1.2 Positive Series

On the contents of the lecture. Infinity is pregnant with paradoxes. Paradoxes throw us down from heavens to the earth. We leave the poetry for prose, and rationalize the "infinity and equality" by working with "finiteness and inequality". We shall lay a solid foundation for a summation theory for positive series. And the reader will find out what $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ precisely means.

Divergent series paradox. Let us consider the series $\sum\limits_{k=0}^{\infty} 2^k$. This is a geometric series. We know how to sum it up by autorecursion. The autorecursion equation is $s = 1 + 2s$. The only number satisfying this equation is -1. The sum of positive numbers turns to be negative!? Something is wrong!

A way to save the situation is to admit infinity as a feasible solution. Infinity is an obvious solution of $s = 1 + 2s$. The sum of any geometric series $\sum\limits_{k=0}^{\infty} q^k$ with denominator $q \geq 1$ is obviously infinite, isn't it?

Indeed, this sum is greater than $1 + 1 + 1 + 1 + \dots$, which symbolizes the infinity. (The autorecursion equation for $1 + 1 + 1 + \dots$ is $s = s + 1$. Infinity is the unique solution of this equation.)

The series $\sum\limits_{k=0}^{\infty} 2^k$ represents Zeno's series in the case of *Mighty Turtle,* which is faster than Achilles. To be precise, this series arises if $v = d_0 = 1$ and $w = 2$. As the velocity of the turtle is greater than the velocity of Achilles he never reaches it. So the infinity is right answer for this problem. But the negative solution -1 also makes sense. One could interpret it as an event in the past. Just the point in time when *the turtle passed Achilles.*

Oscillating series paradoxes. Philosopher Gvido Grandy in 1703 attracted public attention to the series $1-1+1-1+\dots$. He claimed this series symbolized the Creation of Universe from Nothing. Namely, insertion of brackets in one way gives Nothing (that is 0), in another way, gives 1.

$$(1-1)+(1-1)+(1-1)+\cdots = 0+0+0+\cdots = 0$$
$$1-(1-1)-(1-1)-(1-1)\cdots = 1-0-0-0\cdots = 1.$$

On the other hand, this series $1 - 1 + 1 - 1 + 1 - 1 + \dots$ is geometric with negative ratio $q = -1$. Its autorecursion equation $s = 1 - s$ has the unique solution $s = \frac{1}{2}$. Neither $+\infty$ nor $-\infty$ satisfy it. So $\frac{1}{2}$ seems to be its true sum.

Hence we see the Associativity Law dethroned by $1 - 1 + 1 - 1 + \dots$. The next victim is the Commutativity Law. The sum $-1+1-1+1-1+\dots$ is equal to $-\frac{1}{2}$. But the last series is obtained from $1 - 1 + 1 - 1 + \dots$ by transpositions of odd and even terms.

And the third amazing thing: diluting it by zeros changes its sum. The sum $1 + 0 - 1 + 1 + 0 - 1 + 1 + 0 - 1 + \dots$ by no means is $\frac{1}{2}$. It is $\frac{2}{3}$. Indeed, if we denote this sum by s then by shift formulas one gets

$$\begin{aligned} s &= \ \ 1+0-1+1+0-1+1+0-1+1+0-1+\dots \\ s-1 &= \ \ 0-1+1+0-1+1+0-1+1+0-1+1+\dots \\ s-1-0 &= -1+1+0-1+1+0-1+1+0-1+1+0+\dots \end{aligned}$$

Summing the numbers column-wise (i.e., by the Term-wise Addition Formula), we get

$$\begin{aligned} s+(s-1)+(s-1-0) = (1+0-1)+(0-1+1)+(-1+1+0)+ \\ +(1+0-1)+(0-1+1)+(-1+1+0)+\dots \end{aligned}$$

The left-hand side is $3s-2$. The right-hand side is the zero series. That is why $s=\frac{2}{3}$.

