Curves Encomplexed

Oleg Viro

October 31, 2006
Introduction

- Encomplex
- Curves
- curves and complexification

Whitney number

Writhe

Arnold invariants

Encomplexing \( J \)
Many objects studied in geometry are defined in real coordinates by equations.
Encomplex

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The new complex objects are even nicer, although they are less visual.
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I will call this \textit{to encomplex}
Encomplex

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Another option is to consider the original objects embedded into its complexification. More difficult, but nonetheless rewarding!

I will call this to encomplex and try to show its difficulties and advantages on a simple material of curves.
A real plane curve is a generically immersed circle
A real plane curve is a generically immersed circle, immersion $S^1 \hookrightarrow \mathbb{R}^2$
Curves

A real plane curve is a generically immersed circle, immersion $S^1 \hookrightarrow \mathbb{R}^2$, belongs to Differential Geometry,
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One of the most classical of them is the Whitney classification of curves up to regular homotopy.
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The next masterpiece is Arnold’s theory on three first order invariants of generic plane curves.
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The next masterpiece is Arnold’s theory on three first order invariants of generic plane curves.

I am going to encomplex them in this talk.
curves and complexification

A generic immersion $S^1 \hookrightarrow \mathbb{R}^2$ is not assumed to have a complexification.
curves and complexification

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Require algebraicity
curves and complexification

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Require algebraicity

that is assume that the curve-image is defined by a polynomial equation
curves and complexification

A generic immersion $S^1 \hookrightarrow \mathbb{R}^2$ is not assumed to have a complexification.

Require algebraicity, and you get complex points.
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What’s in a complex view?
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What’s in a complex view?

- Geometry hidden in complexification:
curves and complexification

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- Geometry hidden in complexification: genus,
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What's in a complex view?

• Geometry hidden in complexification: genus, moduli, type (of complex conjugation).

Type I: the set of real points divides the set of complex points into two connected components.
curves and complexification

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Type II: the set of real points does not divide the set of complex points.
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- Interaction between real and complex.
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Whitney number is related to complex asymptotes.
curves and complexification

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- Geometry hidden in complexification: genus, moduli, type.
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Arnold’s invariant $J_-$ is related to the number of imaginary intersection points of complex halves.
curves and complexification

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What’s in a complex view?

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- Results on real curves inspired by results on curves with complexification.

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Arnold invariants

Encomplexing $J_-$
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Require algebraicity, and you get complex points.

What’s in a complex view?

• Geometry hidden in complexification: genus, moduli, type.
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• Results on real curves inspired by results on curves with complexification.

Formula for $J_-$:

$$J_-(C) = 1 - \int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{C}}(x))^2 \, d\chi(x).$$
curves and complexification

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Require algebraicity, and you get complex points.

What’s in a complex view?
- Geometry hidden in complexification: genus, moduli, type.
- Interaction between real and complex.
- Results on real curves inspired by results on curves with complexification.
- A world parallel to Real Geometry.
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Arnold’s strangeness of rational real algebraic curves.
curves and complexification

A generic immersion $S^1 \ni \mathbb{R}^2$ is not assumed to have a complexification.

Require algebraicity, and you get complex points.

**What's in a complex view?**

- Geometry hidden in complexification: genus, moduli, type.
- Interaction between real and complex.
- Results on real curves inspired by results on curves with complexification.
- A world parallel to Real Geometry.

The simplest complexification of curves are rational curves: genus zero, no moduli, polynomial parametrization.
Introduction

**Whitney number**
- Whitney number
- choice of curves
- complex line at infinity
- Theorem 1
- in terms of asymptotes
- near a kiss
- proof of Theorem 1
- improving Whitney number

Writhe

Arnold invariants

Encomplexing $J$
Whitney number

For an oriented smooth closed immersed curve $C$ on plane
For an oriented smooth closed immersed curve $C$ on plane $w(C)$, Whitney number.
For an oriented smooth closed immersed curve $C$ on plane $w(C')$, Whitney number $W$ is the rotation number of the velocity vector.
Whitney number

For an oriented smooth closed immersed curve $C$ on plane $w(C)$, Whitney number

= rotation number of the velocity vector
= degree of the Gauss map $C \rightarrow S^1$. 

Arnold invariants

Encomplexing $J$. 
Whitney number

For an oriented smooth closed immersed curve $C$ on plane $w(C)$, Whitney number

$= \text{rotation number of the velocity vector}$

$= \text{degree of the Gauss map } C \to S^1.$

Example.

$$w(C) = +2$$
Whitney number

For an oriented smooth closed immersed curve $C$ on plane $w(C)$, *Whitney number*

= rotation number of the velocity vector

= degree of the Gauss map $C \rightarrow S^1$.

