Research statement

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1 Outline of results

1 Patchworking: constructing real algebraic varieties with controlled topological properties

Although I started in low-dimensional topology, I seem to be best known my contributions to real algebraic geometry. As far as I can judge, my most appreciated invention is the patchwork technique or "Viro method" which allows real algebraic varieties to be constructed by a sort of "cup and paste" technique. It was introduced in order to construct real algebraic varieties with remarkable topological properties. Using it I have completed the classification up to isotopy of non-singular plane projective curves of degree 7 and disproved the classical Ragsdale conjecture formulated in 1906 [13]. See also [16], [17], [18], [20], [23], [34], [42] and [55].

2 The dequantizing of algebraic geometry; tropical geometry

In my talk [47] at the third European Congress of Mathematicians in 2000, I observed that real algebraic geometry can be presented as a quantized (i.e., deformed) piecewise linear geometry. A simple version of the patchwork construction (which builds a real algebraic hypersurface in a toric variety out of a special piecewise-linear hypersurface of Euclidean space) could be understood in terms of this quantization. The generalization of these ideas gave rise to the development (by Kapranov, Kontsevich, Sturmfiels and Mikhalkin) of so called tropical geometry and its applications to problems of classical algebraic geometry.

3 Other results on real algebraic varieties

I found several restrictions on the topology of real algebraic curves (see [10], [17], [18], [27], [38]), explicit elementary constructions of real algebraic surfaces with maximal total Betti numbers [11], and non-singular real projective quartic surfaces of all but one of the possible isotopy types [12]. I generalized complex orientations from the simple case of a real algebraic curve dividing its complexification to real algebraic varieties of high dimensions [15], [39].

Together with Kharlamov I generalized the main topological restrictions on nonsingular plane projective real algebraic curves to singular ones [27]. Recently, in a joint paper [48] with Orevkov, I managed, by applying one of these results on singular curves, to prove Orevkov's conjecture on the topology of nonsingular curves of degree 9.

I studied the Radon transformations based on integrals against the Euler characteristic on real and complex projective spaces, and established its relation to the projective duality for algebraic varieties [26]. This gives new relations between the numerical characteristics of projectively dual varieties.

4 TQFT

Jointly with V. Turaev, I found a (2+1)-dimensional topological quantum field theory based on state sums over triangulations or Heegaard diagrams and involving quantum 6j-symbols, see [35]. This paper had unexpected (to the authors) relations to Physics: it gave the first rigorous realization of the approach by G. Ponzano and T. Redge to 2+1 quantum gravity. In the context of Quantum Topology, it was widely generalized and related to other quantum invariants.

From the algebraic point of view, the invariants introduced in [35] are based on the representation theory of quantum group $U_q sl(2)$ where the parameter q is a root of unity. In [51] I used similar constructions applied to the quantum supergroup $U_q gl(1,1)$ and $U_{\sqrt{-1}} sl(2)$ to study quantum relatives of the Alexander polynomial.

5 Exotic knottings

Jointly with S.Finashin and M.Kreck, I constructed the first infinite series of surfaces smoothly embedded in the 4-sphere, which are pairwise ambiently homeomorphic, but not diffeomorphic, see [24], [25]. The examples come from real algebraic geometry. Namely, the series of Dolgachev surfaces has real structures (that is complex conjugation involutions) the orbit spaces spaces of which are diffeomorphic to the 4-sphere. The real point sets (i.e., the fixed point sets of the involutions) are diffeomorphic to the connected sum of 5 copies of the Klein bottle. They are embedded in the 4-sphere differently from the differential viewpoint, because the Dolgachev surfaces are not diffeomorphic, but in the same way topologically. (In the paper only finiteness of topological types was proven, but later Kreck proved that there is only one type.)

6 Diagrammatic formulas for finite type invariants

In a joint work [40] with M.Polyak, I introduced diagrammatic formulas for Vassiliev knot invariants. Joint work [44] with M. Goussarov and M. Polyak extended these formulas and the notion of Vassiliev invariants to virtual knots.

For the Arnold invariants of generic immersed plane curves, I found counterparts for real plane algebraic curves. This, together with Rokhlin's complex orientations formula for real algebraic curves suggested combinatorial formulas for the Arnold invariants J_- and J_+. The formulas allowed me to prove Arnold's conjecture about the range of these invariants. See [41].

