ into Π_n and transforming the permutation $v \in \mathcal{S}_N$ into the partition, which corresponds to the class of the permutation $\pi_n(w)$ with a random permutation w uniformly distributed on the class containing vg^{-1}. Notice that the study of the induced permutation on the set $[n]$ of a random representative of the class containing vg^{-1} is equivalent to studying the induced permutation of vg^{-1} on a *random* subset $J \subset [N]$ of cardinality n, i.e.,

$$q(v,\lambda) = \frac{1}{\binom{N}{n}} \# \left\{ J \subset [N] \colon \#J = n,\ \pi_J(vg^{-1}) \in [\lambda] \right\}.$$

Let δ_λ denote the indicator function of the class $[\lambda]$. Then

$$\sum_{v\in\mathcal{S}_N} \mu_N(v) q(v,\lambda) = \frac{1}{\binom{N}{n}} \sum_{\substack{J\subset[N]\\ \#J=n}} \sum_{v\in\mathcal{S}_N} \mu_N(v)\, \delta_\lambda\big(\pi_J(vg^{-1})\big).$$

Assume that $N \geqq n+k$. Since $P\mu^g$ depends only on the cycle structure of the permutation g, one can take $g \in \mathcal{S}[n+1,\dots,n+k]$. Let $J = \{j_1,\dots,j_n\}$. Assume that

$$\{j_1,\dots,j_n\} \cap \{n+1,\dots,n+k\} = \varnothing. \tag{8}$$

The probability that a random subset of the set $[N]$ of cardinality n possesses this property is $\binom{N-k}{n}/\binom{N}{n}$. Consider the permutation $h \in \mathcal{S}_N$ such that $h(j_1) = 1, \dots, h(j_n) = n$ and $h(i) = i$ for $i \notin J$. Obviously, h and g commute. Since μ_N is a central measure, it follows that the sum

$$\begin{aligned}\sum_{v\in\mathcal{S}_N} \mu_N(v)\, \delta_\lambda\big(\pi_J(vg^{-1})\big) &= \sum_{v\in\mathcal{S}_N} \mu_N(v)\, \delta_\lambda\big(\pi_J(hvh^{-1}g^{-1})\big)\\ &= \sum_{v\in\mathcal{S}_N} \mu_N(v)\, \delta_\lambda\big(\pi_J(hvg^{-1}h^{-1})\big)\\ &= \sum_{v\in\mathcal{S}_N} \mu_N(v)\, \delta_\lambda\big(\pi_n(vg^{-1})\big)\end{aligned}$$

does not depend on the subset J provided that it satisfies (8). Also since $g \in \mathcal{S}[n+1,\dots,n+k]$, the last sum is equal to $\sum_{v\in\mathcal{S}_{n+k}} \mu_{n+k}(v)\, \delta_\lambda(\pi_n(vg^{-1}))$. Therefore

$$\sigma_N = \frac{\binom{N-k}{n}}{\binom{N}{n}} \sum_{v\in\mathcal{S}_{n+k}} \mu_{n+k}(v)\, \delta_\lambda\big(\pi_n(vg^{-1})\big) + R_N,$$

where $|R_N| \leqq 1 - \binom{N-k}{n}/\binom{N}{n} \to 0$ as $N \to \infty$, which proves this lemma.

5. The main lemma. As noted above, all statements about general central measures on the space of virtual permutations commonly are reduced to the case of saturated measures. In agreement with this principle we prove here first that the Haar measure is the only stationary measure in the class of saturated distributions. To be more exact, an even stronger result holds, two equivalent versions of which follow.

THEOREM 4. *Let T_g be the restriction of the operator $\widetilde{T}_g$ on an invariant subset Σ_1. Denote by T_n the average of the operator T_g over the symmetric group $\mathcal{S}_n$,*

$$T_n = \frac{1}{n!} \sum_{g\in\mathcal{S}_n} T_g.$$

The Poisson–Dirichlet distribution $\mathrm{PD}(1)$ *is the only measure on the simplex* Σ_1 *which is invariant under the family* $\{T_n,\ n \in \mathbf{N}\}$.

