## 2.4 Asymptotics of sums

**On the contents of the lecture.** We become at last acquainted with the fundamental concept of *limit*. We extend the notion of sum of series and discover that change of the order of summands can affect the ultimate sum. Finally we derive the famous Stirling formula for $n!$.

**Asymptotic formulas.** Mercator series shows how useful series can be for evaluating of integrals. In this lecture we will use integrals to evaluate both partial and ultimate sums of series. Rarely one has an explicit formula for partial sums of a series. There are lots of important cases where such a formula does not exist. For example, it is known that partial sums of the Euler series cannot be expressed as a finite combination of elementary functions. When an explicit formula is not available, one tries to find a so-called *asymptotic formulas*. An asymptotic formula for a partial sum $S_n$ of a series is a formula of the type $S_n = f(n) + R(n)$ where $f$ is a known function called *principal part* and $R(n)$ is a *reminder*, which is small, in some sense, with respect to the principal part. Today we will get an asymptotic formula for partial sums of the harmonic series.

**Infinitesimally small sequences.** The simplest asymptotic formula has a constant as its principal part and an infinitesimally small remainder. One says that a sequence $\{z_k\}$ is *infinitesimally small* and writes $\lim z_k = 0$, if $z_k$ tends to 0 as $n$ tends to infinity. That is for any positive $\varepsilon$ eventually (i.e. beginning with some $n$) holds $|z_k| < \varepsilon$. With Iverson notation, this definition can be expressed in the following clear form:

$$[\{z_k\}_{k=1}^{\infty} \text{ is infinitesimally small }] = \prod_{m=1}^{\infty} 2\left|\sum_{n=1}^{\infty}(-1)^n \prod_{k=1}^{\infty}[m[k>n]|z_k| < 1]\right|$$

Three basic properties of infinitesimally small sequences immediately follow from the definition:

- if $\lim a_k = \lim b_k = 0$ then $\lim(a_k + b_k) = 0$
- if $\lim a_k = 0$ then $\lim a_k b_k = 0$ for any bounded sequence $\{b_k\}$
- if $a_k \leq b_k \leq c_k$ for all $k$ and $\lim a_k = \lim c_k = 0$, then $\lim b_k = 0$

The third property is called *squeeze rule.*

Today we needs just one properties of infinitesimally small, that is

**Theorem 1 (Addition theorem).** *If sequences $\{a_k\}$ and $\{b_k\}$ are infinitesimally small, than its sum and its difference are infinitesimally small too.*

*Proof.* Let $\varepsilon$ be a positive number. Then $\varepsilon/2$ also is positive number. And by definition of infinitesimally small, the inequalities $|a_k| < \varepsilon/2$ and $|b_k| < \varepsilon/2$ hold eventually beginning with some $n$. Then for $k > n$ one has $|a_k \pm b_k| \leq |a_k| + |b_k| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. $\square$

**Limit of sequence**

**Definition.** *A sequence* $\{z_k\}$ *of (complex) numbers converges to a number* $z$ *if* $\lim z - z_k = 0$. *The number* $z$ *is called limit of the sequence* $\{z_k\}$ *and denoted by* $\lim z_k$.

The infinite sum represents a particular case of a limit as demonstrate the following.

**Theorem 2.** *The partial sums of an absolute convergent series* $\sum\limits_{k=1}^{\infty} z_k$ *converges to its sum.*

*Proof.* $|\sum\limits_{k=1}^{n-1} z_k - \sum\limits_{k=1}^{\infty} z_k| = |\sum\limits_{k=n}^{\infty} z_k| \leq \sum\limits_{k=n}^{\infty} |z_k|$. Since $\sum\limits_{k=1}^{\infty} |z_k| > \sum\limits_{k=1}^{\infty} |z_k| - \varepsilon$, there is a partial sum such that $\sum\limits_{k=1}^{n-1} |z_k| > \sum\limits_{k=1}^{\infty} |z_k| - \varepsilon$. Then for all $m \geq n$ one has $\sum\limits_{k=m}^{\infty} |z_k| \leq \sum\limits_{k=n}^{\infty} |z_k| < \varepsilon$. □

**Conditional convergence.** The concept of limit of sequence leads to the notion of convergence generalizing the absolute convergence.