The series $1-1+1-1+\dots$ arises as Zeno's series in the case of a blind Achilles directed by a cruel Zeno, who is interested, as always, only in proving his claim, and a foolish, but merciful turtle. The blind Achilles is not fast, his velocity equals the velocity of the turtle. At the first moment Zeno tells the blind Achilles where the turtle is. Achilles starts the rally. But the merciful turtle wishing to help him goes towards him instead of running away. Achilles meets the turtle half-way. But he misses it, being busy to perform the first step of the algorithm. When he accomplishes this step, Zeno orders: "Turn about!" and surprises Achilles by saying that the turtle is on Achilles initial position. The turtle discovers that Achilles turns about and does the same. The situation repeats ad infinitum. Now we see that assigning the sum $\frac{1}{2}$ to the series $1-1+1-1+\dots$ makes sense. It predicts accurately the time of the first meeting of Achilles and turtle.

Positivity. The paradoxes discussed above are discouraging. Our intuition based on handling finite sums fails when we turn to infinite ones. Observe that all paradoxes above involve negative numbers. And to eliminate the evil in its root, let us consider only nonnegative numbers.

We return to the ancient Greeks. They simply did not know what a negative number is. But in contrast to Greeks, we will retain zero. A series with nonnegative terms will be called a *positive* series. We will show that for positive series all familiar laws, including associativity and commutativity, hold true and zero terms do not affect the sum.

Definition of Infinite Sum. Let us consider what Euler's equality (1.2.1) could mean:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{1.2.1}$$

The natural answer is: the *partial sums* $\sum\limits_{k=1}^{n} \frac{1}{k^2}$, which contain more and more reciprocal squares, approach closer and closer the value $\frac{\pi^2}{6}$. Consequently, all partial sums have to be less than $\frac{\pi^2}{6}$, its *ultimate sum*. Indeed, if some partial sum exceeds or coincides with $\frac{\pi^2}{6}$, then all subsequent sums will move away from $\frac{\pi^2}{6}$. Furthermore, any number c which is less than $\frac{\pi^2}{6}$ has to be surpassed by partial sums eventually, when they approach $\frac{\pi^2}{6}$ closer than by $\frac{\pi^2}{6}-c$. Hence the ultimate sum majorizes all partial ones, and any lesser number does not. This means that the ultimate sum is the smallest number which majorizes all partial sums.

Geometric motivation. Imagine a sequence of intervals of real line $[a_{i-1}, a_i]$. Denote by l_i the length of i-th interval. Let $a_0 = 0$ be the left end point of the first interval. Let $[0, A]$ be the smallest interval containing the whole sequence. Its length is naturally interpreted as the sum $\sum_{k=1}^{\infty} l_i$

This motivates the following definition.

Definition. *If the partial sums of the positive series $\sum_{k=1}^{\infty} a_k$ increase without bounds, its sum is defined to be ∞ and the series is called* divergent. *In the opposite case the series called* convergent, *and its sum is defined as the smallest number A such that $A \geq \sum_{k=1}^{n} a_k$ for all n.*

This Definition in equivalent to the following couple of principles. The first principle limits the ultimate sum from below:

Principle (One-for-All). *The ultimate sum of a positive series majorizes all partial sums.*

And the second principle limits the ultimate sum from above:

Principle (All-for-One). *If all partial sums of a positive series do not exceed a number, then the ultimate sum also does not exceed it.*

Term-Wise Addition Formula: $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} (a_k + b_k)$

Proof. The inequality $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty} (a_k + b_k)$. is equivalent to $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{\infty} b_k$. By All-for-One, the last is equivalent to the system of inequalities

$$\sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{\infty} b_k \qquad N = 1, 2, \dots$$

This system is equivalent to the following system

$$\sum_{k=1}^{\infty} b_k \leq \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{N} a_k \qquad N = 1, 2, \dots.$$

Each inequality of the last system, in its turn, is equivalent to the system of inequalities

$$\sum_{k=1}^{M} b_k \leq \sum_{k=1}^{\infty} (a_k + b_k) - \sum_{k=1}^{N} a_k \qquad M = 1, 2, \dots.$$

But these inequalities are true for all N and M, as the following computations show.

$$\sum_{k=1}^{M} b_k + \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{M+N} b_k + \sum_{k=1}^{M+N} a_k = \sum_{k=1}^{M+N} (a_k + b_k) \leq \sum_{k=1}^{\infty} (a_k + b_k)$$

In the opposite direction, we see that any partial sum on the right-hand side $\sum_{k=1}^{n} (a_k + b_k)$ splits into $\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$. And by virtue of the One-for-All principle, this does not exceed $\sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$. Now, the All-for-One principle provides the inequality in the opposite direction. □

