The 16th Hilbert problem, a story of mystery, mistakes and solution.

Oleg Viro

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Read the Sixteenth Hilbert Problem

- **Harnack's inequality**
- Two natures of Harnack inequality
- Relative position of branches
- Harnack's construction
- Hilbert's construction
- Hilbert sextics
- Why impossible?
- Hilbert-Rohn-Gudkov method
- Call for attack
- Solutions
- Solved?

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Breakthrough

Post Solution
Harnack’s inequality

16. Problem of the topology of algebraic curves and surfaces
Harnack’s inequality

16. Problem of the topology of algebraic curves and surfaces

Hilbert started with reminding of a background result:
16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack (Mathematische Annalen, vol. 10).
16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack (Mathematische Annalen, vol. 10).

Here Hilbert referred to the following Harnack inequality.
16. Problem of the topology of algebraic curves and surfaces

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Here Hilbert referred to the following *Harnack inequality*.

The words *Harnack inequality* are confusing: there are other, more famous Harnack inequalities concerning values of a positive harmonic function.
16. Problem of the topology of algebraic curves and surfaces

The maximum number of closed and separate branches which a plane algebraic curve of the n-th order can have has been determined by Harnack (Mathematische Annalen, vol. 10).

Here Hilbert referred to the following Harnack inequality. The number of connected components of a plane projective real algebraic curve of degree \( n \)
Two natures of Harnack inequality

**Harnack’s proof:** Let curve $A$ of degree $n$ has

$$\#(\text{ovals}) > M = \frac{(n-1)(n-2)}{2} + 1.$$
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- Draw a curve $B$ of degree $n - 2$ through $M$ points chosen on $M$ ovals of $A$ and $n - 3$ points on one more oval.
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- Draw a curve $B$ of degree $n-2$ through $M$ points chosen on $M$ ovals of $A$ and $n-3$ points on one more oval.

A curve of degree $n-2$ is defined by an equation with
\[
\frac{(n-1)n}{2} - 1 = \frac{(n-1)(n-2)}{2} + n - 1 - 1 = M + n - 3 \text{ points.}
\]
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- Draw a curve $B$ of degree $n - 2$ through $M$ points chosen on $M$ ovals of $A$ and $n - 3$ points on one more oval.
- Estimate the number of intersection points:
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\[\geq 2M + n - 3\]
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An oval is met even number of times.
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- Estimate the number of intersection points:
  $\geq 2M + n - 3 = (n - 1)(n - 2) + 2 + n - 3 = n^2 - 2n + 1 > n(n - 2)$,
- and apply the Bezout Theorem.
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**Klein’s proof**: apply the following theorem to
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Theorem. Let $S$ be an orientable closed connected surface, $\sigma : S \rightarrow S$ an orientation reversing involution, and $F$ the fixed point set of $\sigma$. 
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Theorem. Let $S$ be an orientable closed connected surface, $\sigma : S \to S$ an orientation reversing involution, and $F$ the fixed point set of $\sigma$. Then
$$\#\text{connected components}(F) \leq \text{genus}(S) + 1.$$
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**Lemma:** $\text{#connected components}(S \setminus F) \leq 2.$
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**Lemma:** \( \#\text{connected components}(S \setminus F) \leq 2 \).

**Proof.** Let \( A \) be a connected component of \( S \setminus F \).
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**Lemma:** $\text{# connected components}(S \setminus F) \leq 2$.

**Proof.** Let $A$ be a connected component of $S \setminus F$. Then $\text{Cl}(A) \cup \sigma(A)$ is a closed surface.
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**Proof.** Let $A$ be a connected component of $S \setminus F$. Then $\text{Cl}(A) \cup \sigma(A)$ is a closed surface. Hence $\text{Cl}(A) \cup \sigma(A) = S$. 
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Hence $\text{Cl}(A) \cup \sigma(A) = S$. If $A \neq \sigma(A)$, then
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\text{#connected components}(S \setminus F) = 2. \quad \text{If } A = \sigma(A), \text{ then}
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Lemma: $\#\text{connected components}(S \setminus F) \leq 2$.

Proof of Theorem. A curve with $> \text{genus}(S) + x$ components divides $S$ to $> x + 1$ components.

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**Proof of Theorem.** A curve with \( > \text{genus}(S) + x \) components divides \( S \) to \( > x + 1 \) components. ☐

Which proof is better?
Let us come back to Hilbert’s text.
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Relative position of branches

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- The depth of each of their nests \( \leq 2 \).
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  \frac{3n^2 - 6n}{8} + 1
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- The depth of each of their nests $\leq 2$.
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In degree 6: 10 outer ovals and 1 inner oval.
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In degree 6: 

![Diagram of a Harnack curve in degree 6]
Harnack’s construction

Take a line and circle:
Harnack’s construction

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Perturb their union:
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And so on…
Hilbert’s construction

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An ellipse does what the line did in Harnack’s construction.
Hilbert sextics

Each Hilbert’s curve of degree 6 has one of the following two configurations of ovals:
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1. the configuration obtained by Harnack:
Hilbert sextics

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1. the configuration obtained by Harnack:
   
   ![Configuration 1](attachment:image1.png)

2. a new configuration, which cannot be realized by Harnack’s construction:
   
   ![Configuration 2](attachment:image2.png)
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1. the configuration obtained by Harnack:
   
   ![Harnack's configuration diagram]

2. a new configuration, which cannot be realized by Harnack’s construction:

![New configuration diagram]

Hilbert worked hard
Hilbert sextics

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```
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Hilbert worked hard, but could not construct curves of degree 6 with 11 connected components positioned with respect to each other in any other way.

