1.5 Telescopic sums

On the content of this lecture. In this lecture we learn the main secret of the elementary summation theory. We will evaluate series via its partial sums. We introduce *factorial powers*, which are easy to sum. Following Stirling we expand $\frac{1}{1+x^2}$ into a series of negative factorial powers and apply this expansion to evaluate the Euler series with Stirling's accuracy 10^{-8}.

The series $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)}$. In the first lecture we have calculated infinite sums directly without invoking partial sums. Now we present a dual approach to summing series. According this approach at first one finds a formula for the n-th partial sum and then substitutes in this formula infinity instead of n. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)}$ gives a simple example for this method. A key to sum it up is the following identity

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Because of this identity $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)}$ turns into the sum of differences

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)+\dots \tag{1.5.1}$$

Its nth partial sum is equal to $1-\frac{1}{n+1}$. Substituting in this formula $n=+\infty$, one gets 1 as its ultimate sum.

Telescopic sums. The sum (1.5.1) represents a *telescopic sum*. This name is used for sums of the form $\sum\limits_{k=0}^{n}(a_k - a_{k+1})$. The value of such a telescopic sum is determined by the values of the first and the last of a_k, similarly to a telescope, whose thickness is determined by the radii of the external and internal rings. Indeed,

$$\sum_{k=0}^{n}(a_k - a_{k+1}) = \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} a_{k+1} = a_0 + \sum_{k=1}^{n} a_k - \sum_{k=0}^{n-1} a_{k+1} - a_{n+1} = a_0 - a_{n+1}.$$

The same arguments for infinite telescopic sum give

$$\sum_{k=0}^{\infty}(a_k - a_{k+1}) = a_0. \tag{1.5.2}$$

But this proof works only if $\sum\limits_{k=0}^{\infty} a_k < \infty$. This is untrue for $\sum\limits_{k=1}^{\infty} \frac{1}{k(k+1)}$, owing to divergence of the Harmonic series. But the equality 1.5.2 holds also if a_k tends to 0 as k tends to infinity. Indeed, in this case a_0 is the least number majorizing all $a_0 - a_n$, the nth partial sums of $\sum\limits_{k=0}^{\infty} a_k$.

Differences. For a given sequence $\{a_k\}$ one denotes by $\{\Delta a_k\}$ the sequence of differences $\Delta a_k = a_{k+1} - a_k$ and calls latter sequence as *difference* of $\{a_k\}$. The main formula of elementary summation theory is

$$\boxed{\sum_{k=0}^{n-1} \Delta a_k = a_n - a_0} \tag{1.5.3}$$

To telescope a series $\sum\limits_{k=0}^{\infty} a_k$ it is sufficient to find a sequence $\{A_k\}$ such that $\Delta A_k = a_k$. On the other hand the sequence of sums $A_n = \sum\limits_{k=0}^{n-1} a_k$ has difference $\Delta A_n = a_n$. Therefore, we see that to telescope a sum is equivalent to find a formula for partial sums. This lead to concept of *telescopic function.* For a function $f(x)$ we introduce its difference $\Delta f(x)$ as $f(x+1) - f(x)$. A function $f(x)$ telescopes $\sum a_k$, if $\Delta f(k) = a_k$ for all k.

Often the sequence $\{a_k\}$ that we would like to telescope has the form $a_k = f(k)$ for some function. Then we are searching for a *telescopic function* $F(x)$ for $f(x)$, i.e. a function such that $\Delta F(x) = f(x)$.

To evaluate the difference of a function is usually much easier than to telescope it. For this reason one has evaluated the differences of all basic functions and organized a *table of differences.* In order to telescope a given function, look in this table to find a table function whose difference coincides with or is close to given function.

For example, the differences of x^n for $n \leq 3$ are $\Delta x = 1$, $\Delta x^2 = 2x + 1$, $\Delta x^3 = 3x^2 + 3x + 1$. To telescope $\sum\limits_{k=1}^{\infty} k^2$ we choose in this table x^3. Then $\frac{\Delta x^3}{3} - x^2 = x + \frac{1}{3} = \frac{\Delta x^2}{2} - \Delta \frac{x}{6}$. Therefore, $x^2 = \Delta\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}\right)$. This immediately implies the following formula for sums of squares:

$$\sum_{k=1}^{n-1} k^2 = \frac{2n^3 - 3n^2 + n}{6} \tag{1.5.4}$$

Factorial powers The usual powers x^n have a complicated differences. So-called *factorial powers* $x^{\underline{k}}$ have simpler differences. For any number x and any natural number k, let $x^{\underline{k}}$ denote $x(x-1)(x-2)\dots(x-k+1)$, and by $x^{\underline{-k}}$ we denote $\frac{1}{(x+1)(x+2)\dots(x+k)}$. At last we define $x^{\underline{0}} = 1$. The factorial power satisfies the following *addition law*

$$\boxed{x^{\underline{k+m}} = x^{\underline{k}}(x-k)^{\underline{m}}} \tag{1.5.5}$$

