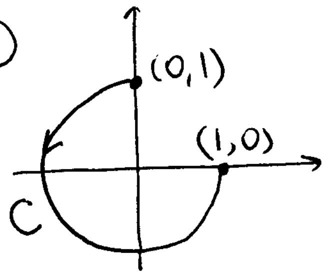
Applied Complex Analysis
MAT 342
Spring 2008

Solutions for Midterm II

(1)



$$\int_C \bar{z}^2 - z\,dz = \int_C \bar{z}^2 dz - \frac{1}{2} z^2 \Big|_i^1 \; (=)$$

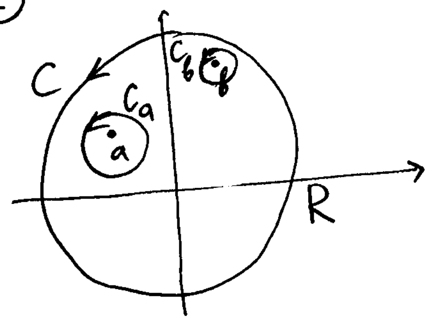
$$z = e^{i\theta}, \quad dz = i e^{i\theta} d\theta$$

$$\bar{z}^2 = (e^{-i\theta})^2 = e^{-2i\theta}$$

$$(=) \int_{\pi/2}^{2\pi} e^{-2i\theta}\, i e^{i\theta} d\theta - \frac{1}{2}(1 - i^2) = \frac{i}{-i} e^{-i\theta}\Big|_{\pi/2}^{2\pi} - 1 =$$

$$= -\Big(\underbrace{e^{-2\pi i}}_{1} - \underbrace{e^{-i\frac{\pi}{2}}}_{-i}\Big) - 1 = \boxed{-2-i}$$

(2)



$$\oint_C \frac{f(z)}{(z-a)(z-b)}\,dz =$$

$$\int_{C_a} \frac{\frac{f(z)}{z-b}}{z-a}\,dz + \int_{C_b} \frac{\frac{f(z)}{z-a}}{z-b}\,dz =$$

[CIF for

$\frac{f(z)}{z-b}$, analytic inside C_a

and

$\frac{f(z)}{z-a}$, analytic inside C

$$= 2\pi i \left[\frac{f(a)}{a-b} + \frac{f(b)}{b-a}\right] = \boxed{2\pi i\,\frac{f(b)-f(a)}{b-a}}$$

(3)

$$\int_{C:\,|z|=1/2} \frac{1}{z^2}\cos\frac{\pi}{z+1}\,dz = 2\pi i \left(\cos\frac{\pi}{z+1}\right)'\Bigg|_{z=0} =$$

Cauchy differentiation formula

$$= 2\pi i \left[-\sin\frac{\pi}{z+1}\cdot\frac{-\pi}{(z+1)^2}\right]_{z=0} = \boxed{0}$$

$\sin\pi = 0$

4) a) $\frac{z}{z+2} = \frac{z+2-2}{z+2} = 1 - \frac{2}{z+2} = 1 - \frac{1}{1+\frac{z}{2}} =$

Geometric series
$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ for $|z|<1$

$= 1 - \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^n =$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{2}\right)^{n+1}$ converges for $\left|\frac{z}{2}\right| < 1 \iff |z| < 2$

b) $\frac{z}{z+2} = 1 - \frac{2}{z+2} = 1 - \frac{2}{3} \frac{1}{1+\frac{z-1}{3}} =$

Geom. series

$= 1 - \frac{2}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-1}{3}\right)^n$ converges for $\left|\frac{z-1}{3}\right| < 1 \iff |z-1| < 3$