Series $\sum\limits_{k=1}^{\infty} a_k$ is called (conditionally) *convergent* if $\lim \sum\limits_{k=1}^{n} a_k = A + \alpha_n$, where $\lim \alpha_n = 0$. The number $A$ is called its ultimate sum.

The following theorem gives a lot of examples of conditionally convergent series, which are not absolutely convergent. By $[[n]]$ we denote the even part of the number $n$ i.e. $[[n]] = 2[n/2]$

**Theorem 3 (Leibniz).** *For any of positive decreasing infinitesimally small sequence* $\{a_n\}$, *the series* $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k$ *converges.*

*Proof.* Denote the difference $a_k - a_{k+1}$ by $\Delta a_k$. The series $\sum\limits_{k=1}^{\infty} \Delta a_{2k-1}$ and $\sum\limits_{k=1}^{\infty} \Delta a_{2k}$ are positive and convergent, because their term-wise sum is $\sum\limits_{k=1}^{\infty} \Delta a_k = a_1$. Hence $S = \sum\limits_{k=1}^{\infty} \Delta a_{2k-1} \leq a_1$. Denote by $S_n$ the partial sum $\sum\limits_{k=1}^{n-1} (-1)^{k+1} a_k$. Then $S_{2n} = \sum\limits_{k=1}^{n-1} \Delta a_{2n-1} = S + \alpha_n$, where $\lim \alpha_n = 0$. Then $S_n = S_{[[n]]} + a_n[n \text{ is odd}] + \alpha_{[[n]]}$. As $a_n[n \text{ is odd}] + \alpha_{[[n]]}$ is infinitesimally small, this implies the theorem. □

**Lemma 4.** *Let* $f$ *be a non-increasing nonnegative function. Then the series* $\sum\limits_{k=1}^{\infty} (f(k) - \int\limits_{k}^{k+1} f(x)\,dx)$ *is positive and convergent and has sum* $c_f \leq f(1)$.

*Proof.* The integration of inequalities $f(k) \geq f(x) \geq f(k+1)$ over $[k, k+1]$ gives $f(k) \geq \int\limits_{k}^{k+1} f(x)\,dx \geq f(n+1)$. This proves the positivity of the series and allows majorize it by the telescopic series $\sum\limits_{k=1}^{\infty} (f(k) - f(k+1)) = f(1)$. □

**Theorem 5 (integral test on convergence).** *If a nonnegative function* $f(x)$ *decreases monotonically on* $[1,+\infty)$*, then* $\sum_{k=1}^{\infty} f(k)$ *converges iff* $\int_1^{\infty} f(x)\,dx < \infty$.

*Proof.* Since $\int_1^{\infty} f(x)\,dx = \sum_{k=1}^{\infty} \int_k^{k+1} f(x)\,dx$, one has $\sum_{k=1}^{\infty} f(k) = c_f + \int_1^{\infty} f(x)\,dx$. □

**Euler constant.** The sum $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \ln(1+\frac{1}{k})\right)$, which is $c_f$ for $f(x) = \frac{1}{x}$, is called *Euler's constant* and denoted by $\gamma$. Its first ten digits are 0.5772156649....