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```
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1. the configuration obtained by Harnack:

   ![Diagram of the configuration obtained by Harnack]

   Hilbert worked hard, but could not construct curves of degree 6 with 11 connected components positioned with respect to each other in any other way.

   He concluded that this is impossible.

2. a new configuration, which cannot be realized by Harnack’s construction:

   ![Diagram of a new configuration which cannot be realized by Harnack’s construction]
Why impossible?

Read the Sixteenth Hilbert Problem

- **Harnack's inequality**
- Two natures of Harnack inequality
- Relative position of branches
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- Hilbert's construction
- Hilbert sextics
- **Why impossible?**
- Hilbert-Rohn-Gudkov method
- Call for attack
- Solutions
- Solved?

Breakthrough

Post Solution
Why impossible?

Hilbert turned to proof of impossibility:
Why impossible?

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As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another,
Hilbert turned to proof of impossibility:

As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.
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In other words, only mutual positions of ovals realized by Harnack’s and Hilbert’s constructions are possible.
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Now it is called _Hilbert-Rohn-Gudkov method_.

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**Why impossible?**

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**Breakthrough**

**Post Solution**
Hilbert-Rohn-Gudkov method involves a detailed analysis of singular curves which could be obtained by continuous deformation from a given nonsingular one.
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In 1954 Gudkov, in his Candidate dissertation (Ph.D.), proved Hilbert’s statement about topology of sextic curves with 11 components.
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In 1954 Gudkov, in his Candidate dissertation (Ph.D.), proved Hilbert’s statement about topology of sextic curves with 11 components.

15 years later, in his Doctor dissertation, Gudkov disproved it and found the final answer.
Call for attack

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Hilbert deeply appreciated this paradigm of the calculus of variations.
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To support this view, they cite also the next piece of Hilbert’s text:
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A “complicated process” could not really satisfy Hilbert. Desperately wishing to understand the real reasons of this very mysterious phenomenon, Hilbert called for attack:

A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.
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The word corresponding is crucial here. Without it, this would really be a mere call to study the topology of real algebraic surfaces.
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Call for attack

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A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.

The word corresponding is crucial here. Without it, this would really be a mere call to study the topology of real algebraic surfaces. So, what is “the corresponding”? Hilbert continues:
A “complicated process” could not really satisfy Hilbert. Desperately wishing to understand the real reasons of this very mysterious phenomenon, Hilbert called for attack:

**A thorough investigation of the relative position of the separate branches when their number is the maximum seems to me to be of very great interest, and not less so the corresponding investigation as to the number, form, and position of the sheets of an algebraic surface in space.** Till now, indeed, it is not even known what is the maximum number of sheets which a surface of the 4-th order in three dimensional space can really have (Cf. Rohn, “Flächen vierter Ordnung” 1886).
Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is 10.
Solutions

Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is 10.

This was proven in 1972 by V.M. Kharlamov in his Master thesis.
Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is *10*.

This was proven in 1972 by V.M.Kharlamov in his Master thesis in the *breakthrough* of 1969-72, which *solved* the sixteenth Hilbert problem.
Solutions

Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is 10.

This was proven in 1972 by V.M.Kharlamov in his Master thesis in the breakthrough of 1969-72, which solved the sixteenth Hilbert problem.

All the questions contained, explicitly or implicitly, in the sixteenth problem have been answered by D.A.Gudkov, V.I.Arnold, V.A.Rokhlin and V.M.Kharlamov in this breakthrough.
Now we know that the maximum number of connected components of a quartic surface in the 3-dimensional projective space is **10**.

This was proven in 1972 by *V.M.Kharlamov* in his Master thesis in the **breakthrough** of 1969-72, which solved the sixteenth Hilbert problem.

In 1969, *D.A.Gudkov* found the final answer to the question about **position of real branches of maximal curves of degree 6**.
Solutions

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In 1969, D.A.Gudkov found the final answer to the question about position of real branches of maximal curves of degree 6. V.I.Arnold and V.A.Rokhlin found in 1971-72 a conceptual cause of the phenomenon which struck Hilbert.
Solutions

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In 1969, D.A. Gudkov found the final answer to the question about position of real branches of maximal curves of degree 6.

V.I. Arnold and V.A. Rokhlin found in 1971-72 a conceptual cause of the phenomenon which struck Hilbert.

Kharlamov completed by 1976 the “corresponding investigation” of nonsingular quartic surfaces.
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem solved.
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.

Unusual?
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.

The solution was initiated by completion of long difficult technical work.
All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.

