Legends

"One of the great mathematical challenges of the early 18th century was to find an expression for the sum of reciprocal squares

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\dots \tag{1}$$

Joh. Bernoulli eagerly sought for this expression for many decades." ([**?**] E.Hairer, G.Wanner, Analysis by Its History, Springer, 1997.)

In 1689 Jac. Bernoulli proved convergence of the series. In 1728-1729 Goldbah and D. Bernoulli evaluated the series with accuracy 0.01. Stirling in 1730 found 8 digits of the sum.

L. Euler in 1734 calculated the first 18 digits (!) of the sum (1) after the decimal point and recognized $\pi^2/6$, which has the same 18 digits. He conjectured that the infinite sum is equal to $\pi^2/6$. In 1735 Euler discovered an expansion of the sine function into an infinite product of polynomials:

$$\frac{\sin x}{x}=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{3^2\pi^2}\right)\left(1-\frac{x^2}{4^2\pi^2}\right)\dots \tag{2}$$

Comparing this presentation with the standard sine series expansion

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\dots \tag{3}$$

Euler not only proved that the sum (1) is equal to $\pi^2/6$, moreover he calculated all sums of the type

$$1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\frac{1}{5^k}+\dots \tag{4}$$

for even k.

Putting $x=\frac{\pi}{2}$ in (2) he got the beautiful Wallis' Product:

$$\frac{\pi}{2}=\frac{2\cdot 2}{1\cdot 3}\frac{4\cdot 4}{3\cdot 5}\frac{6\cdot 6}{5\cdot 7}\frac{8\cdot 8}{7\cdot 9}\dots \tag{5}$$

which had been known since 1655. But Euler's first proof of (2) was not satisfactory. In 1748, in his famous "Introducio in Analysin Infinitorum", he presented a proof which was sufficiently rigorous for the 18th century. The series of reciprocal squares was named the *Euler series*.

If somebody wants to understand all the details of the above legend he had to study a lot of things, up to complex contour integrals. This is why the detailed mathematical exposition of Euler's series legend turns into an entire course of Calculus. The fascinating history of Euler's series is the guiding thread of the present course, "In Euler's footsteps".