7 Real algebraic links

I initiated the topological study of generic configurations of lines in 3-space. See [22], [31] and [32]. For non-singular real algebraic curves in 3-dimensional projective space, I defined a numerical characteristic (the "encomplexed" writhe number) invariant under rigid isotopy, which allows the proof that some real algebraic curves, which are topologically isotopic, cannot be deformed to each other in the class of non-singular real algebraic curves. See [46].

Curves generically immersed in the plane can be considered as the counterpart of links in 3-space, since their natural liftings to the unit tangent bundle and to the projectivized tangent bundle are knots in these 3-manifolds. Arnold's invariants J_+ and J_- are isotopy invariants of the corresponding knots. An even more profound invariant is the Whitney number classifying immersions of the circle into the plane up to regular homotopy. In [53] I found an expression for the Whitney number of a closed real algebraic plane affine curve dividing its complexification and equipped with a complex orientation, in terms of the behavior of its complexification at infinity.

8 Branched coverings

During my study at Leningrad State University, I proved that any closed orientable 3-manifold of genus two is a two-fold branched covering of the 3-sphere branched over a link with 3-bridges [1], [3] (this was proven independently by Joan S.Birman and H.Hilden); I also found interpretations of the signature invariants of a link of codimension 2 as signature invariants of cyclic branched covering spaces of a ball, and proved estimates for the slice genus of links and the genus of non locally flat surfaces in 4-manifolds [4], [6].

2 Research plans

By a research plan one often means either an account of research which has been done, but is yet not documented in papers or preprints, or a statement of research intentions. It provides a sort of research self-portrait of the author, especially valuable at the beginning of his/her career, right after the Ph.D, when a statement on completed research looks short. After 37 years of research in Mathematics an outline of main results says more. In fact, my research plans were never on a par with results. I was never able to foresee really interesting discoveries and new twists of the subject, which ended up drastically changing the direction of my research.

After this disclaimer, let me restrict myself to an indication of wide directions, in which I plan to concentrate. I do have more detailed plans, but my experience suggests that the details most probably will change.

1 Categorification of combinatorial invariants

In low dimensional topology, there is a long standing challenge to bridge the combinatorial topology of knots and 3-manifolds with the gauge-theoretic topology of 4-manifolds.

Two major breakthroughs in low dimensional topology, that happened in the eighties, were motivated by ideas which came from physics. First, S. Donaldson discovered new invariants of differential 4-dimensional manifolds by studying solutions of the Yang-Mills equation. Second, a new class of combinatorial invariants of knots and 3-dimensional manifolds was discovered. This class includes the Jones polynomial and Reshetikhin - Turaev invariant of 3-manifolds and is related to quantum groups.

The topological invariants which emerged in these two revolutions appeared to be quite different. The Donaldson invariants and, closely related to them, the Seiberg-Witten invariants of 4-manifolds are based on heavy analytical techniques from partial differential equations. They led to the discovery of a great variety of smooth structures on 4-manifolds, a phenomenon which distinguishes dimension 4 from all the other dimensions. The analytic nature of these invariants challenges topologists: general, well-known results imply that all the invariants of smooth structures on 4-manifolds should have combinatorial definitions, which are standard for topology. A combinatorial definition would be very useful both for the calculation of the invariants, and for the understanding of their nature.

The quantum invariants of links and 3-manifolds were understood combinatorially from the very beginning. Immediately after their introduction, M. F. Atiyah conjectured that they are related to the Donaldson invariants. This would give a combinatorial perspective on the Donaldson invariants. Although the relation has not yet been found, there has been a real progress in this direction.

In a recent work by M. Khovanov on categorification of the Jones polynomial, the Kauffman state sum construction for the Jones polynomial of a link was elevated to a construction of a bigraded family of homology groups related to the Jones polynomial in the same way as the usual homology groups of a topological space are related to its Euler characteristic. This provided a new approach to the problem under consideration.

