GENERIC IMMERSIONS OF CIRCLE TO SURFACES AND COMPLEX TOPOLOGY OF REAL ALGEBRAIC CURVES

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ABSTRACT. In a recent paper [1] V. I. Arnold introduced three new invariants of a generic immersion of the circle to the plane. These invariants are similar to Vassiliev invariants of classical knots. In a sense they are of degree one. In this paper an investigation based on similar ideas is done for real algebraic plane projective curves. In this more algebraic setting Arnold's invariants have natural counter-parts, two of which admit definitions in terms of the complexification of a curve. On the other hand, the Rokhlin complex orientation formula for a real algebraic curve bounding in its complexification suggests new combinatorial formulas for these two Arnold's invariants. Using the formulas I prove Arnold's conjecture. Arnold's invariants are generalized to generic collections of immersions of the circle to the projective plane and other surfaces. Some invariants of high degrees admitting similar formulas are discussed.

INTRODUCTION

This paper presents an interaction between theories of real algebraic curves and smooth immersions of the circle to plane. I have to acknowledge that the interaction exceeds my primary expectations.

The initial point was Arnold's study [1] of analogues of Vassiliev invariants for immersions of the circle. I started from a straightforward idea to apply the same approach to the theory of real plane projective algebraic curves. I hoped to get invariants which would be useful for description of topology of a real plane algebraic curve with singularities.

Almost immediately it became clear that two of three Arnold's invariants have the same behavior as the following two characteristics of real plane algebraic curve separating its complexification: the number of imaginary self-intersection points of a half of the complexification and the number of imaginary intersection points of the halves. These numbers are involved in versions of Rokhlin complex orientation formulas.

In the situations studied by Arnold there is neither complexification, nor hope to construct its substitute: arbitrary differentiable immersion of the circle to the plane does not admit a complexification.

Nonetheless the analogy started to work. The Rokhlin complex orientation formula suggested to look for its counter-part in the theory of immersions. The formula discovered in this way allowed to prove Arnold's conjecture on the range of values of his invariants. It suggests generalizations of Arnold's invariants to the case of immersions of the circle and several copies of the circle to various other surfaces. A straightforward generalization of the formula provides infinite series of invariants of finite degree for immersions of the circle to the plane.

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1. Arnold's work on immersed circles

1.1. Space of immersions. By a generic immersion of the circle S^1 into the plane $\mathbb{R}^2$ one means an immersion without triple points and points of self-tangency. It has only ordinary double points of transversal self-intersection.

A triple point of an immersion is said to be *ordinary,* if the branches at the point are transversal to each other. A self-tangency point of an immersion is said to be *ordinary,* if the branches have distinct curvatures at the point. A self-tangency point of an immersion is called a point of *direct tangency,* if the velocity vectors are pointing the same direction; otherwise it is called a point of *inverse tangency.*

The space of all immersions is an infinite-dimensional manifold. It consists of infinitely many connected components. The components are in a natural one-to-one correspondence with integers which is provided by the *Whitney index.* The latter is an integer-valued characteristic of an immersion, which is called also *winding number,* and may be defined as the rotation number of the velocity vector, as well as the degree of the Gauss map. It determines the immersion up to a regular homotopy, i.e. path in the space of immersions.

In the space of immersions all nongeneric immersions form a hypersurface called the *discriminant hypersurface* or for short the *discriminant.*

This hypersurface is stratified. There are three main strata (open in the discriminant):

(1) The set of all immersions without triple points, with only one double point which is not transversal, and such that this point is an ordinary direct self-tangency point.
(2) The set of all immersions without triple points, with only one double point which is not transversal, and such that this point is an ordinary inverse self-tangency point.
(3) The set of immersions which have only one triple point, this point is ordinary, and besides this point there are only double points of transversal self-intersection.

A generic path in the space of immersions (i.e. a generic regular homotopy) intersects the discriminant hypersurface in a finite number of points, and these points belong to the main strata. Changes experienced by an immersion when it goes through the strata were called *perestroikas* by Arnold. They are shown in Figures 1.1, 1.2 and 1.3.

By a *coorientation* of a hypersurface one means a choice of one of the two parts separated by the hypersurface in a neighborhood of any of its points. Arnold [1] has constructed natural coorientations of the main strata of the discriminant hypersurface. In Figures 1.1, 1.2 and 1.3 the pointed out direction is positive for these coorientations.

In the case of the self-tangency strata the positive direction is one in which the number of double points increases. The coorientaition of the triple point stratum is defined as follows. A transversal passing through this stratum is positive if the new-born vanishing triangle is positive. A vanishing triangle is a triangle formed by the three branches of a curve close to a curve with a triple point. The sign of a vanishing triangle is defined as follows. The orientation of the curve defines a cyclic ordering of the sides of the vanishing triangle, and hence an orientation

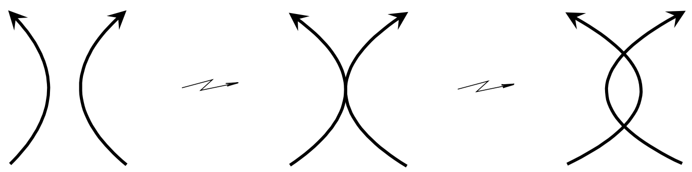

FIGURE 1.1. Direct self-tangency perestroika.

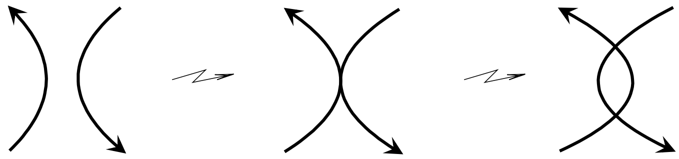

FIGURE 1.2. Inverse self-tangency perestroika.

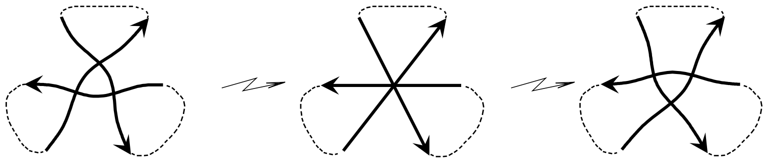

FIGURE 1.3. Triple point perestroika. Dotted curves indicate the scheme of connection of the branches, which is needed to determine the positive direction of the perestroika. On the right hand side $q = 0$, on the left hand side $q = 3$, thus the positive direction is to the right.

of the triangle. Denote by q the number of sides of the vanishing triangle whose orientation as one of a piece of the curve coincides with the orientation defined by the orientation of the triangle. The *sign* of a vanishing triangle is $(-1)^q$. See Figure 1.3.

1.2. Three Arnold's invariants. For a generic immersion Arnold [1] introduced numerical characteristics J^+, J^- and St which[1] are defined by the following properties:

1.2.A. *J^+, J^- and St are invariant under regular homotopy in the class of generic immersions.*

1.2.B. *J^+ does not change while the immersion experiences an inverse self-tangency perestroika or a triple point perestroika, but increases by two under a positive direct self-tangency perestroika.*

1.2.C. *J^- does not change while the immersion experiences a direct self-tangency perestroika or a triple point perestroika, but decreases by 2 under a positive (increasing the number of double points) inverse self-tangency perestroika.*

1.2.D. *St does not change while the immersion experiences a self-tangency perestroika, but increases by 1 under a positive triple point perestroika.*

1.2.E. *For immersions K_i with $i = 0, 1, 2, \dots$ shown in Figure* 1.4

$$\begin{array}{lll} J^+(K_0) = 0, & J^+(K_{i+1}) = -2i & (i = 0, 1, \dots); \\ J^-(K_0) = -1, & J^-(K_{i+1}) = -3i & (i = 0, 1, \dots); \\ St(K_0) = 0, & St(K_{i+1}) = i & (i = 0, 1, \dots). \end{array}$$

[1] The latter is denoted by St because Arnold called it *strangeness*. Really, St seems to be more subtle than $J^{\pm}$.