THEOREM 5. *Let* R_n *be the Markovian operator which shifts by a random permutation with uniform distribution on the group* $\mathcal{S}_n$. *The Haar measure on the space of virtual permutations is the only central measure which satisfies the condition* $\mu = P(R_n \mu),\ n \in \mathbf{N}$.

Because of Lemma 2, Theorems 4 and 5 follow from Main Lemma 3.

For fixed $n, k \in \mathbf{N}$, let $\mathcal{S}^k = \mathcal{S}[n+1, \dots, n+k]$.

MAIN LEMMA 3. *Let* $\{\mu_n\}$ *be a saturated coherent family of central distributions over symmetric groups* $\mathcal{S}_n$ *under the conditions*

$$\sum_{\substack{w \in \mathcal{S}_{n+k} \\ \pi_n(w) = u}} \frac{1}{k!} \sum_{g \in \mathcal{S}^k} \mu_{n+k}(wg) = \mu_n(u) \quad \forall\, n, k \in \mathbf{N},\ \forall u \in \mathcal{S}_n. \tag{9}$$

Then the distribution μ_n *coincides with the Haar measure* m_n *on* $\mathcal{S}_n$ *for any* $n \in \mathbf{N}$.

Proof. Notice that condition (9) can be rewritten in the form

$$\mu_n(u) = \sum_{w \in \mathcal{S}_{n+k}} \mu_{n+k}(w)\, P(w, u),$$

where $P(w, u)$ are the transition probabilities of the Markovian operator acting from $\mathcal{S}_{n+k}$ into $\mathcal{S}_n$ and transforming the permutation $w \in \mathcal{S}_{n+k}$ into the permutation $\pi_n(wg)$ with a random permutation g uniformly distributed on $\mathcal{S}^k$. One obtains for the corresponding distribution $\overline{\mu}_{n+k}$ on the space of partitions Π_{n+k},

$$\mu_n(u) = \sum_{w \in \mathcal{S}_{n+k}} \mu_{n+k}(w)\, P(w,\, u) = \sum_{\lambda \in \Pi_{n+k}} \overline{\mu}_{n+k}(\lambda)\, \overline{P}(\lambda, u),$$

where $\overline{P}(\lambda, u) = (|\lambda|)^{-1} \sum_{w \in [\lambda]} P(w, u)$.

The following result shows that the transition probabilities $\overline{P}(\lambda, u)$ are close to the Haar measure $m_n(u) = 1/n!$ provided that n is fixed, $N = n + k$ is large, and the partition $\lambda \in \Pi_N$ has few unit summands.

LEMMA 4. *Let* $\lambda \in \Pi_N$, *and denote by* $b(\lambda)$ *the number of the unit summands in the partition* λ. *Then for any permutation* $u \in \mathcal{S}_n$

$$\overline{P}(\lambda, u) = m_n(u) + R(\lambda, u),$$

where

$$\bigl|R(\lambda, u)\bigr| \leqq 1 - \frac{(N - b(\lambda))(N - b(\lambda) - 2) \cdots (N - b(\lambda) - 2(n-1))}{N(N-1) \cdots (N - (n-1))}.$$

Proof. Let us call the elements $1, \dots, n$ of the set $[N]$ *junior* and the remaining elements $n+1, \dots, N$ *senior*. Consider the set D of all permutations from $\mathcal{S}_N$ such that the inverse images of the junior elements are senior,

$$D = \bigl\{ w \in \mathcal{S}_N \colon w^{-1}(i) \in \{n+1, \dots, n+k\},\ i = 1, \dots, n \bigr\}.$$