**Harmonic numbers** The sum $\sum_{k=1}^{n} \frac{1}{k}$ is denoted $H_n$ and is called the $n$-th *harmonic number.*

**Theorem 6.** $H_n = \ln n + \gamma + o_n$ *where* $\lim o_n = 0$

*Proof.* Since $\ln n = \sum_{k=1}^{n-1}(\ln(k+1) - \ln k) = \sum_{k=1}^{n-1} \ln(1+\frac{1}{k})$, one has $\ln n + \sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1+\frac{1}{k})\right) = H_{n-1}$. But $\sum_{k=1}^{n-1} \left(\frac{1}{k} - \ln(1+\frac{1}{k})\right) = \gamma + \alpha_n$, where $\lim \alpha_n = 0$. Therefore $H_n = \ln n + \gamma + (\frac{1}{n} + \alpha_n)$. □

**Alternating harmonic series** The *alternating harmonic series* $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is a conditionally convergent series due to Leibniz's theorem, and it is not absolutely convergent. To find its sum we apply our Theorem **6** on the asymptotics of harmonic numbers.

Denote by $S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ the partial sum. Then $S_n = H'_n - H''_n$, where $H'_n = \sum_{k=1}^{n} \frac{1}{k}[k \text{ is odd}]$ and $H''_n = \sum_{k=1}^{n} \frac{1}{k}[k \text{ is even}]$. Since $H''_{2n} = \frac{1}{2}H_n$ and $H'_{2n} = H_{2n} - H''_{2n} = H_{2n} - \frac{1}{2}H_n$ one gets

$$\begin{aligned}(2.4.1)\quad S_{2n} = H_{2n} - \frac{1}{2}H_n - \frac{1}{2}H_n = H_{2n} - H_n = \\ \ln 2n + \gamma + o_{2n} - \ln n - \gamma - o_n = \ln 2 + (o_{2n} - o_n).\end{aligned}$$

Consequently $S_n = \ln 2 + (o_{[[n]]} - o_{[n/2]} + \frac{(-1)^{n+1}}{n}[n \text{ is odd}])$. As the sum in brackets is infinitesimally small, one gets

$$\sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2.$$

The same arguments for a permutated alternating harmonic series give

$$(2.4.2)\qquad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \cdots = \frac{3}{2}\log 2.$$

Indeed, in this case its $3n$-th partial sum is $S_{3n} = H'_{4n} - H''_{2n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n = \log 4n + \gamma + o_{4n} - \frac{1}{2}(\log 2n + \gamma + o_{2n} + \log n + \gamma + o_n) = \log 4 - \frac{1}{2}\log 2 + o'_n = \frac{3}{2}\log 2 + o'_n$, where $\lim o'_n = 0$. Since the difference between $S_n$ and $S_{3m}$ where $m = [n/3]$ is infinitesimally small, this proves (2.4.2).

**Stirling's Formula.** We will try to estimate $\ln n!$. The integration of inequalities $\ln[x] \le \ln x \le \ln[x+1]$ over $[1,n]$ gives $\ln(n-1)! \le \int_1^n \ln x\,dx \le \ln n!$. Let us estimate the difference $D$ between $\int_1^n \ln x\,dx$ and $\frac{1}{2}(\ln n! + \ln(n-1)!)$.

(2.4.3)

$$D = \int_1^n (\ln x - \frac{1}{2}(\ln[x] + \ln[x+1]))\,dx = \sum_{k=1}^{n-1} \int_0^1 \left(\ln(k+x) - \ln\sqrt{k(k+1)}\right)\,dx.$$

To prove that all summands on the left-hand side are nonnegative, we apply the following general lemma.

**Lemma 7.** $\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$ *for any function*

*Proof.* The reflection of the plane across the line $y = \frac{1}{2}$ transforms the curvilinear trapezium of $f(x)$ over $[0,1]$ into curvilinear trapezium of $f(1-x)$ over $[0,1]$. $\square$