The solution was initiated by completion of long difficult technical work.

It looks like a final point.
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem **solved**.

However, I am not aware about any publication, where it is claimed.

The solution was **initiated by completion** of long difficult technical work.

It followed by opening **a bright new world**
All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

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All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.

The solution was initiated by completion of long difficult technical work.

It followed by opening a bright new world with a relation to the complex domain, 4-dimensional topology,
Solved?

All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

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The solution was initiated by completion of long difficult technical work.

It followed by opening a bright new world with a relation to the complex domain, 4-dimensional topology, complex algebraic geometry.
All in all this gives good reasons to consider the sixteenth Hilbert problem **solved**.

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The solution was **initiated by completion** of long difficult technical work.

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The sixteenth Hilbert problem was the **symbol** of the breakthrough.
Solved?

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The sixteenth Hilbert problem was the symbol of the breakthrough.

Nobody wanted to dispose of the symbol.
All in all this gives good reasons to consider the sixteenth Hilbert problem solved.

However, I am not aware about any publication, where it is claimed.

The solution was initiated by completion of long difficult technical work.

It followed by opening a bright new world with a relation to the complex domain, 4-dimensional topology, complex algebraic geometry.

The sixteenth Hilbert problem was the symbol of the breakthrough.

Nobody cared that the puzzle had been solved.
Read the Sixteenth Hilbert Problem

**Breakthrough**
- Isotopy classification of nonsingular sextics
- Gudkov's M-curve
- Gudkov's conjecture
- Arnold's congruence
- Complexification
- In homology
- Proof of Arnold's congruence
- Gudkov-Rokhlin congruence
- The role of complexification
- Mystery of the 16th Hilbert problem
- Second part
- Second part
- The first part success

Post Solution
Isotopy classification of nonsingular sextics

In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.
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A.A. Andronov proposed to Gudkov: develop theory of degrees of coarseness for real algebraic curves.
Isotopy classification of nonsingular sextics

In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6. The project started in 1948.

A.A. Andronov proposed to Gudkov: develop theory of degrees of coarseness for real algebraic curves. Like in the theory of dynamical systems.
In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6. The project started in 1948.

A.A. Andronov proposed to Gudkov: develop theory of degrees of coarseness for real algebraic curves.

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In 1954 Gudkov defended PhD.
Isotopy classification of nonsingular sextics

In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6. The project started in 1948.

A.A. Andronov proposed to Gudkov: develop theory of degrees of coarseness for real algebraic curves.

I.G. Petrovsky suggested to unite this with study of sextics.

In 1954 Gudkov defended PhD. About 12-14 years later he prepared publication.
Isotopy classification of nonsingular sextics

In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:
In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:

\[ p + n : 11 \]

\[ p - n : -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10 \]
Isotopy classification of nonsingular sextics

In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:

The referee did not like it.
In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:

He suggested to make it more symmetric.
In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:

\[ p + n: \begin{array}{c} 11 \\ 10 \\ 9 \\ 8 \\ 7 \\ 6 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{array} \]

\[ p - n: \begin{array}{c} -10 \ -8 \ -6 \ -4 \ -2 \ 0 \ 2 \ 4 \ 6 \ 8 \ 10 \end{array} \]

Gudkov found a mistake
In 1969, D.A. Gudkov completed isotopy classification of nonsingular real algebraic plane projective curves of degree 6.

The summary of results:

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Gudkov found a mistake and the final answer.
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The summary of results:

\[ p + n : 11 \]
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Gudkov found a mistake and the final answer.
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Post Solution
Gudkov’s M-curve

The missing curve
Gudkov’s M-curve

The missing curve

\[
\begin{array}{cccccc}
\circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

disproved Hilbert’s statement.
Gudkov’s M-curve

The missing curve disproved Hilbert’s statement.

As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.
The missing curve disproved Hilbert’s statement.
In the first version Hilbert was more cautious and correct:
As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another, but that one branch must exist in whose interior one branch and in whose exterior nine branches lie, or inversely.
The missing curve disproved Hilbert’s statement. In the first version Hilbert was more cautious and correct: As to curves of the 6-th order, I have satisfied myself—by a complicated process, it is true—that of the eleven branches which they can have according to Harnack, by no means all can lie external to one another.
Gudkov’s conjecture

Symmetric top of the table

\[ p + n: 11 \]

\[ p - n: -10 -8 -6 -4 -2 \]

forced Gudkov to formulate:
Gudkov’s conjecture

Symmetric top of the table

\[
p + n: 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \]

\[
p - n: -10, -8, -6, -4, -2, 0, 2, 4, 6, 8, 10 \]

forced Gudkov to formulate:

**Gudkov’s Conjecture.** For any curve of even degree \( d = 2k \) with maximal number of ovals, \( p - n \equiv k^2 \mod 8 \).
Gudkov’s Conjecture. For any curve of even degree $d = 2k$ with maximal number of ovals, $p - n \equiv k^2 \mod 8$. It was this conjecture that inspired the breakthrough.