Example.

\[ w(C) = +3 \]
For an oriented smooth closed immersed curve $C$ on plane $w(C')$, *Whitney number* 

$w(C') = 0$
Whitney number

For an oriented smooth closed immersed curve $C$ on plane $\mathbb{R}^2$, Whitney number $w(C)$, Whitney number

$= \text{rotation number of the velocity vector}$

$= \text{degree of the Gauss map } C \to S^1.$

**Whitney Theorem.**

$w(C)$ determines $C : S^1 \rightarrow \mathbb{R}^2$ up to regular homotopy.
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- $\mathbb{R}A$ is compact,
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- $\mathbb{R}A$ is compact, real branches don’t go to infinity!
Consider irreducible plane affine real algebraic curves $A$ such that:

- $\mathbb{R}A$ is compact,
- all real singularities are $\times$’s,
choice of curves

Consider irreducible plane affine real algebraic curves $A$ such that:

- $\mathbb{R}A$ is compact,
- all real singularities are $\times$’s, $\mathbb{R}A$ generically immersed.
Consider irreducible plane affine real algebraic curves $A$ such that:

- $\mathcal{R}A$ is compact,
- all real singularities are $\times$'s,
- $\mathcal{R}A$ is zero homologous modulo 2 in $\mathbb{C}A \subset \mathbb{C}P^2$
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to be naturally oriented
choice of curves

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Consider irreducible plane affine real algebraic curves $A$ such that:

- $RA$ is compact,
- all real singularities are $\times$'s,
- $RA$ is zero homologous modulo 2 in $CA \subset CP^2$

If $RA$ is zero homologous in $CA$ then $A$ is said to be of type I.

(Felix Klein)
choice of curves

Consider irreducible plane affine real algebraic curves $A$ such that:

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If $\mathbb{R}A$ is zero homologous in $\mathbb{C}A$ then $A$ is said to be of type I.

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Any real rational curve with infinite $\mathbb{R}A$ is of type I.
Consider irreducible plane affine real algebraic curves $A$ such that:

- $RA$ is compact,
- all real singularities are $\mathbf{X}$’s,
- $RA$ is zero homologous modulo 2 in $CA \subset CP^2$

If $RA$ is zero homologous in $CA$ then $A$ is said to be of type I.  
(Felix Klein)

Any normal $A$ of genus $g$ such that $RA$ has $g + 1$ components is of type I.
choice of curves

Consider irreducible plane affine real algebraic curves $A$ such that:

- $\mathbb{R}A$ is compact,
- all real singularities are $\times$’s,
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If $\mathbb{R}A$ is zero homologous in $\mathbb{C}A$ then $A$ is said to be of type I.

(Felix Klein)

Type I implies:

$$b_0(\mathbb{R} \text{ normalized } A) \equiv \text{genus}(A) + 1 \mod 2.$$
choice of curves

Consider irreducible plane affine real algebraic curves $A$ such that:

- $R_A$ is compact,
- all real singularities are $\times$’s,
- $R_A$ is zero homologous modulo 2 in $CA \subset \mathbb{C}P^2$

If $R_A$ is zero homologous in $CA$ then $A$ is said to be of type I. (Felix Klein)

The orientation of $R_A$ induced from $CA_+ \subset CA$ with $\partial CA_+ = R_A$ is called a complex orientation. (V.A.Rokhlin)
Consider irreducible plane affine real algebraic curves $A$ such that:

- $RA$ is compact,
- all real singularities are $\times$'s,
- $RA$ is zero homologous modulo 2 in $CA \subset CP^2$

If $RA$ is zero homologous in $CA$ then $A$ is said to be of type $I$. (Felix Klein)

The orientation of $RA$ induced from $CA_+ \subset CA$ with $\partial CA_+ = RA$ is called a complex orientation. (V.A.Rokhlin)

Denote $RA$ equipped with the orientation induced from $CA_+ \subset CA$ by $RA_+$. 
<table>
<thead>
<tr>
<th>Introduction</th>
<th>Arnold invariants</th>
<th>Whitney number</th>
<th>proof of Theorem 1</th>
<th>improving Whitney number</th>
<th>choice of curves in terms of asymptotes</th>
<th>complex line at infinity</th>
</tr>
</thead>
</table>

$$\mathbb{C}P^1_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2.$$
complex line at infinity

\[ \mathbb{C}P^1_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2, \quad \mathbb{R}P^1_\infty = \mathbb{R}P^2 \setminus \mathbb{R}^2 \]
complex line at infinity

\[ \mathbb{C}P^1 = \mathbb{C}P^2 \setminus \mathbb{C}^2, \quad \mathbb{R}P^1 = \mathbb{R}P^2 \setminus \mathbb{R}^2 \]

Denote \( \mathbb{R}P^1_\infty \) equipped with the orientation induced by the standard orientation of \( \mathbb{R}^2 \) by \( \mathbb{R}P^1_\infty^+ \).

say, counter-clockwise orientation of \( \mathbb{R}^2 \).
complex line at infinity

\[ \mathbb{C}P^1_\infty = \mathbb{C}P^2 \setminus \mathbb{C}^2, \quad \mathbb{R}P^1_\infty = \mathbb{R}P^2 \setminus \mathbb{R}^2 \]

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Denote by \( \mathbb{C}P^1_\infty^+ \) the hemisphere of \( \mathbb{C}P^1_\infty \) with \( \partial \mathbb{C}P^1_\infty^+ = \mathbb{R}P^1_\infty^+ \).
complex line at infinity

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\[ \partial \mathbb{C}P^1_{\infty+} = \mathbb{R}P^1_{\infty+}. \]
Theorem 1

Let $A$ be a plane affine real algebraic curve of type I,
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Then $w(\mathbb{R}A_+) = CA_+ \circ CP^1_{\infty+} - CA_+ \circ CP^1_{\infty-}$. 
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Let $A$ be a plane affine real algebraic curve of type I, such that

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Corollary. $|w(\mathbb{R}A)| \leq \frac{1}{2} \deg A.$
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Corollary. $|w(\mathbb{R}A)| \leq \frac{1}{2} \deg A$.