Shift Formula. The Shift Formula immediately follows from the Term-Wise Addition. To be precise, immediately from the Definition, one gets the following: $a_0 + 0 + 0 + 0 + 0 + \cdots = a_0$ and that $0 + a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k$. Term-Wise Addition of these series gives

$$a_0 + \sum_{k=1}^{\infty} a_k = (a_0 + 0) + (0 + a_1) + (0 + a_2) + (0 + a_3) + \cdots = \sum_{k=0}^{\infty} a_k$$

Term-Wise Multiplication Formula. $\lambda \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \lambda a_k$. For any partial sum from the right-hand side one has $\sum_{k=1}^{n} \lambda a_k = \lambda \sum_{k=1}^{n} a_k \leq \lambda \sum_{k=1}^{\infty} a_k$ by the Distributivity Law for finite sums and One-for-All. This implies the inequality $\lambda \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{\infty} \lambda a_k$ by All-for-One. The opposite inequality is equivalent to $\sum_{k=1}^{\infty} a_k \geq \frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$. As any partial sum $\sum_{k=1}^{n} a_k$ is equal to $\frac{1}{\lambda} \sum_{k=1}^{n} \lambda a_k$, which does not exceed $\frac{1}{\lambda} \sum_{k=1}^{\infty} \lambda a_k$, one gets the opposite inequality.

Geometric series. We have to return to the geometric series, because the Autorecursion equation produced by shift and multiplication formulas says nothing about convergence. So we have to prove convergence for $\sum_{k=0}^{\infty} q^k$ with positive $q < 1$. It is sufficient to prove the following inequality for all n

$$1 + q + q^2 + q^3 + \cdots + q^n < \frac{1}{1-q}.$$

Multiplying both sides by $1 - q$ one gets on the left-hand side

$$(1-q) + (q - q^2) + (q^2 - q^3) + \cdots + (q^{n-1} - q^n) + (q^n - q^{n+1}) = \\ 1 - q + q - q^2 + q^2 - q^3 + q^3 - \cdots - q^n + q^n - q^{n+1} = 1 - q^{n+1}$$

and 1 on the right-hand side. The inequality $1 - q^{n+1} < 1$ is obvious. Hence we have proved the convergence. Now the autorecursion equation $x = 1 + qx$

for $\sum_{k=0}^{\infty} q^k$ is constructed in usual way by Shift and Term-Wise multiplication. It leaves only two possibilities for $\sum_{k=0}^{\infty} q^k$, either $\frac{1}{q-1}$ or ∞. For $q < 1$ we have proved convergence, and for $q \geq 1$ infinity is the true answer.

Let us pay special attention to the case $q = 0$. We adopt a common convention:

$$0^0 = 1. \tag{1.2.2}$$

This means that the series $\sum_{k=0}^{\infty} 0^k$ satisfies the common formula for a convergent geometric series $\sum_{k=0}^{\infty} 0^k = \frac{1}{1-0} = 1$. Finally we state the theorem, which essentially due to Eudoxus, who proved the convergence of the geometric series with ratio $q < 1$.

Theorem 1 (Eudoxus). *For every nonnegative q one has*

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q} \quad \textit{for } q < 1, \quad \textit{and} \quad \sum_{k=0}^{\infty} q^k = \infty \quad \textit{for } q \geq 1.$$

Comparison of series Quite often exact summation of series is too difficult, and for practical purposes it is enough to know the sum approximatively. In this case one usually compares the series with another one which sum is known. Such comparison is based on the following *Term-Wise Comparison Principle*, which immediately follows from the Definition of Sum.

Principle (Term-Wise Comparison). *If $a_k \leq b_k$ for k, then $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$*

The only series we have so far to compare with are the geometric ones. The following lemma is very useful for this purposes.