Symmetric top of the table

$$p + n: 11 \quad 10 \quad 9 \quad 8 \quad 7 \quad 6 \quad 5 \quad 4 \quad 3 \quad 2 \quad 1 \quad 0$$

$$p - n: -10 \quad -8 \quad -6 \quad -4 \quad -2 \quad 0 \quad 2 \quad 4 \quad 6 \quad 8 \quad 10$$

forced Gudkov to formulate:

Gudkov’s Conjecture.
Arnold’s congruence

In 1971 Arnold proved a *half* of Gudkov’s conjecture:
Arnold’s congruence

In 1971 Arnold proved a half of Gudkov’s conjecture:

What is a half of congruence

\[ p - n \equiv k^2 \mod 8 \]
Arnold’s congruence

In 1971 Arnold proved a *half* of Gudkov’s conjecture: the same congruence, but *modulo 4*
Arnold’s congruence

In 1971 Arnold proved a half of Gudkov’s conjecture: the same congruence, but modulo 4: \( p - n \equiv k^2 \mod 4 \).
Arnold’s congruence

In 1971 Arnold proved a *half* of Gudkov’s conjecture: the same congruence, but *modulo* 4: \( p - n \equiv k^2 \mod 4 \). Arnold’s proof works for a larger class of curves:
Arnold’s congruence

In 1971 Arnold proved a half of Gudkov’s conjecture: the same congruence, but modulo 4: \( p - n \equiv k^2 \mod 4 \).

Arnold’s proof works for a larger class of curves: for any nonsingular curve of type I.
Arnold’s congruence

In 1971 Arnold proved a half of Gudkov’s conjecture: the same congruence, but modulo 4: \( p - n \equiv k^2 \mod 4 \).

Arnold’s proof works for a larger class of curves: for any nonsingular curve of type I – a curve whose real ovals divide the Riemann surface of its complex points.
Arnold’s congruence

In 1971 Arnold proved a half of Gudkov’s conjecture: the same congruence, but modulo 4: \( p - n \equiv k^2 \mod 4 \).

Arnold’s proof works for a larger class of curves: for any nonsingular curve of type I – a curve whose real ovals divide the Riemann surface of its complex points.

Arnold’s proof relies on the topology of the configuration formed in the complex projective plane \( \mathbb{C}P^2 \) by the complexification \( \mathbb{C}A \) of the curve and the real projective plane \( \mathbb{R}P^2 \).
Complexification

Curve $A$ of degree $d = 2k$, 

- Isotopy classification of nonsingular sextics
- Gudkov's M-curve
- Gudkov's conjecture
- Arnold's congruence
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- In homology
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- Second part
- The first part success

Post Solution
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane,
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. 
Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles.
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$. 

Read the Sixteenth Hilbert Problem

Breakthrough

- Isotopy classification of nonsingular sextics
- Gudkov's M-curve
- Gudkov's conjecture
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- Second part
- Second part
- The first part success
Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. 

**Complexification**
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. They are well-defined, as $F(\lambda x) = \lambda^{2k} F(x)$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable.
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{RP}^2$, a collection of smooth ovals in $\mathbb{RP}^2$ and $\mathbb{C}A \subset \mathbb{CP}^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{RP}^2$ into $\mathbb{RP}^2_+$, where $F(x) \geq 0$, and $\mathbb{RP}^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{RP}^2_+$ orientable. $p - n = \chi(\mathbb{RP}^2_+)$.
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $R^A \subset R P^2$, a collection of smooth ovals in $R P^2$ and $C^A \subset C P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $R^A$ divides $R P^2$ into $R P^2_+$, where $F(x) \geq 0$, and $R P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $R P^2_+$ orientable. $p - n = \chi(R P^2_+)$. $p$ is the number of even ovals.
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$. $p$ is the number of even ovals, the number of components of $\mathbb{R}P^2_+$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $RA \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $CA \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $RA$ divides $\mathbb{R}P^2$ into $\mathbb{R}P_+^2$, where $F(x) \geq 0$, and $\mathbb{R}P_-^2$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P_+^2$ orientable. $p - n = \chi(\mathbb{R}P_+^2)$. $p$ is the number of even ovals, the number of components of $\mathbb{R}P_+^2$. $n$ is the number of odd ovals, the number of holes in $\mathbb{R}P_+^2$. 

Post Solution
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. \( p - n = \chi(\mathbb{R}P^2_+) \).

How to complexify $\mathbb{R}P^2_+$?
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where

$F$ is a real homogeneous polynomial of degree $d$.

If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines

$\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$

and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$.

Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$.

How to complexify $\mathbb{R}P^2_+$?

How to complexify the notion of manifold with boundary?
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$.

If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2$, where $F(x) \geq 0$, and $\mathbb{R}P^2$, where $F(x) \leq 0$.

Choose $F$ to have $\mathbb{R}P^2$ orientable. $p - n = \chi(\mathbb{R}P^2)$. How to complexify $\mathbb{R}P^2$?

How to complexify the notion of manifold with boundary? How to complexify inequality $F(x) \geq 0$?
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$. 

Arnold: Complexification of inequality is two-fold branched covering!
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$. Arnold: Complexification of inequality is two-fold branched covering!