Indeed, $|w(\mathbb{R}A)| = |CA_+ \circ CP^1_{\infty+} - CA_+ \circ CP^1_{\infty-}|$
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$\leq |CA_+ \circ \mathbb{C}P^1_{\infty+} + CA_+ \circ \mathbb{C}P^1_{\infty-}|$
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**Corollary.** $|w(\mathbb{R}A)| \leq \frac{1}{2} \deg A$.

Indeed, $|w(\mathbb{R}A)| = |CA_+ \circ CP_{\infty+} - CA_+ \circ CP_{\infty-}|$

$\leq |CA_+ \circ CP_{\infty+} + CA_+ \circ CP_{\infty-}|$

$= |CA_+ \circ CP_{\infty}|$
Theorem 1

Let $A$ be a plane affine real algebraic curve of type I, such that

- $\mathbb{R}A$ is compact,
- all real singularities are $\times$'s.

Then $w(\mathbb{R}A_{+}) = CA_{+} \circ CP^{1}_{\infty+} - CA_{+} \circ CP^{1}_{\infty-}$.

Corollary. $|w(\mathbb{R}A)| \leq \frac{1}{2} \deg A$.

Indeed, $|w(\mathbb{R}A)| = |CA_{+} \circ CP^{1}_{\infty+} - CA_{+} \circ CP^{1}_{\infty-}|$

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$= \frac{1}{2} \deg A$. 
in terms of asymptotes

If $CA \cap \mathbb{CP}^1$, then each point of $CA \cap \mathbb{CP}^1$ corresponds to an asymptote of $CA \cap \mathbb{C}^2$. 
in terms of asymptotes

If $\mathbb{C} A \cap \mathbb{C} P^1$, then each point of $\mathbb{C} A \cap \mathbb{C} P^1$ corresponds to an asymptote of $\mathbb{C} A \cap \mathbb{C}^2$. Asymptotes are imaginary.
in terms of asymptotes

If $\mathcal{C}A \cap \mathbb{C}P^1$, then each point of $\mathcal{C}A \cap \mathbb{C}P^1$ corresponds to an asymptote of $\mathcal{C}A \cap \mathbb{C}^2$.

Asymptotes are imaginary.

An imaginary line disjoint with $\mathbb{R}P^1_\infty$ meets either $\mathbb{C}P^1_\infty+$ or $\mathbb{C}P^1_\infty-$.
in terms of asymptotes

If $CA \cap CP^1_\infty$, then each point of $CA \cap CP^1_\infty$ corresponds to an asymptote of $CA \cap C^2$. Asymptotes are imaginary. An imaginary line disjoint with $RP^1_\infty$ meets either $CP^1_\infty^+$ or $CP^1_\infty^-$. Theorem 1 says:

$w(RA_+)$ equals the difference between the numbers of the asymptotes of $CA_+ \cap C^2$ of these two sorts.
near a kiss

Lemma
near a kiss

**Lemma**  (Imaginary intersection after a real kiss)
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Let $A$ and $B$ be curves of type I,
Lemma (Imaginary intersection after a real kiss)

Let $A$ and $B$ be curves of type I, with $R_A^+$ and $R_B^+$ almost kissing each other near a point $p$. 
Lemma  (Imaginary intersection after a real kiss)

Let $A$ and $B$ be curves of type I, with $RA_+$ and $RB_+$ almost kissing each other near a point $p$. 

\[
RA_+ \quad \overset{p}{\quad} \quad RB_+
\]
Lemma (Imaginary intersection after a real kiss)

Let $A$ and $B$ be curves of type I, with $\mathbb{R}A_+$ and $\mathbb{R}B_+$ almost kissing each other near a point $p$.

Then $\mathbb{C}A_+$ meets $\mathbb{C}B_+$ at an imaginary point near $p$. 
Lemma  (Imaginary intersection after a real kiss)
Let $A$ and $B$ be curves of type I, with $R_A^+$ and $R_B^+$ almost kissing each other near a point $p$.

Then $C_A^+$ meets $C_B^+$ at an imaginary point near $p$, while $C_A^-$ does not.
**Lemma (Imaginary intersection after a real kiss)**

Let $A$ and $B$ be curves of type I, with $\mathbb{R}A_+$ and $\mathbb{R}B_+$ almost kissing each other near a point $p$.

Then $\mathcal{C}A_+$ meets $\mathcal{C}B_+$ at an imaginary point near $p$, while $\mathcal{C}A_-$ does not.

**Proof.** Look at the scene *complexly* from the left hand side.
Lemma (Imaginary intersection after a real kiss)

Let $A$ and $B$ be curves of type I, with $R_A^+$ and $R_B^+$ almost kissing each other near a point $p$.

Then $C_A^+$ meets $C_B^+$ at an imaginary point near $p$, while $C_A^-$ does not.

Proof. Look at the scene complexly from the left hand side.
Lemma  (Imaginary intersection after a real kiss)

Let $A$ and $B$ be curves of type I, with $RA_+$ and $RB_+$ almost kissing each other near a point $p$.

\[
\begin{align*}
&\text{Then } CA_+ \text{ meets } CB_+ \text{ at an imaginary point near } p, \\
&\text{while } CA_- \text{ does not.}
\end{align*}
\]

Proof. Look at the scene complexly from the left hand side.