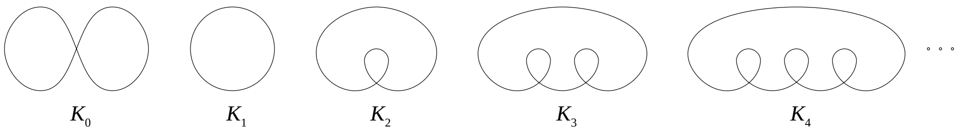

FIGURE 1.4. The standard curves of Whitney indices 0, ± 1, ± 2, ...

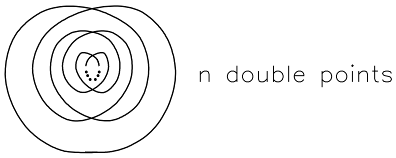

FIGURE 1.5. Curve A_{n+1}

At first glance the normalization provided by 1.2.E looks artificial. It is motivated by desire to have invariants with nice properties: it is the only normalization giving invariants additive with respect to connected sum.

1.3. Arnold's conjecture. In [1] Arnold formulated several conjectures on the range of values of his invariants. In this paper I prove one of them. It was formulated as follows.

1.3.A CONJECTURE. *The minimal values of $J^{\pm}$ on all generic curves with n double points is attained only on the curve A_{n+1} of Figure* 1.5*:*

$$J^+ \geq -n^2 - n, \qquad J^- \geq -n^2 - 2n$$

2. REAL ALGEBRAIC VARIATIONS ON THEME OF $J^{\pm}$

2.1. Curves under consideration. The closest real algebraic counter-parts of immersions $S^1 \to \mathbb{R}^2$ are real plane projective rational curves with infinite set of real points. However, if one has in mind only $J^{\pm}$, it is not difficult to consider essentially wider situation. (For a counter-part of St see my preprint [9], where it is defined only for real plane projective rational curves with infinite set of real points.)

Namely, consider irreducible plane projective curves of degree m and genus g. To distinguish direct and inverse selftangencies one needs an orientation. Especially if curves may have several connected component, which may happen when $g > 1$. A natural orientation of the set of real points of an algebraic curve appears if the set of real points is zero homologous in the complexification. Curves with this property are called *curves of type* I. See e.g. [5].

If a curve of type I is irreducible, the real part of its normalization divides the set of complex points of the normalization into two halves. The images of the halves of the normalization in the set of points of the original curve may intersect each other. However, I will call these images *the halves of the curve.*

Each of the halves of the normalization is oriented (as a piece of a complex curve) and induces an orientation on the real part as on its boundary. These two orientations are opposite to each other. They are called *complex orientations* of the real curve.

We will consider irreducible plane projective curves of degree m genus g and type I with a distinguished complex orientation. The latter means that we will consider curves with a selected half of its complexification.

Curves of this kind comprise a finite dimensional stratified real algebraic variety. A curve all whose singular points are ordinary double will be called a *generic* curve. As is well known, generic curves comprise Zariski open set in the space of all curves described above.

2.2. Singularities of a generic curve. A generic curve has only ordinary double singularities. They are equivalent from the viewpoint of complex algebraic geometry. The real algebraic geometry distinguishes several types of them.

First, a singular point may be real or imaginary.

Second, a real double point may belong to two real branches or to two imaginary branches conjugate to each other. I will call a real ordinary double point a *crossing,* if it is of the former type, and a *solitary* double point, if it is of the latter type.

Third, an imaginary double point may be a self-intersection point of one of the halves, or an intersection point of different halves. Denote the number of points of the former type by τ, the number of points of the latter type by σ. (Certainly, both τ and σ are even.)

In a solitary ordinary double point the choice of a half of the complexification determines a local orientation of $\mathbb{R}P^2$. It can be defined as the local orientation such that the imaginary branch of the curve belonging to the chosen half intersects at this point $\mathbb{R}P^2$ equipped with this local orientation with intersection number $+1$.

(Another, equivalent definition: one can perturb the curve keeping type I and converting the solitary point into an oval. The oval receives the complex orientation. The latter induces an orientation of the disk bounded by the oval. This orientation coincides with the local orientation of $\mathbb{R}P^2$ above. To prove the coincidence it is enough to consider a model example. Say, a conic $x^2+y^2=0$ and its perturbation $x^2+y^2=\varepsilon$ with $\varepsilon>0$.)

2.3. The Rokhlin formula. Curves of type I satisfy Rokhlin's complex orientation formula. For the sake of simplicity, I formulate it below only for a generic curve. I preface it with several definitions.

For a generic curve A of type I by a smoothing $\widetilde{\mathbb{R}A}$ of its real part $\mathbb{R}A$ we will understand a smooth oriented 1-dimensional submanifold of $\mathbb{R}P^2$ obtained from $\mathbb{R}A$ by modification at each real double point determined by the complex orientation as shown in Figure 2.1.

For an oriented closed 1-dimensional submanifold C of $\mathbb{R}P^2$ and a point $x\in\mathbb{R}P^2\smallsetminus C$, there is the index $ind_C(x)$ of the point with respect to the curve. It is a nonnegative integer defined as follows. Draw a line L on $\mathbb{R}P^2$ through x transversal to C. Equip it with a normal vector field vanishing only at x. For such a vector field one may take the velocity field of a rotation of the line around x. At each intersection point of L and C there are two directions transversal to L: the direction of the vector belonging to the normal vector field and the direction defined by the local orientation of C at the point. Denote the number of intersection points where the directions are faced to the same side of L by i_+ and the number of intersection

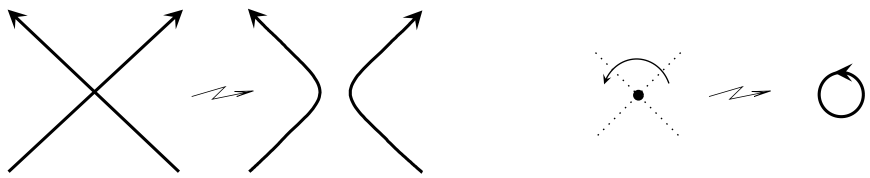

FIGURE 2.1. Smoothing of ordinary double points determined by an orientation. A solitary double point is shown on the right hand side in an old-fashioned way (complex conjugate imaginary branches are indicated by dashed lines). At this point the local orientation of the real plane is shown by a circle arrow.

points where the directions are faced to the opposite sides of L by i_-. Then set [1] $ind_C(x) = |i_+ - i_-|/2$. It is easy to check that it is well defined: it does not depend neither on the choice of L, nor on the choice of the normal vector field.

The second prerequisite notion is a sort of unusual integration: integration with respect to Euler characteristics, in which the Euler characteristics plays the role of measure. It is well known that Euler characteristics shares an important property of measures: it is additive in the sense that for any sets A, B such that Euler characteristics χA, χB, $\chi(A \cap B)$ and $\chi(A \cup B)$ are defined,

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

However, Euler characteristic is neither σ-additive, nor positive. Thus the usual theory of integral can not be developed for it. It can be done though if one restricts to a very narrow class of functions. Namely, to functions which are finite linear combinations of characteristic functions of sets belonging to some algebra of subsets of a topological space such that each element of the algebra has a well defined Euler characteristic. For a function $f = \sum_{i=1}^{r} \lambda_i \mathbb{1}_{S_i}$ set

$$\int f(x)\, d\chi(x) = \sum_{i=1}^{r} \lambda_i \chi(S_i).$$

For details and other applications of this notion see [8].

