Khovanov homology of framed and signed chord diagrams.

Oleg Viro

December 2, 2006
A *knot* is a smooth simple closed curve in the 3-space.
A knot is a smooth simple closed curve in the 3-space. That is a circle smoothly embedded into $\mathbb{R}^3$. 

Classical link diagrams

Knots and links
- Classical link diagrams
- 1D-picture
- Gauss diagram
- Reconstruction of knot

Virtual links

Moves

Kauffman bracket
Gauss diagrams of a poor man

Khovanov homology
Orientation of chord diagrams
Khovanov complex of framed chord diagram
Classical link diagrams

A knot is a smooth simple closed curve in the 3-space. A link is a union of several disjoint knots.
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A **link** is a union of several disjoint knots.
To describe a knot graphically, project it to a plane.
Classical link diagrams

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A link diagram:
1D-picture

A knot diagram
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In many cases 1D picture serves better.
1D-picture

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1D picture comes from a parameterization.
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1D picture comes from a parameterization.
Gauss diagram

Decorate the source:
Gauss diagram

Decorate the source:
- with arrows from overpass to underpass,
Gauss diagram

Decorate the source:
- with arrows from overpass to underpass,
Gauss diagram

Decorate the source:
- with arrows from overpass to underpass,
- with the signs of crossings
Gauss diagram

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Signs:
Gauss diagram

Decorate the source:
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Signs: positive
**Gauss diagram**

Decorate the source:

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- with the **signs** of crossings

---

**Signs:** positive $\times\times$, negative $\times\times$. 
Gauss diagram

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Gauss diagram

Decorate the source:
- with arrows from overpass to underpass,
- with the signs of crossings

Signs: positive \(\rightarrow\), negative \(\leftarrow\). The result, 

is called a Gauss diagram of the knot.
Take any such diagram, say, and try to reconstruct the knot.
Reconstruction of knot

Start with crossings:
Reconstruction of knot

Start with crossings:
Reconstruction of knot

Connect them step by step:
Reconstruction of knot

Connect them step by step:
Reconstruction of knot

The next step does not work!
Reconstruction of knot

But let us continue!
Reconstruction of knot

Yet another obstruction!
Reconstruction of knot

We did it!
Reconstruction of knot

Classical link diagrams
1D-picture
Gauss diagram
Reconstruction of knot

Virtual links

Moves

Kauffman bracket
Gauss diagrams of a poor man

Khovanov homology
Orientation of chord diagrams
Khovanov complex of framed chord diagram

We did it! But what is the result?
We did it! But what is the result?

The result is called a virtual knot diagram.
Virtual links

- Virtual knot diagrams
- Diagram on a surface

Moves

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Virtual knot diagrams

A virtual knot diagram has crossings of 2 types:
A virtual knot diagram has crossings of 2 types: classical
A virtual knot diagram has crossings of 2 types: classical or real.
Virtual knot diagrams

A virtual knot diagram has crossings of 2 types: classical or real decorated like in a knot diagram.
A virtual knot diagram has crossings of 2 types: classical or real decorated like in a knot diagram and virtual.
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A virtual knot diagram has crossings of 2 types: classical or real decorated like in a knot diagram and virtual not decorated at all.
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Who can help to get rid of virtual crossings?

Handles!
Virtual knot diagrams

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*Handles!*
Diagram on a surface

A knot diagram drawn on orientable surface $S$
Diagram on a surface

A knot diagram drawn on orientable surface $S$, instead of the plane
A knot diagram drawn on orientable surface $S$, instead of the plane, defines a knot in a thickened surface $S \times I$. 
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*Any Gauss diagram appears in this way.*
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For each Gauss diagram there is the **smallest** surface
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Virtual knot diagrams emerge as projections to plane
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The surfaces is not unique:
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A knot diagram drawn on orientable surface $S$, instead of the plane, defines a knot in a thickened surface $S \times I$. It defines also a Gauss diagram. *Any Gauss diagram appears in this way.*

For each Gauss diagram there is the **smallest** surface with a knot diagram defining this Gauss diagram.

Virtual knot diagrams emerge as projections to plane of knot diagrams on a surface. The surfaces is not unique: one can add handles.
Knots and links

Virtual links

Moves
- Moves
- Moves of virtual link diagram
- Moves of Gauss diagrams
- Combinatorial incarnation of knot theory
- Topological meaning of virtual knot theory
- Isotopy problem

Kauffman bracket

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Orientation of chord diagrams

Khovanov complex of framed chord diagram
Moves

What happens to a link diagram, when the link moves?
Moves

What happens to a link diagram, when the link moves? Link diagram moves, too.
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Link diagram moves, too.

Reidemeister moves:
Moves

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Reidemeister moves:

(R1):
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# Moves

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**Reidemeister moves:**

(R1): 

![Reidemeister Move (R1)](image)
Moves

What happens to a link diagram, when the link moves? Link diagram moves, too.

Reidemeister moves:

(R1):

(R2):
Moves

What happens to a link diagram, when the link moves?
Link diagram moves, too.

Reidemeister moves:

(R1): \[ \begin{array}{c}
\text{Diagram 1} \\
\end{array} \]

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Moves

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Moves

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Reidemeister moves:

(R1): 

(R2): 

(R3): 

Moves of virtual link diagram

A virtual link diagram
(i.e., a plane projection of a link diagram on a surface)
Moves of virtual link diagram

A virtual link diagram moves like this:

```
0/0/0/0/0/0/0/0/0/0/0/0/0
0/0/0/0/0/0/0/0/0/0/0/0/0
0/0/0/0/0/0/0/0/0/0/0/0/0
0/0/0/0/0/0/0/0/0/0/0/0/0
1/1/1/1/1/1/1/1/1/1/1/1/1
1/1/1/1/1/1/1/1/1/1/1/1/1
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1/1/1/1/1/1/1/1/1/1/1/1/1
1/1/1/1/1/1/1/1/1/1/1/1/1
1/1/1/1/1/1/1/1/1/1/1/1/1
1/1/1/1/1/1/1/1/1/1/1/1/1
```

A virtual link diagram moves like this:

Reidemeister moves:
Moves of virtual link diagram

A virtual link diagram moves like this:

Reidemeister moves:

Virtual moves:
Moves of virtual link diagram

A virtual link diagram moves like this:

Reidemeister moves:

Virtual moves:

All virtual moves can be replaced by detour moves:
Moves of Gauss diagrams

Gauss diagrams has nothing to do with virtual crossings!
Moves of Gauss diagrams

Gauss diagrams has nothing to do with virtual crossings! They do not change under virtual moves.
Reidemeister moves acts on Gauss diagram:
## Moves of Gauss diagrams

Reidemeister moves acts on Gauss diagram:

<table>
<thead>
<tr>
<th>Move’s name</th>
<th>Reidemeister move</th>
<th>Its action on Gauss diagram</th>
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<tbody>
<tr>
<td>Positive first move</td>
<td><img src="image" alt="Positive First Move" /></td>
<td><img src="image" alt="Positive First Move" /></td>
</tr>
<tr>
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**Moves of Gauss diagrams**