We leave to the reader to check this rule for all integers m, k. The power $n^{\underline{n}}$ for a natural n coincides with the factorial $n! = 1 \cdot 2 \cdot 3 \cdots n$. The main property of factorial powers is given by

$$\boxed{\Delta x^{\underline{n}} = n x^{\underline{n-1}}} \tag{1.5.6}$$

The proof is straightforward:

$$(x+1)^{\underline{k}} - x^{\underline{k}} = (x+1)^{\underline{1+(k-1)}} - x^{\underline{(k-1)+1}} = (x+1)x^{\underline{k-1}} - x^{\underline{k-1}}(x-k+1) = kx^{\underline{k-1}}$$

Applying this formula one can easily telescope any *factorial polynomial,* i.e. an expression of the form $a_0 + a_1x^{\underline{1}} + a_2x^{\underline{2}} + a_3x^{\underline{3}} + \dots a_nx^{\underline{n}}$. Indeed, the explicit formula for the telescoping function is $a_0x^{\underline{1}} + \frac{a_1}{2}x^{\underline{2}} + \frac{a_2}{3}x^{\underline{3}} + \frac{a_3}{4}x^{\underline{4}} + \dots \frac{a_n}{n+1}x^{\underline{n+1}}$. Therefore, another strategy to telescope x^k is to represent it as a factorial polynomial.

For example, to represent x^2 as factorial polynomial, consider $a + bx + cx^{\underline{2}}$, a general factorial polynomial of degree 2. We are looking for $x^2 = a + bx + cx^{\underline{2}}$. Substituting $x = 0$ in this equality one gets $a = 0$. Substituting $x = 1$, one gets $1 = b$, and finally for $x = 2$ one has $4 = 2 + 2c$. Hence $c = 1$. As result $x^2 = x + x^{\underline{2}}$. And the telescoping function is given by $\frac{x^{\underline{2}}}{2} + \frac{x^{\underline{3}}}{3} = \frac{1}{2}(x^2 - x) + \frac{1}{3}(x(x^2 - 3x + 2)) = \frac{1}{6}(2x^3 - 3x^2 + x)$. And we have once again proved the formula (1.5.4).

Stirling Estimation of the Euler series. We will expand $\frac{1}{(1+x)^2}$ into a series of negative factorial powers in order to telescope it. A natural first approximation to $\frac{1}{(1+x^2)}$ is $x^{\underline{-2}} = \frac{1}{(x+1)(x+2)}$. We represent $\frac{1}{(1+x)^2}$ as $x^{\underline{-2}} + R_1(x)$, where $R_1(x) = \frac{1}{(1+x)^2} - x^{\underline{-2}} = \frac{1}{(x+1)^2(x+2)}$. The remainder $R_1(x)$ is in a natural way approximated by $x^{\underline{-3}}$. If $R_1(x) = x^{\underline{-3}} + R_2(x)$ then $R_2(x) = \frac{2}{(x+1)^2(x+2)(x+3)}$. Further, $R_2(x) = 2x^{\underline{-4}} + R_3(x)$, where $R_3(x) = \frac{2\cdot 3}{(x+1)^2(x+2)(x+3)(x+4)} = \frac{3!x^{\underline{-4}}}{x+1}$. The above calculations led to conjecture

$$(1.5.7) \qquad \frac{1}{(1+x)^2} = \sum_{k=0}^{n-1} k!x^{\underline{-k-2}} + \frac{n!x^{\underline{-n-1}}}{x+1}.$$

This conjecture is easily proved by induction. The remainder $R_n(x) = \frac{n!x^{\underline{-n-1}}}{x+1}$ represents the difference $\frac{1}{(1+x)^2} - \sum\limits_{k=0}^{n-1} k!x^{\underline{-2-k}}$. Owing to the inequality $x^{\underline{-1-n}} \leq \frac{1}{(n+1)!}$, which is valid for all $x \geq 0$, the remainder decreases to 0 as n increases to infinity. This implies

Theorem 1. *For all $x \geq 0$ one has*

$$(1.5.8) \qquad \frac{1}{(1+x)^2} = \sum_{k=0}^{\infty} k!x^{\underline{-2-k}}$$

