2.6 Virtually monotone functions

Monotonization of integrand. Let us say that a pair of functions f_1, f_2 *monotonize* a function f, if f_1 is a non-negative and non-decreasing, f_2 is a non-positive and non-increasing and $f = f_1 + f_2$.

Lemma 1. *Let $f = f_1 + f_2$ and $f = f_1' + f_2'$ be two monotonizations of f. Then for any monotone h one has $f_1\,dh + f_2\,dh = f_1' dh + f_2' dh$.*

Proof. Our equality is equivalent to $f_1\,dh - f_2'\,dh = f_1'\,dh - f_2\,dh$. By the sign rule this turns into $f_1\,dh + (-f_2')\,dh = f_1'\,dh + (-f_2)\,dh$. Now all integrands are nonnegative and for non-decreasing h we can apply the Addition Theorem and transform the inequality into $(f_1 - f_2')\,dh = (f_1' - f_2)\,dh$. This is true because $(f_1 - f_2') = (f_1' - f_2)$.

The case of non-increasing differand is reduced to the case of non-decreasing one by the transformation $f_1\,d(-h) + f_2\,d(-h) = f_1' d(-h) + f_2' d(-h)$, which is based on the Sign Rule.

□

A function which has a monotonization is called *virtually monotone.*

We define the integral $\int_a^b f\,dg$ for any virtually monotone integrand f and any continuous monotone differand g via a monotonization $f = f_1 + f_2$ by

$$\int_a^b f\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg. \tag{2.6.1}$$

Lemma **1** demonstrates that this definition does not depend on the choice of a monotonization.

Lemma 2. *Let f and g be virtually monotone functions; then $f+g$ is virtually monotone and $fdh + gdh = (f+g)dh$ for any continuous monotone h.*

Proof. Let h be nondecreasing. Consider monotonizations $f = f_1 + f_2$ and $g = g_1 + g_2$. Then $fdh + gdh = f_1dh + f_2dh + g_1dh + g_2dh$ by definition via monotonization of the integrand. By virtue of the Addition Theorem **3** this turns into $(f_1+g_1)dh + (f_2+g_2)dh$. But the pair of brackets monotonize $f+g$. Hence $f+g$ is proved to be virtually monotone and the later expression is $(f+g)dh$ by definition, via monotonization of integrand. The case of non-increasing h is reduced to the previous case via $-fd(-h) - gd(-h) = -(f+g)d(-h)$. □

Lemma on locally constant functions. Let us say that a function $f(x)$ is *locally constant* at a point x if $f(y) = f(x)$ for all y sufficiently close to x, i.e for all y from an interval $(x-\varepsilon, x+\varepsilon)$.

Lemma 3. *A function f which is locally constant at any point of an interval is constant.*

Proof. Suppose $f(x)$ is not constant on $[a,b]$. We will construct by induction a sequence of intervals $I_k = [a_k, b_k]$, such that $I_0 = [a,b]$, $I_{k+1} \subset I_k$, $|b_k - a_k| \geq 2|b_{k+1} - a_{k+1}|$ and the function f is not constant on each I_k. First step. Let $c = (a+b)/2$, as f is not constant $f(x) \neq f(c)$ for some x. Then choose $[x,c]$ or

$[c, x]$ as for $[a_1, b_1]$. On this interval f is not constant. The same are all further steps. The intersection of the sequence is a point such that any its neighborhood contains some interval of the sequence. Hence f is not locally constant at this point. $\square$

Lemma 4. *If $f(x)$ is a continuous monotone function and $a < f(x) < b$ then $a < f(y) < b$ for all y sufficiently close to x.*

Proof. If f takes values greater than b, than it takes value b and if $f(x)$ takes values less than a then it takes value a due to continuity. Then $[f^{-1}(a), f^{-1}(b)]$ is the interval where inequalities hold. $\square$

Lemma 5. *Let g_1, g_2 be continuous* comonotone *functions. Then $g_1 + g_2$ is continuous and monotone, and for any virtually monotone f one has*

$$fdg_1 + fdg_2 = fd(g_1 + g_2) \tag{2.6.2}$$

Proof. Suppose $g_1(x) + g_2(x) < p$, let $\varepsilon = p - g_1(x) - g_2(x)$. Then $g_1(y) < g_1(y) + \varepsilon/2$ and $g_2(y) < g_2(y) + \varepsilon/2$ for all y sufficiently close to x. Hence $g_(y) + g_2(y) < p$ for all y sufficiently close to x. The same is true for the opposite inequality. Hence $\operatorname{sgn}(g_1(x) + g_2(x) - p)$ is locally constant at all points where it is not 0. But it is not constant if p is an intermediate value, hence it is not locally constant, hence it takes value 0. At this point $g_1(x) + g_2(x) = p$ and the continuity of $g_1 + g_2$ is proved.