Pictures:
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P^1 \) and
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_1^\infty$ and rotate oriented real line $L$ around $p$. 
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P_1^1 \) and rotate oriented real line \( L \) around \( p \).
(In \( \mathbb{R}^2 \) rotation around a point at infinity is a translation.)
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \).
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $\mathcal{C}A_+ \circ \mathcal{C}L_+ - \mathcal{C}A_+ \circ \mathcal{C}L_-$. We consider only imaginary intersection points.
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P^1_{\infty} \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \).

We consider only imaginary intersection points.

(Although we start with \( RL = \mathbb{R}P^1_{\infty} \) and \( \mathbb{R}P^1_{\infty} \cap RA = \emptyset \), \( RL \) sweeps the whole \( RA \) while moving).
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $CA_+ \circ CL_+ - CA_+ \circ CL_-$. $CA_+ \circ CL_+ - CA_+ \circ CL_-$ changes, when $RL$ gets tangent to $RA$. 
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_1^\infty$ and rotate oriented real line $L$ around $p$ counting changes of $CA_+ \circ CL_+ - CA_+ \circ CL_-$. $CA_+ \circ CL_+ - CA_+ \circ CL_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \to S^1$. 
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \). \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \to S^1 \).

Consider, for example, the following curve:
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-$. $\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \rightarrow S^1$.

$\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+) = -1,$
$\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_-) = 0,$
$\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-) = -1$

$ldeg = +1$
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}^1_{\infty} \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \). \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \to S^1 \).

\[
\Delta(CA_+ \circ CL_+) = -1,
\Delta(CA_+ \circ CL_-) = 0,
\Delta(CA_+ \circ CL_+ - CA_+ \circ CL_-) = -1
\]

\( ldeg = +1 \)
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( \mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_- \). \( \mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \to S^1 \).

\[
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+) = 0, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_-) = -1, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-) = +1 \\
ldeg = -1
\]
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}^1_{\infty} \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \).
\( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \).
At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

\[
\Delta(CA_+ \circ CL_+) = 0,
\Delta(CA_+ \circ CL_-) = -1,
\Delta(CA_+ \circ CL_+ - CA_+ \circ CL_-) = +1
\]
\( ldeg = -1 \)
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( \mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_- \). \( \mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

\[
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+) = +1, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_-) = 0, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-) = +1 \\
ldeg = -1
\]
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $CA_+ \circ CL_+ - CA_+ \circ CL_-$. $CA_+ \circ CL_+ - CA_+ \circ CL_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \rightarrow S^1$.

$\Delta(\text{CA}_+ \circ \text{CL}_+) = +1$,  
$\Delta(\text{CA}_+ \circ \text{CL}_-) = 0$,  
$\Delta(\text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_-) = +1$  
$ldeg = -1$
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{R}P^1_\infty$ and rotate oriented real line $L$ around $p$ counting changes of $\text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_-$. $\text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \to S^1$.

$\Delta(\text{CA}_+ \circ \text{CL}_+) = 0$,
$\Delta(\text{CA}_+ \circ \text{CL}_-) = +1$,
$\Delta(\text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_-) = -1$

$ldeg = +1$
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $CA_+ \circ CL_+ - CA_+ \circ CL_-$. $CA_+ \circ CL_+ - CA_+ \circ CL_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \rightarrow S^1$.

$\Delta(CA_+ \circ CL_+) = 0$,

$\Delta(CA_+ \circ CL_-) = +1$,

$\Delta(CA_+ \circ CL_+ - CA_+ \circ CL_-) = -1$

$ldeg = +1$
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}^1_\infty \) and rotate oriented real line \( L \) around \( p \) counting changes of \( \mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_- \).

At these moments, evaluate also local degree \( ldeg \) of Gauss map \( \mathbb{RA} \rightarrow S^1 \).

\[
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+) = 0, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_-) = +1, \\
\Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-) = -1 \\
ldeg = +1
\]

At each of the moments, \( \Delta(\mathcal{CA}_+ \circ \mathcal{CL}_+ - \mathcal{CA}_+ \circ \mathcal{CL}_-) = -ldeg \).
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P^1_\infty \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \).
\( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \).
At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \to S^1 \).

\[
\Delta(CA_+ \circ CL_+) = 0,
\Delta(CA_+ \circ CL_-) = +1,
\Delta(CA_+ \circ CL_+ - CA_+ \circ CL_-) = -1
\]
\( ldeg = +1 \)

At each of the moments, \( \Delta(CA_+ \circ CL_+ - CA_+ \circ CL_-) = -ldeg \).
The full change of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) is \( -2w(\mathcal{RA}) \), since we have summed up \( -ldeg \) over the preimages of 2 points.
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( \text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_- \). \( \text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

The full change of \( \text{CA}_+ \circ \text{CL}_+ - \text{CA}_+ \circ \text{CL}_- \) is \(-2w(RA)\).
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \). 

\( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

The full change of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) is \(-2w(\mathbb{RA})\).

On the other hand, the full change is

\[-2(CA_+ \circ CP^1_{\infty+} - CA_+ \circ CP^1_{\infty-})\]
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \). \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \to S^1 \).