Then Ozsvath and Szabo constructed a version of Floer homology theory and, using it, made a categorification of the Alexander-Conway polynomial. Their construction was not entirely combinatorial, but in 2006 Manolescu, Ozsvath and Sarkar proposed a purely combinatorial construction. This categorification looks very similar to Khovanov homology. These two homology theories share so many properties that one could even suspect the existence of a homology theory, which would generalize both of them, as the HOMFLY polynomial generalizes both underlying polynomials, the Alexander and the Jones polynomials. Indeed, in a recent preprints by Khovanov and Rozansky, the HOMFLY polynomial was categorified, and conjecturally both the Khovanov homology and the Ozsvath-Szabo Heegaard Floer homology can be recovered from this categorification.

Besides the conceptual simplification and clarification of the subject, one may expect that the new homology theories will bring new results and new simpler proofs of known geometric results. To some extent, these hopes have started to come true: recently Rasmussen, Livingston and Shumakovitch managed to find new proofs, based on these new homology theories, for the most exciting results obtained originally via gauge theory on the slice genus of knots.

There are many open problems in this direction. First of all, the major problem is to extend the combinatorial categorifications of quantum polynomials to invariants of 3- and 4-manifolds. Second, categorifications of quantum polynomial link invariants should be generalized to links in an arbitrary 3-manifold. During the last few years I have worked in this direction. So far only partial results have been obtained. Khovanov homology has been generalized to virtual links, and links in thickened surfaces. At the first stage, unexpectedly, a full-fledged categorification with integer coefficients required an additional assumption on the virtual links. As a by-product, I constructed Khovanov homology for links in projective space. In May 2006 V. O. Manturov managed to eliminate these additional assumptions on virtual links. This opens a new perspective for work in this direction. I mean to continue this research.

The state sum models that I studied in [51] provide opportunities for combinatorial categorification of invariants related to the Alexander polynomial. This is a promising, although technically involved, problem that I am working on now.

2 Real algebraic knot theories

In classical knot theory by a link one means a smooth closed 1-dimensional submanifold of the 3-dimensional sphere S^3, i. e. several disjoint circles smoothly embedded in S^3.

A classical link may emerge as the set of real points of a real algebraic curve. First, this gives rise to questions about relations between the invariants of the same curve provided by link theory and by algebraic geometry. Second, this suggests to develop a theory parallel to the classical link theory, but taking into account the algebraic nature of the objects. From this viewpoint it is more natural to consider real algebraic links up to an isotopy consisting of real algebraic links belonging to the same continuous family of algebraic curves, rather than up to smooth isotopy in the class of classical links. An isotopy of the former kind is called a *rigid isotopy.*

Of course, there is a forgetful functor: any real algebraic link can be considered as a classical link and a rigid isotopy as a smooth isotopy. It is interesting, how much is lost in this transition.

There is another forgetful functor: a real algebraic link can be considered over the field of complex numbers. A rigid isotopy gives rise to a path in the moduli space of the corresponding non-singular complex algebraic curves embedded into the complexification of the ambient 3-manifold. These moduli spaces are known to be numerous and complicated. A path-connected component of moduli space containing the complexification of a real algebraic link is invariant under rigid isotopy of that link. Thus instead of a single real algebraic knot theory we see a special real algebraic link theory for each path-connected component of moduli space. The most elementary of those components consist of rational curves.

The tools coming from the classical knot theory did not apply easily and naturally to real algebraic knots. However finite type invariants and especially their recent geometric interpretations seem to fit better.

3 Tropical geometry

This is a geometry based on the group of real affine transformations with linear part belonging to $SL(n, \mathbb{Z})$. It can be viewed also as an algebraic geometry over the *semi-field* of real numbers with operations of taking maximum (for addition) and addition (for multiplication). It is closely related to algebraic geometry over the complex or real numbers and, especially, over the fields of Puiseux series. Although tropical varieties have appeared in many situations and proved to be useful, the main notions related to them are still to be developed. The combinatorial patchworking that I discovered about 25 years ago is now considered as real tropical hypersurfaces. Other classes of tropical varieties, for instance curves in 3-space, are yet to be studied.

4 Differential spaces

I plan to revise the foundations of differential topology and geometry. The goal is to make them more similar to the foundations of algebraic geometry. Their present state does not allow discussion of manifolds with singularities, quotient spaces, etc. I believe that a more flexible system of notions is needed in research, teaching, and applications.

In the sixties a few attempts were made to introduce the category of differential spaces, but they failed to be accepted by the mainstream research community. I believe this will require additional research, and I plan to take a look in this direction.

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