Consider a monotonization $f = f_1 + f_2$. Let g_i be nondecreasing. By definition via monotonization of the integrand, the left-hand side of (2.6.2) turns into $(f_1dg_1 + f_2dg_1) + (f_1dg_2 + f_2dg_2) = (f_1dg_1 + f_1dg_2) + (f_2dg_1 + f_2dg_2)$. By the Addition Theorem **3** $f_1dg_1 + f_1dg_2 = f_1d(g_1 + g_2)$. And the equality $f_2dg_1 + f_2dg_2 = f_2d(g_1 + g_2)$ follows from $(-f_2)dg_1 + (-f_2)dg_2 = (-f_2)d(g_1 + g_2)$ by the Sign Rule. Hence the left-hand side is equal to $f_1d(g_1 + g_2) + f_2d(g_1 + g_2)$, which coincides with the right-hand side of (2.6.2) by definition via monotonization of integrand. The case of non-increasing differands is taken care of via transformation of (2.6.2) by the Sign Rule into $fd(-g_1) + fd(-g_2) = fd(-g_1 - g_2)$. $\square$

Lemma 6. *Let $g_1 + g_2 = g_3 + g_4$ where all $(-1)^k g_k$ are non-increasing continuous functions. Then $fdg_1 + fdg_2 = fdg_3 + fdg_4$ for any virtually monotone f*

Proof. Our equality is equivalent to $fdg_1 - fdg_4 = fdg_3 - fdg_2$. By the Sign Rule it turns into $fdg_1 + fd(-g_4) = fdg_3 + fd(-g_2)$. Now all differands are nondecreasing and by Lemma **5** it transforms into $fd(g_1 - g_4) = fd(g_3 - g_2)$. This is true because $g_1 - g_4 = g_3 - g_2$. $\square$

Monotonization of differand. A monotonization by continuous functions is called continuous. Virtually monotone function which has a continuous monotonization is called *continuous.* The integral for any virtually monotone integrand f against a virtually monotone continuous differand g is defined via a continuous virtualization $g = g_1 + g_2$ of the differand

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2 \tag{2.6.3}$$

The integral is well-defined because of Lemma **6**.

Theorem 7 (Addition Theorem). *For any virtually monotone functions f, f' and any virtually monotone continuous g, g' hold $fdg + f'dg = (f + f')dg$ and $fdg + fdg' = fd(g + g')$*

Proof. To prove $fdg + f'dg = (f + f')dg$, consider a continuous monotonization $g = g_1 + g_2$. Then by definition of integral for virtually monotone differand this equality turns into $(fdg_1 + fdg_2) + (f'dg_1 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$. After rearranging it turns into $(fdg_1 + f'dg_1) + (fdg_2 + f'dg_2) = (f + f')dg_1 + (f + f')dg_2$ But this is true due to Lemma **2**.

To prove $fdg + fdg' = fd(g + g')$, consider monotonizations $g = g_1 + g_2$, $g' = g_1' + g_2'$. Then $(g_1 + g_1') + (g_2 + g_2')$ is monotonization for $g + g'$. And by the definition of integral for virtually monotone differand our equality turns into $fdg_1 + fdg_2 + fdg_1' + fdg_2'$ □

Change of variable.

Lemma 8. *If f is virtually monotone and g is monotone, then $f(g(x))$ is virtually monotone.*

Proof. Let $f_1 + f_2$ be a monotonization of f. If h is non-decreasing then $f_1(h(x)) + f_2(h(x))$ gives a monotonization of $f(g(x))$. If h is decreasing the monotonization is given by $(f_2(h(x))+c)+(f_1(h(x))-c)$ where c is a sufficiently large constant, to provide positivity of the first brackets and negativity of the second one. □

The following natural convection is applied to define the integral with reversed limits.

$$\int_a^b f(x)\,dg(x) = -\int_b^a f(x)\,dg(x)$$

Theorem 9 (on change of variable). *If $h\colon [a,b] \to [h(a), h(b)]$ is monotone, $f(x)$ is virtually monotone, and $g(x)$ is virtually monotone continuous, then $\int_a^b f(h(t))\,dg(h(t)) = \int_{h(a)}^{h(b)} f(x)\,dg(x)$*