The full change of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) is \(-2w(RA)\).

On the other hand, the full change is

\[
-2(CA_+ \circ CP_\infty^1_+ - CA_+ \circ CP_\infty^1_-)
\]

Indeed, we have turned \( \mathbb{RP}_\infty^1 \) by \( \pi \),
proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{RP}_\infty^1 \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \).

\( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \).

At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

The full change of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) is \( -2w(RA) \).

On the other hand, the full change is
\[-2(CA_+ \circ CP_{\infty_+}^1 - CA_+ \circ CP_{\infty_-}^1)\]

Indeed, we have turned \( \mathbb{RP}_\infty^1 \) by \( \pi \), its orientation has reversed,
proof of Theorem 1

Choose a generic point $p$ on $\mathbb{RP}_\infty^1$ and rotate oriented real line $L$ around $p$ counting changes of $CA_+ \circ CL_+ - CA_+ \circ CL_-$. $CA_+ \circ CL_+ - CA_+ \circ CL_-$ changes, when $RL$ gets tangent to $RA$. At these moments, evaluate also local degree $ldeg$ of Gauss map $RA \to S^1$.

The full change of $CA_+ \circ CL_+ - CA_+ \circ CL_-$ is $-2w(RA)$. On the other hand, the full change is

$$-2(\text{CA}_+ \circ CP_{\infty+}^1 - \text{CA}_+ \circ CP_{\infty-}^1)$$

Indeed, we have turned $\mathbb{RP}_\infty^1$ by $\pi$, its orientation has reversed, and $CA_+ \circ CL_+ - CA_+ \circ CL_-$ evolved from $\text{CA}_+ \circ CP_{\infty+}^1 - \text{CA}_+ \circ CP_{\infty-}^1$ to $\text{CA}_+ \circ CP_{\infty-}^1 - \text{CA}_+ \circ CP_{\infty+}^1 = -(\text{CA}_+ \circ CP_{\infty+}^1 - \text{CA}_+ \circ CP_{\infty-}^1)$. 


proof of Theorem 1

Choose a generic point \( p \) on \( \mathbb{R}P^1_\infty \) and rotate oriented real line \( L \) around \( p \) counting changes of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \). \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) changes, when \( RL \) gets tangent to \( RA \). At these moments, evaluate also local degree \( ldeg \) of Gauss map \( RA \rightarrow S^1 \).

The full change of \( CA_+ \circ CL_+ - CA_+ \circ CL_- \) is \(-2w(RA)\).

On the other hand, the full change is
\[
-2(CA_+ \circ CP^1_\infty_+ - CA_+ \circ CP^1_\infty_-)
\]
Thus,
\[
w(RA) = CA_+ \circ CP^1_\infty_+ - CA_+ \circ CP^1_\infty_-
\]
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself:
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The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:
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However, this move is impossible for algebraic curves of type I.
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. **It takes two, to become imaginary!**
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{R}A$ with imaginary complex conjugate branches.
The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

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improving Whitney number

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However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{RA}$ with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1.
The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{RA}$ with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1. Allow such points,
The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{R}A$ with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1. Allow such points, but take into account their contribution to $w(\mathbb{R}A_+)$. 
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C)$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{RA}$ with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1. Allow such points, but take into account their contribution to $w(\mathbb{RA}_+)$. Only one of the branches passing through it, belongs to $\mathbb{CA}_+$. 
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not $C^1$, but $C^0$. $w(C')$ changes by 1, when $C$ moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on $\mathbb{R}A$ with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1. Allow such points, but take into account their contribution to $w(\mathbb{R}A_+)$. Only one of the branches passing through it, belongs to $\mathbb{C}A_+$. Its intersection number with $\mathbb{R}^2$ is to be added to $w(\mathbb{R}A_+)$.
improving Whitney number

The expression provided by Theorem 1 for Whitney number seems to be more stable than the Whitney number itself: not \( C^1 \), but \( C^0 \). \( w(C) \) changes by 1, when \( C \) moves like that:

However, this move is impossible for algebraic curves of type I. A real double point cannot disappear by becoming imaginary alone. Instead, it can turn into a double point isolated on \( RA \) with imaginary complex conjugate branches. A double real point with imaginary branches is not allowed in Theorem 1. Allow such points, but take into account their contribution to \( w(RA_+) \). Only one of the branches passing through it, belongs to \( CA_+ \). Its intersection number with \( \mathbb{R}^2 \) is to be added to \( w(RA_+) \). Improved \( w(RA_+) \) is more invariant, and Theorem 1 holds true for it.
Writhe

• wrihte of a knot diagram
• encomplexed wrihte

Arnold invariants

Encomplexing $J$
writhe of a knot diagram

At a crossing an oriented link diagram looks either like this: ✻
writhe of a knot diagram

At a crossing an oriented link diagram looks either like this: ⤡ or like that: ⤣.
writhe of a knot diagram

At a crossing an oriented link diagram looks either like this: \(\xleftarrow{\xrightarrow{}}\) or like that: \(\xleftarrow{\xrightarrow{}}\).

(Local) writhe: \(w(\xleftarrow{\xrightarrow{}}) = +1\), \(w(\xleftarrow{\xrightarrow{}}) = -1\).
writhe of a knot diagram

At a crossing an oriented link diagram looks either like this: \( \frown \) or like that: \( \frown \).

