

The 16th Hilbert problem, a story of mystery, mistakes and solution.

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Harnack's inequality

16. Problem of the topology of algebraic curves and surfaces

Hilbert started with reminding of a background result:

The maximum number of closed and separate branches which a plane algebraic curve of the n -th order can have has been determined by Harnack (Mathematische Annalen, vol. 10).

Here Hilbert referred to the following *Harnack inequality*.

The words *Harnack inequality* are confusing: there are other, more famous Harnack inequalities concerning values of a positive harmonic function.

The number of connected components of a plane projective real algebraic curve of degree n

$$\leq \frac{(n-1)(n-2)}{2} + 1.$$

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Two natures of Harnack inequality

Harnack's proof: Let curve A of degree n has $\#(ovals) > M = \frac{(n-1)(n-2)}{2} + 1$.

- Draw a curve B of degree $n - 2$ through M points chosen on M ovals of A and $n - 3$ points on one more oval.

A curve of degree $n - 2$ is defined by an equation with $\frac{(n-1)n}{2}$ coefficients. Hence it can be drawn through $\frac{(n-1)n}{2} - 1 = \frac{(n-1)(n-2)}{2} + n - 1 - 1 = M + n - 3$ points. • Estimate the number of intersection points:

$$\geq 2M + n - 3 \geq 2M + n - 3 = (n - 1)(n - 2) + 2 + n - 3 = n^2 - 2n + 1 > n(n - 2) ,$$

An oval is met even number of times.

- and apply the Bezout Theorem.

Klein's proof: apply the following theorem to

- the complexification of the curve and
- the complex conjugation involution on it:

Theorem. Let S be an orientable closed connected surface,

$\sigma : S \rightarrow S$ an orientation reversing involution,

and F the fixed point set of σ . Then $\#\text{connected components}(F) \leq \text{genus}(S) + 1$.

Lemma: $\#\text{connected components}(S \setminus F) \leq 2$.

Proof. Let A be a connected component of $S \setminus F$.

Then $\text{Cl}(A) \cup \sigma(A)$ is a closed surface.

Hence $\text{Cl}(A) \cup \sigma(A) = S$. If $A \neq \sigma(A)$, then $\#\text{connected components}(S \setminus F) = 2$. If

$A = \sigma(A)$, then $\#\text{connected components}(S \setminus F) = 1$. □ **Proof of Theorem.** A curve with

$> \text{genus}(S) + x$ components divides S to $> x + 1$ components. □

Which proof is better?

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Relative position of branches

Let us come back to Hilbert's text. He continued:

There arises the further question as to the relative position of the branches in the plane.

This question was raised by Hilbert in his paper

Über die reellen Züge algebraischen Curven,

Mathematische Annalen 38 (1891), 115–138.

Harnack, in the paper mentioned by Hilbert, constructed curves with the maximal number of components for each degree.

His curves are very special:

- The depth of each of their nests ≤ 2 .

- A Harnack curve of degree n has

$\frac{3n^2-6n}{8} + 1$ outer and $\frac{n^2-6n}{8} + 1$ inner ovals.

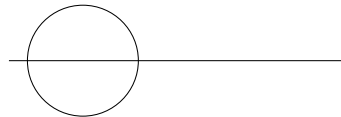
In degree 6: 10 outer ovals and 1 inner oval.

In degree 6: 

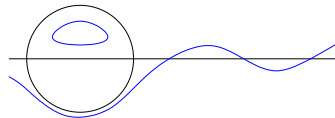
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Harnack's construction

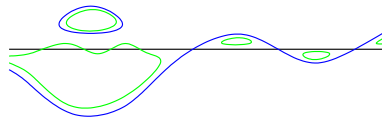
Take a line and circle:



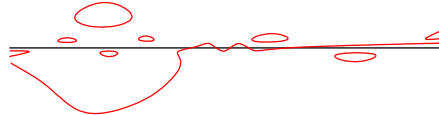
Perturb their union:



Perturb the union of the result and the line:



Perturb the union of the result and the line:

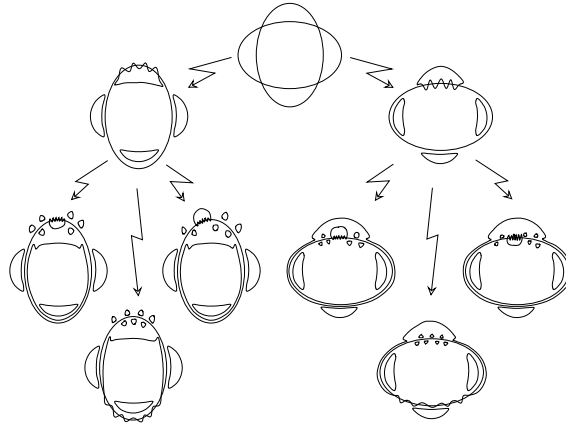


And so on...

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Hilbert's construction

Hilbert, in his paper of 1891, suggested another construction:



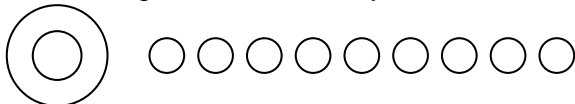
An ellipse does what the line did in Harnack's construction.

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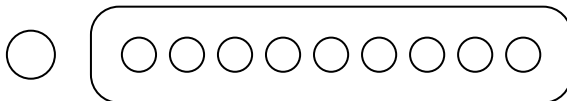
Hilbert sextics

Each Hilbert's curve of degree 6 has one of the following two configurations of ovals:

1. the configuration obtained by Harnack:



2. a new configuration, which cannot be realized by Harnack's construction:



Hilbert **worked hard**, but **could not** construct curves of degree 6 with 11 connected components **positioned** with respect to each other **in any other way**.

He concluded that **this is impossible**.

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Why impossible?

Hilbert turned to proof of impossibility:

As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.

In other words, only mutual positions of ovals realized by Harnack's and Hilbert's constructions are possible.

Hilbert's "complicated process" allows one to answer to virtually all questions on topology of curves of degree 6.

Now it is called *Hilbert-Rohn-Gudkov method*.

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Hilbert-Rohn-Gudkov method

involves a detailed analysis of singular curves which could be obtained by continuous deformation from a given nonsingular one.

The Hilbert-Rohn-Gudkov method required complicated fragments of singularity theory, which had not been elaborated at the time of Hilbert.

Hilbert's arguments were full of gaps.

His approach was realized completely only 69 years later by D.A.Gudkov

In 1954 Gudkov, in his Candidate dissertation (Ph.D.), proved Hilbert's statement about topology of sextic curves with 11 components.

15 years later, in his Doctor dissertation, Gudkov disproved it and found the final answer.

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Call for attack

A “**complicated process**” **could not** really **satisfy** Hilbert.

Desperately wishing to understand the real reasons of this very mysterious phenomenon, Hilbert called for attack:

A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest,

Why did Hilbert distinguish curves with maximal number of branches?

Extremal cases of inequalities had been known to be of **extreme interest**.

Hilbert deeply appreciated this paradigm of the **calculus of variations**.

Now people (especially, specialists) tend to widen the content of Hilbert's 16th problem. as just a call for

study of the topology of all real algebraic varieties.

To support this view, they cite also the next piece of Hilbert's text: *and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space. and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.*

The word **corresponding** is crucial here. Without it, this would really be a mere call to study the topology of real algebraic surfaces. So, what is “**the corresponding**”? Hilbert continues: *Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4-th order in three dimensional space can really have* (Cf. Rohn, “*Flächen vierter Ordnung*” 1886).

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Solutions

Now we know that **the maximum number of connected components of a quartic surface in the 3-dimensional projective space is 10**. Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is **10**.

This was proven in 1972 by **V.M.Kharlamov** in his Master thesis

in the **breakthrough** of 1969-72, which **solved** the sixteenth Hilbert problem.

All the questions contained, explicitly or implicitly, in the sixteenth problem have been answered by D.A.Gudkov, V.I.Arnold, V.A.Rokhlin and V.M.Kharlamov in this breakthrough.

In 1969, **D.A.Gudkov** found the final answer to the question about **position of real branches of maximal curves of degree 6**.