**Lemma 8.** $\int_0^1 \ln(k+x)\,dx \ge \ln\sqrt{k(k+1)}$

*Proof.* Due to Lemma **7** one has

$$\int_0^1 \ln(k+x)\,dx = \int_0^1 \ln(k+1-x)\,dx = \int_0^1 \frac{1}{2}(\ln(k+x) + \ln(k+1-x))\,dx =$$

$$= \int_0^1 \ln\sqrt{(k+x)(k+1-x)}\,dx = \int_0^1 \ln\sqrt{k(k+1)+x-x^2}\,dx \ge$$

$$\int_0^1 \ln\sqrt{k(k+1)}\,dx = \ln\sqrt{k(k+1)}$$

$\square$

Integration of the inequality $\ln(1+x/k) \le x/k$ over $[0,1]$ gives

$$\int_0^1 \ln(1+x/k)\,dx \le \int_0^1 \frac{x}{k}\,dx = \frac{1}{2k}.$$

This estimate together with the inequality $\ln(1+1/k) \ge 1/(k+1)$ allow us to estimate the summands from the right-hand side of (2.4.3) in the following way:

(2.4.4)

$$\int_0^1 \ln(k+x) - \ln\sqrt{k(k+1)}\,dx = \int_0^1 \ln(k+x) - \ln k - \frac{\ln(k+1) - \ln k}{2}\,dx =$$

$$= \int_0^1 \ln\left(1+\frac{x}{k}\right) - \frac{1}{2}\ln\left(1+\frac{1}{k}\right)\,dx \le \frac{1}{2k} - \frac{1}{2(k+1)}$$

We see that $D_n \le \sum\limits_{k=1}^{\infty} \frac{1}{2k} - \frac{1}{2(k+1)} = \frac{1}{2}$ for all $n$. Denote by $D_\infty$ the sum (2.4.3) for infinite $n$. Then $R_n = D_\infty - D_n = \frac{\theta}{2n}$ for some nonnegative $\theta < 1$, and we get

$$(2.4.5)\quad D_\infty - \frac{\theta}{2n} = \int_1^n \ln x\, dx - \frac{1}{2}(\ln n! + \ln(n-1)!) = \int_1^n \ln x\, dx - \ln n! + \frac{\ln n}{2}$$

Substituting in (2.4.5) the value of the integral $\int\limits_1^n \ln x\, dx = \int\limits_1^n d(x\ln x - x) = (n\ln n - n) - (1\ln 1 - 1) = n\ln n - n + 1$, one gets

$$(2.4.6)\qquad \ln n! = n\ln n - n + \frac{\ln n}{2} + (1 - D_\infty) + \frac{\theta}{2n}$$

Now we know that $1 \ge (1 - D_\infty) \ge \frac{1}{2}$, but it is possible to evaluate the value of $D_\infty$ with more accuracy. Later we will prove that $1 - D_\infty = \sqrt{2\pi}$.

**Problems.**

1. Does $\sum\limits_{k=1}^{\infty} \sin k$ converge?
2. Does $\sum\limits_{k=1}^{\infty} \sin k^2$ converge?
3. Evaluate $1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots - \frac{2}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} \cdots$
4. If $\lim \frac{a_{n+1}}{a_n} < 1$, then $\sum\limits_{k=1}^{\infty} a_k$ converge.
5. If $\sum\limits_{k=1}^{\infty} |a_k - a_{k-1}| < \infty$, then $\{a_k\}$ converges.
6. Prove convergence $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{[\sqrt{k}]}}{k}$
7. Convergence $\sum\limits_{k=1}^{\infty} \frac{1}{\ln^3 k}$
8. Convergence $\sum\limits_{k=1}^{\infty} \frac{1}{k\ln k\sqrt{\ln\ln k}}$
9. Convergence $\sum\limits_{k=1}^{\infty} \frac{1}{k\ln k(\ln\ln k)^2}$
10. Convergence $\sum\limits_{k=1}^{\infty} \frac{1}{k\ln k}$ and asymptotic formula?
11. Convergence $\sum\limits_{k=1}^{\infty} \frac{1}{k\ln^2 k}$

12. Which partial sum of above series is 0.01 close to its ultimate sum?

13. Evaluate $\sum\limits_{k=1}^{\infty} \frac{1}{k \ln^2 k}$ with precision 0.01

14. Evaluate $\int\limits_{1}^{3} \ln x \, d[x]$

15. Express the Stirling constant via the Wallis product $\frac{\pi}{2} = \prod\limits_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1}$