Proof. Let $f = f_1 + f_2$ and $g = g_1 + g_2$ be a monotonization and a continuous monotonization of f and g respectively. The $\int_a^b f(h(t))\,dg(h(t))$ splits into sum of four integrals. $\int_a^b f_i(h(t))\,dg_j(h(t))$ where f_i are of constant sign and g_j are monotone continuous. This integrals coincide with corresponding integrals $\int_{h(a)}^{h(b)} f_i(x)\,dg_i(x)$. Indeed its absolute values are the area of the same curvilinear trapezium. And their signs determined by the Sign Rule are the same. □

Integration by parts. We have established the Integration by parts formula for non-negative and non-decreasing differential forms. Now we extend it to the case of continuous monotone forms. In the first case f and g are non-decreasing. In this case choose a positive constant c sufficiently large to provide positivity of $f+c$ and $g+c$ on the interval of integration. Then $d(f+c)(g+c) =$

$(f+c)d(g+c)+(g+c)d(f+c)$. On the other hand $d(f+c)(g+c) = dfg+cdf+cdg$ and $(f+c)d(g+c)+(g+c)d(f+c) = fdg+cdg+cdf$. Compare these results to get $dfg = fdg+gdf$. Now if f is increasing and g decrease the $-g$ increase and we get $-dfg = df(-g) = fd(-g)+(-g)df = -fdg-gdf$, which lead to $dfg = fdg+gdf$. The other cases: f decreasing, g increasing and both decreasing are proved by the same arguments. The extension of the Integration by Parts to piece-wise monotone forms immediately follows by the Partition Rule.

Variation. Define *variation of a sequence* of numbers $\{x_k\}_{k=1}^n$ as the sum $\sum_{k=1}^{\infty} |x_{k+1}-x_k|$. Define variation of a function f along a sequence $\{x_k\}_{k=0}^n$ as the variation of sequence $\{f(x_k)\}_{k=0}^n$. Define a *chain* on an interval $[a,b]$ as a nondecreasing sequence $\{x_k\}_{k=0}^n$ such that $x_0 = a$ and $x_n = b$. Define a *partial variation* of f on an interval $[a,b]$ as its variation along a chain on the interval.

The least number surpassing all partial variations function f over $[a,b]$ is called *the (ultimate) variation of a function* $f(x)$ on an interval $[a,b]$ and denote by $\operatorname{var}_f[a,b]$.

Lemma 10. *For any function f one has inequality* $\operatorname{var}_f[a,b] \geq |f(b)-f(a)|$. *If f is a monotone function on $[a,b]$, then* $\operatorname{var}_f[a,b] = |f(b)-f(a)|$

Proof. Inequality $\operatorname{var}_f[a,b] \geq |f(b)-f(a)|$ follows immediately from the definition because $\{a,b\}$ is a chain. For monotone f this all partial variations are telescopic sums equal to $|f(b)-f(a)|$ □

Theorem 11 (additivity of variation). $\operatorname{var}_f[a,b]+\operatorname{var}_f[b,c] = \operatorname{var}_f[a,c]$

Proof. Consider a chain $\{x_k\}_{k=0}^n$ of $[a,c]$, which contains b. In this case the variation of f along $\{x_k\}_{k=0}^n$ splits into sums of partial variations of f along $[a,b]$ and along $[b,c]$. As a partial variations does not exceed an ultimate. We get that in this case the variation of f along $\{x_k\}_{k=0}^n$ does not exceed $\operatorname{var}_f[a,b]+\operatorname{var}_f[b,c]$.

If $\{x_k\}_{k=0}^n$ does not contain b, let us add b to the chain. Then in the sum expressing the partial variation of f, summand $|f(x_{i+1}-f(x_i)|$ changes by sum $|f(b)-f(x_i)|+|f(x_{i+1}-f(b)|$ which is greater or equal. Hence the variation does not decrease after such modification. But the variation along the modified chain does not exceed $\operatorname{var}_f[a,b]+\operatorname{var}_f[b,c]$ as was proved above. As all partial variations of f over $[a,c]$ do not exceed $\operatorname{var}_f[a,b]+\operatorname{var}_f[b,c]$, the same is true for the ultimate variation.