2.3.A. *Let A be a generic real plane projective algebraic curve of degree m and type I. Then*

$$\frac{m^2}{4} = \sigma + \int_{\mathbb{R}P^2 \smallsetminus \widetilde{\mathbb{R}A}} (ind_{\widetilde{\mathbb{R}A}}(x))^2\, d\chi(x)$$

where σ is (as above) the number of imaginary double points of A where different halves of the complexification meet each other.

[1]Division by 2 appears here to make this notion generalizing the well-known notion for an affine plane curve. In the definition for the affine situation one uses a ray instead of an entire line. In the projective situation there is no natural way to divide a line into rays, but we still have an opportunity to divide the result by 2. Another distinction from the affine situation is that there the index may be negative. It is related to the fact that the affine plane is orientable, while the projective plane is not.

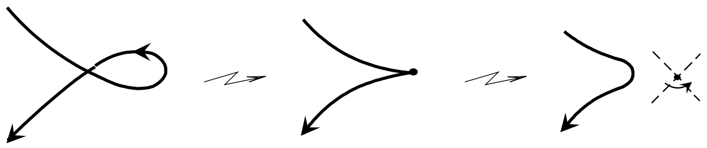

FIGURE 2.2. Cusp perestroika: σ and τ do not change.

This theorem has a rather long history. Its first special case was discovered by V. A. Rokhlin [4] in 1974. Then it was stated only for nonsingular plane projective curves of even degree m with maximal number of ovals (equal to $\frac{(m-1)(m-2)}{2}+1$). The case of nonsingular curves of odd degrees with the maximal number of ovals was done by N. M. Mishachev [3]. For nonsingular curves of type I it was stated by V. A. Rokhlin [5]. In terms of integral against Euler characteristic the Rokhlin formula of [5] was rewritten in [8] (implicitly it was done earlier by R. W. Sharpe [6]). Double points and σ appeared in [7] and the most general formula in [10]. All versions of the formula are proved in the following way. Take a half of the complexification of the curve. Complete it with a chain contained in $\mathbb{R}P^2$ to a 2-cycle. Calculate the intersection number of this cycle with its image under the complex conjugation involution. This calculation may be done geometrically (putting the cycles in general position to each other and studying the intersection) and homologically (finding the homology classes of the cycles, which are in fact $m/2[\mathbb{C}P^1]$ and $-m/2[\mathbb{C}P^1]$). Comparison of the results gives rise to the formula.

Theorem 2.3.A provides a tool for understanding what happens with σ and τ when a curve experiences various perestroikas.

2.4. Perestroikas. The complement of the set of generic curves in the variety of all real plane projective algebraic curves of degree m genus g and type I is a sort of discriminant hypersurface. It contains six main strata. Each of them consists of curves having only one singular point which is not an ordinary double point. The type of that singular point defines the stratum. Here is the list of them:

(1) real cusp;
(2) real point of direct ordinary tangency;
(3) real point of inverse ordinary tangency;
(4) real point of ordinary tangency of two imaginary branches;
(5) real ordinary triple point of intersection of three real branches;
(6) real ordinary triple point of intersection of a real branch and two conjugate imaginary branches.

These singularities and the corresponding perestroikas are shown in Figures 2.2, 1.1, 1.2, 2.3, 1.3 and 2.4, respectively. Behavior of the local orientations at solitary double points under the cusp perestroika and solitary self-tangency perestroika shown in Figures 2.2 and 2.3 follows from Rokhlin's formula 2.3.A.

In self-tangency perestroikas and the perestroika of Figure 2.4 imaginary double points are involved. Theorem 2.3.A and the fact that $\sigma+\tau$ is the number of all imaginary double points and that the total number of double points (real and imaginary) is constant $(=(m-1)(m-2)/2-g)$ imply that σ and τ change under the five perestroikas as follows:

(1) cusp perestroika (Figure 2.2): σ and τ do not change;

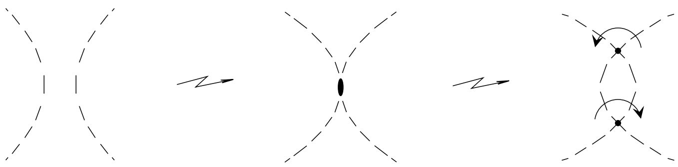

FIGURE 2.3. In this picture everything but one point in the center and two points in the right hand side is not real. This is a solitary self-tangency perestroika: two solitary double points come from the world of imaginary. At the moment of their arrival they show up as a single solitary self-tangency point. σ decreases by 2, while τ is constant.

(2) direct self-tangency perestroika (Figure 1.1): σ is constant, while τ decreases by 2;
(3) inverse self-tangency perestroika (Figure 1.2): σ decreases by 2, while τ is constant;
(4) solitary self-tangency perestroika (Figure 2.3): σ decreases by 2, while τ is constant;
(5) Triple point perestroika (Figure 1.3): σ and τ do not change;
(6) Triple point with imaginary branches perestroika (Figure 2.4): σ increases by 2, while τ decreases by 2.

This suggests that *σ is a counter-part of J^-, while $-\tau$ is a counter-part of J^+.*

3. Back to immersed circle

3.1. Rokhlin type formula. Since for generic real algebraic curves by 2.3.A

$$\sigma = \frac{m^2}{4} - \int_{\mathbb{R}P^2 \smallsetminus \widetilde{\mathbb{R}A}} (ind_{\widetilde{\mathbb{R}A}}(x))^2 \, d\chi(x),$$

and $m^2/4$ does not change under perestroikas, the integral

$$-\int_{\mathbb{R}P^2 \smallsetminus \widetilde{\mathbb{R}A}} (ind_{\widetilde{\mathbb{R}A}}(x))^2 \, d\chi(x)$$

has the same behavior under direct and inverse self-tangency perestroikas and triple point perestroika as σ and J^-.

This suggests to compare $J^-(C)$ with

$$-\int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x)$$

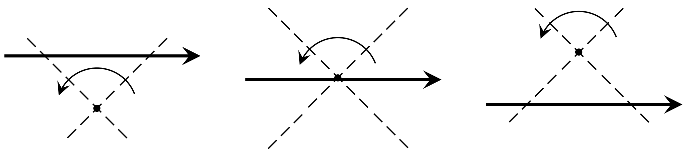

FIGURE 2.4. Triple point with imaginary branches perestroika: σ increases by 2, while τ decreases by 2.