V.I.Arnold and **V.A.Rokhlin** found in 1971-72 a **conceptual cause** of the phenomenon which struck Hilbert.

Kharlamov completed by 1976 the “corresponding investigation” of **nonsingular quartic surfaces**.

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Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem **solved**.

However, I am not aware about any publication, where it is claimed.

Unusual? The solution was initiated by completion of long difficult technical work.

It looks like a final point.

It followed by opening **a bright new world** with a relation to the complex domain, 4-dimensional topology, complex algebraic geometry.

The sixteenth Hilbert problem was the symbol of the breakthrough. Nobody wanted to **dispose of the symbol**. Nobody cared that the puzzle had been solved.

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Isotopy classification of nonsingular sextics

In 1969, **D.A.Gudkov** completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The project **started** in 1948.

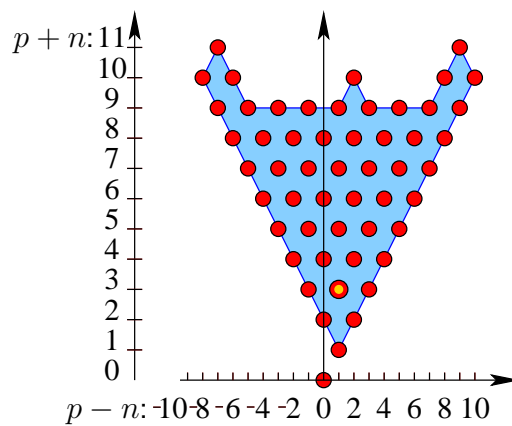
The project started in 1948.

A.A.Andronov proposed to Gudkov: **develop theory of degrees of coarseness for real algebraic curves. Like in the theory of dynamical systems.**

I.G.Petrovsky suggested to unite this with study of sextics.

In 1954 Gudkov defended PhD.

About 12-14 years later he prepared publication. The summary of results:



The referee **did not like it.**He suggested to make it **more symmetric.**Gudkov found a **mistake** and the **final answer.**

Gudkov's M-curve



The missing curve disproved Hilbert's statement.

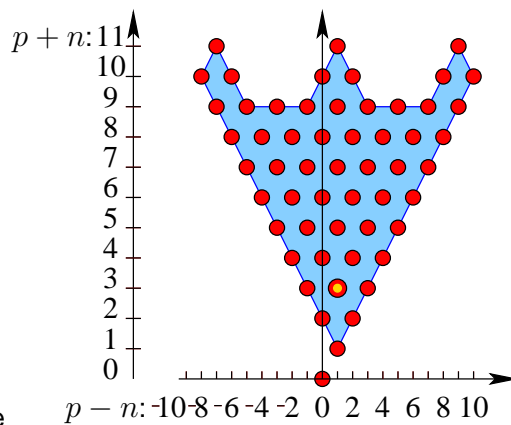
In the first version Hilbert was **more cautious and correct**:

As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.

As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another.

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Gudkov's conjecture



Symmetric top of the table forced Gudkov to formulate:

Gudkov's Conjecture. For any curve of even degree $d = 2k$ with maximal number of ovals, $p - n \equiv k^2 \pmod{8}$.

It was this conjecture that inspired the breakthrough.

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Arnold's congruence

In 1971 Arnold proved a *half* of Gudkov's conjecture:

What is a half of congruence

$p - n \equiv k^2 \pmod{8}$? the same congruence, but modulo 4: $p - n \equiv k^2 \pmod{4}$.

Arnold's proof works for a larger class of curves:

for any nonsingular curve of *type I* – a curve whose real ovals divide the Riemann surface of its complex points.

Arnold's proof relies on the topology of the configuration formed in the complex projective plane $\mathbb{C}P^2$ by the complexification $\mathbb{C}A$ of the curve and the real projective plane $\mathbb{R}P^2$.

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Complexification

Curve A of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where F is a real homogeneous polynomial of degree d .

If F is generic, then $F(x_0, x_1, x_2) = 0$ defines

$\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$

and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since d is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into

$\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$.