Assume that $w \in D$. Let $j_1 = w^{-1}(1), \dots, j_n = w^{-1}(n)$, and fix the isomorphism of groups $\mathcal{S}_n$ and $\mathcal{S}[j_1, \dots, j_n]$ identifying i and j_i. This isomorphism also identifies $\pi_n(w)$ and $\pi_{\{j_1, \dots, j_n\}}(w)$. Moreover, for any permutation $g \in \mathcal{S}^k$, one gets $(wg)^{-1}(i) =$

$w^{-1}(i)$ for all $i = 1, \dots, n$. Indeed, the junior elements $1, \dots, n$ are fixed points of the permutation g. For this reason $\pi_n(wg)$ is identified with $\pi_{\{j_1,\dots,j_n\}}(wg)$ for any permutation $g \in \mathcal{S}^k$. Thus the distribution of the induced permutation $\pi_n(wg)$ on the group $\mathcal{S}_n$ coincides in the sense of this isomorphism with the distribution of the permutation $\pi_{\{j_1,\dots,j_n\}}(wg)$ on $\mathcal{S}[j_1, \dots, j_n]$. But if a random permutation g is uniformly distributed on $\mathcal{S}^k$, then the permutation $\pi_{\{j_1,\dots,j_n\}}(wg)$ is uniformly distributed on $\mathcal{S}[j_1, \dots, j_n]$.

We have proved that for $w \in D$, $P(w, u) = m_n(u)$. It follows that

$$\overline{P}(\lambda,\, u) = \frac{1}{|\lambda|} \sum_{w \in [\lambda]} P(w,\, u) = \frac{\#(D \cap |\lambda|)}{|\lambda|}\, m_n(u) + r(\lambda,\, u),$$

where

$$r(\lambda, u) = \frac{1}{|\lambda|} \sum_{w \in [\lambda] \setminus D} P(w, u) \leqq 1 - \frac{\#(D \cap |\lambda|)}{|\lambda|}.$$

Therefore

$$\bigl|R(\lambda,\, u)\bigr| \leqq \left| r(\lambda, u) - m_n(u) \left(1 - \frac{\#(D \cap |\lambda|)}{|\lambda|}\right)\right| \leqq 1 - \frac{\#(D \cap |\lambda|)}{|\lambda|}.$$

Thus, all we have to do is estimate the probability that a random permutation $w \in [\lambda]$ belongs to D.

To obtain a uniformly distributed permutation $w \in [\lambda]$ we use the following procedure: Consider the Young diagram of the partition λ and allocate stepwise the elements $1, \dots, n$ into the cells of this diagram, assuming at each stage that the empty cells all have the same probability. The rows of the completed diagram we interpret as cycles. In this way one obtains a random permutation $w \in [\lambda]$. Obviously it has the uniform distribution on the class $[\lambda]$.

At the first stage of this procedure we call *bad* the cells of the diagram which form the rows of length 1. The number of such cells, as we recall, was denoted by $b(\lambda)$. The probability that the element 1 will be allocated into a good (i.e., not a bad) cell is $(N - b(\lambda))/N$. After element 1 has been allocated, we call bad the cell of the diagram which is adjacent to the left from 1 in the cyclic order. Thus, at the second stage we have $N - 1$ empty cells with no more than $b(\lambda) + 1$ bad cells. The probability that the elements 1 and 2 will be allocated into good cells is at least

$$\frac{N - b(\lambda)}{N} \cdot \frac{N - b(\lambda) - 2}{N - 1}\,.$$

Continue this procedure by adding after each stage to the bad cells of the diagram the left neighbors of the most recently occupied cell. At each step the number of bad cells of the diagram can increase by at most one, and thus the probability that the elements $1, 2, \dots, n$ will be allocated into good cells is at least

$$\frac{N - b(\lambda)}{N} \cdot \frac{N - b(\lambda) - 2}{N - 1} \cdot \dots \cdot \frac{N - b(\lambda) - 2(n-1)}{N - (n-1)}.$$