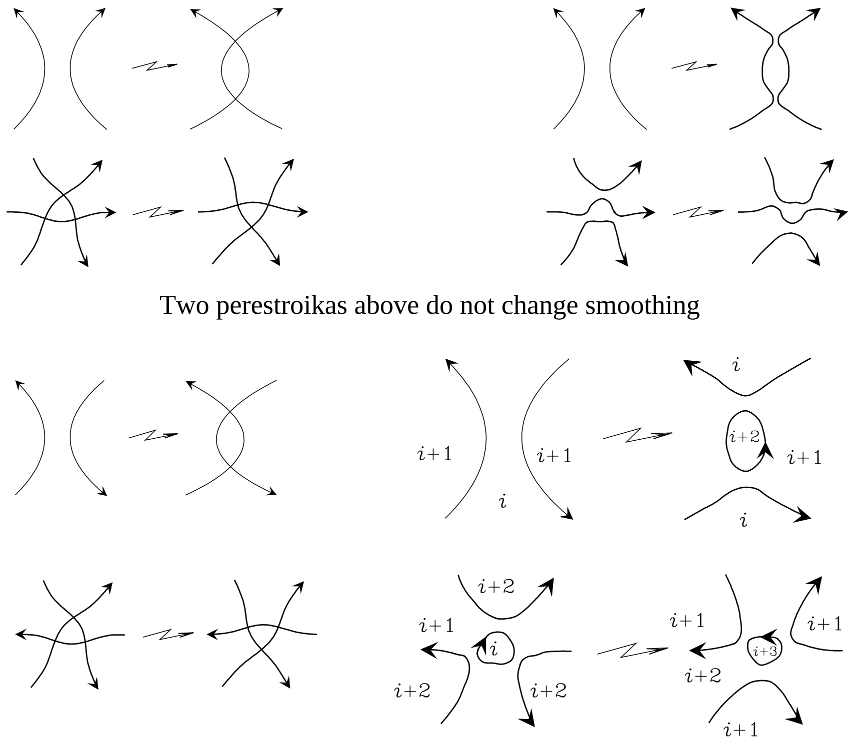

FIGURE 3.1. Smoothings of perestroikas

in the original Arnold's situation: for a generic immersed circle C. Here $\tilde{C}$ means smoothing of C defined exactly as in Section 2.3. The integral is defined in the same way, too.

3.1.A. *For any generic immersed circle C*

$$J^-(C) = 1 - \int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x).$$

PROOF. It is easy to see that in this nonalgebraic situation

$$-\int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x)$$

changes under perestroikas of C as J^-. See Figure 3.1, where smoothings of the fragments involved into Arnold's perestroikas are shown.

Furthermore,

$$\int_{\mathbb{R}P^2 \smallsetminus \tilde{K}_m} (ind_{\tilde{K}_m}(x))^2 \, d\chi(x) = \begin{cases} 2, \text{ if } m = 0 \\ (1-(m-1)) + 4(m-1) = 3m-2, \text{ if } m > 0. \end{cases}$$

See Figure 3.2.

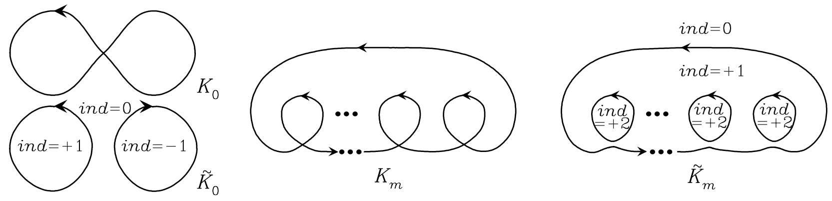

FIGURE 3.2. Calculation of $\int_{\mathbb{R}^2\smallsetminus\tilde{K}_m}(ind_{\tilde{K}_m}(x))^2\,d\chi(x)$. On the left hand side one can see that $\int_{\mathbb{R}^2\smallsetminus\tilde{K}_0}(ind_{\tilde{K}_0}(x))^2\,d\chi(x)=1+1=2$. On the right hand side after smoothing there are $m-1$ interior ovals. Therefore $\int_{\mathbb{R}^2\smallsetminus\tilde{K}_m}(ind_{\tilde{K}_m}(x))^2\,d\chi(x) = 1(1-(m-1))+4(m-1)=3m-2$.

On the other hand, by 1.2.E

$$J^-(K_0)=-1, \qquad J^-(K_m)=-3m+3 \quad (m=1,2,\dots).$$

Comparing, we obtain the desired result. □

3.1.B COROLLARY. *For any generic immersed circle C with n double points*

$$J^+(C)=1+n-\int_{\mathbb{R}^2\smallsetminus\tilde{C}}(ind_{\tilde{C}}(x))^2\,d\chi(x).$$

PROOF. Indeed, $J^+=J^-+n$ □

3.2. A version of 3.1.A without $\tilde{C}$. Let C be a generic immersed circle. On each connected component E of the complement $R^2\smallsetminus C$ the function assigning to a point $x\in E$ its index with respect to C is constant. Denote its value by $ind_C(E)$.

A double point V of C is adjacent to four angles of $\mathbb{R}^2\smallsetminus C$. Denote by $ind_C(V)$ the arithmetic mean of the values taking by ind_C on these four angles.

3.2.A OBVIOUS LEMMA. *For a generic immersed circle C*

$$\int_{\mathbb{R}^2\smallsetminus\tilde{C}}(ind_{\tilde{C}}(x))^2\,d\chi(x) = \sum_{E \text{ a component of } \mathbb{R}^2\smallsetminus C}(ind_C(E))^2-\sum_{V \text{ a double point of } C}(ind_C(V))^2 \tag{3.1}$$

□

3.2.B COROLLARY. *For a generic immersed circle C*

$$J^-(C)=1-\sum_{E \text{ a component of } \mathbb{R}^2\smallsetminus C}(ind_C(E))^2+\sum_{V \text{ a double point of } C}(ind_C(V))^2,$$

$$J^+(C)=1-\sum_{E \text{ a component of } \mathbb{R}^2\smallsetminus C}(ind_C(E))^2+\sum_{V \text{ a double point of } C}(1+(ind_C(V))^2)$$

□

FIGURE 3.3. Positive and negative injective pairs of circles

3.3. A version of 3.1.A in terms of mutual position of components of $\tilde{C}$. Two disjoint circles embedded into $\mathbb{R}^2$ compose an *injective pair,* if one of them is contained in a disk bounded by the other. An injective pair of oriented circles is said to be positive if the orientations of the circles are induced by an orientation of the annulus bounded by the circles. Otherwise it is said to be negative. See Figure 3.3.

Given a generic immersed circle C, denote by l the number of components of its smoothing $\tilde{C}$, by Π the number of injective pairs of components of $\tilde{C}$, by Π^+ the number of positive injective pairs of components of $\tilde{C}$ and by Π^- the number of negative injective pairs of components of $\tilde{C}$.

3.3.A EASY LEMMA.

$$\int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2\, d\chi(x) = l - 2\Pi^+ + 2\Pi^-$$

□

In fact this presentation of $\int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2\, d\chi(x)$ is due to Rokhlin [5]. He used it instead of the integral in the original formulation of his formula.

3.3.B COROLLARY. *For a generic immersed circle C*

$$J^- = 1 - l + 2\Pi^+ - 2\Pi^-.$$

□

3.4. Proof of Arnold's Conjecture 1.3.A. By 3.3.B

$$J^- = 1 - l + 2\Pi^+ - 2\Pi^-.$$

Since $\Pi = \Pi^+ + \Pi^-$, one has $2\Pi^+ - 2\Pi^- \geq -2\Pi$ and therefore

$$J^- \geq 1 - l - 2\Pi.$$

The number of injective pairs Π is not greater than the number of all pairs of components of $\tilde{C}$, which is equal to $\frac{l^2 - l}{2}$. Therefore

$$J^- \geq 1 - l - l^2 + l = 1 - l^2.$$

On the other hand, $l \leq n + 1$ obviously. Consequently,

$$J^- \geq 1 - (n+1)^2 = -n^2 - 2n.$$

If the equality $J^- = -n^2 - 2n$ holds, then any two components of $\tilde{C}$ comprise an injective pair, this pair is negative, and the number of double points of C is equal to $l - 1$. These conditions imply that $C = A_{n+1}$.