They are well-defined, as $F(\lambda x) = \lambda^{2k} F(x)$.

Choose F to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$.

p is the number of even ovals, the number of components of $\mathbb{R}P^2_+$. n is the number of odd ovals, the number of holes in $\mathbb{R}P^2_+$.

How to complexify $\mathbb{R}P^2_+$?

How to complexify the notion of manifold with boundary?

How to complexify inequality $F(x) \geq 0$? Arnold: Complexification of inequality is two-fold branched covering!

Indeed, $F(x) \geq 0 \Leftrightarrow \exists y \in \mathbb{R} : F(x) = y^2$. $F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{C}Y$ in 3-variety

$E = (\mathbb{C}^3 \setminus 0) \times \mathbb{C} / (x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^k y)$. Projection

$\mathbb{C}Y \rightarrow \mathbb{C}P^2 : [x_0, x_1, x_2, y] \mapsto [x_0 : x_1 : x_2]$ is a two-fold covering branched over $\mathbb{C}A$.

It maps $\mathbb{R}Y$ onto $\mathbb{R}P^2_+$. Automorphism $\tau : \mathbb{C}Y \rightarrow \mathbb{C}Y$, involution with $\text{fix}(\tau) = \mathbb{C}A$.

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In homology

$\pi_1(\mathbb{C}Y) = 0$. This simplifies algebra, makes it commutative.

$$H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0.$$

$H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}$, our **scene of algebraic action**.; **decorations**: Intersection form

$$H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta,$$

symmetric bilinear unimodular form. .

Involution $\tau_* : H_2(\mathbb{C}Y) \rightarrow H_2(\mathbb{C}Y)$.

Form of involution $\tau : H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta)$, which is also a symmetric bilinear unimodular form. Homology classes $[\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y)$.

We orient $\mathbb{R}Y$.

$[\infty]$ is the preimage of a generic projective line under $\mathbb{C}Y \rightarrow \mathbb{C}P^2$.

$$[\mathbb{C}A] \circ_\tau \xi \equiv \xi \circ_\tau \xi \pmod{2} \text{ for any } \xi.$$

Because $X \cap \tau(X)$

$$= (X \cap \mathbb{C}A) \cup (\text{even number of points}).$$

$$[\mathbb{C}A] = k[\infty]; k[\infty] \equiv [\mathbb{R}Y] \pmod{2}, \text{ if } \mathbb{R}A \text{ divides } \mathbb{C}A.$$

Hence

$$[\mathbb{R}Y] \circ_\tau \xi \equiv \xi \circ_\tau \xi \pmod{2} \text{ for any } \xi, \text{ if } \mathbb{R}A \text{ divides } \mathbb{C}A.$$

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Proof of Arnold's congruence

Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \rightarrow \mathbb{Z}$ be a unimodular symmetric bilinear form.

$w \in \mathbb{Z}^r$ is a **characteristic class** of Φ ,

if $\Phi(x, x) \equiv \Phi(x, w) \pmod{2}$ for any $x \in \mathbb{Z}^r$.

Any unimodular symmetric bilinear form has a characteristic class. Any two characteristic classes are congruent modulo 2.

Lemma. For any two characteristic classes w, w' of a form Φ

$$\Phi(w', w') \equiv \Phi(w, w) \pmod{8}$$

Proof. $w' = w + 2x$ for some $x \in \mathbb{Z}^r$.

Hence $\Phi(w', w') = \Phi(w, w) + 4\Phi(x, w) + 4\Phi(x, x)$,

but $\Phi(x, x) \equiv \Phi(x, w) \pmod{2}$.

Therefore $\Phi(w', w') \equiv \Phi(w, w) + 8\Phi(x, x) \pmod{8}$. \square **Back to CY:** As we have seen $[CA]$ and $[\mathbb{R}Y]$ are characteristic for \circ_τ , if $\mathbb{R}A$ divides $\mathbb{C}A$.