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<td>$\leftarrow \uparrow +$</td>
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<td>$\leftarrow \downarrow \rightarrow$</td>
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Combinatorial incarnation of knot theory

Knots and links

Virtual links

Moves
- Moves
- Moves of virtual link diagram
- Moves of Gauss diagrams
  - Combinatorial incarnation of knot theory
- Topological meaning of virtual knot theory
- Isotopy problem

Kauffman bracket

Gauss diagrams of a poor man

Khovanov homology

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Combinatorial incarnation of knot theory

Classical Links $\rightarrow$ Link diagrams
Combinatorial incarnation of knot theory

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Classical Links → Link diagrams
Isotopies → Reidemeister moves
Combinatorial incarnation of knot theory

- Classical Links → Link diagrams
- Isotopies → Reidemeister moves

Combinatorial incarnations of virtual knot theory

- Moves
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### Combinatorial incarnation of knot theory

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<th>Knots and links</th>
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#### Combinatorial incarnations of virtual knot theory

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Topological meaning of virtual knot theory

Third incarnation of virtual knot theory is provided by Kuperberg’s theorem.
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Virtual links up to virtual isotopies = Irreducible links in thickened orientable surfaces up to orientation preserving homeomorphisms.
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Implies that virtual links generalize classical ones.
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Virtual links up to virtual isotopies

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Bridges combinatorics
Topological meaning of virtual knot theory

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Virtual links up to virtual isotopies = Irreducible links in thickened orientable surfaces up to orientation preserving homeomorphisms.

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Bridges combinatorics (= 1D topology)
Topological meaning of virtual knot theory

Third incarnation of virtual knot theory is provided by Kuperberg’s theorem.

Virtual links up to virtual isotopies

\[ \text{Irreducible links in thickened orientable surfaces up to orientation preserving homeomorphisms.} \]

Implies that virtual links generalize classical ones.

Bridges combinatorics with (3D-) topology.
Isotopy Problem:

- Moves
  - Moves
  - Moves of virtual link diagram
  - Moves of Gauss diagrams
  - Combinatorial incarnation of knot theory
  - Topological meaning of virtual knot theory
  - Isotopy problem

- Kauffman bracket
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- Khovanov homology
  - Orientation of chord diagrams
  - Khovanov complex of framed chord diagram
Isotopy Problem: Are given two classical links isotopic?
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Combinatorial reformulation:
Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation: Can given two Gauss diagrams be related by moves?
Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
Can given two Gauss diagrams be related by moves?

Virtual Isotopy Problem:
Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
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Invariants needed!
Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
Can given two Gauss diagrams be related by moves?

Virtual Isotopy Problem:
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Invariants needed!
The most classical link invariant is the link group.
Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
Can given two Gauss diagrams be related by moves?

Virtual Isotopy Problem:
Can given two Gauss diagrams be related by moves?

Invariants needed!
The most classical link invariant is the link group, the fundamental group of the link complement $\mathbb{R}^3 \setminus \text{link}$.
Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
Can given two Gauss diagrams be related by moves?

Virtual Isotopy Problem:
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Invariants needed!
The most classical link invariant is the *link group*. It was generalized.
Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
Can given two Gauss diagrams be related by moves?

Virtual Isotopy Problem:
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Invariants needed!
The most classical link invariant is the *link group*. It was generalized, even in two ways!
Isotopy problem

Isotopy Problem: Are given two classical links isotopic?

Combinatorial reformulation:
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Invariants needed!
The most classical link invariant is the link group.
It was generalized: upper and lower!
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The most classical link invariant is the *link group*.
It was generalized: *upper and lower*!
In terms of links in a thickened surface this is the fundamental
group of the complement, but with one of two sides of the
boundary contracted to a point.
Isotopy problem

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The most classical link invariant is the link group. It was generalized: upper and lower!

In terms of links in a thickened surface this is the fundamental group of the complement, but with one of two sides of the boundary contracted to a point.

Kauffman bracket is more practical and elementary invariant.
Kauffman bracket

Example

Kauffman state sum model for Gauss diagrams

Gauss diagrams of a poor man

Khovanov homology

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]
Kauffman bracket

\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]

(a Laurent polynomial in $A$ with integer coefficients).
Kauffman bracket

\( \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \)

\( \langle \text{unknot} \rangle = \)
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \langle \bigcirc \rangle = \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = -A^2 - A^{-2} \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = - A^2 - A^{-2} \]

Example

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\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = - A^2 - A^{-2} \]

\[ \langle \text{Hopf link} \rangle = \]

\[ \langle \bigcirc \bigcirc \rangle = \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \quad \langle \bigcirc \rangle = - A^2 - A^{-2} \]

\[ \langle \text{Hopf link} \rangle = \quad \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \langle \bigcirc \rangle = -A^2 - A^{-2} \]

\[ \langle \text{Hopf link} \rangle = \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]

\[ \langle \text{empty link} \rangle = \langle \bigcirc \rangle = \]

---

Knots and links

Virtual links

Moves

Kauffman bracket

- Kauffman bracket
- Kauffman state sum. I
- Kauffman state sum. II
- Example
- Kauffman state sum model for Gauss diagrams

Gauss diagrams of a poor man

Khovanov homology

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[
\begin{align*}
\langle \text{unknot} \rangle &= \langle \bigcirc \rangle = -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \bigcirc \bigcirc \rangle = 1
\end{align*}
\]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = -A^2 - A^{-2} \]

\[ \langle \text{Hopf link} \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]

\[ \langle \text{empty link} \rangle = 1 \]


\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[
\begin{align*}
\langle \text{unknot} \rangle &= \langle \circ \rangle &= -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle &= A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \rangle &= 1 \\
\langle \text{trefoil} \rangle &= \langle \bigcirc \bigcirc \bigcirc \rangle
\end{align*}
\]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[
\begin{aligned}
\langle \text{unknot} \rangle &= \langle \bigcirc \rangle = -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \bigcirc \bigcirc \rangle = 1 \\
\langle \text{trefoil} \rangle &= \langle \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \rangle = A^7 + A^3 + A^{-1} - A^{-9}
\end{aligned}
\]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \]
\[ \langle \text{Hopf link} \rangle = \]
\[ \langle \text{empty link} \rangle = \]
\[ \langle \text{trefoil} \rangle = \]
\[ \langle \text{figure-eight knot} \rangle = \]

\[ \langle \bigcirc \rangle = -A^2 - A^{-2} \]
\[ \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]
\[ \langle \bigcirc \rangle = 1 \]
\[ \langle \bigcirc \bigcirc \bigcirc \rangle = A^7 + A^3 + A^{-1} - A^{-9} \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \quad \langle \bigcirc \rangle = -A^2 - A^{-2} \]
\[ \langle \text{Hopf link} \rangle = \quad \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]
\[ \langle \text{empty link} \rangle = \quad \langle \bigcirc \rangle = 1 \]
\[ \langle \text{trefoil} \rangle = \quad \langle \bigotimes \rangle = A^7 + A^3 + A^{-1} - A^{-9} \]
\[ \langle \text{figure-eight knot} \rangle = \quad \langle \bigotimes \rangle = \]
Kauffman bracket

\langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}]

\langle \text{unknot} \rangle = \langle \bigcirc \rangle = -A^2 - A^{-2}

\langle \text{Hopf link} \rangle = \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6}

\langle \text{empty link} \rangle = \langle \bigcirc \rangle = 1

\langle \text{trefoil} \rangle = \langle \bigcirc \bigcirc \bigcirc \rangle = A^7 + A^3 + A^{-1} - A^{-9}

\langle \text{figure-eight knot} \rangle = \langle \bigcirc \bigcirc \rangle = -A^{10} - A^{-10}
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[
\begin{align*}
\langle \text{unknot} \rangle &= \langle \bigcirc \rangle = -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \rangle = 1 \\
\langle \text{trefoil} \rangle &= \langle \bigotimes \rangle = A^7 + A^3 + A^{-1} - A^{-9} \\
\langle \text{figure-eight knot} \rangle &= \langle \bigotimes \bigotimes \rangle = -A^{10} - A^{-10}
\end{align*}
\]

Kauffman bracket is defined by the following properties:
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[
\begin{align*}
\langle \text{unknot} \rangle &= \langle \bigcirc \rangle = -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \bigcirc \bigcirc \rangle = 1 \\
\langle \text{trefoil} \rangle &= \langle \bigotimes \bigotimes \bigotimes \rangle = A^7 + A^3 + A^{-1} - A^{-9} \\
\langle \text{figure-eight knot} \rangle &= \langle \bigotimes \bigotimes \bigotimes \bigotimes \rangle = -A^{10} - A^{-10}
\end{align*}
\]

Kauffman bracket is defined by the following properties:

1. \[ \langle \bigcirc \rangle = -A^2 - A^{-2}, \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \quad \langle \bigcirc \rangle = -A^2 - A^{-2} \]

\[ \langle \text{Hopf link} \rangle = \quad \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]

\[ \langle \text{empty link} \rangle = \quad \langle \bigcirc \rangle = 1 \]

\[ \langle \text{trefoil} \rangle = \quad \langle \bigotimes \rangle = A^7 + A^3 + A^{-1} - A^{-9} \]

\[ \langle \text{figure-eight knot} \rangle = \quad \langle \bigotimes \rangle = -A^{10} - A^{-10} \]

Kauffman bracket is defined by the following properties:

1. \[ \langle \bigcirc \rangle = -A^2 - A^{-2} \]

2. \[ \langle D \bigcirc \rangle = \langle -A^2 - A^{-2} \rangle \langle D \rangle \]
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\[ \langle \text{unknot} \rangle = \quad \langle \bigcirc \rangle = -A^2 - A^{-2} \]
\[ \langle \text{Hopf link} \rangle = \quad \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \]
\[ \langle \text{empty link} \rangle = \quad \langle \bigcirc \rangle = 1 \]
\[ \langle \text{trefoil} \rangle = \quad \langle \bigcirc \bigcirc \bigcirc \rangle = A^7 + A^3 + A^{-1} - A^{-9} \]
\[ \langle \text{figure-eight knot} \rangle = \quad \langle \bigcirc \bigcirc \bigcirc \bigcirc \rangle = -A^{10} - A^{-10} \]

Kauffman bracket is defined by the following properties:
1. \[ \langle \bigcirc \rangle = -A^2 - A^{-2} , \]
2. \[ \langle D \bigcirc \bigcirc \rangle = (-A^2 - A^{-2})\langle D \rangle , \]
3. \[ \langle \bigotimes \bigotimes \rangle = A\langle \bigotimes \rangle + A^{-1}\langle \bigotimes \bigotimes \rangle \ (\text{Kauffman Skein Relation}) . \]
Kauffman bracket

\( \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \)

\[
\begin{align*}
\langle \text{unknot} \rangle &= \langle \bigcirc \rangle = -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= \langle \bigcirc \bigcirc \rangle = A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= \langle \bigcirc \bigcirc \bigcirc \rangle = 1 \\
\langle \text{trefoil} \rangle &= \langle \bigcirc \bigotimes \bigcirc \rangle = A^7 + A^3 + A^{-1} - A^{-9} \\
\langle \text{figure-eight knot} \rangle &= \langle \bigcirc \bigotimes \bigcirc \rangle = -A^{10} - A^{-10}
\end{align*}
\]

Kauffman bracket is defined by the following properties:

1. \( \langle \bigcirc \rangle = -A^2 - A^{-2} \),
2. \( \langle D \coprod \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle \),
3. \( \langle \bigotimes \bigotimes \rangle = A \langle \bigotimes \bigotimes \rangle + A^{-1} \langle \bigotimes \bigotimes \rangle \) (Kauffman Skein Relation).

Uniqueness is obvious.
Kauffman bracket

\[ \langle \text{Link diagram} \rangle \in \mathbb{Z}[A, A^{-1}] \]

\begin{align*}
\langle \text{unknot} \rangle &= -A^2 - A^{-2} \\
\langle \text{Hopf link} \rangle &= A^6 + A^2 + A^{-2} + A^{-6} \\
\langle \text{empty link} \rangle &= 1 \\
\langle \text{trefoil} \rangle &= A^7 + A^3 + A^{-1} - A^{-9} \\
\langle \text{figure-eight knot} \rangle &= -A^{10} - A^{-10}
\end{align*}

Kauffman bracket is defined by the following properties:

1. \( \langle \bigcirc \rangle = -A^2 - A^{-2} \),
2. \( \langle D \amalg \bigcirc \rangle = (-A^2 - A^{-2}) \langle D \rangle \),
3. \( \langle \bigotimes \rangle = A \langle \bigcirc \rangle + A^{-1} \langle \bigotimes \rangle \) (Kauffman Skein Relation).

Uniqueness is obvious.

