1.4 Infinite Products

On the contents of the lecture In this lecture we become acquainted with infinite products. The famous Euler Identity will be proved. We will find out that $\zeta(2)$ is another name for the Euler series. And we will see how the Euler's decomposition of the sine into product works to sum up the Euler Series.

Definition. *Product of an infinite sequence of numbers* $\{a_k\}$, *such that* $a_k \geq 1$ *for all* k *is defined as the least number majorizing all partial products* $\prod_{k=1}^{n} a_k = a_1 a_2 \dots a_n$

A sequence of natural numbers is called *essentially finite* if all but finitely many of its elements are equal to zero. Denote by $\mathbb{N}^\infty$ the set of all essentially finite sequences of natural numbers.

Theorem 1. *For any given sequence of positive series* $\sum\limits_{k=0}^{\infty} a_k^j$, $j = 1, 2, \dots$, *such that* $a_0^j = 1$ *for all* j *one has*

$$\prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^\infty} \prod_{j=1}^{\infty} a_{k_j}^j \tag{1.4.1}$$

Summands on the right-hand side of (1.4.1) usually contain factors which are less than one. But each of the summands contains only finitely many factors different from 1. So the summands are in fact finite products.

Proof. For a sequence $\{k_j\} \in \mathbb{N}^\infty$ define its *length* as maximal j for which $k_j \neq 0$ and its *maximum* as the value of its maximal term. The length of zero sequence is defined as 0.

Consider a finite subset $S \subset \mathbb{N}^\infty$. Consider the partial sum $\sum\limits_{\{k_j\} \in S} \prod_{k=1}^{\infty} a_{k_j}^j$. To estimate it, denote by L the maximal length of elements of S and denote by M the greatest of maxima of $\{k_j\} \in S$. In this case

$$\sum_{\{k_j\} \in S} \prod_{j=1}^{\infty} a_{k_j}^j = \sum_{\{k_j\} \in S} \prod_{j=1}^{L} a_{k_j}^j \leq \sum_{\{k_j\} \in \mathbb{N}_M^L} \prod_{j=1}^{L} a_{k_j}^j = \prod_{j=1}^{L} \sum_{k=0}^{M} a_k^j \leq \prod_{j=1}^{\infty} \sum_{k=0}^{\infty} a_k^j,$$

where $\mathbb{N}_M^L$ denotes the set of all finite sequences $\{k_1, k_2, \dots k_L\}$ of natural numbers such that $k_i \leq M$. By All-for-One this implies one of the required inequalities, namely, $\geq$.

To prove the opposite inequality, we prove that for any natural L one has

$$\prod_{j=1}^{L} \sum_{k=0}^{\infty} a_k^j = \sum_{\{k_j\} \in \mathbb{N}^L} \prod_{j=1}^{L} a_{k_j}^j, \tag{1.4.2}$$

where $\mathbb{N}^L$ denotes the set of all finite sequences $\{k_1, \dots k_L\}$ of natural numbers. The proof is by induction on L.

Lemma 2. *For any families* $\{a_i\}_{i \in I}$ $\{b_j\}_{j \in J}$ *of nonnegative numbers, one has*

$$\sum_{i \in I} a_i \sum_{j \in J} b_j = \sum_{(i,j) \in I \times J} a_i b_j \tag{1.4.3}$$

Proof. Since $I \times J = \bigsqcup_{i \in I} \{i\} \times J$ by the Sum Partition Theorem one gets:

(1.4.4)
$$\sum_{(i,j)\in I\times J} a_i b_j = \sum_{i\in I} \sum_{(i,j)\in\{i\}\times J} a_i b_j = \sum_{i\in I}\sum_{j\in J} a_i b_j = \sum_{i\in I} a_i \sum_{j\in J} b_j = \sum_{j\in J} b_j \sum_{i\in I} a_i$$

□

Case $L = 2$ follows from Lemma **2**, because $\mathbb{N}^2 = \mathbb{N}\times\mathbb{N}$. The induction step is done as follows

$$\prod_{j=1}^{L+1}\sum_{k=0}^{\infty} a_k^j = \sum_{k=0}^{\infty} a_k^{L+1}\prod_{j=1}^{L}\sum_{k=0}^{\infty} a_k^j = \sum_{k\in\mathbb{N}} a_k^{L+1} \sum_{\{k_j\}\in\mathbb{N}^L}\prod_{j=1}^{L} a_{k_j}^j = \sum_{\{k_j\}\in\mathbb{N}^{L+1}}\prod_{j=1}^{L+1} a_{k_j}^j$$

The left-hand side of (1.4.2) is a partial product for the left-hand side of (**1**) and the right-hand side of (1.4.2) is a subsum of the right-hand side of (**1**). Consequently, all partial products of the right-hand side in (**1**) do not exceed its left-hand side. This proves the inequality $\leq$.