\((Local)\text{ writhe}:\) \( w(\frown) = +1, \ w(\frown) = -1. \)

\text{Writhe} of an oriented link diagram is the sum of local writhes over all crossings.
writhe of a knot diagram

At a crossing an oriented link diagram looks either like this: \( \heartsuit \) or like that: \( \bowtie \).

(Local) writhe: \( w(\heartsuit) = +1 \), \( w(\bowtie) = -1 \).

Writhe of an oriented link diagram is the sum of local writhes over all crossings. It is not invariant: the first Reidemeister move changes it by one.
encomplexed writhe

Introduction

Whitney number

Writhe
- writhe of a knot diagram
  - encomplexed writhe

Arnold invariants

Encomplexing $J_-$

For an algebraic link the move cannot happen.
For an algebraic link the move cannot happen.
The first real algebraic Reidemeister move looks like that:
For an algebraic link the move
cannot happen.

The first real algebraic Reidemeister move looks like that:
A crossing turns into a solitary real crossing of
two complex conjugate imaginary branches.
encomplexed writhe

For an algebraic link the move

\[
\begin{array}{c}
\rightarrow \\
\downarrow
\end{array}
\]

cannot happen.

The first real algebraic Reidemeister move looks like that:

A crossing turns into a solitary real crossing of two complex conjugate imaginary branches.

There is a writhe of a solitary crossing such that the total writhe does not change.
Arnold invariants

- genericity of immersions
- main strata of discriminant
- perestroikas
- Arnold's invariants

Encomplexing $J$
genericity of immersions

An immersion $S^1 \looparrowright \mathbb{R}^2$ is \textit{generic}, if it has neither triple point, nor a point of self-tangency.
genericity of immersions

An immersion $S^1 \hookrightarrow \mathbb{R}^2$ is \textit{generic}, if has neither triple point, nor a point of self-tangency. It has \textbf{only ordinary double points} of transversal self-intersection.
genericity of immersions

An immersion \( S^1 \hookrightarrow \mathbb{R}^2 \) is generic, if has neither triple point, nor a point of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is ordinary, if the branches at the point are transversal to each other.
genericity of immersions

An immersion $S^1 \hookrightarrow \mathbb{R}^2$ is *generic*, if has neither triple point, nor a point of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is *ordinary*, if the branches at the point are transversal to each other.

A self-tangency point of an immersion is *ordinary*, if the branches have distinct curvatures at the point.
An immersion $S^1 \to \mathbb{R}^2$ is generic, if has neither triple point, nor a point of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is ordinary, if the branches at the point are transversal to each other.

A self-tangency point of an immersion is ordinary, if the branches have distinct curvatures at the point.

A self-tangency point of an immersion is called direct, if the velocity vectors are pointing the same direction;
genericity of immersions

An immersion $S^1 \hookrightarrow \mathbb{R}^2$ is *generic*, if has neither triple point, nor a point of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is *ordinary*, if the branches at the point are transversal to each other.

A self-tangency point of an immersion is *ordinary*, if the branches have distinct curvatures at the point.

A self-tangency point of an immersion is called *direct*, if the velocity vectors are pointing the same direction; otherwise it is *inverse*. 
An immersion \( S^1 \rightarrow \mathbb{R}^2 \) is *generic*, if has neither triple point, nor a point of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is *ordinary*, if the branches at the point are transversal to each other.

A self-tangency point of an immersion is *ordinary*, if the branches have distinct curvatures at the point.

A self-tangency point of an immersion is called *direct*, if the velocity vectors are pointing the same direction; otherwise it is *inverse*.

All non-generic immersions form a *discriminant hypersurface*, or just discriminant in the space of all immersions.
main strata of discriminant

The discriminant is stratified.
main strata of discriminant

The discriminant is stratified. There are 3 main open strata:
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions
The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points,
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point,
The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
- the set $ST_-$ of all immersions
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
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The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
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The discriminant is stratified. There are 3 main strata:

- The set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
- The set $ST_-$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary inverse self-tangency point.
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
- the set $ST_-$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary inverse self-tangency point.
- the set $TP$ of all immersions
main strata of discriminant

The discriminant is stratified. There are 3 main strata:

- the set $ST_+$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary direct self-tangency point.
- the set $ST_-$ of all immersions without triple points, with only one non-transversal double point, and this is an ordinary inverse self-tangency point.
- the set $TP$ of all immersions which have only one triple point,
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Arnold’s invariants

For generic $C : S^1 \hookrightarrow \mathbb{R}^2$, Arnold introduced numerical characteristics $J^+(C)$, $J^-(C)$ and $St(C)$ defined by the following properties:
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- For curves $K_0$, $K_1$, $K_2$, $K_3$, $K_4$, …

the invariants take the following values:
Arnold’s invariants

For generic $C : S^1 \hookrightarrow \mathbb{R}^2$, Arnold introduced numerical characteristics $J^+(C)$, $J^-(C)$ and $St(C)$ defined by the following properties:

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$J^+(K_0) = 0$, $J^+(K_{i+1}) = −2i$ ($i = 0, 1, \ldots$);
$J^−(K_0) = −1$, $J^−(K_{i+1}) = −3i$ ($i = 0, 1, \ldots$);
$St(K_0) = 0$, $St(K_{i+1}) = i$ ($i = 0, 1, \ldots$).
**Introduction**

