2.3 Stieltjes Integral

On the contents of the lecture. The Stieltjes relativization of integral makes the integral flexible. We learn the main transformations of integrals. They allow us to evaluate a lot of integrals.

Basic rules. *Parametric curve* is a mapping of an interval into the plane. In cartesian coordinates a parametric curve can be presented as a pair of functions $x(t), y(t)$. The first function $x(t)$ represents the value of abscises at the moment t, and the second $y(t)$ is ordinate at the same moment. We define the integral $\int_a^b f(t)\,dg(t)$ for a nonnegative function f called *integrand* and with respect to a nondecreasing *continuous* function g, called *differand* as the area below the curve $f(t), g(t) \mid t \in [a,b]$.

A monotone function f is called *continuous* over interval $[a,b]$ if it takes all intermediate values, that is the image $f[a,b]$ of $[a,b]$ coincides with $[f(a), f(b)]$. If it is not continuous for some $y \in [f(a), f(b)] \setminus f[a,b]$, there is a point $x(y) \in [a,b]$ with the following property: $f(x) < y$ if $x < x(y)$ and $f(x) > y$ if $x > x(y)$. Let us define a *generalized preimage* $f^{[-1]}(y)$ of a point $y \in [f(a), f(b)]$ either as its usual preimage $f^{-1}(y)$ if it is not empty, or as $x(y)$ in the opposite case.

Now the curvilinear trapezium below the curve $f(t), g(t)$ over $[a,b]$ is defined as $\{(x,y) \mid 0 \le y \le g(f^{[-1]}(x))\}$.

The basic rules for relative integrals transforms into

Rule of constant	$\int_a^b f(t)\,dg(t) = c(g(b) - g(a))$,	if $f(t) = c$	for $t \in (a,b)$
Rule of inequality	$\int_a^b f_1(t)\,dg(t) \le \int_a^b f_2(t)\,dg(t)$,	if $f_1(t) \le f_2(t)$	for $t \in (a,b)$
Rule of partition	$\int_a^c f(t)\,dg(t) = \int_a^b f(t)\,dg(t) + \int_b^c f(t)\,dg(t)$		for $b \in (a,c)$

Addition theorem. The proof of others properties of integral is based on the piece-wise constant functions. For any number x, let us define its ε-integral part as $\varepsilon[x/\varepsilon]$. Immediately from the definition one gets.

Lemma 1. *For any monotone nonnegative function f on the interval $[a,b]$ and for any $\varepsilon > 0$, the function $[f]_\varepsilon$ is piecewise constant such that $[f(x)]_\varepsilon \le f(x) \le [f(x)]_\varepsilon + \varepsilon$ for all x.*

Theorem 2 (on multiplication). *For any nonnegative monotone f, and continuous nondecreasing g and any positive constant c one has*

$$\int_a^b cf(x)\,dg(x) = c\int_a^b f(x)\,dg(x) = \int_a^b f(x)\,dcg(x) \tag{2.3.1}$$

Proof. For the piecewise constant $f_\varepsilon = [f]_\varepsilon$, the proof is by a direct calculation. Hence

$$\int_a^b cf_\varepsilon(x)\,dg(x) = c\int_a^b f_\varepsilon(x)\,dg(x) = \int_a^b f_\varepsilon(x)\,dcg(x) = I_\varepsilon. \tag{2.3.2}$$

Now let us estimate the differences between integrals from (2.3.1) and their approximations from (2.3.2). For example, for the right-hand side integrals one has:

$$\int_a^b f\,dcg - \int_a^b f_\varepsilon\,dcg = \int_a^b (f - f_\varepsilon)\,dcg \le \int_a^b \varepsilon\,dcg = \varepsilon(cg(b) - cg(a)). \tag{2.3.3}$$