□

Euler's Identity Our next goal is to prove the *Euler Identity*

(1.4.5)
$$\boxed{\sum_{k=1}^{\infty}\frac{1}{k^\alpha} = \prod_{p=1}^{\infty}\left(1-\frac{1}{p^\alpha}\right)^{-[p \text{ is prime}]}}$$

Here α is any rational (or even irrational) positive number.

The product on the right-hand side is called the *Euler Product.* The series on the left-hand side is called *Dirichlet series.* Each factor of the Euler Product expands into geometric series $\sum_{k=0}^{\infty}\dfrac{1}{p^{k\alpha}}$. By Theorem 1.4.1 the product of these geometric series is equal to the sum of products of the type $p_1^{-k_1\alpha}p_2^{-k_2\alpha}\dots p_n^{-k_n\alpha} = N^{-\alpha}$. Here $\{p_i\}$ are different prime numbers, $\{k_i\}$ are positive natural numbers and $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$. But each product $p_1^{k_1}p_2^{k_2}\dots p_n^{k_n} = N$ is a natural number, different products represent different numbers and any natural number has a unique representation of this sort. This is exactly what is called Principal Theorem of Arithmetics. That is the above decomposition of the Euler product expands in Dirichlet series.

Convergence of Dirichlet series

Theorem 3. *The Dirichlet series* $\sum_{k=1}^{\infty}\frac{1}{n^s}$ *converges iff* $s > 1$

Proof. Consider $\{2_k\}$ packing of the series. Then the n-th term of the packed series one estimates from the up as $\sum_{k=2^n}^{2^{n+1}-1}\frac{1}{k^s} \leq \sum_{k=2^n}^{2^{n+1}-1}\frac{1}{(2^n)^s} = 2^n\frac{1}{2^{ns}} = 2^{n-ns} = (2^{1-s})^n$. If $s > 1$ then $2^{1-s} < 1$ and the packed series is term-wise majorized by convergent geometric progression. Hence it converges. In case the of the Harmonic series ($s = 1$) the n-th term of its packing one estimates from below as $\sum_{k=2^n}^{2^{n+1}-1} \frac{1}{k} \geq \sum_{k=2^n}^{2^{n+1}-1} \frac{1}{2^{n+1}} = 2^n \frac{1}{2^{n+1}} = \frac{1}{2}$. That is why the harmonic series diverges. A Dirichlet series for $s < 1$ term-wise majorizes the Harmonic series and so diverges. □

Riemann ζ-function Function $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{n^s}$ is called *Riemann ζ-function.* It is of great importance in the number theory.

The simplest application of the Euler's Identity represents Euler's proof of infinity of the set of primes. The divergence of the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ implies the Euler Product has to contain infinitely many factors to diverge.

Euler proved essentially more exact result: the series of reciprocal primes diverges $\sum \frac{1}{p} = \infty$.

Summing via multiplication Multiplication of series gives rise to a new approach to evaluating their sums. Consider the geometric series $\sum_{k=0}^{\infty} x^k$. Then

$$\left(\sum_{k=0}^{\infty} x^k\right)^2 = \sum_{j,k\in\mathbb{N}^2} x^j x^k = \sum_{m=0}^{\infty} \sum_{j+k=m} x^j x^k = \sum_{m=0}^{\infty} (k+1)x^m$$

As $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ one gets $\sum_{k=0}^{\infty} (k+1)x^k = \frac{1}{(1-x)^2}$.