**Whitney number**

**Writhe**

**Arnold invariants**

**Encomplexing $J_-$**
- choice of curves
- new perestrojkas
- Smoothing of curve
- Index of point
- Complex orientation formula
- intersection of complex halves
- encomplexing $J_-$
- back to immersed circles
- $J_+$
- last slide

**Encomplexing $J_-$**
choice of curves

Consider irreducible real curves of degree $d$
Consider irreducible real plane projective curves of degree $d$. 

choice of curves
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I
Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

A generic curve $A$ of this kind has only non-degenerate double singular points.
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Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

A generic curve $A$ of this kind has only non-degenerate double singular points, they can be of the following 4 types:

- real double points with two real branches
- solitary real double point with two imaginary conjugate branches

isolated point in $\mathbb{R}A$, local normal form $x^2 + y^2 = 0$. 
**choice of curves**

Consider irreducible real plane projective curves of degree \( d \), genus \( g \) and type I, equipped with complex orientations.

A generic curve \( A \) of this kind has only non-degenerate double singular points, they can be of the following 4 types:
- real double points with two real branches \( \times \),
- solitary real double point with two imaginary conjugate branches,

At a solitary ordinary double point, the choice of \( CA_+ \) determines a local orientation of \( \mathbb{RP}^2 \).
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

A generic curve $A$ of this kind has only non-degenerate double singular points, they can be of the following 4 types:

- real double points with two real branches x,
- solitary real double point with two imaginary conjugate branches,

At a solitary ordinary double point, the choice of $C A_+$ determines a local orientation of $\mathbb{R}P^2$ such that $\mathbb{R}P^2$ equipped with this local orientation intersects $C A_+$ at this point with intersection number $+1$. 
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

A generic curve $A$ of this kind has only non-degenerate double singular points, they can be of the following 4 types:

- real double points with two real branches $\times$,
- solitary real double point with two imaginary conjugate branches.

At a solitary ordinary double point, the choice of $\mathbb{C}A_+$ determines a local orientation of $\mathbb{R}P^2$.

Another way to get the local orientation: perturb the curve keeping type I and converting the solitary point into an oval.
choice of curves

Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

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Another way to get the local orientation: perturb the curve keeping type I and converting the solitary point into an oval. The complex orientation of this oval gives the local orientation of $\mathbb{R}P^2$. 
Consider irreducible real plane projective curves of degree $d$, genus $g$ and type I, equipped with complex orientations.

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- imaginary double point of self-intersection of $\mathcal{CA}_+$,
choice of curves

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- imaginary intersection point of $CA_+$ and $CA_-$.
choice of curves

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- real double points with two real branches $\times$,
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- imaginary double point of self-intersection of $CA_+$,
- imaginary intersection point of $CA_+$ and $CA_-$. Denote the number of the latter points by $\sigma$. 
new perestrojcas

Generic RA experiences perestrojcas considered above plus the following three new ones.
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Solitary self-tangency perestrojka.
new perestrojkas

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new perestrojkas

Generic $\mathbb{R}A$ experiences perestrojkas considered above plus the following three new ones.

Solitary self-tangency perestrojka.
new perestrojkas

Generic RA experiences perestrojkas considered above plus the following three new ones.

Solitary self-tangency perestrojka.

Triple point perestrojka with two imaginary branches.
new perestrojkas

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Solitary self-tangency perestrojka.

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new perestrojkas

Generic $\mathbb{RA}$ experiences perestrojkas considered above plus the following three new ones.

- Solitary self-tangency perestrojka.
- Triple point perestrojka with two imaginary branches.
- Cusp perestrojka.

- choice of curves
- new perestrojkas
- Smoothing of curve
- Index of point
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new perestrojkas

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Solitary self-tangency perestrojka.

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Cusp perestrojka.
Smoothing of curve

Smoothen a generic curve of type I according to the complex orientation: $A \mapsto \tilde{A}$
Smoothing of curve

Smoothen a generic curve of type I according to the complex orientation: $A \leftrightarrow \tilde{A}$
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\[ \longleftrightarrow \]
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Index of point

For oriented closed curve $C \subset \mathbb{R}P^2$ and $x \in \mathbb{R}P^2 \setminus C$,
Index of point

For oriented closed curve \( C \subset \mathbb{R}P^2 \) and \( x \in \mathbb{R}P^2 \setminus C \), define non-negative integer or half-integer \( \text{ind}_C(x) \):
Index of point

For oriented closed curve $C \subset \mathbb{R}P^2$ and $x \in \mathbb{R}P^2 \setminus C$, define non-negative integer or half-integer $\text{ind}_C(x)$: $C$ realizes $2 \cdot \text{ind}_C(x)$-fold generator of $H_1(\mathbb{R}P^2 \setminus \{x\}) = \mathbb{Z}$. 
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Examples:

1. $\text{ind}_{\mathbb{R}P^1}(x) = \frac{1}{2}$
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Examples:

1. $\text{ind}_{\mathbb{R}P^1}(x) = \frac{1}{2}$

2. If $C$ is a circle $x_1^2 + x_2^2 = x_0^2$ and $x$ is a point in the disk bounded by $C$, then $\text{ind}_C(x) = 1$ independently on orientation of $C$. 
Index of point