To prove the opposite inequality we consider a *relaxed* inequality $\operatorname{var}_f[a,b]+\operatorname{var}_f[b,c] \leq \operatorname{var}_f[a,c]+\varepsilon$ where ε is an positive number. Choose chains $\{x_k\}_{k=0}^n$ on $[a,b]$ and $\{y_k\}_{k=0}^m$ on $[b,c]$ such that corresponding partial variation of f be $\geq \operatorname{var}_f[a,b]+\varepsilon/2$ and $\geq \operatorname{var}_f[b,c]+\varepsilon/2$ respectively. As the union of these chains is a chain on $[a,c]$ the sum of these partial variations is a partial variation of f on $[a,c]$. Consequently this sum is less or equal to $\operatorname{var}_f[a,c]$. On the other hand it is greater or equal to $\operatorname{var}_f[a,b]+\varepsilon/2+\operatorname{var}_f[b,c]+\varepsilon/2$. Comparing this results gives just the relaxed inequality. As relaxed inequality is proved for all $\varepsilon > 0$ it also holds for $\varepsilon = 0$. □

Lemma 12. *For any functions f, g one has inequality* $\operatorname{var}_{f+g}[a,b] \leq \operatorname{var}_f[a,b]+\operatorname{var}_g[a,b]$

Proof. Since $|f(x_{k+1})+g(x_{k+1})-f(x_k)-g(x_k)| \leq |f(x_{k+1})-f(x_k)|+|g(x_{k+1})-g(x_k)|$, the variation of $f+g$ along any sequence does not exceed the sum of the variations of f and g along the sequence. Hence all partial variations of $f+g$ do not exceed $\operatorname{var}_f[a,b] + \operatorname{var}_g[a,b]$, and so the same is true for the ultimate variation. □

Lemma 13. *For any function of finite variation on* $[a,b]$, *functions* $\operatorname{var}_f[a,x]$ *and* $\operatorname{var}_f[a,x] - f(x)$ *are both nondecreasing functions of* x.

Proof. The nondecreasing of $\operatorname{var}_f[a,x]$ follows from nonnegativity and additivity of variation. If $x > y$ then the inequality $\operatorname{var}_f[a,x] - f(x) \geq \operatorname{var}_f[a,y] - f(y)$ is equivalent to $\operatorname{var}_f[a,x] - \operatorname{var}_f[a,y] \geq f(x) - f(y)$. This is true because $\operatorname{var}_f[a,x] - \operatorname{var}_f[a,y] = \operatorname{var}_f[x,y] \geq |f(x) - f(y)|$. □

Lemma 14. $\operatorname{var}_{f^2}[a,b] \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b]$

Proof. For all $x,y \in [a,b]$ one has $|f(x) + f(y)| = |2f(a) + f(x) - f(a) + f(y) - f(a)| \leq 2|f(a)| + \operatorname{var}_f[a,x] + \operatorname{var}_f[a,y] \leq 2|f(a)| + 2\operatorname{var}_f[a,b]$. Hence $\sum\limits_{k=0}^{n-1} |f^2(x_{k+1}) - f^2(x_k)| = \sum\limits_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)||f(x_{k+1} + f(x_k)| \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \sum\limits_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| \leq 2(|f(a)| + \operatorname{var}_f[a,b]) \operatorname{var}_f[a,b]$ □

Lemma 15. *If* $\operatorname{var}_f[a,b] < \infty$ *and* $\operatorname{var}_g[a,b] < \infty$, *then* $\operatorname{var}_{fg}[a,b] < \infty$

Proof. $4fg = (f+g)^2 - (f-g)^2$ □

Theorem 16. *The function* f *is virtually monotone on* $[a,b]$ *if and only if it has a finite variation.*

Proof. Since monotone functions have finite variation on finite intervals, and the variation of a sum does not exceed the sum of variations one gets that all virtually monotone functions have finite variation. On the other hand if f has a finite variation then $f = (\operatorname{var}_f[a,x] + c) + (f(x) - \operatorname{var}_f[a,x] - c)$, the functions in the brackets are monotone due to Lemma **13**, and choosing constant c sufficiently large one obtains that the second bracket is negative. □

Problems.

1. Evaluate $\int\limits_1^i z^2\,dz$
2. Prove that $1/f(x)$ has finite variation if it is bounded.
3. Prove $\int\limits_a^b f(x)\,dg(x) \leq \max_{[a,b]} f \operatorname{var}_g[a,b]$