Indeed, $F(x) \geq 0 \Leftrightarrow \exists y \in \mathbb{R} : F(x) = y^2$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$.

$F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{C}Y$ in 3-variety $E = (\mathbb{C}^3 \setminus 0) \times \mathbb{C}/(x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^ky)$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$.

$F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{C}Y$ in 3-variety $E = (\mathbb{C}^3 \setminus 0) \times \mathbb{C}/(x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^ky)$. Projection $\mathbb{C}Y \to \mathbb{C}P^2 : [x_0, x_1, x_2, y] \mapsto [x_0 : x_1 : x_2]$ is a two-fold covering branched over $\mathbb{C}A$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{RA} \subset \mathbb{RP}^2$, a collection of smooth ovals in $\mathbb{RP}^2$ and $\mathbb{CA} \subset \mathbb{CP}^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{RA}$ divides $\mathbb{RP}^2$ into $\mathbb{RP}^2_+$, where $F(x) \geq 0$, and $\mathbb{RP}^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{RP}^2_+$ orientable. $p - n = \chi(\mathbb{RP}^2_+)$.

$F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{CY}$ in 3-variety $E = (\mathbb{C}^3 \setminus 0) \times \mathbb{C}/(x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^ky)$. Projection $\mathbb{CY} \to \mathbb{CP}^2 : [x_0, x_1, x_2, y] \mapsto [x_0:x_1:x_2]$ is a two-fold covering branched over $\mathbb{CA}$. It maps $\mathbb{RY}$ onto $\mathbb{RP}^2_+$. 
Complexification

Curve $A$ of degree $d = 2k$, is defined by equation $F(x_0, x_1, x_2) = 0$ on projective plane, where $F$ is a real homogeneous polynomial of degree $d$. If $F$ is generic, then $F(x_0, x_1, x_2) = 0$ defines $\mathbb{R}A \subset \mathbb{R}P^2$, a collection of smooth ovals in $\mathbb{R}P^2$ and $\mathbb{C}A \subset \mathbb{C}P^2$, a smooth sphere with $g = \frac{(d-1)(d-2)}{2}$ handles. Since $d$ is even, $\mathbb{R}A$ divides $\mathbb{R}P^2$ into $\mathbb{R}P^2_+$, where $F(x) \geq 0$, and $\mathbb{R}P^2_-$, where $F(x) \leq 0$. Choose $F$ to have $\mathbb{R}P^2_+$ orientable. $p - n = \chi(\mathbb{R}P^2_+)$. $F(x_0, x_1, x_2) = y^2$ defines a surface $\mathbb{C}Y$ in 3-variety $E = (\mathbb{C}^3 \setminus 0) \times \mathbb{C}/(x_0, x_1, x_2, y) \sim (tx_0, tx_1, tx_2, t^ky)$. Projection $\mathbb{C}Y \to \mathbb{C}P^2 : [x_0, x_1, x_2, y] \mapsto [x_0:x_1:x_2]$ is a two-fold covering branched over $\mathbb{C}A$. It maps $\mathbb{R}Y$ onto $\mathbb{R}P^2_+$. Automorphism $\tau : \mathbb{C}Y \to \mathbb{C}Y$, involution with $\text{fix}(\tau) = \mathbb{C}A$. 
\[ \pi_1(\mathbb{C}Y) = 0. \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0 \]. This simplifies algebra, makes it commutative.
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4} \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0 \]. This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2 - 6k + 4}, \text{our scene of algebraic action.} \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{ our scene of algebraic action;} \]

decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{ our scene of algebraic action;} \]

decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta, \]

symmetric bilinear unimodular form.
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{ our scene of algebraic action;} \]

**decorations:** Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{our scene of algebraic action;} \]
decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau^* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau^*(\beta) \]
In homology

\[ \pi_1(\mathbb{CY}) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{CY}) = H_4(\mathbb{CY}) = \mathbb{Z}, \quad H_1(\mathbb{CY}) = H_3(\mathbb{CY}) = 0. \]

\[ H_2(\mathbb{CY}) = \mathbb{Z}^{4k^2 - 6k + 4}, \] our scene of algebraic action;

**decorations:** Intersection form

\[ H_2(\mathbb{CY}) \times H_2(\mathbb{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{CY}) \to H_2(\mathbb{CY}). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{CY}) \times H_2(\mathbb{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta), \]

which is also a symmetric bilinear unimodular form.
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \]  This simplifies algebra, makes it commutative.
\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \]  our scene of algebraic action;

**decorations:** Intersection form
\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \rightarrow H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)
\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_{\tau} \beta = \alpha \circ \tau_*(\beta) \]

Homology class
\[ [CA] \in H_2(\mathbb{C}Y). \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \] our scene of algebraic action;

**decorations:** Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \rightarrow H_2(\mathbb{C}Y) \).

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \rightarrow \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_{\tau} \beta = \alpha \circ \tau_* (\beta) \]

Homology classes \([\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y)\).