Invariant under R2 and R3, under R1 multiplies by \(-A^{\pm 3}\).
Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.
Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram:
A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: \[ \includegraphics[width=0.2\textwidth]{knot_diagram} \] and its states:
Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: \( \includegraphics{example_knot} \) and its states: \( \includegraphics{example_states} \)
Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: \[
\begin{array}{c}
\begin{tikzpicture}
\draw[thick, blue] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
\draw[thick, blue] (0,0) .. controls (-1,1) and (-1,-1) .. (0,0);
\end{tikzpicture}
\end{array}
\]
and its states:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw[thick, blue] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
\draw[thick, blue] (0,0) .. controls (-1,1) and (-1,-1) .. (0,0);
\fill[green] (0,0) circle (2pt);
\end{tikzpicture}
\end{array},
\begin{array}{c}
\begin{tikzpicture}
\draw[thick, blue] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
\draw[thick, blue] (0,0) .. controls (-1,1) and (-1,-1) .. (0,0);
\fill[green] (0,0) circle (2pt);
\end{tikzpicture}
\end{array},
\begin{array}{c}
\begin{tikzpicture}
\draw[thick, blue] (0,0) .. controls (1,1) and (1,-1) .. (0,0);
\draw[thick, blue] (0,0) .. controls (-1,1) and (-1,-1) .. (0,0);
\fill[green] (0,0) circle (2pt);
\end{tikzpicture}
\end{array},
\end{array}
\]
Kauffman state sum. I

A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: \[\text{and its states:}\]

Gauss diagrams of a poor man

Khovanov complex of framed chord diagram
A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: and its states:
A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: and its states:

\[
\begin{array}{c}
\text{Knot diagram:} \\
\hspace{0.5cm} \\
\hspace{0.5cm} \\
\end{array}
\]
A *state* of diagram is a distribution of *markers* over all crossings.

Knot diagram: and its states:

Totally $2^c$ states, where $c$ is the number of crossings.
Kauffman state sum. II

Three numbers associated to a state $s$:
Kauffman state sum. II

Three numbers associated to a state $s$:
1. the number $a(s)$ of positive markers,
Kauffman state sum. II

Three numbers associated to a state \( s \):

1. the number \( a(s) \) of \textit{positive} markers,

2. the number \( b(s) \) of \textit{negative} markers,
Kauffman state sum. II

Three numbers associated to a state $s$:
1. the number $a(s)$ of \textit{positive} markers,
2. the number $b(s)$ of \textit{negative} markers,
3. the number $|s|$ of components of the curve obtained by smoothing along the markers:
Kauffman state sum. II

Three numbers associated to a state $s$:
1. the number $a(s)$ of positive markers,
2. the number $b(s)$ of negative markers,
3. the number $|s|$ of components of the curve obtained by smoothing along the markers:

\[
\begin{array}{c}
\text{Example} \\
\text{Kauffman state sum} \\
\text{model for Gauss diagrams} \\
\text{Gauss diagrams of a poor man} \\
\text{Khovanov homology} \\
\text{Orientation of chord diagrams} \\
\text{Khovanov complex of framed chord diagram}
\end{array}
\]
Kauffman state sum. II

Three numbers associated to a state \( s \):

1. the number \( a(s) \) of positive markers,

2. the number \( b(s) \) of negative markers,

3. the number \(|s|\) of components of the curve obtained by smoothing along the markers:

\[ s \rightarrow \text{smoothing}(s) = \]
Kauffman state sum. II

Three numbers associated to a state $s$:
1. the number $a(s)$ of positive markers,
2. the number $b(s)$ of negative markers,
3. the number $|s|$ of components of the curve obtained by smoothing along the markers:

$$s = \begin{array}{c}
\begin{array}{c}
\text{smoothing}(s) = \end{array}
\end{array}$$

$$|s| = 2$$
Kauffman state sum. II

Three numbers associated to a state $s$:
1. the number $a(s)$ of positive markers,
2. the number $b(s)$ of negative markers,
3. the number $|s|$ of components of the curve obtained by smoothing along the markers:

$$s = \quad \text{smoothing}(s) =$$

$$|s| = 2$$

State Sum: $$\langle D \rangle = \sum_{s \text{ state of } D} A^{a(s) - b(s)} \left(-A^2 - A^{-2}\right)^{|s|}$$
Example

Hopf link,
Example

Hopf link, \( \langle \bigcirc \circ \bigcirc \rangle = \)

- Knots and links
- Virtual links
- Moves
- Kauffman bracket
  - Kauffman bracket
  - Kauffman state sum. I
  - Kauffman state sum. II
- Example
  - Kauffman state sum model for Gauss diagrams
- Gauss diagrams of a poor man
- Khovanov homology
  - Orientation of chord diagrams
- Khovanov complex of framed chord diagram
Example

Hopf link, \[ \langle \text{Hopf link} \rangle = \]

\[ \langle \text{Hopf link} \rangle + \langle \text{Hopf link} \rangle + \langle \text{Hopf link} \rangle + \langle \text{Hopf link} \rangle = \]
Example

Hopf link, \( \bigcirc \bigcirc \)

\[
\langle \bigcirc \bigcirc \bigcirc \rangle = \langle \bigcirc \bigcirc \bigcirc \rangle + \langle \bigcirc \bigcirc \bigcirc \rangle + \langle \bigcirc \bigcirc \bigcirc \rangle + \langle \bigcirc \bigcirc \bigcirc \rangle = A^2(-A^2 - A^{-2})^2 + 2(-A^2 - A^{-2}) + A^{-2}(-A^2 - A^{-2})^2
\]
Example

Hopf link, \hspace{1cm}

\[
\langle \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}\rangle =
\langle \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}\rangle + \langle \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}\rangle + \langle \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}\rangle + \langle \begin{array}{c}
\begin{array}{c}
\bigcirc \\
\bigcirc
\end{array}
\end{array}\rangle =
\]

\[
A^2(-A^2-A^{-2})^2 + 2(-A^2-A^{-2}) + A^{-2}(-A^2-A^{-2})^2 =
\]

\[
A^6 + A^2 + A^{-2} + A^{-6}
\]
Kauffman state sum model for Gauss diagrams

Crossing $\leftrightarrow$ arrow.

Gauss diagrams of a poor man

Khovanov homology

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Kauffman state sum model for Gauss diagrams

Crossing $\leftrightarrow$ arrow.

Smoothing of a crossing $\leftrightarrow$ a surgery along the arrow.
Kauffman state sum model for Gauss diagrams

Crossing $\leftrightarrow$ arrow.

Smoothing of a crossing $\leftrightarrow$ a surgery along the arrow.

positive marker, positive crossing

negative marker, negative crossing
Kauffman state sum model for Gauss diagrams

Crossing $\leftrightarrow$ arrow.

Smoothing of a crossing $\leftrightarrow$ a surgery along the arrow.

positive marker, positive crossing

negative marker, positive crossing

positive marker, negative crossing

negative marker, positive crossing
Kauffman state sum model for Gauss diagrams

Crossing \(\leftrightarrow\) arrow.

\[
\begin{array}{c}
\text{positive marker, positive crossing} \\
\text{negative marker, positive crossing} \\
\text{positive marker, negative crossing} \\
\text{negative marker, negative crossing}
\end{array}
\]

Smoothing of a crossing \(\leftrightarrow\) a surgery along the arrow.

Smoothing depends only on the signs of marker and crossing.
Crossing $\leftrightarrow$ arrow.

Smoothing of a crossing $\leftrightarrow$ a surgery along the arrow.

Positive marker, positive crossing

Negative marker, negative crossing

Negative marker, positive crossing

Positive marker, negative crossing

Smoothing depends only of the signs of marker and crossing.

No need in direction of the arrow!
Kauffman state sum model for Gauss diagrams

Crossing $\leftrightarrow$ arrow.