Therefore $[CA] \circ_\tau [CA] \equiv [\mathbb{R}Y] \circ_\tau [\mathbb{R}Y] \pmod{8}$.

$$[CA] \circ_\tau [CA] = [CA] \circ [CA] = k[\infty] \circ k[\infty] = k^2[\infty] \circ [\infty] = 2k^2.$$

$$[\mathbb{R}Y] \circ_\tau [\mathbb{R}Y] = -[\mathbb{R}Y] \circ [\mathbb{R}Y] = -(-\chi(\mathbb{R}Y)) = \chi(\mathbb{R}Y) = 2\chi(\mathbb{R}P_+^2) = 2(p - n).$$

Because multiplication by $\sqrt{-1}$ is antiisomorphism between tangent and normal fibrations of $\mathbb{R}A + \text{Poincaré-Hopf}$.

Finally, we get $2k^2 \equiv 2(p - n) \pmod{8}$,

that is $p - n \equiv k^2 \pmod{4}$. **Provided $\mathbb{R}A$ bounds in $\mathbb{C}A$. In particular, if $p + n = g + 1$.** \square

Gudkov-Rokhlin congruence

Soon after Arnold's paper, Rokhlin published a paper "Proof of Gudkov's conjecture". Soon after Arnold's paper, Rokhlin published a paper "Proof of Gudkov's conjecture". He extended his famous topological theorem on divisibility of signature by 16, and deduced the Gudkov congruence. The deduction was **wrong**.

Few months later Rokhlin published a generalization of Gudkov conjecture to maximal varieties of any dimension with a **simple and correct general proof**.

Rokhlin's Theorem. *Let A be a non-singular real algebraic variety of even dimension with $\dim_{\mathbb{Z}_2} H_*(\mathbb{R}A; \mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H_*(\mathbb{C}A; \mathbb{Z}_2)$.*

($\dim_{\mathbb{Z}_2} H_(\mathbb{R}A; \mathbb{Z}_2) \leq \dim_{\mathbb{Z}_2} H_*(\mathbb{C}A; \mathbb{Z}_2)$ for any A)*

Then $\chi(\mathbb{R}A) \equiv \sigma(\mathbb{C}A) \pmod{16}$.

Between the two papers by Rokhlin, there was a paper by Kharlamov with the **upper bound (=10)** for the number of connected components of a quartic surface.

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The role of complexification

Hilbert's puzzle had been solved!

The answer is in the **complexification**.

Gudkov's conjecture and its high-dimensional generalization proven by Rokhlin **explain** all the phenomena which had struck Hilbert and motivated his sixteenth problem.

They are **real** manifestations of fundamental topological phenomena located in the **complex**.

Hilbert never showed a slightest sign that he had expected a progress via **getting out of the real world into the realm of complex**. **Felix Klein** did. consciously looked for interaction of real and complex pictures as early as in 1876.

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Mystery of the 16th Hilbert problem

that emerged when the problem was solved.

This is its **number**!

The number **sixteen** plays a very special role in the topology of real algebraic varieties.

Rokhlin's paper with his proof of Gudkov's conjecture and its generalizations is entitled:

"Congruences modulo **sixteen**
in the **sixteenth** Hilbert problem".

Many of subsequent results in this field have also the form of **congruences modulo 16**.

It is **difficult to believe** that **Hilbert was aware** of phenomena that would not be discovered until some seventy years later.

Nonetheless, **16** was the number chosen by Hilbert.

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Second part

Hilbert's sixteenth problem does not stop where I stopped citation, it has the second half: In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré's boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where X and Y are rational integral functions of the n th degree in x and y .

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Second part

Written homogeneously, this is

$$X \left(y \frac{dz}{dt} - z \frac{dy}{dt} \right) + Y \left(z \frac{dx}{dt} - x \frac{dz}{dt} \right) + Z \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0,$$

where X , Y , and Z are rational integral homogeneous functions of the n th degree in x , y , z , and the latter are to be determined as functions of the parameter t .

There is still almost no progress in the second half of the sixteenth problem. Hilbert's [hope](#) for a similarity between the two halves has not realized.

[Finiteness](#) for the number of limit cycles for each individual equation has been [proven](#). But even for $n = 2$, the [maximal number](#) of limit cycles is still unknown.