In this situation the elements $1, \dots, n$ do not belong to the rows of length one (the cells of one-lines are definitely bad cells). Also the inverse images of these elements belong to the set $\{n+1, \dots, n+k\}$ (the cells positioned to the left in the cyclic order from the cells containing junior elements are bad, and must be occupied by senior elements). Thus the corresponding permutation must belong to the set D. Therefore

$$\frac{\#(D \cap |\lambda|)}{|\lambda|} \geqq \frac{N-b(\lambda)}{N} \cdot \frac{N-b(\lambda)-2}{N-1} \cdot \dots \cdot \frac{N-b(\lambda)-2(n-1)}{N-(n-1)},$$

which was to be proven.

To apply the estimate of Lemma 4 one has to show that for the saturated measures the probability that a random permutation has many fixed points asymptotically vanishes.

LEMMA 5. *Let*

$$A_N(\delta) = \left\{\lambda \in \Pi_N \colon \frac{b(\lambda)}{N} < \delta\right\}, \qquad 0 < \delta < 1.$$

If $\{\mu_n\}$ *is a saturated coherent family of central distributions over symmetric groups, then for any* $\delta \in (0,1)$

$$\overline{\mu}_N\big(A_N(\delta)\big) \to 1 \quad \text{as } N \to \infty.$$

Proof. Fix $\delta \in (0,1)$ and let $A_n = A_n(\delta)$. Consider the vector

$$\phi_N(w) = \left(\frac{l_1(w)}{N}, \frac{l_2(w)}{N}, \dots\right) \in \Sigma$$

of the normalized cycle lengths of the permutation $w \in \mathcal{S}_N$ written in nonincreasing order. Let ν_N be the image of the measure ν_N under the mapping ϕ_N. Then the sequence of distributions ν_N weakly converges to the distribution ν of the vector of the normalized cycle lengths of virtual permutations with regard to the limiting measure $\mu = \varprojlim \mu_N$. By the definition of saturated measures, the distribution ν is supported by the simplex Σ_1 of sequences with the unit sum.

Denote by $F_N \colon \Sigma \to [0,1]$ the function $F_N(x) = \sum_{i\colon x_i \geqq 2/N} x_i$. It is clear that $F_N(x) \nearrow \sum_{i=1}^{\infty} x_i$ as $N \to \infty$, so that $F_N(x) \nearrow 1$ almost surely with regard to the measure ν. Let $B_N = \{x \in \Sigma \colon F_N(x) \geqq 1-\delta\}$. Since $B_N \nearrow B = \{x \colon \sum x_i \geqq 1-\delta\}$ one has $\nu B_N \nearrow 1$.

Clearly, $\lambda \in A_N$ if and only if for any permutation $w \in [\lambda]$, $\phi_N(w) \in B_N$. Thus $\overline{\mu}_N A_N = \nu_N B_N$. Since the sequence of distributions ν_N weakly converges to the distribution ν, the sets B_N form an increasing sequence and $\nu B_N \nearrow 1$, one has $\nu_N B_N \to 1$, and the conclusion of the lemma follows.

Now we can conclude the proof of Main Lemma 3. Fix $\varepsilon > 0$. For a partition $\lambda \in \pi_N$, put

$$f(\lambda) = \frac{(N-b(\lambda))(N-b(\lambda)-2)\cdots(N-b(\lambda)-2(n-1))}{N(N-1)\cdots(N-(n-1))}.$$