Smoothing of a crossing $\leftrightarrow$ a surgery along the arrow.

Positive marker, positive crossing

Negative marker, negative crossing

Negative marker, positive crossing

Positive marker, negative crossing

Smoothing depends only of the signs of marker and crossing.

No need in direction of the arrow!

Kauffman state sum is defined for signed chord diagrams.
Gauss diagrams of a poor man

- Signed chord diagrams
- State of signed chord diagram
- Smoothing of a signed chord diagram
- Framing
- Framed chord diagrams
- Signed to framed
- Orientable thickenings of non-orientable surfaces
- Abstract construction of an orientable thickening
- A link in orientable thickening of a non-orientable surface

Khovanov homology

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Signed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\)
A chord diagram \((B, c_1, \ldots, c_n)\)
(a closed 1-manifold \(B\) (base), and
**Signed chord diagrams**

A chord diagram \((B, c_1, \ldots, c_n)\)

( a closed 1-manifold \(B\) (base), and disjoint chords \(c_1, \ldots, c_n\) with end points on the base.)
Signed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\) (a closed 1-manifold \(B\) (base), and disjoint chords \(c_1, \ldots, c_n\) with end points on the base.)

- in which \(B\) is oriented and
Signed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\)

(a closed 1-manifold \(B\) (base), and disjoint chords \(c_1, \ldots, c_n\) with end points on the base.)

• in which \(B\) is oriented and

• each chord is equipped with a sign
Signed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\) (a closed 1-manifold \(B\) (base), and disjoint chords \(c_1, \ldots, c_n\) with end points on the base.)

- in which \(B\) is oriented and
- each chord is equipped with a sign

is called a signed chord diagram.
State of signed chord diagram

A *state* of the signed chord diagram is a distribution of another collection of signs over the set of all chords.
State of signed chord diagram

A *state* of the signed chord diagram is a distribution of another collection of signs over the set of all chords. These are *marker signs,*
A state of the signed chord diagram is a distribution of another collection of signs over the set of all chords. These are *marker signs*, the original signs are *structure signs*. 
Smoothing of a signed chord diagram

A *smoothing* of a chord diagram \((B, c_1, \ldots, c_n)\) is the result of Morse modifications of index 1 performed on \(B\) along each of its chords.
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Smoothing of a signed chord diagram

A *smoothing* of a chord diagram \((B, c_1, \ldots, c_n)\) is the result of Morse modifications of index 1 performed on \(B\) along each of its chords.

Morse modification at a chord depends on its signs.
A smoothing of a chord diagram \((B, c_1, \ldots, c_n)\) is the result of Morse modifications of index 1 performed on \(B\) along each of its chords.

Morse modification at a chord depends on its signs. Denote by \(\sigma\) the product of the structure and the marker signs.
Smoothing of a signed chord diagram

A smoothing of a chord diagram \((B, c_1, \ldots, c_n)\) is the result of Morse modifications of index 1 performed on \(B\) along each of its chords.

Morse modification at a chord depends on its signs. Denote by \(\sigma\) the product of the structure and the marker signs. If \(\sigma = +\), the Morse modification preserves the structure orientation.
A *smoothing* of a chord diagram \((B, c_1, \ldots, c_n)\) is the result of Morse modifications of index 1 performed on \(B\) along each of its chords.

Morse modification at a chord depends on its signs. Denote by \(\sigma\) the product of the structure and the marker signs.

If \(\sigma = +\), the Morse modification preserves the structure orientation.

If \(\sigma = -\), the Morse modification destroys the orientation.
Framing

A sign of an arrow in Gauss diagram of a classical link depends on orientation of the link.
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If the link is not oriented, specify the framing on the chords giving positive smoothing.

shorthand notation
A chord diagram \((B, c_1, \ldots, c_n)\)
Framed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\)
Framed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\) in which each chord is equipped with a framing.
Framed chord diagrams

A chord diagram \( (B, c_1, \ldots, c_n) \)

in which each chord is equipped with a framing
A chord diagram \((B, c_1, \ldots, c_n)\) in which each chord is equipped with a framing is called a **framed chord diagram**.
A chord diagram \((B, c_1, \ldots, c_n)\) in which each chord is equipped with a framing is called a framed chord diagram.

Kauffman bracket state sum is defined for a framed chord diagram.
A chord diagram \((B, c_1, \ldots, c_n)\) in which each chord is equipped with a framing is called a \textit{framed chord diagram}.

Kauffman bracket state sum is defined for a framed chord diagram.

A \textit{state} is a distribution of signs over the set of chords.
Framed chord diagrams

A chord diagram \((B, c_1, \ldots, c_n)\) in which each chord is equipped with a **framing** is called a *framed chord diagram*.

Kauffman bracket state sum is defined for a framed chord diagram.

A **state** is a distribution of signs over the set of chords. The **smoothing** defined by a state is according to the framing along the chords marked with \(+\) and the opposite one otherwise.
Signed to framed

A signed chord diagram turns canonically to a framed one:
Signed to framed

A signed chord diagram turns canonically to a framed one: On a chord with $+$ take the framing surgery along which preserves the orientation.
Signed to framed

A signed chord diagram turns canonically to a framed one: On a chord with $+$ take the framing surgery along which preserves the orientation, on a chord with $-$ take the framing surgery along which reverses the orientation.
Signed to framed

A signed chord diagram turns canonically to a framed one: On a chord with $+$ take the framing surgery along which preserves the orientation, on a chord with $-$ take the framing surgery along which reverses the orientation. Forget the orientation.
Orientable thickenings of non-orientable surfaces

Non-orientable surface can be thickened to an oriented 3-manifold!
Orientable thickenings of non-orientable surfaces

Non-orientable surface can be thickened to an oriented 3-manifold!
Example:
Thicken a Möbius band \( M \) in \( \mathbb{R}^3 \).
Orientable thickenings of non-orientable surfaces

Non-orientable surface can be thickened to an oriented 3-manifold!

Example:
Thicken a Möbius band $M$ in $\mathbb{R}^3$. 

![Diagram of thicken Möbius band](image-url)
Orientable thickenings of non-orientable surfaces

Non-orientable surface can be thickened to an oriented 3-manifold!

**Example:**

Thicken a Möbius band $M$ in $\mathbb{R}^3$.