Since $J^+ = J^- + n$, the statement on J^+ follows from the statement on J^-. □

4. Generalizations of $J^{\pm}$

Theorem 3.1.A can be used not only as a tool for calculations. It gives a new proof of existence of J^- (i.e. existence of a generic immersed curve characteristic satisfying 1.2.A and 1.2.C).

In this Section the problem of generalizing of $J^{\pm}$ to new situations is considered. The main object is a generic collection C of k circles immersed into a surface F. Here by genericity I mean basically the same as in the case of a single immersed circle: intersections and self-intersections are transversal and at each point of F there are at most two branches of C. By a generalization of J^+ (respectively, J^-) I mean a numerical characteristic of a generic collection which is invariant under regular homotopy in the class of generic collections of immersed circles and satisfies 1.2.B (respectively, 1.2.C).

The most interesting question about this is the one on existence of such a characteristic for collections of curves of a given regular homotopy class. It can be solved by studying the stratification of the space of such collections, as it was done by Arnold [1] for single immersions of circle to plane. In this Section another approach suggested by Theorem 3.1.A is used.

The question of uniqueness has an obvious solution: to a characteristic of a generic collection of immersed circles one can add any function of a regular homotopy class to get another characteristic satisfying the same conditions, and any characteristic satisfying these conditions can be obtained in this way.

It is sufficient to consider the problems only for J^-, since for a characteristic J^- satisfying the definition above for a generalization of J^- the characteristic

$$J^-(C) + \text{ the number of double points of } C$$

satisfies our definition for a generalization of J^+.

I make special efforts to construct generalizations of J^- with the most interesting additional properties. There are indications, e.g. Theorem 4.4.A, that sometimes this goal is achieved.

4.1. In the affine plane. For a collection C of k circles immersed to $\mathbb{R}^2$ set

$$J^-(C) = k - \int_{\mathbb{R}^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x).$$

It has the properties of J^- and is additive under both connected sum and disjoint sum.

4.2. In the projective plane. The same approach can be applied to more general situations. For example, we can extend this definition to the class of generic collections of circles immersed to the projective plane. Given a generic collections C of k circles immersed to $\mathbb{R}P^2$, denote by $J^-(C)$ the number

$$1 - \int_{\mathbb{R}P^2 \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x).$$

It is easy to see that $J^-(C)$ is invariant under regular homotopy in the class of generic immersions and 1.2.C holds true for it. It is a generalization of Arnold's J^- in the sense that if C is a composition of an immersion C' of S^1 to $\mathbb{R}^2$ and an embedding $\mathbb{R}^2 \to \mathbb{R}P^2$, then $J^-(C) = J^-(C')$.

4.3. A straightforward generalization: zero-homologous curves in a surface with trivial 2-homology. Generalizations of J^- given in Sections 4.1 and 4.2 are based on the notion of index of a point with respect to a curve. In both cases it has a simple homological sense.

In the case of $\mathbb{R}^2$ a curve C is zero homologous and it bounds a chain. Since $H_2(\mathbb{R}^2) = 0$, the homology class of the chain in the relative homology $H_2(\mathbb{R}^2, C)$ is unique. The index of a point $x \in \mathbb{R}^2 \smallsetminus C$ is the image of this class in the local homology group $H_2(\mathbb{R}^2, \mathbb{R}^2 \smallsetminus x) = \mathbb{Z}$. It is really a well-defined integer, since the isomorphism $H_2(\mathbb{R}^2, \mathbb{R}^2 \smallsetminus x) \to \mathbb{Z}$ is determined by the orientation of $\mathbb{R}^2$.

In the case of $\mathbb{R}P^2$ all this arguments are still valid besides the following two points: first, C may be non-homologous to zero; second, the projective plane is not orientable therefore the index is defined up to multiplication by -1. Fortunately, both obstructions are easy to overcome. To overcome the first one it is enough to take rational homology instead of integer one, or even homology with $\mathbb{Z}[\frac{1}{2}]$ coefficients. As for the second one, we need only the square of the index.

This approach is easy to apply to generic immersions of a collection of circles into a connected surface F with $H_2(F) = 0$ realizing the zero rational homology class.

Indeed, for a generic collection C of k circles immersed to a connected surface F and realizing the zero element of $H_1(F; \mathbb{Q})$, we construct a smoothing $\tilde{C}$, then find a homology class ξ in $H_2(F, \tilde{C}; \mathbb{Q})$ whose image under the boundary homomorphism $\partial : H_2(F, \tilde{C}; \mathbb{Q}) \to H_1(\tilde{C}; \mathbb{Q})$ is the fundamental class of $\tilde{C}$. Since $H_2(F; \mathbb{Q}) = 0$, such class ξ is unique. For each $x \in F \smallsetminus \tilde{C}$ we take the image of ξ under the relativization homomorphism $H_2(F, \tilde{C}; \mathbb{Q}) \to H_2(F, F \smallsetminus x; \mathbb{Q})$. Denote the absolute value of this image under an isomorphism $H_2(F, F \smallsetminus x; \mathbb{Q}) \to \mathbb{Q}$ by $ind_{\tilde{C}}(x)$ and denote by $J^-(C)$ the number

$$k - \int_{F \smallsetminus \tilde{C}} (ind_{\tilde{C}}(x))^2 \, d\chi(x).$$

It is easy to see that $J^-(C)$ is invariant under regular homotopy in the class of generic collections of immersions and 1.2.C holds true for it. It is a generalization of Arnold's J^- in the sense that if C is a composition of an immersion C' of S^1 to $\mathbb{R}^2$ and an embedding $\mathbb{R}^2 \to F$, then $J^-(C) = J^-(C')$.

4.4. In sphere. In the case of sphere, although any curve C is zero homologous, the homology class in $H_2(S^2, C)$ of a chain bounded by C is not unique, because one can add to such a class the fundamental class of S^2 multiplied by any integer.

Thus, contrary to the case of affine and projective planes, in the sphere, at first glance, there is no index of a point with respect to a 1-cycle. However, for a 1-cycle, which is a collection of immersed circles, there is a nice replacement for the index.

Recall that for a collection C of circles immersed to R^2 there is a well-defined number $w(C)$ which is called *winding number* [1] and, in the case of a single immersed circle, defines the immersion up to regular isotopy.

Given a point $x \in S^2$ and a collection of immersed circles $C \subset S^2$ with $x \notin C$, set $w_C(x)$ to be the winding number of C in $S^2 \smallsetminus x$. Local behavior of $w_C(x)$ and $ind_C(x)$ are similar: when x jumps over a branch of C from the left hand side to

[1] It is called also the Whitney index, it is the rotation number of the velocity vector as well as the degree of the Gauss map.

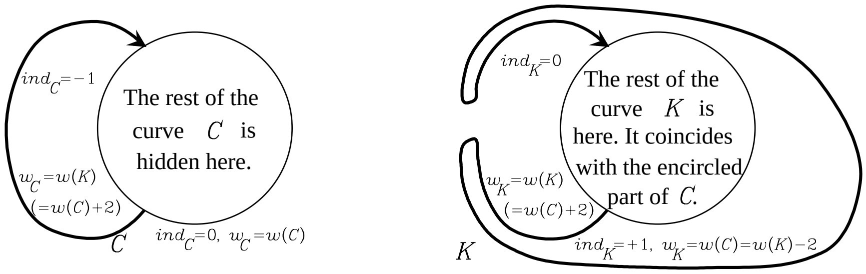

FIGURE 4.1. Behavior of $ind_C(x)$ and $w_C(x)$ when x jumps over a branch of C. To visualize it, we move around the sphere (through the point of infinity of the plane of picture) the piece of C which is jumped over. The curve K obtained has the winding number which differs by 2 from the winding number of C.

the right hand side, $ind_C(x)$ is decreased by 1 while $w_C(x)$ is increased by 2. See Figure 4.1.