We orient \( \mathbb{R}Y \).
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{our scene of algebraic action}; \]

decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z}: (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_*: H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z}: (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta) \]

Homology classes \[ [\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y). \]

\[ [\infty] \text{ is the preimage of a generic projective line under} \]

\[ \mathbb{C}Y \to \mathbb{C}P^2. \]
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \] our scene of algebraic action;

**decorations:** Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

**Involution** \( \tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y) . \)

**Form of involution** \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta) \]

**Homology classes** \([\infty], [\mathcal{R}Y], [\mathcal{C}A] \in H_2(\mathbb{C}Y) . \)

\[ [\mathcal{C}A] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2 \] for any \( \xi . \)
In homology

\[ \pi_1(\mathbb{C}Y) = 0 \]  . This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \text{our scene of algebraic action;} \]

**decorations:** Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_{\tau} \beta = \alpha \circ \tau_*(\beta) \]

Homology classes \([\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y).\]

\[ [\mathbb{C}A] \circ_{\tau} \xi \equiv \xi \circ_{\tau} \xi \mod 2 \text{ for any } \xi. \]

Because \( X \cap \tau(X) \)
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2-6k+4}, \] our scene of algebraic action;

decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_* (\beta) \]

Homology classes \([\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y).\)

\[ [\mathbb{C}A] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2 \] for any \( \xi. \)

Because \( X \cap \tau(X) \)

\[ = (X \cap \mathbb{C}A) \cup (\text{even number of points}). \]
\[ \pi_1(\mathbb{CY}) = 0 \]. This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{CY}) = H_4(\mathbb{CY}) = \mathbb{Z}, \quad H_1(\mathbb{CY}) = H_3(\mathbb{CY}) = 0. \]

\[ H_2(\mathbb{CY}) = \mathbb{Z}^{4k^2 - 6k + 4} , \text{our scene of algebraic action;} \]

**decorations:** Intersection form

\[ H_2(\mathbb{CY}) \times H_2(\mathbb{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{CY}) \to H_2(\mathbb{CY}) \).

Form of involution \( \tau \)

\[ H_2(\mathbb{CY}) \times H_2(\mathbb{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta) \]

Homology classes [\( \infty \)], [\( \mathbb{R}Y \)], [\( \mathbb{CA} \)] \( \in H_2(\mathbb{CY}) \).

\[ [\mathbb{CA}] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2 \text{ for any } \xi. \]

\[ [\mathbb{CA}] = k[\infty] \]
In homology

\[ \pi_1(\mathcal{CY}) = 0 . \] This simplifies algebra, makes it commutative.

\[ H_0(\mathcal{CY}) = H_4(\mathcal{CY}) = \mathbb{Z}, \quad H_1(\mathcal{CY}) = H_3(\mathcal{CY}) = 0. \]

\[ H_2(\mathcal{CY}) = \mathbb{Z}^{4k^2 - 6k + 4}, \] our scene of algebraic action;

**decorations:** Intersection form

\[ H_2(\mathcal{CY}) \times H_2(\mathcal{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta . \]

Involution \( \tau_* : H_2(\mathcal{CY}) \to H_2(\mathcal{CY}) . \)

Form of involution \( \tau \)

\[ H_2(\mathcal{CY}) \times H_2(\mathcal{CY}) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_\tau \beta = \alpha \circ \tau_*(\beta) \]

Homology classes \([\infty], [\mathbb{R}Y], [\mathcal{C}A] \in H_2(\mathcal{CY}) . \)

\[ [\mathcal{C}A] \circ_\tau \xi \equiv \xi \circ_\tau \xi \mod 2 \] for any \( \xi . \)

\[ [\mathcal{C}A] = k[\infty] ; \ k[\infty] \equiv [\mathbb{R}Y] \mod 2 , \] if \( R \) divides \( \mathcal{C}A . \)
In homology

\[ \pi_1(\mathbb{C}Y) = 0. \] This simplifies algebra, makes it commutative.

\[ H_0(\mathbb{C}Y) = H_4(\mathbb{C}Y) = \mathbb{Z}, \quad H_1(\mathbb{C}Y) = H_3(\mathbb{C}Y) = 0. \]

\[ H_2(\mathbb{C}Y) = \mathbb{Z}^{4k^2 - 6k + 4}, \] our scene of algebraic action;

decorations: Intersection form

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ \beta. \]

Involution \( \tau_* : H_2(\mathbb{C}Y) \to H_2(\mathbb{C}Y). \)

Form of involution \( \tau \)

\[ H_2(\mathbb{C}Y) \times H_2(\mathbb{C}Y) \to \mathbb{Z} : (\alpha, \beta) \mapsto \alpha \circ_{\tau} \beta = \alpha \circ \tau_*(\beta) \]

Homology classes \( [\infty], [\mathbb{R}Y], [\mathbb{C}A] \in H_2(\mathbb{C}Y). \)

\[ [\mathbb{C}A] \circ_{\tau} \xi \equiv \xi \circ_{\tau} \xi \mod 2 \] for any \( \xi. \)

\[ [\mathbb{C}A] = k[\infty]; k[\infty] \equiv [\mathbb{R}Y] \mod 2, \text{ if } RA \text{ divides } CA. \]

Hence

\[ [\mathbb{R}Y] \circ_{\tau} \xi \equiv \xi \circ_{\tau} \xi \mod 2 \] for any \( \xi, \) if \( RA \) divides \( CA. \)
Proof of Arnold’s congruence

Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \rightarrow \mathbb{Z}$ be a unimodular symmetric bilinear form.
Proof of Arnold’s congruence

Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form.