A neighborhood of $M$ in $\mathbb{R}^3$ is orientable and fibers over $M$. 
Abstract construction of an orientable thickening

Thicken a non-orientable surface $S$: 
Abstract construction of an orientable thickening

Thicken a non-orientable surface $S$:

1. Find an orientation change line $C$ (like International date line) on $S$. 
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Abstract construction of an orientable thickening

Thicken a non-orientable surface $S$:
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2. Cut $S$ along $C$: $S \leftrightarrow S_C$

Knots and links
Virtual links
Moves
Kauffman bracket
Gauss diagrams of a poor man
- Signed chord diagrams
- State of signed chord diagram
- Smoothing of a signed chord diagram
- Framing
- Framed chord diagrams
- Signed to framed diagrams
- Orientable thickenings of non-orientable surfaces
- Abstract construction of an orientable thickening
- A link in orientable thickening of a non-orientable surface

Khovanov homology
Orientation of chord diagrams

Khovanov complex of framed chord diagram

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Abstract construction of an orientable thickening

Thicken a non-orientable surface $S$:

1. Find an *orientation change line* $C$ (like *International date line*) on $S$.

2. Cut $S$ along $C$: $S \leftrightarrow S \prec C$

3. Thicken: $(S \prec C) \times \mathbb{R}$.
Abstract construction of an orientable thickening

Thicken a non-orientable surface $S$:

1. Find an orientation change line $C$ (like International date line) on $S$.

2. Cut $S$ along $C$: $S \mapsto S \# C$

3. Thicken: $(S \# C) \times \mathbb{R}$.

4. Paste over the sides of the cut $(x_+, t) \sim (x_-, -t)$. 
A link in orientable thickening of a non-orientable surface

A diagram on the surface.
A link in orientable thickening of a non-orientable surface

A diagram on the surface.

Reidemeister moves plus two more moves:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\draw[very thick,red] (0,0) circle (0.5);
\draw[very thick,blue] (0.5,0) circle (0.5);
\draw[very thick,red] (-0.5,0) circle (0.5);
\draw[very thick,blue] (1,0) circle (0.5);
\end{tikzpicture}
\end{array}
& \quad \begin{array}{c}
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\end{tikzpicture}
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\draw[very thick,red] (0,0) circle (0.5);
\draw[very thick,blue] (0.5,0) circle (0.5);
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\end{array}
\end{align*}
\]
A link in orientable thickening of a non-orientable surface

A diagram on the surface.
Reidemeister moves plus two more moves:

Twisted Gauss diagram = Gauss diagram with a finite set of dots marked on the circle.
A link in orientable thickening of a non-orientable surface

A diagram on the surface.

Reidemeister moves plus two more moves:

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Two more moves:
A link in orientable thickening of a non-orientable surface

A diagram on the surface.
Reidemeister moves plus two more moves:

Twisted Gauss diagram = Gauss diagram with a finite set of dots marked on the circle.

Two more moves:

Forgetting dots and arrows turns a twisted Gauss diagram into a signed chord diagram.
A link in orientable thickening of a non-orientable surface

A diagram on the surface. Reidemeister moves plus two more moves:

Twisted Gauss diagram = Gauss diagram with a finite set of dots marked on the circle.

Two more moves:

Forgetting dots and arrows turns a twisted Gauss diagram into a signed chord diagram. (together with moves)
A link in orientable thickening of a non-orientable surface

A diagram on the surface.
Reidemeister moves plus two more moves:

Twisted Gauss diagram = Gauss diagram with a finite set of dots marked on the circle.

Two more moves:

Forgetting dots and arrows turns a twisted Gauss diagram into a signed chord diagram.

Corollary (Bourgoin). Links in orientable thickenings of surfaces have well-defined Kauffman bracket.
Khovanov homology

• Khovanov homology
• Enhanced states
• Khovanov complex
• More algebraic construction
• What about virtual links?

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Khovanov homology categorifies Jones polynomial.
Khovanov homology categorifies Jones polynomial. Here we will deal with a version of Khovanov homology, which categorifies Kauffman bracket.
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\[ D \mapsto H_{p,q}(D), \quad \langle D \rangle = \sum_{p,q} (-1)^p A^q \text{rk } H_{p,q}(D). \]
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Relation to the original Khovanov homology:
Khovanov homology categorifies Jones polynomial. Here we will deal with a version of Khovanov homology, which categorifies Kauffman bracket.

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Relation to the original Khovanov homology:

\[ H_{p,q}(D) = \mathcal{H}^{w(D)-q-2p, 3w(D)-q}(D) \]

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\[ D \mapsto H_{p,q}(D), \quad \langle D \rangle = \sum_{p,q} (-1)^p A^q \operatorname{rk} H_{p,q}(D). \]

Relation to the original Khovanov homology:

\[ H_{p,q}(D) = \mathcal{H} \frac{w(D) - q - 2p}{2}, \frac{3w(D) - q}{2} (D), \]

or

\[ \mathcal{H}^{i,j}(D) = H_{j - i - w(D), 3w(D) - 2j}(D). \]
Khovanov homology categorifies Jones polynomial. Here we will deal with a version of Khovanov homology, which categorifies Kauffman bracket.

\[ D \mapsto H_{p,q}(D), \quad \langle D \rangle = \sum_{p,q} (-1)^p A^q \ \text{rk} \ H_{p,q}(D). \]

Relation to the original Khovanov homology:

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\[ \mathcal{H}^{i,j}(D) = H_{j-i-w(D), 3w(D)-2j}(D). \]

In other words: \( H_{p,q}(D) = \mathcal{H}^{i,j}(D) \) iff \( q + 2j = 3w(D) \) and \( j - i + p = w(D) \).
Enhanced states

Enhance states involved in the Kauffman state sum by attaching a sign to each component of the smoothing along the state.
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Enhance states involved in the Kauffman state sum by attaching a sign to each component of the smoothing along the state.

For example: state with smoothing gives rise to 4 enhanced states
Enhanced states

Enhance states involved in the Kauffman state sum by attaching a sign to each component of the smoothing along the state.

For example: state with smoothing gives rise to 4 enhanced states
Khovanov complex

For enhanced state $S$, set $\tau(S) = \#(\text{pluses}) - \#(\text{minuses})$ and $\langle S \rangle = (-1)^{\tau(S)} A^{a(S) - b(S) - 2\tau(S)}$. 
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\langle S \rangle = (-1)^{\tau(S)} A^{a(S)-b(S)-2\tau(S)}.
$$

$$
\langle D \rangle = \sum S \text{ enhanced state of } D \langle S \rangle
$$
Khovanov complex

For enhanced state $S$, set $\tau(S) = \#(\text{pluses}) - \#(\text{minuses})$ and $\langle S \rangle = (-1)^{\tau(S)} A^{a(S) - b(S) - 2\tau(S)}$.

$$\langle D \rangle = \sum_{S \text{ enhanced state of } D} \langle S \rangle$$

Let $C_{p,q}(D)$ be a free abelian group generated by enhanced states $S$ of $D$ with:

$\tau(S) = p$ and $a(S) - b(S) - 2\tau(S) = q$. 
Khovanov complex

For enhanced state \( S \), set \( \tau(S) = \#(\text{pluses}) - \#(\text{minuses}) \) and
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Khovanov complex

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Then
\[
\langle D \rangle = \sum_{p,q} (-1)^p A^q \rk C_{p,q}(D).
\]
Any differential $\partial : C_{p,q}(D) \rightarrow C_{p-1,q}(D)$ gives homology
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H_{p,q}(D) \quad \text{with} \quad \langle D \rangle = \sum_{p,q} (-1)^p A^q \rk H_{p,q}(D).
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Invariance of $H_{p,q}(D)$ under Reidemeister moves wanted!
Khovanov complex

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$\partial(S) = \sum \pm T$ with $T$, which differ from $S$ by a single marker and appropriate signs on the circles passing near the vertex.
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Invariance of $H_{p,q}(D)$ under Reidemeister moves wanted!