Clearly $f(\lambda) \to 1$ as $N \to \infty$ and $b(\lambda)/N \to 0$. Hence, one can find $N_1 \in \mathbf{N}$ and $\delta > 0$ such that $1-\varepsilon \leqq f(\lambda) \leqq 1$ for $N \geqq N_1$ and $b(\lambda)/N \leqq \delta$. By Lemma 5 there exists $N \geqq N_1$ for which $\overline{\mu}_N(A_N(\delta)) \geqq 1-\varepsilon$. By Lemma 4, if $\lambda \in A_N(\delta)$, then $P(\lambda, u) = m_n(u) + R(\lambda, u)$ with $|R(\lambda, u)| \leqq \varepsilon$. Therefore

$$\mu_n(u) = m_n(u) + R_1 + R_2 - R_3,$$

where

$$R_1 = \sum_{\lambda \in A_N(\delta)} \overline{R}(\lambda, u)\, \overline{\mu}_N(\lambda) \leqq \varepsilon \overline{\mu}_N\big(A_N(\delta)\big) \leqq \varepsilon,$$

$$R_2 = \sum_{\lambda \in \Pi_N \setminus A_n(\delta)} \overline{\mu}_N(\lambda)\, \overline{P}(\lambda, u) \leqq \overline{\mu}_N\big(\Pi_n \setminus A_N(\delta)\big) \leqq \varepsilon,$$

$$R_3 = m_n(u)\Big(1 - \overline{\mu}_N\big(A_N(\delta)\big)\Big) \leqq \varepsilon.$$

Thus, $|\mu_n(u) - m_n(u)| \leqq 2\varepsilon$, which concludes the proof of Main Lemma 3.

6. Reduction to saturated measures. In this section we conclude the proof of the main theorem by reducing it to Theorem 4.

As was noted in section 3, the sets Σ_t are invariant under $\widetilde{T}_g$ for all $g \in \mathcal{S}_\infty$. The case when $t = 0$ is trivial, as the set Σ_0 has only one point, and we can assume that $t \neq 0$.

The definition of the operator $\widetilde{T}_g$ and the construction of ergodic distributions P^x show that after identification of Σ_t with the standard simplex Σ the restriction of the operator $\widetilde{T}_g$ takes the form

$$T_g^t = \Gamma_t^{-1}\Big(\widetilde{T}_g\Big|_{\Sigma_t}\Big) = \sum_{k=0}^{n} t^k (1-t)^{n-k} \sum_{\substack{J \subset [n] \\ |J| = k}} T_{\pi_J(g)}.$$

Averaging over $g \in \mathcal{S}_n$ one obtains

$$\begin{aligned} T_n^t &= \frac{1}{n!} \sum_{g \in \mathcal{S}_n} T_g^t = \sum_{k=0}^{n} t^k (1-t)^{n-k} \sum_{\substack{J \subset [n] \\ |J| = k}} \frac{1}{n!} \sum_{g \in \mathcal{S}_n} T_{\pi_J(g)} \\ &= \sum_{k=0}^{n} t^k (1-t)^{n-k} \sum_{\substack{J \subset [n] \\ |J| = k}} \frac{1}{k!} \sum_{h \in \mathcal{S}[J]} T_h. \end{aligned}$$

Since the operator T_g depends only on the class of the permutation g, the last normalized sum is equal to T_k. Therefore

$$T_n^t = \sum_{k=0}^{n} t^k (1-t)^{n-k} \sum_{\substack{J \subset [n] \\ |J| = k}} T_k = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k T_k. \tag{10}$$

Let ν be a $\{\widetilde{T}_g\}$-invariant measure and let ν^t be the conditional distribution of ν on Σ_t. Notice that $T_0^t = T_1^t = E$, where E denotes the identity operator on Σ_1. By (10), $T_2^t = t^2 T_2 + (1 - t^2)E$. Since the measure $\overline{\nu}^t = \Gamma_t^{-1}\nu^t$ is invariant under the operator T_2^t, the last formula implies that this measure is T_2-invariant. A similar argument shows by induction that the measure $\overline{\nu}^t$ is invariant under the family $\{T_n\}$, $n = 1, 2, \ldots$. According to Theorem 4 for any $t \in [0, 1]$, $\overline{\nu}^t = \mathrm{PD}(1)$, which completes the proof.

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