Therefore

$$-\int_{S^2 \smallsetminus \tilde{C}} \left(\frac{w_{\tilde{C}}(x)}{2}\right)^2 d\chi(x)$$

changes under self-tangency and triple point perestroikas exactly like J^-. This suggests the following definition.

For a generic collection C of k circles immersed to S^2 set

$$J^-(C) = k - \frac{1}{4}\int_{S^2 \smallsetminus \tilde{C}} (w_{\tilde{C}}(x))^2 \, d\chi(x). \tag{4.1}$$

Note that it is not a generalization of J^- of plane curves in the sense above. It is a counter-part rather than a generalization. It is related to J^- of plane curves, but the relation is a bit more complicated, namely:

4.4.A. *Let C be a collection of circles immersed to plane, $i : \mathbb{R}^2 \to S^2$ be an embedding and C' be $i(C)$. Then*

$$J^-(C') = J^-(C) + \frac{(w(C))^2}{2}.$$

V. I. Arnold [2] observed that $J^-(C) + \frac{(w(C))^2}{2}$ is an invariant of C'. He called $J^-(C) + \frac{(w(C))^2}{2}$ a *conformal invariant* of C (it is invariant under conformal transformations). Theorem 4.4.A means that

$$J^-(C) = k - \frac{1}{4}\int_{S^2 \smallsetminus \tilde{C}} (w_{\tilde{C}}(x))^2 \, d\chi(x)$$

coincides with the conformal invariant.

To prove 4.4.A, we need a more explicit relation between ind_C and w_C.

4.4.B LEMMA 1. *Let C be a collection of circles immersed to plane, $i : \mathbb{R}^2 \to S^2$ be an embedding and C' be $i(C)$. Then for any $x \in \mathbb{R}^2 \smallsetminus C$*

$$ind_C(x) = \frac{1}{2}(w(C) - w_{C'}(i(x))).$$

Proof. Observe that the formula is correct for x belonging to the outer component of $\mathbb{R}^2 \smallsetminus C$ (where both sides are equal to zero) and that both sides changes by the same quantity when x jumps over a branch of C. $\square$

4.4.C Lemma 2. *Let K be a smooth closed oriented 1-manifold in S^2. Then*

$$\int_{S^2\smallsetminus K} w_K(x)\,d\chi(x) = 0.$$

Proof. In the case of one circle it is obvious, since the complement $S^2 \smallsetminus K$ consists of 2 open disks, on one of them w_K is equal to 1, while on the other it is -1. The general case follows, because $w_{A\cup B}(x) = w_A(x) + w_B(x)$ and the integral is additive. $\square$

Proof of 4.4.A. By the definition given in Section 4.1

$$J^-(C) = k - \int_{\mathbb{R}^2\smallsetminus\tilde C} (ind_{\tilde C}(x))^2\,d\chi(x).$$

By Lemma 1 the right hand side is equal to

$$k - \int_{\mathbb{R}^2\smallsetminus\tilde C} \frac14 (w_{\widetilde{C}'}(i(x)))^2\,d\chi(x) + \int_{\mathbb{R}^2\smallsetminus\tilde C} \frac12 w(C) w_{\widetilde{C}'}(i(x))\,d\chi(x) - \frac14 (w(C))^2.$$

Since for any function $\varphi : S^2 \to \mathbb{R}$ which is constant on $S^2 \smallsetminus i(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2\smallsetminus\tilde C} \varphi(x)\,d\chi(x) = \int_{S^2\smallsetminus\widetilde{C}'} \varphi(i(x))\,d\chi(x) - \varphi(S^2\smallsetminus i(\mathbb{R}^2)),$$

we can rewrite the expression of $J^-(C)$ obtained above as follows:

$$\begin{aligned}
J^-(C) = k - \int_{S^2\smallsetminus\widetilde{C}'} &\frac14 (w_{\widetilde{C}'}(i(x)))^2\,d\chi(x) + \frac14 (w(C))^2 \\
&+ \frac12 w(C) \int_{S^2\smallsetminus\widetilde{C}'} w_{\widetilde{C}'}(i(x))\,d\chi(x) - \frac12 (w(C))^2 \\
&\qquad - \frac14 (w(C))^2 \\
&= k - \int_{S^2\smallsetminus\widetilde{C}'} \frac14 (w_{\widetilde{C}'}(i(x)))^2\,d\chi(x) + \\
&\qquad + \frac12 w(C) \int_{S^2\smallsetminus\widetilde{C}'} w_{\widetilde{C}'}(i(x))\,d\chi(x) - \frac12 (w(C))^2
\end{aligned}$$

By Lemma 2, it is equal to

$$k - \int_{S^2\smallsetminus\widetilde{C}'} \frac14 (w_{\widetilde{C}'}(i(x)))^2\,d\chi(x) - \frac12 (w(C))^2.$$

By the definition of $J^-(C')$ it is

$$J^-(C') - \frac12 (w(C))^2$$

$\square$

Lemma 2 suggests a way to characterize $\frac{1}{2}w_K(x)$ as a reasonable choice for index without appeal to differential topology: consider all functions ind_K obtained as above from chains with boundary K and select one of them satisfying the identity

$$\int_{S^2 \smallsetminus K} ind_K(x)\, d\chi(x) = 0.$$

This rule assumes that K is a collection of disjoint embedded circles. In general case, one has to either smooth singularities, or change the integral extending the function ind_K over K and taking the integral over the whole S^2.

The same natural chain in S^2 with a given boundary has been constructed by V. I. Arnold [2] in a different way.

Consider now any chain in S^2 with boundary K and the function ind_K related with this chain. From the natural chain related with $\frac{1}{2}w_K(x)$ this one differs by $c[S^2]$ for some $c \in \mathbb{Z}$, therefore $ind_K(x) = \frac{1}{2}w_K(x) + c$. The number c can be found as follows: integrating the latter relation

$$\int_{S^2 \smallsetminus K} ind_K(x)\, d\chi(x) = \frac{1}{2}\int_{S^2 \smallsetminus K} w_K(x)\, d\chi(x) + c\chi(S^2 \smallsetminus K)$$

and taking into account Lemma 2, one obtains $c = \frac{1}{2}\int_{S^2 \smallsetminus K} ind_K(x)\, d\chi(x)$. Therefore

$$\frac{1}{2}w_K(x) = ind_K(x) - \frac{1}{2}\int_{S^2 \smallsetminus K} ind_K(u)\, d\chi(u).$$

The latter formula allows to rewrite our definition (4.1) of $J^-(C)$ for a spherical curve in terms of arbitrary chain with boundary $\widetilde{C}$:

$$\begin{aligned}
(4.2) \quad J^-(C) &= k - \int_{S^2 \smallsetminus \widetilde{C}} (ind_{\widetilde{C}}(x) - \frac{1}{2}\int_{S^2 \smallsetminus \widetilde{C}} ind_{\widetilde{C}}(u)\, d\chi(u))^2\, d\chi(u) \\
&= k - \int_{S^2 \smallsetminus \widetilde{C}} (ind_{\widetilde{C}}(x))^2\, d\chi(x) + \frac{1}{2}\left(\int_{S^2 \smallsetminus \widetilde{C}} ind_{\widetilde{C}}(x)\, d\chi(x)\right)^2.
\end{aligned}$$

This formula has a real algebraic counter-part. Namely, for a generic real algebraic curve A of type I on sphere S^2 which is the intersection of S^2 and a surface of degree d

$$\frac{d^2}{2} = \int_{S^2 \smallsetminus \widetilde{\mathbb{R}A}} (ind_{\widetilde{\mathbb{R}A}}(x))^2\, d\chi(x) - \frac{1}{2}\left(\int_{S^2 \smallsetminus \widetilde{\mathbb{R}A}} ind_{\widetilde{\mathbb{R}A}}(x)\, d\chi(x)\right)^2 + \sigma.$$

4.5. Zero-homologous curve in an orientable closed surface with non-zero Euler characteristic. Consider now a generic collection of k immersed circles in an orientable closed connected surface F with $\chi(F) \neq 0$ realizing the zero of $H_1(F)$.