$\partial(S) = \sum \pm T$ with $T$, which differ from $S$ by a single marker and appropriate signs on the circles passing near the vertex.

$(|T| - |S|) = 1$ is needed to have $\tau(T) = \tau(S) - 1$. 
More algebraic construction

Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ generated by $1$ and $X$ with $X^2 = 0$. 
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Grading: $\deg(1) = 0$, $\deg(X) = 2$. 

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More algebraic construction

Let $A$ be an algebra over $\mathbb{Z}$ generated by $1$ and $X$ with $X^2 = 0$.

**Grading:** $\deg(1) = 0$, $\deg(X) = 2$.

**Comultiplication:**
\[
\Delta : A \to A \otimes A,
\Delta(1) = X \otimes 1 + 1 \otimes X,
\Delta(X) = X \otimes X.
\]
More algebraic construction

Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ generated by $1$ and $X$ with $X^2 = 0$.

**Grading:** $\deg(1) = 0$, $\deg(X) = 2$.

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\[ \Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \quad \Delta(1) = X \otimes 1 + 1 \otimes X, \]
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For a state $s$ of a link diagram $D$

associate a copy of $\mathcal{A}$ with each component of $D_s$. 
More algebraic construction

Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ generated by 1 and $X$ with $X^2 = 0$.

Grading: $\deg(1) = 0$, $\deg(X) = 2$.

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For a state $s$ of a link diagram $D$ associate a copy of $\mathcal{A}$ with each component of $D_s$.

Denote by $V_s$ the tensor product of these copies of $\mathcal{A}$.
More algebraic construction

Let $\mathcal{A}$ be an algebra over $\mathbb{Z}$ generated by $1$ and $X$ with $X^2 = 0$.

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For a state $s$ of a link diagram $D$ associate a copy of $\mathcal{A}$ with each component of $D_s$.

Denote by $V_s$ the tensor product of these copies of $\mathcal{A}$.

Equip $V_s$ with the second grading equal to the first grading shifted by $a(s) - b(s) - |s|$.

More algebraic construction

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Then

$$\bigoplus_{p,q} C_{p,q}(D) = \bigoplus_s V_s$$
More algebraic construction

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For a state $s$ of a link diagram $D$ associate a copy of $\mathcal{A}$ with each component of $D_s$.

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Equip $V_s$ with the second grading equal to the first grading shifted by $a(s) - b(s) - |s|$.

Then
$$\bigoplus_{p,q} C_{p,q}(D) = \bigoplus_s V_s$$

Differentials are defined by the multiplication and co-multiplication in $\mathcal{A}$. 
What about virtual links?

This works for classical links, but does not for virtual!
What about virtual links?

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For virtual links, it works with $\mathbb{Z}_2$ coefficients.
What about virtual links?

Over integers $d^2 \neq 0!$
What about virtual links?

Over integers $d^2 \neq 0$!

Consider virtual diagram of the unknot: 

\[ \text{Virtual diagram of the unknot} \]
What about virtual links?

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Consider virtual diagram of the unknot: 

There are 4 states contributing to Kauffman bracket as follows:
What about virtual links?

Over integers $d^2 \neq 0$!

Consider virtual diagram of the unknot:

There are 4 states contributing to Kauffman bracket as follows:

\[ - A^4 - 1 \longrightarrow - A^2 - A^{-2} \]

\[ A^4 + 2 + A^{-4} \longrightarrow -1 - A^{-4} \]
What about virtual links?

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\end{align*}
\]

Differentials are obvious in all $A$-components but the one corresponding to $A^0$. 
What about virtual links?

Over integers \( d^2 \neq 0! \)

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Consider virtual diagram of the unknot:

There are 4 states contributing to Kauffman bracket as follows:

Differentials are obvious in all $A$-components but the one corresponding to $A^0$. 

\[ 1 \otimes X + X \otimes 1 \rightarrow 2 \times 1 \]
What about virtual links?

Over integers $d^2 \neq 0$!

Consider virtual diagram of the unknot: 

There are 4 states contributing to Kauffman bracket as follows:

This does not happen if the chord diagram is *orientable*!
Orientation of chord diagrams

- Orientation of a chord diagram
- Obstruction to orientability
- Orientation of a smoothened chord diagram

Khovanov homology

Knots and links

Virtual links

Moves

Kauffman bracket

Gauss diagrams of a poor man

Orientation of chord diagrams

Khovanov complex of framed chord diagram
Orientation of a chord diagram

= orientations of chords and arcs
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients

$$\sum \text{arcs} + \sum 2 \text{chords}$$
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients

\[ \sum \text{arcs} + \sum 2 \text{chords} \]

is a cycle.
Orientation of a chord diagram

= orientations of chords and arcs
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\[ \sum \text{arcs} + \sum 2 \text{chords} \]

is a cycle.

That is

\[ \partial (\sum \text{arcs} + \sum 2 \text{chords}) = 0. \]
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients
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A chord diagram is called **orientable** if it admits an orientation.
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients
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A chord diagram is called orientable if it admits an orientation. Orientability of chord diagram with connected base is equivalent to the following condition known to K.-F. Gauss:
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients

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\[ \partial (\sum \text{arcs} + \sum 2 \text{chords}) = 0. \]

A chord diagram is called *orientable* if it admits an orientation. Orientability of chord diagram with connected base is equivalent to the following condition known to K.-F. Gauss:

The number of endpoints of chords on each arc bounded by endpoints of a chord is even.
Orientation of a chord diagram

= orientations of chords and arcs such that the chain with integer coefficients

\[ \sum \text{arcs} + \sum 2 \text{chords} \]

is a cycle.

That is

\[ \partial (\sum \text{arcs} + \sum 2 \text{chords}) = 0. \]

A chord diagram is called \textit{orientable} if it admits an orientation. Orientability of chord diagram with connected base is equivalent to the following condition known to K.-F.Gauss:

The number of endpoints of chords on each arc bounded by endpoints of a chord is even.

The simplest nonorientable chord diagram: \( \otimes \).
Try to orient a chord diagram.
Obstruction to orientability

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The obstruction to orientability of a chord diagram \( (B, c_1, \ldots, c_n) \) is an element of \( H^1(B, \bigcup_{i=1}^{n} \partial c_i; \mathbb{Z}_2) \).
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The obstruction to orientability of a chord diagram $(B, c_1, \ldots, c_n)$ is an element of $H^1(B, \bigcup_{i=1}^n \partial c_i; \mathbb{Z}_2)$. Dual class belongs to $H_0(B \setminus \bigcup_{i=1}^n \partial c_i; \mathbb{Z}_2)$. 