Hence $\int_a^b f\,dcg = I_\varepsilon + \varepsilon_1$, where $\varepsilon_1 \le c\varepsilon(g(b) - g(a))$. The same argument proves $c\int_a^b f\,dg = I_\varepsilon + \varepsilon_2$ and $\int_a^b cf\,dg = I_\varepsilon + \varepsilon_3$, where $\varepsilon_2, \varepsilon_3 \le c\varepsilon(g(b) - g(a))$. Then the pairwise differences between integrals of (2.3.1) do not exceed $2c\varepsilon(g(b) - g(a))$. Consequently they are less than any positive number, that is, they are zeros. □

Theorem 3 (Addition Theorem). *Let f_1, f_2 be nonnegative monotone functions and g_1, g_2 be a nondecreasing continuous functions over $[a, b]$, then*

$$\int_a^b (f_1(t) + f_2(t))\,dg_1(t) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_2(t)\,dg_1(t) \tag{2.3.4}$$

$$\int_a^b f_1(t)\,d(g_1(t) + g_2(t)) = \int_a^b f_1(t)\,dg_1(t) + \int_a^b f_1(t)\,dg_2(t) \tag{2.3.5}$$

Proof. For piecewise constant integrands the both equalities follows from the Rule of Constant and the Rule of Partition. To prove (2.3.4) replace f_1 and f_2 in both parts by $[f_1]_\varepsilon$ and $[f_2]_\varepsilon$. We get equality and denote by I_ε the common value of both sides of this equality. Then by (2.3.3) both integral on the right-hand side differs from they approximation at most by $\varepsilon(g_1(b) - g_1(a))$, therefore the right-hand side of (2.3.4) differs from I_ε at most by $2\varepsilon(g_1(b) - g_1(a))$. The same is true for the left-hand side of (2.3.4). This follows immediately from (2.3.3) in case $f = f_1 + f_2$, $f_\varepsilon = [f_1]_\varepsilon + [f_2]_\varepsilon$ and $g = g_1$. Consequently, the difference between left-hand and right-hand sides of the (2.3.4) does not exceed $4\varepsilon(g_1(b) - g_1(a))$. As ε can be chosen arbitrarily small this difference has to be zero.

The proof of (2.3.5) is even simpler. Denote by I_ε the common value of both parts of (2.3.5) where f_1 is changed by $[f_1]_\varepsilon$. By (2.3.3) one can estimate the differences between integrals of (2.3.5) and their approximations as being $\le \varepsilon(g_1(b) + g_2(b) - g_1(a) - g_2(a))$ for the left-hand side, and as $\le \varepsilon(g_1(b) - g_1(a))$ and $\le \varepsilon(g_2(b) - g_2(a))$ for corresponding integrals of the right-hand side of (2.3.5). So both sides differs from I_ε at most on $\le \varepsilon(g_1(b) - g_1(a) + g_2(b) - g_2(a))$. Hence they difference vanishes. □

Differential forms An expression of the type $f_1dg_1 + f_2dg_2 + \cdots + f_ndg_n$ is called a *differential form.* One can add differential forms and multiply it by functions. The integral of a differential form $\int_a^b (f_1\,dg_1 + f_2dg_2 + \cdots + f_ndg_n)$

is defined as sum of integrals $\sum\limits_{k=1}^{n}\int\limits_a^b f_k\,dg_k$. Two differential forms are called equivalent on the interval $[a,b]$ if their integrals are equal for all subintervals of $[a,b]$. For the sake of brevity we denote the differential form $f_1dg_1 + f_2dg_2 + \dots + f_ndg_n$ by FdG, where $F = \{f_1,\dots,f_n\}$ is a collection of integrands and $G = \{g_1,\dots,g_n\}$ is a collection of differands.