$w \in \mathbb{Z}^r$ is a characteristic class of $\Phi$, if $\Phi(x, x) \equiv \Phi(x, w) \mod 2$ for any $x \in \mathbb{Z}^r$. 
Proof of Arnold’s congruence

Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form.

$w \in \mathbb{Z}^r$ is a characteristic class of $\Phi$, if $\Phi(x, x) \equiv \Phi(x, w) \mod 2$ for any $x \in \mathbb{Z}^r$.

Any unimodular symmetric bilinear form has a characteristic class.
Proof of Arnold’s congruence

Arithmetics digression. Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form.

$w \in \mathbb{Z}^r$ is a characteristic class of $\Phi$, if $\Phi(x, x) \equiv \Phi(x, w) \mod 2$ for any $x \in \mathbb{Z}^r$.

Any unimodular symmetric bilinear form has a characteristic class. Any two characteristic classes are congruent modulo 2.
Proof of Arnold’s congruence

**Arithmetics digression.** Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \rightarrow \mathbb{Z}$ be a unimodular symmetric bilinear form.

$w \in \mathbb{Z}^r$ is a characteristic class of $\Phi$, if $\Phi(x, x) \equiv \Phi(x, w) \mod 2$ for any $x \in \mathbb{Z}^r$.

**Lemma.** For any two characteristic classes $w, w'$ of a form $\Phi$

$$\Phi(w', w') \equiv \Phi(w, w) \mod 8$$
Proof of Arnold’s congruence

### Arithmetics digression.
Let \( \Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z} \) be a unimodular symmetric bilinear form.

\( w \in \mathbb{Z}^r \) is a characteristic class of \( \Phi \), if \( \Phi(x, x) \equiv \Phi(x, w) \mod 2 \) for any \( x \in \mathbb{Z}^r \).

**Lemma.** For any two characteristic classes \( w, w' \) of a form \( \Phi \)

\[
\Phi(w', w') \equiv \Phi(w, w) \mod 8
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**Proof.** \( w' = w + 2x \) for some \( x \in \mathbb{Z}^r \).
Proof of Arnold’s congruence

**Arithmetics digression.** Let $\Phi : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a unimodular symmetric bilinear form.

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Hence $\Phi(w', w') = \Phi(w, w) + 4\Phi(x, w) + 4\Phi(x, x)$
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Hence \( \Phi(w', w') = \Phi(w, w) + 4\Phi(x, w) + 4\Phi(x, x) \),
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**Lemma.** For any two characteristic classes $w, w'$ of a form $\Phi$ $\Phi(w', w') \equiv \Phi(w, w) \mod 8$

**Back to CY:** As we have seen $[CA]$ and $[RY]$ are characteristic for $\circ_\tau$, if $RA$ divides $CA$. 

Proof of Arnold’s congruence

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Proof of Arnold’s congruence

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Therefore \([CA] \circ_{\tau} [CA] \equiv [RY] \circ_{\tau} [RY] \mod 8\).

\([CA] \circ_{\tau} [CA] = [CA] \circ [CA]\)
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$$[CA] \circ_{\tau} [CA] = [CA] \circ [CA] = k[\infty] \circ k[\infty]$$
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\( [RY] \circ_\tau [RY] = -[RY] \circ [RY] = -(-\chi(\text{RY})) \)

Because multiplication by \( \sqrt{-1} \) is antiisomorphism between tangent and normal fibrations of \( RA \) + Poincaré-Hopf.
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Proof of Arnold’s congruence

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Finally, we get \( 2k^2 \equiv 2(p - n) \mod 8 \)
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Finally, we get $2k^2 \equiv 2(p - n) \mod 8$,

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Finally, we get $2k^2 \equiv 2(p-n) \mod 8$, that is $p-n \equiv k^2 \mod 4$. In particular, if $p+n = g+1$. $\square$
Soon after Arnold’s paper, Rokhlin published a paper ”Proof of Gudkov’s conjecture”.
Gudkov-Rokhlin congruence

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Few months later Rokhlin published a generalization of Gudkov conjecture to maximal varieties of any dimension.
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Then $\chi(\mathbb{R}A) \equiv \sigma(\mathbb{C}A) \mod 16$.