Obstruction to orientability

Try to orient a chord diagram.

We have met an obstruction. The obstruction to orientability of a chord diagram \((B, c_1, \ldots, c_n)\) is an element of \(H^1(B, \bigcup_{i=1}^{n} \partial c_i; \mathbb{Z}_2)\). Dual class belongs to \(H_0(B \setminus \bigcup_{i=1}^{n} \partial c_i; \mathbb{Z}_2)\). Orient the complement of the 0-cycle realizing it, to get vice-orientation of the chord diagram.
Orientation of a smoothened chord diagram

If a chord diagram
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![Chord Diagram]
Orientation of a smoothened chord diagram

If a chord diagram is oriented,

\[
\begin{array}{c}
(+) \quad (+) \\
(-) \quad (-) \\
(+) \quad (+)
\end{array}
\]
Orientation of a smoothened chord diagram

If a chord diagram is oriented,

its orientation induces an orientation of each result of its smoothing.
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Theorem (Manturov, Viro)  *Definition of the Khovanov complex extended straightforwardly to an oriented framed chord diagram gives a complex invariant under Reidemeister moves preserving the orientation.*
Khovanov complex of framed chord diagram

- Structure used in the construction
- Involution in the Frobenius algebra
- Space associated to a state
- Partial differential
Structure used in the construction

1. Framed chord diagram.
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2. Vice-orientation of the chord diagram.
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3. At each chord one of two arcs adjacent to its arrowhead is marked.
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Knots and links
Virtual links
Moves
Kauffman bracket
Gauss diagrams of a poor man
Khovanov homology
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The chain groups are the same as in the Khovanov construction:
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$$\bigoplus_{p,q} C_{p,q}(D) = \bigoplus_s V_s$$
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algebraically (up to isomorphisms).
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The structure is needed for a collection of the isomorphisms.
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The structure is needed for a collection of the isomorphisms needed for construction of differentials.
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The structure is needed for a collection of the isomorphisms needed for construction of differentials.

Homology does not depend on the structure.
Involution in the Frobenius algebra

Remind that \( \mathcal{A} \) is a Frobenius algebra generated by 1 and \( X \) with \( X^2 = 0 \).

with Grading: \( \text{deg}(1) = 0 \), \( \text{deg}(X) = 2 \).

and Comultiplication:
\[
\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}, \quad \Delta(1) = X \otimes 1 + 1 \otimes X, \\
\Delta(X) = X \otimes X.
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Involution \( \text{conj} : \mathcal{A} \to \mathcal{A} : 1 \mapsto 1, X \mapsto -X \).
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Notice: $\text{conj}(ab) = \text{conj}(a) \text{conj}(b)$.
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But $\Delta(\text{conj}(1)) = \Delta(1) = X \otimes 1 + 1 \otimes X$
$= -\Delta(X) \otimes \Delta(1) - \Delta(1) \otimes \Delta(X)$. 
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and

$\Delta(\text{conj}(X)) = \Delta(-X) = -X \otimes X = -\Delta(X) \otimes \Delta(X)$.
Space associated to a state

Given a state $s$ of a framed chord diagram $D$. 
Space associated to a state

Given a state $s$ of a framed chord diagram $D$. Orient each connected component of $D_s$. 
Given a state \( s \) of a framed chord diagram \( D \).
Orient each connected component of \( D_s \).
Order the set of components.
Space associated to a state

Given a state $s$ of a framed chord diagram $D$. Orient each connected component of $D_s$. Order the set of components. Associate a copy of $A$ to each component of $D_s$. 
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Given a state \( s \) of a framed chord diagram \( D \).
Orient each connected component of \( D_s \).
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Associate a copy of \( \mathcal{A} \) to each component of \( D_s \).
Denote by \( V_s \) the tensor product of these copies of \( \mathcal{A} \).
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Given a state $s$ of a framed chord diagram $D$. Orient each connected component of $D_s$. Order the set of components. Associate a copy of $A$ to each component of $D_s$. Denote by $V_s$ the tensor product of these copies of $A$. This construction depends on the orientations and ordering.
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Reversing of orientation of a component corresponds to \( \text{conj} \) in the corresponding copy of \( \mathcal{A} \).
Space associated to a state

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This construction depends on the orientations and ordering.
The results corresponding to the different choices of them are related by isomorphisms:
Reversing of orientation of a component corresponds to \( \text{conj} \) in the corresponding copy of \( A \).
Permutations of the components corresponds to the permutation isomorphism of the tensor product multiplied by the sign of the permutation.
Let \( s \) and \( t \) be adjacent states of a framed chord diagram \( D \) which is equipped with a vice orientation and markers.
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$. 
Partial differential

Let \( s \) and \( t \) be adjacent states of a framed chord diagram \( D \) which is equipped with a vice orientation and markers. Let \( t \) differs from \( s \) only by a marker sign at chord \( c \), positive in \( s \) and negative at \( t \).
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$. Construct $V_s \to V_t$. 
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$. Construct $V_s \rightarrow V_t$. Put it to be 0 if $|s| = |t|$.
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Construct $V_s \to V_t$.

Put it to be 0 if $|s| = |t|$.

Otherwise, order the components of $D_s$ and $D_t$ so that:
Partial differential

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Construct \( V_s \rightarrow V_t \).

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Otherwise, order the components of \( D_s \) and \( D_t \) so that:

- The first component passes through the marker at \( c \).
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$. Construct $V_s \rightarrow V_t$.

Put it to be 0 if $|s| = |t|$. Otherwise, order the components of $D_s$ and $D_t$ so that:

- The first component passes through the marker at $c$.
- On the second place put the other component passes though $c$ (if there is one).
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$. Construct $V_s \to V_t$. Put it to be 0 if $|s| = |t|$. Otherwise, order the components of $D_s$ and $D_t$ so that:

- The first component passes through the marker at $c$.
- On the second place put the other component passes though $c$ (if there is one).
- Other components (which are common for $D_s$ and $D_t$) are to be ordered coherently.
Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$. Construct $V_s \rightarrow V_t$. Put it to be $0$ if $|s| = |t|$. Otherwise, order the components of $D_s$ and $D_t$ so that:

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Orient the first components according to the vice orientation at $c$. 
Partial differential

Let $s$ and $t$ be adjacent states of a framed chord diagram $D$ which is equipped with a vice orientation and markers. Let $t$ differs from $s$ only by a marker sign at chord $c$, positive in $s$ and negative at $t$.

Construct $V_s \rightarrow V_t$.

Put it to be $0$ if $|s| = |t|$.

Otherwise, order the components of $D_s$ and $D_t$ so that:

- The first component passes through the marker at $c$.
- On the second place put the other component passes though $c$ (if there is one).
- Other components (which are common for $D_s$ and $D_t$) are to be ordered coherently.

Orient the first components according to the vice orientation at $c$. In these representations of $V_s$ and $V_t$, define the map by multiplication or co-multiplication.