For oriented closed curve $C \subset \mathbb{R}P^2$ and $x \in \mathbb{R}P^2 \setminus C$, define non-negative integer or half-integer $\text{ind}_C(x)$:

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Examples:

1. $\text{ind}_{\mathbb{R}P^1}(x) = \frac{1}{2}$
2. If $C$ is a circle $x_1^2 + x_2^2 = x_0^2$ and $x$ is a point in the disk bounded by $C$, then $\text{ind}_C(x) = 1$.
3. If $C$ consists of two concentric circles, and $x$ is their common center, then $\text{ind}_C(x)$ is either 0 or 2.
Let $A$ be generic real plane projective algebraic curve of degree $d$ and type I.
Complex orientation formula

Let $A$ be generic real plane projective algebraic curve of degree $d$ and type I.

Then

$$\frac{d^2}{4} = \sigma + \int_{\mathbb{RP}^2 \setminus \widetilde{RA}} (\text{ind}_{RA}(x))^2 \, d\chi(x)$$
Complex orientation formula

Let $A$ be generic real plane projective algebraic curve of degree $d$ and type I. Then

$$\frac{d^2}{4} = \sigma + \int_{\mathbb{R}P^2 \setminus \tilde{RA}} \left( \text{ind}_{\mathbb{R}A}(x) \right)^2 d\chi(x)$$

here $\sigma$ is the number of imaginary double points of $A$, where $\mathbb{C}A_+$ and $\mathbb{C}A_-$ meet,
Complex orientation formula

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Complex orientation formula

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Integral \( \int f(x) \, d\chi(x) \) is defined for \( f \) which is a finite linear combination of characteristic functions,
Complex orientation formula

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\[
f = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{S_i},
\]

by formula
Complex orientation formula

Let $A$ be generic real plane projective algebraic curve of degree $d$ and type I.

Then

$$\frac{d^2}{4} = \sigma + \int_{\mathbb{R}P^2 \setminus \mathcal{R}A} (\text{ind}_{\mathcal{R}A}(x))^2 \, d\chi(x)$$

here $\sigma$ is the number of imaginary double points of $A$, where $\mathcal{C}A_+$ and $\mathcal{C}A_-$ meet, and the integral is against the Euler characteristic.

Integral $\int f(x) \, d\chi(x)$ is defined for $f$ which is a finite linear combination of characteristic functions,

$$f = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{S_i},$$

by formula

$$\int f(x) \, d\chi(x) = \sum_{i=1}^{r} \lambda_i \chi(S_i).$$
Denote by $\sigma$ the number of imaginary intersection points of $CA_+$ and $CA_-$ and study its behavior under perestrojkas.
intersection of complex halves

Denote by $\sigma$ the number of imaginary intersection points of $\mathcal{C}A_+$ and $\mathcal{C}A_-$ and study its behavior under perestroikas.

$\sigma$ does not change.
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**encomplexing \( J_- \)**

Notice that \( \sigma \) behaves in the same way as \( J_- \) under direct and inverse self-tangency and triple point perestrojkas with only real branches involved.
encomplexing $J_-$

Notice that $\sigma$ behaves in the same way as $J_-$ under direct and inverse self-tangency and triple point perestrojkas with only real branches involved. Thus, $\sigma$ can be considered as an encomplexed $J_-$. 
**encomplexing \( J_- \)**

Notice that \( \sigma \) behaves in the same way as \( J_- \) under direct and inverse self-tangency and triple point perestrojkas with only real branches involved.

Thus, \( \sigma \) can be considered as an encomplexed \( J_- \).

Complex orientation formula can be rewritten as a formula for \( \sigma \):

\[
\sigma = \frac{d^2}{4} - \int_{\mathbb{R}P^2 \setminus \mathbb{RA}} (\text{ind}_{\mathbb{RA}}(x))^2 \, d\chi(x).
\]
back to immersed circles

Integral \(- \int_{\mathbb{R}P^2 \setminus \mathcal{R}_A} (\text{ind}_{\mathcal{R}_A}(x))^2 \, d\chi(x)\) has the same behavior under direct and inverse self-tangency and triple point perestrojkas as \(\sigma\) and \(J_\_\).
Integral $-\int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{C}}(x))^2 \, d\chi(x)$ has the same behavior under direct and inverse self-tangency and triple point perestrojkas as $\sigma$ and $J_-$. This suggests to compare $J_-(C')$ with

$$-\int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{C}}(x))^2 \, d\chi(x)$$

for a generic immersed circle $C'$. 

back to immersed circles
Integral $-\int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{A}}(x))^2 \, d\chi(x)$ has the same behavior under direct and inverse self-tangency and triple point perestroikas as $\sigma$ and $J_-$. This suggests to compare $J_-(C')$ with

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for a generic immersed circle $C'$.

**Theorem.** *For any generic immersed circle $C'$*

$$J_-(C') = 1 - \int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{C}}(x))^2 \, d\chi(x).$$
Corollary. For any generic immersed circle $C$ with $n$ double points

$$J_+(C) = 1 + n - \int_{\mathbb{R}^2 \setminus \tilde{C}} (\text{ind}_{\tilde{C}}(x))^2 \, d\chi(x).$$
The beginning of the story
last slide

The beginning of the story, or the end of it?