Theorem 4 (on multiplication). *Let FdG and $F'dG'$ be two differential forms with positive increasing integrand and continuous increasing differands, which are equivalent on $[a,b]$. Then their products by any increasing function f on $[a,b]$ are equivalent on $[a,b]$ too.*

Proof. If f is constant then the statement follows from the multiplication formula. If f is piece-wise constant, then divide $[a,b]$ into intervals where it is constant and prove the equality for parts and after collect the results by the Partition Rule. In the general case,

$$0 \le \int\limits_a^b fF\,dG - \int\limits_a^b [f]_\varepsilon F\,dG \le \int\limits_a^b \varepsilon F\,dG = \varepsilon\int\limits_a^b F\,dG.$$

Since $\int\limits_a^b [f]_\varepsilon F'\,dG' = \int\limits_a^b [f]_\varepsilon F\,dG$, one concludes that

$$\left|\int\limits_a^b fF'\,dG' - \int\limits_a^b fF\,dG\right| \le \varepsilon\int\limits_a^b F\,dG + \varepsilon\int\limits_a^b F'\,dG'$$

The right-hand side of this inequality can be made arbitrarily small. Hence the left-hand side is 0. □

Integration by parts.

Theorem 5. *If f and g are continuous nondecreasing nonnegative functions on $[a,b]$ then $d(fg)$ is equivalent to $fdg + gdf$.*

Proof. Consider $[c,d] \subset [a,b]$. The integral $\int\limits_c^d f\,dg$ represents the area below the curve $(f(t),g(t))_{t\in[c,d]}$. And the integral $\int\limits_c^d g\,df$ represents the area on the left of the same curve. Its union is equal to $[0,f(d)]\times[0,g(d)]\setminus[0,f(c)]\times[0,g(c)]$. The area of this union is equal to $(f(d)g(d) - f(c)g(c) = \int\limits_c^d dfg$. On the other hand the area of this union is the sum of the areas of curvilinear trapezia representing the integrals $\int\limits_c^d f\,dg$ and $\int\limits_c^d g\,df$. □

Change of variable Consider a Stieltjes integral $\int\limits_a^b f(\tau)\,dg(\tau)$ and suppose there is a continuous nondecreasing mapping $\tau\colon [t_0,t_1] \to [a,b]$, such that

$\tau(t_0) = a$ and $\tau(t_1) = b$. The composition $g(\tau(t))$ is a continuous nondecreasing function and the curve $\{(f(\tau(t), g(\tau(t))) \mid t \in [t_0, t_1]\}$ just coincides with the curve $\{(f(\tau), g(\tau)) \mid \tau \in [a, b]$. Hence, the following equality holds; it is known as the Formula of Change of Variable:

$$\int_{t_0}^{t_1} f(\tau(t))\, dg(\tau(t)) = \int_{\tau(t_0)}^{\tau(t_1)} f(\tau)\, dg(\tau) \tag{2.3.6}$$

For differentials this means that the equality $F(x)dG(x) = F'(x)dG'(x)$ conserves if one substitutes instead of an independent variable x a function.

Differential Transformations

dx^n. Integration by parts for $f(t) = g(t) = t$ gives $dt^2 = tdt + tdt$. Hence $tdt = \frac{t^2}{2}$. If we already know that $dx^a = adx^{a-1}$, then $dx^{a+1} = d(xx^a) = xdx^a + x^a dx = axx^{a-1}dx + x^a dx = (a+1)x^a dx$. This proves the Fermat Theorem for natural a.

$d\sqrt[n]{x}$. To evaluate $d\sqrt[n]{x}$ substitute $x = y^n$ into equality $dy^n = ny^{n-1}dy$. One gets $dx = \frac{nx\,d\sqrt[n]{x}}{\sqrt[n]{x}}$, hence $d\sqrt[n]{x} = \frac{1}{n}\frac{\sqrt[n]{x}dx}{x}$.

$\ln x dx$. We know $d\ln x = \frac{dx}{x}$. Integration by parts gives $\ln x dx = d(x\ln x) - xd\ln x = d(x\ln x) - dx = d(x\ln x - x)$.

Problems.

1. $dx^{2/3}$
2. dx^{-1}
3. $x\ln x\, dx$
4. $d\ln^2 x$
5. $\ln^2 x\, dx$
6. de^x
7. Investigate the convergence of $\sum\limits_{k=1}^{\infty} \frac{1}{k\ln k}$.