Between the two papers by Rokhlin, there was a paper by Kharlamov with the upper bound (=10) for the number of connected components of a quartic surface.
The role of complexification

Hilbert’s puzzle had been solved!
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Gudkov’s conjecture and its high-dimensional generalization proven by Rokhlin explain all the phenomena which had struck Hilbert and motivated his sixteenth problem.
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They are real manifestations of fundamental topological phenomena located in the complex.
The role of complexification

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Hilbert never showed a slightest sign that he had expected a progress via getting out of the real world into the realm of complex.
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Hilbert never showed a slightest sign that he had expected a progress via getting out of the real world into the realm of complex. Felix Klein did.
The role of complexification

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Hilbert never showed a slightest sign that he had expected a progress via getting out of the real world into the realm of complex. Felix Klein consciously looked for interaction of real and complex pictures as early as in 1876.
Mystery of the 16th Hilbert problem

Read the Sixteenth Hilbert Problem

Breakthrough
- Isotopy classification of nonsingular sextics
- Gudkov’s M-curve
- Gudkov’s conjecture
- Arnold’s congruence
- Complexification
- In homology
- Proof of Arnold’s congruence
- Gudkov-Rokhlin congruence
- The role of complexification
- Mystery of the 16th Hilbert problem
- Second part
- Second part
- The first part success

Post Solution
Mystery of the 16th Hilbert problem

that emerged when the problem was solved.
Mystery of the 16th Hilbert problem

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The number **sixteen** plays a very special role in the topology of real algebraic varieties.
Mystery of the 16th Hilbert problem

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The number sixteen plays a very special role in the topology of real algebraic varieties. Rokhlin’s paper with his proof of Gudkov’s conjecture and its generalizations is entitled: “Congruences modulo sixteen in the sixteenth Hilbert problem”.
Mystery of the 16th Hilbert problem

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The number **sixteen** plays a very special role in the topology of real algebraic varieties. Rokhlin’s paper with his proof of Gudkov’s conjecture and its generalizations is entitled:

“Congruences modulo **sixteen**

in the **sixteenth** Hilbert problem”.

Many of subsequent results in this field have also the form of congruences modulo **16**.
Mystery of the 16th Hilbert problem

that emerged when the problem was solved. This is its number! The number sixteen plays a very special role in the topology of real algebraic varieties. Rokhlin’s paper with his proof of Gudkov’s conjecture and its generalizations is entitled: “Congruences modulo sixteen in the sixteenth Hilbert problem”.

Many of subsequent results in this field have also the form of congruences modulo 16.

It is difficult to believe that Hilbert was aware of phenomena that would not be discovered until some seventy years later.
that emerged when the problem was solved. This is its number!

The number *sixteen* plays a very special role in the topology of real algebraic varieties. Rokhlin’s paper with his proof of Gudkov’s conjecture and its generalizations is entitled: “Congruences modulo *sixteen* in the *sixteenth* Hilbert problem”.

Many of subsequent results in this field have also the form of congruences modulo *16*.

It is difficult to believe that Hilbert was aware of phenomena that would not be discovered until some seventy years later. Nonetheless, *16* was the number chosen by Hilbert.
Second part

Hilbert’s sixteenth problem does not stop where I stopped citation, it has the second half:
In connection with this purely algebraic problem, I wish to bring forward a question which, it seems to me, may be attacked by the same method of continuous variation of coefficients, and whose answer is of corresponding value for the topology of families of curves defined by differential equations. This is the question as to the maximum number and position of Poincaré’s boundary cycles (cycles limites) for a differential equation of the first order and degree of the form

$$\frac{dy}{dx} = \frac{Y}{X},$$

where $X$ and $Y$ are rational integral functions of the $n$th degree in $x$ and $y$. 
Second part

Written homogeneously, this is

\[
X \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) + Y \left( z \frac{dx}{dt} - x \frac{dz}{dt} \right) + \\
Z \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0,
\]

where \( X, Y, \) and \( Z \) are rational integral homogeneous functions of the \( n \)th degree in \( x, y, z \), and the latter are to be determined as functions of the parameter \( t \).
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X \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) + Y \left( z \frac{dx}{dt} - x \frac{dz}{dt} \right) + Z \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0,
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where \( X \), \( Y \), and \( Z \) are rational integral homogeneous functions of the \( n \)th degree in \( x \), \( y \), \( z \), and the latter are to be determined as functions of the parameter \( t \).

There is still almost no progress in the second half of the sixteenth problem.
Second part

Written homogeneously, this is

\[ X \left( y \frac{dz}{dt} - z \frac{dy}{dt} \right) + Y \left( z \frac{dx}{dt} - x \frac{dz}{dt} \right) + \left( x \frac{dy}{dt} - y \frac{dx}{dt} \right) = 0, \]

where \( X \), \( Y \), and \( Z \) are rational integral homogeneous functions of the \( n \)th degree in \( x, y, z \), and the latter are to be determined as functions of the parameter \( t \).

There is still almost no progress in the second half of the sixteenth problem. Hilbert’s hope for a similarity between the two halves has not realized.
Written homogeneously, this is

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where \( X, Y, \) and \( Z \) are rational integral homogeneous functions of the \( n \) th degree in \( x, y, z, \) and the latter are to be determined as functions of the parameter \( t. \)

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Finiteness for the number of limit cycles for each individual equation has been proven.
Second part

Written homogeneously, this is

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This direction has little chances to be completed. As a “thorough investigation”, the problem can hardly be solved.
<table>
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