As above, construct a smoothing $\tilde{C}$ of C, then consider homology classes $\xi \in H_2(F, \tilde{C}; \mathbb{Q})$ whose image under the boundary homomorphism $\partial : H_2(F, \tilde{C}; \mathbb{Q}) \to H_1(\tilde{C}; \mathbb{Q})$ is the fundamental class of $\tilde{C}$. Any two ξ's of this sort can be obtained one from another one by adding the image of $q[F]$ with $q \in \mathbb{Q}$ under the relativization homomorphism $\rho : H_2(F; \mathbb{Q}) \to H_2(F, \tilde{C}; \mathbb{Q})$. For each $x \in F \smallsetminus \tilde{C}$ take the image of ξ under the relativization homomorphism $H_2(F, \tilde{C}; \mathbb{Q}) \to H_2(F, F \smallsetminus x; \mathbb{Q})$. Denote the absolute value of this image under an isomorphism $H_2(F, F \smallsetminus x; \mathbb{Q}) \to \mathbb{Q}$ by $ind_\xi(x)$. Consider

$$\int_{F \smallsetminus \tilde{C}} ind_\xi(x)\, d\chi(x).$$

It is a rational number and if we replace ξ with $\xi + \rho(q[F])$ then it changes by $q\chi(F)$. Since by assumption $\chi(F) \neq 0$, there exists a unique ξ_0 such that

$$\int_{F \smallsetminus \tilde{C}} ind_{\xi_0}(x)\, d\chi(x) = 0.$$

Pick up this ξ_0 and denote by $J^-(C)$ the number

$$k - \int_{F \smallsetminus \tilde{C}} (ind_{\xi_0}(x))^2\, d\chi(x).$$

This generalizes the construction of the previous section. It is easy to prove that this is invariant under regular isotopy in the class of generic collections of immersed circles and satisfies 1.2.C.

4.6. Curves in torus and collections of contractible curves. Since the Euler characteristic of torus is zero, it is impossible to apply the construction of the previous section to a generic collection of circles immersed into torus. This "unfortunate" property of torus can be partially compensated by commutativity of its fundamental group. Partially means that it allows to deal with generic collections of immersed circles in which each circle is zero-homologous, but not with generic collections with the sum of the classes realized by all the immersed circles equal to zero.

Consider a generic collection C of circles $C_1, C_2, \dots, C_k$ immersed into torus T such that each C_i realizes $0 \in H_1(T)$. Since the fundamental group is commutative, each C_i is homotopic to zero. Therefore C_i can be presented as the composition of a generic immersion C_i^* of the circle into plane $\mathbb{R}^2$ and the universal covering $\pi : \mathbb{R}^2 \to T$.

Take the function $x \mapsto ind_{\bigcup_{i=1}^k C_i^*}(x)$ defined on $\mathbb{R}^2 \smallsetminus \bigcup_{i=1}^k C_i^*$. Since for any $x \in T \smallsetminus \bigcup_{i=1}^k C_i$ only in a finite number of points of $\pi^{-1}(x)$ this function does not vanish, there exists its direct image

$$T \smallsetminus C \to \mathbb{Z} : x \mapsto \sum_{y \in \pi^{-1}(x)} ind_{\bigcup_{i=1}^k C_i^*}(y).$$

Denote the latter function by ind_C. It is easy to see that ind_C does not depend on the choice of the liftings C_i^*.

For a smoothing $\tilde{C}$ of C there is a unique locally constant function on $T \smallsetminus \tilde{C}$ coinciding with ind_C outside the neighborhoods of double points of C where the smoothing takes place. Denote this function by $ind_{\tilde{C},C}$. It does depend not only on $\tilde{C}$, but on C.

Set

$$J^-(C) = k - \int_{T \smallsetminus \tilde{C}} (ind_{\tilde{C},C}(x))^2\, d\chi(x).$$

It is invariant under regular isotopy in the class of generic collections of immersed circles and satisfies 1.2.C.

The same construction can be applied to a generic collection of circles immersed in any surface (not only torus) such that each of the circles is contractible in the surface. If the surface is orientable, but not T then one can apply the construction of the previous section. The results differ.

5. Some high degree invariants of plane generic curves

5.1. Momenta of index. From formula (4.2) and Theorem 4.4.A it follows that for a generic plane curve C

$$w(C) = \int_{\mathbb{R}^2 \smallsetminus \widetilde{C}} ind_{\widetilde{C}}(x)\, d\chi(x).$$

This formula is easy to prove ab ovo, too. It suggests to consider all "momenta" $\int_{\mathbb{R}^2 \smallsetminus \widetilde{C}} (ind_{\widetilde{C}}(x))^r\, d\chi(x)$ of $ind_{\widetilde{C}}$.

Given a generic plane curve C, denote by $M_r(C)$ the integral

$$\int_{\mathbb{R}^2 \smallsetminus \widetilde{C}} (ind_{\widetilde{C}}(x))^r\, d\chi(x).$$

By (4.2) $M_1(C) = w(C)$. It is invariant under regular homotopy. Thus it can be considered as an invariant of degree 0.

As follows from the definition of Section 4.1, $M_2(C) = k - J^-(C)$ where k is the number of components of the immersing curve. Thus it is an invariant of degree 1. This suggests that M_r may be an invariant of degree $r-1$. Below it is proved to be the case.

Following to the general scheme of definition of finite degree invariants, for any characteristic of generic immersions of the circle to the plane which is locally constant on the space of generic immersions, one defines its *first derivative.* It is a characteristic of immersions having only one double point which is not ordinary and this point is either ordinary triple or ordinary self-tangency point. On such a curve the first derivative of the original characteristic is defined to be the difference between the values of the original characteristic on the adjacent generic immersions. In other words, it is the jump of the original characteristic happening at the corresponding perestroika. Of course, to define it, one has to specify a direction of the perestroika (a coorientation of the stratum of the discriminant hypersurface). In the case of self-tangency there is a natural direction: from curves with less double points to curves with more double points. In the case of triple points a coorientation was defined by Arnold [1], see Section 1.1 above. However we will need another local coorientation.

5.1.A. *For any r a direct self-tangency perestroika of C does not change $M_r(C)$.*

This is obvious: $\widetilde{C}$ does not change under a direct self-tangency perestroika. See Figure 4.1. □

Theorem 5.1.A says that the first derivative of any M_r vanishes on curves with a direct self-tangency point.

Studying M_r one has to distinguish two kinds of the triple point perestroikas. At the moment of perestroika at the triple point take vectors tangent to the branches and directed according to their orientations. If one of the vectors can be presented as a linear combination of the other two vectors with positive coefficients, the perestroika is said to be *weak,* otherwise it is said to be *strong.* See Figure 5.1.