Sine-product Now we are ready to understand how two formulas

$$\text{(1.4.6)} \qquad \frac{\sin x}{x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right) \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(} 2k+1)!$$

which appeared in Legends, yield an evaluation of the Euler Series. Since at the moment we do not know how to multiply infinite sequences of numbers which are less than one, we invert the product in the first formula. We get

$$\text{(1.4.7)} \qquad \frac{x}{\sin x} = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)^{-1} = \prod_{k=1}^{\infty} \sum_{j=0}^{\infty} \frac{x^{2j}}{k^{2j}\pi^{2j}}$$

To avoid negative numbers, we interpret the series $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ in the second of formulas (1.4.6) as the difference $\sum_{k=0}^{\infty} \frac{x^{4k+1}}{(4k+1)!} - \sum_{k=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!}$. Substituting this expression for $\sin x$ in $\frac{x}{\sin x}$ and cancelling out x, we get

$$\text{(1.4.8)} \qquad \frac{x}{\sin x} = \frac{1}{1 - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k+1)!}} = \sum_{j=0}^{\infty} \left(\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k}}{(2k+1)!}\right)^j$$

All terms on the right-hand side starting with $j = 2$ are divisible by x^4. Consequently the only summands with x^2 on the right-hand side is $\frac{x^2}{6}$. On the other hand in (1.4.7) after expansion into sum by Theorem 1.4.1 terms with x^2 give the series $\sum\limits_{k=1}^{\infty} \frac{x^2}{k^2\pi^2}$. Comparing these results, one gets $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

Problems.

1. Prove $\prod_{n=1}^{\infty} 1.1 = \infty$
2. Prove identity $\prod_{n=1}^{\infty} a_n^2 = (\prod_{n=1}^{\infty} a_n)^2$ $(a_n \geq 1)$
3. Does $\prod_{n=1}^{\infty}(1 + \frac{1}{n})$ converge?
4. Evaluate $\prod_{n=2}^{\infty} \frac{n^2}{n^2-1}$
5. Prove divergence $\prod\limits_{1}^{\infty}(1 + \frac{1}{k})^{[\text{k is prime}]}$
6. Evaluate $\prod_{n=3}^{\infty} \frac{n(n+1)}{(n-2)(n+3)}$
7. Evaluate $\prod_{n=2}^{\infty} \frac{n^2-1}{n^2-4}$
8. Evaluate $\prod_{n=1}^{\infty}(1 + \frac{1}{n(n+2)})$
9. Evaluate $\prod_{n=1}^{\infty} \frac{(2n+1)(2n+7)}{(2n+3)(2n+5)})$
10. Evaluate $\prod_{n=2}^{\infty} \frac{n^3+1}{n^3-1}$
11. Prove inequality $\prod_{k=2}^{\infty}(1 + \frac{1}{k^2}) \geq \sum\limits_{k=2}^{\infty} \frac{1}{k^2}$
12. Prove convergence of the Wallis product $\prod \frac{4k^2}{4k^2-1}$
13. Evaluate $\sum\limits_{k=1}^{\infty} \frac{1}{k^4}$ applying (1.4.6).
14. Prove $\prod_{n=2}^{\infty} \frac{n^2+1}{n^2} < \infty$
15. Multiply geometric series by itself and get a power series expansion for $(1-x)^{-2}$
16. Define $\tau(n)$ as the number of divisors of n, then $\zeta^2(x) = \sum\limits_{k=1}^{\infty} \dfrac{\tau(n)}{n^x}$
17. Define $\phi(n)$ as the number of numbers which are less than n are relatively prime to n, then $\dfrac{\zeta(x-1)}{\zeta(x)} = \sum\limits_{k=1}^{\infty} \dfrac{\phi(n)}{n^x}$
18. Define $\mu(n)$ (Möbius function) as follows: $\mu(1) = 1$, $\mu(n) = 0$, if n is divisible by the square of a prime number, $\mu(n) = (-1)^k$, if n is the product of k different prime numbers. Then $\dfrac{1}{\zeta(x)} = \sum\limits_{k=1}^{\infty} \dfrac{\mu(n)}{n^x}$

19. * Prove $\sum\limits_{k=1}^{\infty} \frac{[k \text{ is prime}]}{k} = \infty$

20. * Prove identity $\prod_{n=0}^{\infty}(1 + x^{2^n}) = \frac{1}{1-x}$