To calculate $\sum\limits_{k=p}^{\infty} \frac{1}{(1+k)^2}$, replace all summands by the expressions (1.5.7). We will get $\sum\limits_{k=p}^{\infty}\left(\sum\limits_{m=0}^{n-1} m!k^{\underline{-2-m}} + \frac{n!k^{\underline{-1-n}}}{k+1}\right)$. Changing the order of summation we have $\sum\limits_{m=0}^{n-1} m! \sum\limits_{k=p}^{\infty} k^{\underline{-2-m}} + \sum\limits_{k=p}^{\infty} \frac{n!k^{\underline{-1-n}}}{k+1}$. Since $\frac{x^{\underline{-1-m}}}{1+m}$ telescopes the sequence $\{k^{\underline{-2-m}}\}$, we have $\sum\limits_{k=p}^{\infty} k^{\underline{-2-m}} = \frac{p^{\underline{-1-m}}}{1+m}$. Denote the sum of remainders $\sum\limits_{k=p}^{\infty} \frac{n!k^{\underline{-1-n}}}{k+1}$ by $R(n,p)$. Then for all natural p and n one has

$$(1.5.9) \qquad \sum_{k=p}^{\infty} \frac{1}{(1+k)^2} = \sum_{m=0}^{n-1} m!\frac{p^{\underline{-1-m}}}{1+m} + R(n,p)$$

For $p=0$ and $n=+\infty$, the right-hand side turns into the Euler series, and one could get a false impression that we get nothing new. But $k^{\underline{-2-n}} \leq \frac{k^{\underline{-1-n}}}{k+1} \leq (k-1)^{\underline{-2-n}}$, hence

$$\frac{n!p^{\underline{-1-n}}}{1+n} = \sum_{k=p}^{\infty} n!k^{\underline{-2-n}} \leq R(n,p) \leq \sum_{k=p}^{\infty} n!(k-1)^{\underline{-2-n}} = \frac{n!(p-1)^{\underline{-1-n}}}{1+n}.$$

Since $(p-1)^{\underline{-1-n}} - p^{\underline{-1-n}} = (1+n)(p-1)^{\underline{-2-n}}$, there is a $\theta \in (0,1)$ such that

$$R(n,p) = \frac{n!p^{\underline{-1-n}}}{1+n} + \theta n!(p-1)^{\underline{-2-n}}.$$

Finally we get:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=0}^{p-1} \frac{1}{(1+k)^2} + \sum_{k=0}^{n-1} k! \frac{p^{\underline{-1-k}}}{1+k} + \theta n!(p-1)^{\underline{-2-n}} \tag{1.5.10}$$

For $p=n=3$ this formula turns into

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{4} + \frac{1}{40} + \frac{1}{180} + \frac{\theta}{420}$$

For $p=n=10$ one gets $R(10,10) \leq 10!9^{\underline{-12}}$. After cancellations one has $\frac{1}{2\cdot 11\cdot 12\cdot 13\cdot 14\cdot 15\cdot 17\cdot 19}$. This is approximately $2\cdot 10^{-8}$. Therefore $\sum_{k=0}^{10-1} \frac{1}{(k+1)^2} + \sum_{k=0}^{10-1} \frac{k!10^{\underline{-1-k}}}{1+k}$ is inferior than the sum of the Euler series by only $2\cdot 10^{-8}$. In such a way one can in one hour calculate eight digits of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ after the decimal point. It is not a bad result, but it is still far from Euler's 18 digits. For $p=10$ to provide 18 digits one has to sum essentially more than 100 terms of the series. This is a bit too much for a person, but is possible for a computer.

Problems.

1. Telescope $\sum k^3$
2. Represent x^4 as a factorial polynomial.
3. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+2)}$
4. Evaluate $\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)}$
5. If $\Delta a_k \geq \Delta b_k$ for all k and $a_1 \geq b_1$ then $a_k \geq b_k$ for all k
6. $\Delta(x+a)^{\underline{n}} = n(x+a)^{\underline{n-1}}$
7. Prove Archimedes's inequality $\frac{n^3}{3} \leq \sum_{k=1}^{n-1} k^2 \leq \frac{(n+1)^3}{3}$

8. Telescope $\sum\limits_{k=1}^{\infty} \frac{k}{2^k}$

9. Prove the inequalities $\frac{1}{n} \geq \sum\limits_{k=n+1}^{\infty} \frac{1}{k^2} \geq \frac{1}{n+1}$

10. Prove that the degree of $\Delta P(x)$ is less than the degree of $P(x)$ for any polynomial $P(x)$.

11. Relying on $\Delta 2^n = 2^n$, prove that $P(n) < 2^n$ eventually for any polynomial $P(x)$.

12. Prove $\sum\limits_{k=0}^{\infty} k!(x-1)^{\underline{-1-k}} = \frac{1}{x}$