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The first part success

Contrary to this, the first half was [extremely successful](#):

It contained [difficult concrete problems](#) (maximal sextic curves, number of components of a quartic surface) which have been [solved](#).

It attracted attention to a [difficult field in the core of Mathematics](#).

[Topological](#) problems are the [roughest](#) and allow one to [treat complicated objects](#) unavailable for investigation from more refined viewpoints.

This direction has little chances to be completed. As a "thorough investigation", the problem can hardly be solved.

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What has happened since then?

Use of complexification made possible to find numerous *restrictions on the topology of real algebraic varieties*.

Besides the congruence modulo 4, Arnold proved in the same paper several *inequalities* on numerical characteristics of mutual position of ovals.

He found several useful ways to translate geometric phenomena in the real domain to the complex domain and back.

Kharlamov, Gudkov, Krakhnov, Nikulin, Fiedler and Mikhalkin proved *congruences modulo various powers of 2* similar to the Gudkov-Rokhlin congruence.

Rokhlin observed that a curve of type I brings a distinguished pair of *orientations* which come from the complexification and discovered a topological restriction on them. He suggested to change the main object of study: *Add to topology of the real variety the topology of its position in the complexification.*

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To what degree?

Often people ask: **To what degree the Hilbert problem has been solved?**

The problem was not to give topological classification of real algebraic curves of some specific degree.

However, one may ask: *For what degrees the classification problems on topology of real algebraic varieties are solved?*

Isotopy classification problem of nonsingular plane projective curves of degree n has been solved for $n \leq 7$.

For $n \leq 5$ it was easy, solved in XIX century.

For $n = 6$ in 1969 by Gudkov.

For $n = 7$ in 1979 by Viro.

For *maximal* curves the *isotopy* classification has almost been done in degree 8.

Only 6 isotopy types are questionable.

For *pseudoholomorphic M-curves* the *isotopy* classification has been done in degree 8 by Orevkov.

Rigid isotopy classification of nonsingular plane projective curves of degree n has been solved for $n \leq 6$.

For $n \leq 4$ in XIX century by Zeuthen, Klein.

For $n = 5$ in 1981 by Kharlamov.

For $n = 6$ in 1979 by Nikulin.

For nonsingular *surfaces* in the projective 3-space all the problems have been solved for degree ≤ 4 .

For $n \leq 2$ see textbooks on Analytic Geometry.

For $n = 3$ by Klein.

For $n = 4$ in the seventies by Nikulin and Kharlamov.

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Other objects

of real algebraic geometry also were studied: Curves on surfaces. Curves with symmetries. Degenerations of curves and surfaces. Surfaces of classical types (like rational, Abelian, Enriques and K3 surfaces), rational 3-varieties, singular points of real polynomial vector fields, critical points of real polynomials, real algebraic knots and links, amoebas of real and complex algebraic varieties, real pseudoholomorphic curves, tropical varieties, . . .

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Open problems

1. The **second half** of the sixteenth Hilbert problem! 1. The second half of the sixteenth Hilbert problem!
2. How many connected **components** can a **surface of degree 5** in the real projective 3-space have? 2. How many connected components can a surface of degree 5 in the real projective 3-space have?
3. **Rigid isotopy** classification of curves for degree **7** . 3. Rigid isotopy classification of curves for degree **7** .
4. Are all nonsingular real projective curves of a given **odd degree** with **connected** set of real points **rigid isotopic** to each other? 4. Are all nonsingular real projective curves of a given odd degree with connected set of real points rigid isotopic to each other?
5. **Algebraic expressions** for basic topological invariants of a real algebraic curve (and, further, hypersurface, . . .) **in terms of its equation**. 5. Algebraic expressions for basic topological invariants of a real algebraic curve (and, further, hypersurface, . . .) in terms of its equation.
6. Sharp estimates in the theory of **fewnomials**. 6. Sharp estimates in the theory of fewnomials.
7. Real **algebraic knot** theories. 7. Real algebraic knot theories.
8. Metric characteristics of real algebraic curves.
9. Formulate counter-parts of topological questions about real algebraic varieties for varieties over **other** non algebraically closed **fields**, **and solve them!**

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