Lemma 2 (Ratio Test). *If $a_{k+1} \leq q a_k$ holds for some $q < 1$ and all terms of series $\sum_{k=0}^{\infty} a_k$ than $\sum_{k=0}^{\infty} a_k \leq \frac{a_0}{1-q}$*

Proof. By induction one proves inequality $a_k \leq a_0 q^k$. Now by Term-Wise Comparison one estimates $\sum_{k=0}^{\infty} a_k$ from above by the geometric series $\sum_{k=0}^{\infty} a_0 q^k = \frac{a_0}{1-q}$ □

If the series under consideration satisfies an autorecursion equation, to prove its convergence usually mean to evaluate it exactly. For proving of convergence the Term-Wise Comparison Principle can be strengthened. Let us say that the series $\sum_{k=1}^{\infty} a_k$ is *eventually* majorized by the series $\sum_{k=1}^{\infty} b_k$, if the inequality $b_k \geq a_k$ holds for each k starting from $k = n$ for some n. The following lemma is very useful to prove convergence.

Principle (Eventual Comparison). *A series $\sum_{k=1}^{\infty} a_k$, which is eventually majorized by a convergent series $\sum_{k=1}^{\infty} b_k$, is convergent.*

Proof. Consider a tail $\sum_{k=n}^{\infty} b_k$, which term-wise majorizes $\sum_{k=n}^{\infty} a_k$. Then $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} a_k \leq \sum_{k=1}^{n-1} a_k + \sum_{k=n}^{\infty} b_k \leq \sum_{k=1}^{n-1} a_k + \sum_{k=1}^{\infty} b_k < \infty$. □

Example Consider series $\sum_{k=1}^{\infty} k2^{-k}$. The ratio of two successive terms $\frac{a_{k+1}}{a_k}$ of the series is $\frac{k+1}{2k}$. This ratio is less or equal to $\frac{2}{3}$ starting with $k = 3$. Hence this series is eventually majorities by geometric one $\sum_{k=0}^{\infty} a_3 \frac{2^k}{3^k}$, $(a_3 = \frac{2}{3})$. This proves its convergence. And now by autorecursion equation one gets its sum.

Harmonic series paradox. Now we have a solid background to evaluate positive series. Nevertheless, we must be careful about infinity! Consider the following calculation:

$$\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = \sum_{k=1}^{\infty} \frac{1}{2k-1} - \sum_{k=1}^{\infty} \frac{1}{2k} = \sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k} =$$
$$\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{2k-1} + \sum_{k=1}^{\infty} \frac{1}{2k}\right) =$$
$$= \left(\sum_{k=1}^{\infty} \frac{1}{2k-1} - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k-1}\right) - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k} =$$
$$= \frac{1}{2}\left(\sum_{k=1}^{\infty} \frac{1}{2k-1}\right) - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)}$$

We get that the sum of $\sum_{k=1}^{\infty} \frac{1}{(2k-1)2k}$ satisfies the equation $s = \frac{s}{2}$. This equation has two roots 0 and ∞. But s satisfies inequalities $\frac{1}{2} < s < \frac{\pi^2}{6}$. What is wrong?

Problems.

1. Prove $\sum_{k=1}^{\infty} 0 = 0$
2. Prove $\sum_{k=1}^{\infty} 0^k = 1$
3. Prove $\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} a_{2k} + \sum_{k=0}^{\infty} a_{2k+1}$
4. Prove $\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k$ for convergent series
5. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$
6. Prove $(1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \dots = 1 - [(\frac{1}{2} - \frac{1}{3}) + (\frac{1}{4} - \frac{1}{5}) + \dots$

7. Prove the convergence of $\sum\limits_{k=0}^{\infty} \frac{2^k}{k!}$
8. Prove the convergence of $\sum\limits_{k=1}^{\infty} \frac{1000^k}{k!}$
9. Prove the convergence of $\sum\limits_{k=1}^{\infty} \frac{k^{1000}}{2^k}$
10. Prove that $q^n < \frac{1}{n(1-q)}$ for $0 < q < 1$
11. Prove that for any positive $q < 1$ there is an n that $q^n < \frac{1}{2}$
12. Prove $\sum\limits_{k=1}^{\infty} \frac{1}{k!} \leq 2$
13. Evaluate $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+2)}$
14. Prove the convergence of the Euler series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$
15. * Prove that $\sum\limits_{i=1}^{\infty} \sum\limits_{j=1}^{\infty} a_{ij} = \sum\limits_{j=1}^{\infty} \sum\limits_{i=1}^{\infty} a_{ij}$ for $a_{ij} \geq 0$