FIGURE 5.1. Weak (on the left hand side) and strong (on the right hand side) triple point perestroikas.

5.1.B. *For any r a weak triple point perestroika of C does not change $M_r(C)$.*

This holds true for the same reason as 5.1.A: a weak triple point perestroika does not change $\widetilde{C}$. See Figure 4.1. $\square$

Theorem 5.1.B says that the first derivative of any M_r vanishes on curves with a weak triple point.

$M_r(C)$ changes under an inverse self tangency perestroika if $r > 1$, and under a strong triple point perestroika if $r > 2$. To describe the change, let me introduce the following notion.

For a multiple point p of a circle C immersed to the plane let *index* of p be the minimal number i such that there exists a small perturbation C' of C and a point p' in $\mathbb{R}^2 \setminus C$" arbitrary close to p and having index i with respect to C'.

For example, the index of an inverse self-tangency point is equal to the index of the narrow adjacent domains if the latter is smaller than the index of the adjacent wide domains. Otherwise it is smaller by 2 than the index of the adjacent narrow domains. See Figure 5.2.

FIGURE 5.2. Inverse self-tangency point with index i of the two types.

5.1.C. *For any r an inverse self-tangency perestroika of C changes $M_r(C)$ by $(i+2)^r - 2(i+1)^r + i^r$ where i is the index of the self-tangency point.*

PROOF. The corresponding perestroika of $\widetilde{C}$ replaces a vanishing arc by two arcs and the disk bounding by the vanishing newborn oval. See Figure 5.3. It means that there is a homeomorphism mapping the complement of the arc onto the complement of the two arcs and the disk and mapping $\widetilde{C}$ before the perestroika to $\widetilde{C}$ after it. The homeomorphism preserves index. Thus the difference between the integrals is the integral over the newborn disk and two arcs minus the integral over the vanishing arc. It is easy to see that it is $(i+2)^r - 2(i+1)^r + i^r$. $\square$

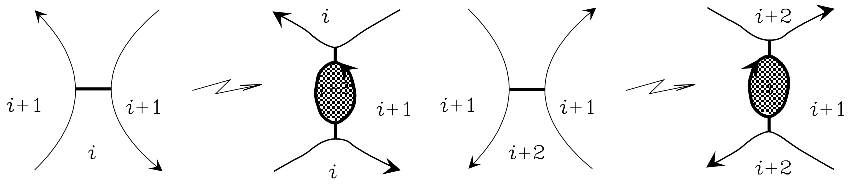

FIGURE 5.3

In the case of a strong triple point its index is i if the adjacent domains have indices $i+1$ and $i+2$, because then there is a perturbation with a vanishing triangle borning in the place of the point with having index i. There are only two topologically distinct perturbations. In the other perturbation the vanishing triangle has index $i+3$. See Figure 5.4

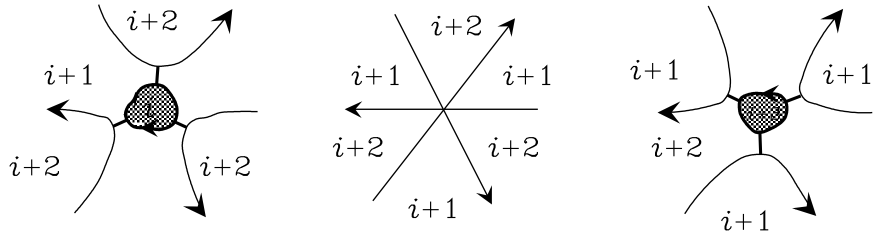

FIGURE 5.4. Strong triple point of index i and the smoothings of its perturbations. The vanishing arcs and disks are shown thicker.

By the positive direction of a strong triple point perestroika we will call the direction in which points of area inside the newborn triangle has index $i+3$ where i is the index of the triple point.

5.1.D. *For any r a strong triple point perestroika of C changes $M_r(C)$ by $(i+3)^r - 3(i+2)^r + 3(i+1)^r - i^r$ where i is the index of the triple point.*

The proof is similar to the proof of 5.1.C above. □

On the set of plane curves with a single nonordinary multiple point, the index of this point is an invariant of degree 1: a generic perestroika changes it by a constant depending only on the local structure of the perestroika. Therefore polynomial functions of the index of this point are invariants of finite degree. Thus the theorems of this section imply that $M_r(C)$ are invariants of C of finite degree.

5.2. A polynomial of immersion. For a generic plane curve C consider formal power series

$$P_C(h) = \sum_{r=0}^{\infty} \frac{M_r(C)h^r}{r!} \tag{5.1}$$

It provides a possibility to reformulate in the following concise form the results of the preceding section.

5.2.A. *Direct self-tangency and weak triple point perestroikas of C do not change P_C. An inverse self-tangency perestroika at a point of index i adds $e^{ih}(e^h-1)^2$ to $P_C(h)$. A strong triple point perestroika at a point of index i adds $e^{ih}(e^h-1)^3$ to $P_C(h)$.* $\square$

There is more compact formula for $P_C(h)$:

5.2.B. *For a generic plane curve C*

$$P_C(h)=\int_{\mathbb{R}^2\smallsetminus\widetilde{C}} e^{ind_{\widetilde{C}}(x)h}\,d\chi(x) \tag{5.2}$$

PROOF. Indeed,

$$\int_{\mathbb{R}^2\smallsetminus\widetilde{C}} e^{ind_{\widetilde{C}}(x)h}\,d\chi(x)=\int_{\mathbb{R}^2\smallsetminus\widetilde{C}}\sum_{r=0}^{\infty}\frac{(ind_{\widetilde{C}}(x))^r h^r}{r!}\,d\chi(x)= \tag{5.3}$$
$$=\sum_{r=0}^{\infty}\frac{h^r}{r!}\int_{\mathbb{R}^2\smallsetminus\widetilde{C}}(ind_{\widetilde{C}}(x))^r\,d\chi(x)=\sum_{r=0}^{\infty}\frac{h^r}{r!}M_r(C)=$$
$$=P_C(h)$$

$\square$

Proposition 5.2.B suggests to introduce a variable $q=e^h$. The power series $P_C(h)$ turns into a Laurent polynomial $P_C(q)$ in q defined by

$$\int_{\mathbb{R}^2\smallsetminus\widetilde{C}} q^{ind_{\widetilde{C}}(x)}\,d\chi(x) \tag{5.4}$$

Coefficients of $P_C(q)$ have a simple geometric sense: if $P_C(q)=\sum_{r=-\infty}^{\infty}p_rq^r$, then p_r is equal to the Euler characteristic of the subset of $\mathbb{R}^2\smallsetminus\widetilde{C}$ where $ind_{\widetilde{C}}(x)=r$. The changes of P_C under perestroikas also look simpler: an inverse self-tangency perestroika at point of index i adds $q^i(q-1)^2$ and a strong triple point perestroika at point of index i adds $q^i(q-1)^3$.

5.3. Analogy with knot polynomial invariants. Thus the polynomial $P_C(q)$ is very similar to the quantum knot polynomial invariants like the Jones polynomial. Substituting instead of q the exponent e^h provides the power series in h whose coefficients are invariants of finite degree. Behavior of $P_C(q)$ under perestroikas of C is similar to the skein relations. It allows to calculate $P_C(q)$ inductively, using any regular homotopy connecting C with an immersion whose polynomial is known. Formula (5.4) can be viewed as an analogue of face state sum formulas for knot quantum polynomials.

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